

ORDERS OF NIKSHYCH'S HOPF ALGEBRA

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ABSTRACT. Let p be an odd prime number and K a number field having a primitive p -th root of unity ζ_p . We prove that Nikshych's non group-theoretical Hopf algebra H_p , which is defined over $\mathbb{Q}(\zeta_p)$, admits a Hopf order over the ring of integers \mathcal{O}_K if and only if there is an ideal I of \mathcal{O}_K such that $I^{2(p-1)} = (p)$. This condition does not hold in a cyclotomic field. Hence this gives an example of a semisimple Hopf algebra over a number field not admitting a Hopf order over any cyclotomic ring of integers. Moreover, we show that, when a Hopf order over \mathcal{O}_K exists, it is unique and we describe it explicitly.

INTRODUCTION

Many results in the Representation Theory of Finite Groups exploit the fact that the complex group algebra $\mathbb{C}G$ of a finite group G is defined over the integers or, more generally, over the ring of integers \mathcal{O}_K of a number field K . In other terms, $\mathcal{O}_K G$ is an algebra order of $\mathbb{C}G$; indeed a Hopf (algebra) order. A prominent role is played by cyclotomic fields: for example, the celebrated Brauer's splitting field theorem states that any irreducible representation of KG can be realized in $K(\omega)$, with ω a primitive root of unity of order equals $\exp G$ (see [3, Theorem 15.16, Corollary 15.18]).

Kaplansky's sixth conjecture, still unsolved, is a generalization of Frobenius Theorem for groups. It asserts that in a complex semisimple Hopf algebra H the dimension of every irreducible representation of H divides the dimension of H . Larson gave a positive answer in [6] if H admits a Hopf order over a number ring. Motivated by this result, in [1] we addressed the question as to whether any complex semisimple Hopf algebra admits a Hopf order over a number ring. In the dimensions less than 36 in which the classification is complete (24 and 32 are still open) it turns out that all semisimple Hopf algebras are defined over cyclotomic rings of integers, see [1, Subsection 2.4] for an account. However, we exhibited in [1] an example in dimension 36 that does not admit a Hopf order over any number ring, although it satisfies the conjecture.

As a continuation of our previous work we investigate in this paper the problem of definability of semisimple Hopf algebras over cyclotomic ring of integers. Let H be a semisimple Hopf algebra over a number field K and suppose that H has a Hopf order over some number ring. Does H admit a Hopf order over a cyclotomic ring of integers contained in K ? Our main result gives a negative answer for the family of non group-theoretical semisimple Hopf algebras $\{H_p\}$, with p an odd prime, constructed by Nikshych (see [11]). The dimension of H_p is $4p^2$ (so in particular the dimension of H_3 is 36). These Hopf algebras were not constructed explicitly but

through a tensor category and a fiber functor. The representation category $\text{Rep}(H_p)$ was obtained by equivariantization by C_2 from $\text{Rep}(A_p)$, with A_p the Hopf algebra studied by Masuoka in [7]. Using Tannaka reconstruction, in Section 3 we describe H_p completely as follows:

Theorem 1. *Let $\zeta_p \in \mathbb{C}$ be a primitive p -th root of unity. The Hopf algebra H_p is generated, as an algebra over \mathbb{C} , by the elements $e_0, e_1, u_a, u_b, v_a, v_b$ and g subject to the following relations:*

$$\begin{aligned} e_0 + e_1 &= 1, & e_0 e_1 &= e_1 e_0 = 0, \\ u_a^p &= u_b^p = e_0, & e_0 u_a &= u_a, & e_0 u_b &= u_b, & u_a u_b &= u_b u_a, \\ v_a^p &= v_b^p = e_1, & e_1 v_a &= v_a, & e_1 v_b &= v_b, & v_a v_b &= \zeta_p v_b v_a, \\ g^2 &= 1, & g u_a &= u_b g, & g u_b &= u_a g, & g v_a &= v_a g, & g v_b &= v_b g. \end{aligned}$$

The comultiplication, counit, and antipode of H_p is given by the following formulas:

$$\begin{aligned} \Delta(u_a) &= u_a \otimes u_a + v_a \otimes v_a, & \varepsilon(u_a) &= 1, & S(u_a) &= u_a^{p-1}, \\ \Delta(u_b) &= u_b \otimes u_b + v_b \otimes v_b^{p-1}, & \varepsilon(u_b) &= 1, & S(u_b) &= u_b^{p-1}, \\ \Delta(v_a) &= u_a \otimes v_a + v_a \otimes u_a, & \varepsilon(v_a) &= 0, & S(v_a) &= v_a^{p-1}, \\ \Delta(v_b) &= u_b \otimes v_b + v_b \otimes u_b^{p-1}, & \varepsilon(v_b) &= 0, & S(v_b) &= v_b. \end{aligned}$$

The comultiplication of g is given by

$$\begin{aligned} \Delta(g) &= \frac{1}{p^2} \sum_{i,j,k,l} \zeta_p^{kj-il} g u_a^i u_b^j \otimes g u_a^k u_b^l + \frac{1}{p} \sum_{k,l} \zeta_p^{-(k+l)k} g u_a^k u_b^l \otimes g v_a^{k+l} v_b^{k+l} \\ &\quad + \frac{1}{p} \sum_{k,l} \zeta_p^{k(k+l)} g v_a^{k+l} v_b^{(p-1)(k+l)} \otimes g u_a^k u_b^l + \frac{1}{p} \sum_{k,l} g v_a^k v_b^l \otimes g v_a^{(p-1)l} v_b^k. \end{aligned}$$

The counit and antipode of g are $\varepsilon(g) = 1$ and $S(g) = g$.

In Section 4 we delve into the structure of H_p : we describe its irreducible (co)representations and attached (co)characters, its Hopf automorphisms, and we show that H_p is self-dual.

The set

$$\mathcal{B} := \{u_a^i u_b^j\} \cup \{v_a^i v_b^j\} \cup \{g u_a^i u_b^j\} \cup \{g v_a^i v_b^j\}$$

is a basis of H_p . All structure constants of H_p in this basis belong to $\mathbb{Q}(\zeta_p)$. Hence H_p is defined over $\mathbb{Q}(\zeta_p)$. Our main result states:

Theorem 2. *Let K be a number field containing a primitive p -th root of unity ζ_p . Consider H_p as defined over K . Then, H_p admits a Hopf order over \mathcal{O}_K , which must be unique, if and only if there is an ideal I of \mathcal{O}_K such that $I^{2(p-1)} = (p)$. In particular, K can not be a cyclotomic field (nor an abelian extension of \mathbb{Q}) if a Hopf order exists.*

This theorem implies that Nikshych's Hopf algebras behave rather differently than group algebras. Firstly, all group algebras are already defined over \mathbb{Z} . Secondly, the number of Hopf orders of a group algebra over \mathcal{O}_K depends on K , and in some cases it is not bounded (see for example the classification of orders of the group algebras of the cyclic groups of prime orders in Section 2).

The main result is contained in Section 5. We outline the strategy to prove it and construct the Hopf order. The element $h := u_a + v_a$ is a group-like element of H_p and generate a Hopf subalgebra isomorphic to KC_p . If X is a Hopf order of H_p over \mathcal{O}_K , then $X \cap KC_p$ is a Hopf order of KC_p . The Hopf orders of the latter are known by the works of Greither, Larson, Tate and Oort (we review their description in Section 2, after the preliminaries). They are given by ideals I of \mathcal{O}_K containing $\zeta_p - 1$, see Formula 2.1. Denoting by $H(I)$ the corresponding Hopf order, the \mathcal{O}_K -submodule of integrals of $H(I)$ is $\frac{1}{p}I^{p-1} \sum_i h^i$. This determines uniquely the Hopf orders of KC_p . On the other hand, any Hopf order must contain certain elements arising from characters and cocharacters. The proof of the main result is based on the interaction between the order X of H_p and the order $X \cap KC_p$ of KC_p . We exhibit certain elements which must be in X . We then conclude that necessarily $\frac{1}{\sqrt{p}} \sum_i h^i \in X \cap KC_p$, and by the classification of orders mentioned above, we conclude that some more elements must lie in $X \cap KC_p$ and therefore in X . We then show that these elements generate an order of H_p , which thus must be a minimal order. We then use the self-duality of H_p and conclude that there is also a maximal order. A result of Larson (see Proposition 1.4) now implies that the two orders must be equal, and therefore we only have one order. The necessity of the existence of an ideal I of \mathcal{O}_K such that $I^{2(p-1)} = (p)$ arises from the following consideration: We prove that the set of integrals of $X \cap KC_p$ is exactly $\mathcal{O}_K(\frac{1}{\sqrt{p}} \sum_i h^i)$. We write $J = \{x \in K \mid x(h-1) \in X\}$. By the classification in Section 2 we find out that $I := J^{-1}$ must satisfy $I^{2(p-1)} = (p)$. The unique Hopf order of H_p is the \mathcal{O}_K -subalgebra of H_p generated by $e_0, e_1, g, J(u_a - e_0), J(u_b - e_0), J(v_a - e_1)$, and $J(v_b - e_1)$.

In Section 6 we study the problem of definability over cyclotomic ring of integers of H_p but now considered as a complex Hopf algebra. Since H_p is already defined over a number field K , the question reads now as follows. Let L/K be a Galois extension. Could a L/K -form of H_p admit an order over some cyclotomic ring of integers? Namely, could there be another Hopf algebra H'_p over K such that $H'_p \otimes_K L \simeq H_p \otimes_K L$ and H'_p admits an order over some cyclotomic ring of integers? The following result gives a number theoretical condition under which the answer is affirmative:

Theorem 3. *Let $\zeta_n \in \mathbb{C}$ be a primitive n -th root of unity, with n divisible by p . Consider H_p as defined over $\mathbb{Q}(\zeta_n)$. Let $w \in \mathbb{Z}[\zeta_n]$ and $t \in \mathbb{C}$ be such that w is invertible and $t^2 = w(\zeta_p - 1)$. Assume that there is $d \in \mathbb{Z}[\zeta_n]$ such that $\frac{1}{2}(d+t) \in \mathcal{O}_{\mathbb{Q}(\zeta_n, t)}$. Then, H_p admits a $\mathbb{Q}(\zeta_n, t)/\mathbb{Q}(\zeta_n)$ -form H'_p which in turn admits an order over $\mathbb{Z}[\zeta_n]$.*

For $p = 7$ and $n = 28$ we construct elements w, t and d satisfying this condition. So, H_7 , as a complex Hopf algebra, admits an order over the cyclotomic ring of integers $\mathbb{Z}[\zeta_{28}]$.

The following questions on the definability over cyclotomic ring of integers of complex semisimple Hopf algebras remain open:

Questions. *Does there exist a value of p for which Nikshych's Hopf algebra H_p , as defined over the complex numbers, does not admit an order over any cyclotomic ring*

of integers? More generally, does there exist a complex semisimple Hopf algebra which admits an order over a number ring but not over any cyclotomic ring of integers?

1. PRELIMINARIES

Throughout H is a finite-dimensional Hopf algebra over a ground field K . Unless otherwise stated, vector spaces, linear maps, and unadorned tensor products are over K . The identity element of H is denoted by 1_H and the comultiplication, counit, and antipode by Δ, ε , and S respectively. Our main references for Hopf algebra theory are [9] and [12].

We next collect from [1, Subsection 1.2] several notions and results on Hopf orders that we will need later. We refer the reader to there for the proofs.

1.1. Hopf orders. Let $R \subset K$ be a subring and V a finite-dimensional K -vector space. Recall that an order of V over R is a finitely generated and projective R -submodule X of V such that the natural map $X \otimes_R K \rightarrow V$ is an isomorphism. We view X inside V as the image of $X \otimes_R R$. A *Hopf order of H over R* is an order X of H such that $1_H \in X$, $XX \subseteq X$, $\Delta(X) \subseteq X \otimes_R X$, $\varepsilon(X) \subseteq R$ and $S(X) \subseteq X$. (Note that $X \otimes_R X$ can be identified naturally as an R -submodule of $H \otimes H$.) Equivalently, a Hopf order of H over R is a Hopf algebra X over R , which is finitely generated and projective as an R -module, such that $X \otimes_R K \simeq H$ as Hopf algebras over K . We will assume throughout this subsection that K is a number field and $R = \mathcal{O}_K$. A Hopf order without indication of the ground ring means a Hopf order over R .

Proposition 1.1. *Let X be a Hopf order of H .*

- (i) *The dual order $X^* := \{\varphi \in H^* : \varphi(X) \subseteq R\}$ is a Hopf order of H^* .*
- (ii) *The natural isomorphism $H \simeq H^{**}$ induces an isomorphism of Hopf orders $X \simeq X^{**}$.*
- (iii) *If A is a Hopf subalgebra of H , then $X \cap A$ is a Hopf order of A .*
- (iv) *If $f : H \rightarrow B$ is a surjective Hopf algebra map, then $f(X)$ is a Hopf order of B .*

An important fact in our study of Hopf orders is that they contain certain elements arising from the characters and cocharacters of the Hopf algebra.

Proposition 1.2. *Let X be a Hopf order of H . Any character of H belongs to X^* . As a consequence, any character of H^* belongs to X .*

We will also need the following two results by Larson:

Proposition 1.3. [6, Proposition 2.2] *Let H be a semisimple Hopf algebra over K and X a Hopf order of H . Denote by Λ_X and Λ_{X^*} the R -submodule of left integrals of X and X^* respectively. Then $\varepsilon(\Lambda_X)\varepsilon(\Lambda_{X^*}) = (\dim H)$ as ideals in R .*

Proposition 1.4. [6, Corollary 3.2] *With hypotheses as before, assume that X and Y are Hopf orders of H such that $X \subseteq Y$. If $\varepsilon(\Lambda_X) = \varepsilon(\Lambda_Y)$, then $X = Y$.*

2. CLASSIFICATION OF HOPF ORDERS OF KC_p

Let p be a prime number and ζ a primitive p -th root of unity. Let K be a number field containing ζ and $R := \mathcal{O}_K$. Let σ denote a generator of the cyclic group C_p . We will describe here all Hopf orders of KC_p . Tate and Oort classified all group schemes of order p over R in [14, Theorem 3]. Their result is more general than classifying Hopf orders over R . However, we will combine it with Greither's result [4, Lemma 1.2, page 40] to give a more explicit description of all Hopf orders of KC_p .

We begin with the following observation:

Lemma 2.1. *Let X be a Hopf order of KC_p . Consider the fractional ideal*

$$J = \{\alpha \in K : \alpha(\sigma - 1) \in X\}.$$

Then $R \subseteq J \subseteq R_{\frac{1}{\zeta-1}}$.

Proof. By Proposition 1.2, $\psi(X) \subseteq R$ for any character ψ of C_p . Using the character mapping σ to ζ we see that $J(\zeta - 1) \subseteq R$. Hence $J \subseteq R_{\frac{1}{\zeta-1}}$. For the other inclusion, notice that σ is a character of $(KC_p)^*$. Then $\sigma \in X$ again by Proposition 1.2, and $R(\sigma - 1) \subseteq X$. \square

The above lemma leads us to the following definition:

Definition 2.2. Let I be an ideal of R containing $\zeta - 1$. The *global Larson order associated to I* is the R -submodule of KC_p

$$(2.1) \quad H(I) = \bigoplus_{i=0}^{p-1} I^i (\zeta - 1)^{-i} (\sigma - 1)^i.$$

The name global Larson order will make sense in a few paragraphs. Notice that if $(\zeta - 1) \subseteq I \subseteq I'$, then $H(I) \subseteq H(I')$. Even though the Larson orders are orders of the cyclic group algebra, they will play a decisive role in the classification of orders of Nikshych's Hopf algebra in Section 5.

Lemma 2.3. *The global Larson orders are Hopf orders of KC_p . The set of integrals in $H(I)$ is $\frac{1}{p} I^{p-1} \sum_i \sigma^i$.*

Proof. We first show that $H(I)$ is closed under multiplication. For this, it is enough to prove that $I^p (\zeta - 1)^{-p} (\sigma - 1)^p \subseteq H(I)$. This follows from the fact that the element $x := \frac{1}{\zeta-1} (\sigma - 1)$ satisfies a monic polynomial over R of degree p . We have:

$$1 = \sigma^p = ((\zeta - 1)x + 1)^p = \sum_{k=0}^p \binom{p}{k} (\zeta - 1)^k x^k \implies \sum_{k=1}^p \binom{p}{k} (\zeta - 1)^{k-1} x^k = 0.$$

The coefficient of x^p is $(\zeta - 1)^{p-1}$ and this equals $p\xi$ for some $\xi \in R$ invertible. Multiplying by $p^{-1}\xi^{-1}$ we obtain the desired polynomial. On the other hand, it is clear that $1 \in H(I)$, $\varepsilon(H(I)) \subseteq R$, and $S(H(I)) \subseteq H(I)$. It remains to prove that $\Delta(H(I)) \subset H(I) \otimes_R H(I)$. Since Δ is an algebra map and $H(I)$ is closed under multiplication, it suffices to check that $\Delta(rx) \in H(I) \otimes_R H(I)$ for every $r \in I$. A direct calculation reveals that

$$\Delta(rx) = rx \otimes 1 + 1 \otimes rx + (\zeta - 1)x \otimes rx.$$

The first two summands clearly belong to $H(I) \otimes_R H(I)$ and the third summand too because $\zeta - 1 \in I$.

To prove the statement about the integrals, notice that the integral $\frac{1}{p} \sum_i \sigma^i$ equals an invertible element times a monic polynomial f of degree $p - 1$ in x . This can be seen by the following calculation:

$$(2.2) \quad \frac{1}{p} \sum_i \sigma^i = \frac{1}{p} \frac{((\zeta - 1)x + 1)^p - 1}{(\zeta - 1)x} = \sum_{k=1}^p \frac{1}{p} \binom{p}{k} (\zeta - 1)^{k-1} x^{k-1}.$$

The fractional expression is just symbolic as $(\zeta - 1)x$ is not necessarily invertible. The powers of x in the right-hand side term have coefficients in R . Observe that p divides $\binom{p}{k}$ for $k = 1, \dots, p - 1$. For $k = p$ the coefficient of x^{p-1} is $(\zeta - 1)^{p-1} = p\xi$ with $\xi \in R$ invertible. If $r \in I^{p-1}$, then $\frac{r}{p} \sum_i \sigma^i$ is an integral in $H(I)$ by (2.2), since $\zeta - 1 \in I$. For the reverse inclusion, observe that by construction we have $I^{p-1} = \{\alpha \in K : \alpha x^{p-1} \in H(I)\}$. Let f be an integral in $H(I)$. There is $\lambda \in K$ such that $f = \frac{\lambda}{p} \sum_i \sigma^i$. Then $\lambda \xi x^{p-1} \in H(I)$ by (2.2) and thus $\lambda \in I^{p-1}$. \square

We will next prove that all Hopf orders of KC_p are global Larson orders. Over a local ring, this is a theorem by Greither, see [4, Lemma 1.2, page 40]. We will use the local to global result of Tate and Oort [14, Lemma 4] to pass to the number field case.

Let $\mathfrak{p} \subset R$ be a prime ideal such that $p \in \mathfrak{p}$. Consider the corresponding valuation ν , scaled so that $\nu(p) = 1$ (we find more convenient to write here the valuation in additive terms). Then it is easy to see that $\nu(1 - \zeta) = \frac{1}{p-1}$ because $(\zeta - 1)^{p-1} = (p)$.

Definition 2.4. [5, Section 3] Let $b \in R_{\mathfrak{p}}$ be such that $0 \leq \nu(b) \leq \frac{1}{p-1}$. Set $s = \nu(b)$. The *Larson order* $H(s)$ is the $R_{\mathfrak{p}}$ -subalgebra of $K_{\mathfrak{p}}C_p$ generated by $\frac{1}{b}(\sigma - 1)$.

One can see, exactly as in Lemma 2.3, that Larson orders are indeed Hopf orders, and that $H(s)$ does not depend on the choice of b . Notice that $H(s)$ is defined only if there is an element with valuation s in $R_{\mathfrak{p}}$. We have the following classification result by Greither, see [15, Theorem 3.0.0] and [4, Lemma 1.2, page 40].

Theorem 2.5 (Greither). *All Hopf orders of $K_{\mathfrak{p}}C_p$ over $R_{\mathfrak{p}}$ are Larson orders.*

We recall the following result of Tate and Oort:

Proposition 2.6. [14, Lemma 4] *For any commutative ring T , let $E(T)$ denote the set of isomorphism classes of group schemes of order p over T . Then, the square*

$$\begin{array}{ccc} E(R) & \longrightarrow & \prod_{\mathfrak{p} \in \text{Spec}(R)} E(R_{\mathfrak{p}}) \\ \downarrow & & \downarrow \\ E(K) & \longrightarrow & \prod_{\mathfrak{p} \in \text{Spec}(R)} E(K_{\mathfrak{p}}) \end{array}$$

where the maps are given by extension of scalars, is cartesian.

With this in hand we can establish:

Theorem 2.7. *Every Hopf order of KC_p over R is a global Larson order.*

Proof. A Hopf order X of KC_p over R can be viewed as a group scheme of order p . Proposition 2.6 tells us that giving X is the same as giving its extension of scalars to K and $R_{\mathfrak{p}}$ for every $\mathfrak{p} \in \text{Spec}(R)$, in a compatible way. The extension of scalars of X to K will be just KC_p , and thus we know the extension of scalars to all $K_{\mathfrak{p}}$. Furthermore, if $\mathfrak{p} \in \text{Spec}(R)$ satisfies $p \notin \mathfrak{p}$, then we only have one Hopf order over $R_{\mathfrak{p}}$. This is because all primitive idempotents will be contained in any Hopf order.

The different orders will differ only by their extension of scalars to $R_{\mathfrak{p}}$ with $p \in \mathfrak{p}$. We know by Greither's Theorem that $X \otimes_R R_{\mathfrak{p}}$ is a Larson order over $R_{\mathfrak{p}}$. Let $\mathfrak{q}_1^{r_1} \cdots \mathfrak{q}_l^{r_l}$ be the prime decomposition of $(\zeta - 1)$ in R . Assume that $X \otimes_R R_{\mathfrak{q}_i}$ is isomorphic to $H(s_i)$ over $R_{\mathfrak{q}_i}$. Consider the ideal $I = \prod_i \mathfrak{q}_i^{(p-1)r_i s_i}$. One can now see that the Larson order $H(I)$ will give rise to exactly the same localizations as X at \mathfrak{q}_i . Since the square in Proposition 2.6 is cartesian, this means that $X = H(I)$. \square

We know how the integrals inside Larson orders look like by Lemma 2.3. As a consequence:

Corollary 2.8. *A Hopf order $H(J)$ of KC_p over R which contains $\frac{1}{p}I^{p-1} \sum_i \sigma^i$ contains the Hopf order $H(I)$.*

Proof. Using the prime decomposition of ideals, $I^{p-1} \subseteq J^{p-1}$ implies $I \subseteq J$. \square

The computation of the submodule of integrals in Lemma 2.3 together with Theorem 2.7 has the following outcome, from which we will derive the necessary condition in our main theorem:

Corollary 2.9. *Let X be a Hopf order of KC_p .*

- (i) *Suppose that the R -submodule of integrals of X is generated by $\frac{1}{\sqrt{p}} \sum_i \sigma^i$. Then there exists an ideal I of R such that $I^{2(p-1)} = (p)$.*
- (ii) *Suppose that $\frac{1}{\sqrt{p}} \sum_i \sigma^i \in X$ and there is $\pi \in K$ such that $\pi^2 = \zeta - 1$. Then $\frac{1}{\pi}(\sigma - 1) \in X$.*

Proof. (i) In view of Theorem 2.7, X is isomorphic to $H(I)$ for some ideal I of R containing $\zeta - 1$. By hypothesis and Lemma 2.3 the submodule of integrals is

$$R \left(\frac{1}{\sqrt{p}} \sum_i \sigma^i \right) = \frac{1}{p} I^{p-1} \sum_i \sigma^i.$$

Then $I^{p-1} = (\sqrt{p})$ and thus $I^{2(p-1)} = (p)$.

(ii) From the hypothesis and Lemma 2.3, we obtain $(\sqrt{p}) \subseteq I^{p-1}$. We know that $(\zeta - 1)^{p-1} = (p)$. Using the prime factorization of ideals, we have $(\pi)^{p-1} = (\sqrt{p}) \subseteq I^{p-1}$. This implies that $(\pi) \subseteq I$. Then the element $\frac{\pi}{\zeta-1}(\sigma - 1) = \frac{1}{\pi}(\sigma - 1) \in X$ by the construction of $H(I)$. \square

3. AN EXPLICIT DESCRIPTION OF NIKSHYCH'S HOPF ALGEBRA

The goal of this section will be to write in an explicit way Nikshych's Hopf algebra.

For an odd prime number p , Nikshych constructed in [11] a finite-dimensional, semisimple, weakly group-theoretical and non group-theoretical Hopf algebra H_p of dimension $4p^2$. It was defined in terms of a tensor category and a fiber functor. The

representation category $\text{Rep}(H_p)$ is constructed from the representation category of another Hopf algebra, A_p , by means of equivariantization by C_2 . As fusion categories, $\text{Rep}(H_p) \simeq \text{Rep}(A_p)^{C_2}$. The Hopf algebra A_p first appeared in the work of Masuoka [7]. The above equivalence implies that H_p fits into the short exact sequence

$$K \rightarrow A_p \rightarrow H_p \rightarrow KC_2 \rightarrow K.$$

To describe explicitly the structure of H_p we need to write the structure of A_p , the action of the generator g of C_2 on A_p , and the comultiplication of g .

From now on we abbreviate A_p to A and H_p to H . In this section we assume that K is algebraically closed of characteristic zero.

The main result of this section is the following:

Theorem 3.1. *Let $\zeta \in K$ be a primitive p -th root of unity. The Hopf algebra H is generated, as an algebra over K , by the elements $e_0, e_1, u_a, u_b, v_a, v_b$ and g subject to the following relations:*

$$\begin{aligned} e_0 + e_1 &= 1, & e_0 e_1 &= e_1 e_0 = 0, \\ u_a^p &= u_b^p = e_0, & e_0 u_a &= u_a, & e_0 u_b &= u_b, & u_a u_b &= u_b u_a, \\ v_a^p &= v_b^p = e_1, & e_1 v_a &= v_a, & e_1 v_b &= v_b, & v_a v_b &= \zeta_p v_b v_a, \\ g^2 &= 1, & g u_a &= u_b g, & g u_b &= u_a g, & g v_a &= v_a g, & g v_b &= v_b g. \end{aligned}$$

The comultiplication, counit, and antipode of H is given by the following formulas:

$$(3.1) \quad \begin{aligned} \Delta(u_a) &= u_a \otimes u_a + v_a \otimes v_a, & \varepsilon(u_a) &= 1, & S(u_a) &= u_a^{p-1}, \\ \Delta(u_b) &= u_b \otimes u_b + v_b \otimes v_b^{p-1}, & \varepsilon(u_b) &= 1, & S(u_b) &= u_b^{p-1}, \\ \Delta(v_a) &= u_a \otimes v_a + v_a \otimes u_a, & \varepsilon(v_a) &= 0, & S(v_a) &= v_a^{p-1}, \\ \Delta(v_b) &= u_b \otimes v_b + v_b \otimes u_b^{p-1}, & \varepsilon(v_b) &= 0, & S(v_b) &= v_b. \end{aligned}$$

The comultiplication of g is given by

$$(3.2) \quad \begin{aligned} \Delta(g) &= \frac{1}{p^2} \sum_{i,j,k,l} \zeta^{kj-il} g u_a^i u_b^j \otimes g u_a^k u_b^l + \frac{1}{p} \sum_{k,l} \zeta^{-(k+l)k} g u_a^k u_b^l \otimes g v_a^{k+l} v_b^{k+l} \\ &+ \frac{1}{p} \sum_{k,l} \zeta^{k(k+l)} g v_a^{k+l} v_b^{(p-1)(k+l)} \otimes g u_a^k u_b^l + \frac{1}{p} \sum_{k,l} g v_a^k v_b^l \otimes g v_a^{(p-1)l} v_b^k. \end{aligned}$$

The counit and antipode of g are $\varepsilon(g) = 1$ and $S(g) = g$.

The rest of this section will be devoted to prove Theorem 3.1.

3.1. The algebra A . As an algebra, A is the direct sum

$$K(C_p \times C_p) \oplus K^c(C_p \times C_p),$$

where $c : (C_p \times C_p) \times (C_p \times C_p) \rightarrow K^\times$ is the 2-cocycle given by

$$c(a^i b^j, a^k b^l) = \zeta^{-jk}, \quad 0 \leq i, j, k, l < p.$$

Here a, b are generators of $C_p \times C_p$. We present the group algebra $K(C_p \times C_p)$ by generators u_a, u_b and defining relations $u_a^p = u_b^p = 1, u_a u_b = u_b u_a$. The twisted group algebra $K^c(C_p \times C_p)$ is presented by generators v_a, v_b and relations $v_a^p = v_b^p = 1, v_a v_b = \zeta v_b v_a$. Notice that $K^c(C_p \times C_p)$ is isomorphic to the matrix algebra $M_p(K)$.

To shorten, we set $A_0 = K(C_p \times C_p)$ and $A_1 = K^c(C_p \times C_p)$. We denote the units of A_0 and A_1 by e_0 and e_1 respectively. So $1_A = e_0 + e_1$ and $e_0e_1 = e_1e_0 = 0$. Unless otherwise specified, the inverses are taking inside either A_0 or A_1 . For example, u_a^{-1} means u_a^{p-1} .

The comultiplication, counit, and antipode of A are described in (3.1) above.

3.2. The algebra H . As an algebra, H is the crossed product $A * KC_2$, where g acts as an algebra automorphism on A by:

$$g(u_a) = u_b, \quad g(u_b) = u_a, \quad g(v_a) = v_a, \quad g(v_b) = v_b.$$

In H we have the relations:

$$gu_a = u_b g, \quad gu_b = u_a g, \quad gv_a = v_a g, \quad gv_b = v_b g.$$

The hard part in the description of H is the formula for $\Delta(g)$. Recall from [11] that H is constructed as follows: the automorphism g induces an autoequivalence

$$F : \text{Rep}(A) \rightarrow \text{Rep}(A), \quad V \mapsto {}_g V.$$

Here ${}_g V = V$ as a vector space, with new action $x \cdot v = g(x)v$ for all $x \in A, v \in V$. The functor F is a tensor equivalence. Moreover, $F^{-1} = F$. To compute $\Delta(g)$ we will need to describe the tensor structure of F . For this, we first need to consider the irreducible representations of A .

3.3. Irreducible representations of A . Every irreducible representation of A is an irreducible representation of either A_0 or A_1 .

The algebra A_0 has p^2 one-dimensional irreducible representations, which we denote by $K^{i,j}$ with $0 \leq i, j < p$. As a K -vector space, $K^{i,j} = K$. The action of u_a and u_b on $K^{i,j}$ is given by:

$$u_a \cdot 1 = \zeta^i 1 \quad u_b \cdot 1 = \zeta^j 1.$$

The algebra A_1 has only one irreducible representation, of dimension p , which we denote by M . Let $\{m_i : 0 \leq i < p\}$ be a basis for M . The action of A_1 on M is

$$v_a \cdot m_i = \zeta^i m_i, \quad v_b \cdot m_i = m_{i+1} \quad (\text{indices are taken mod. } p).$$

3.4. Tensor structure on F . For any $V, W \in \text{Rep}(A)$ irreducible we must establish an isomorphism $\theta_{V,W} : F(V \otimes W) \rightarrow F(V) \otimes F(W)$ satisfying the unit and associativity constraints. We do need to calculate these isomorphisms explicitly, as we will use them later to compute $\Delta(g)$. Observe that at the level of representations $F(K^{i,j}) = K^{j,i}$ and $F(M) = M$.

3.4.1. Isomorphisms between certain representations of A . Given $x \in A_1$ invertible, ${}_x M$ stands for the following representation of A_1 : as a vector space, ${}_x M = M$, and the action is given by

$$y \cdot m = (x^{-1}yx)m \quad \forall y \in A_1, m \in M.$$

We have an isomorphism

$${}_x M \rightarrow M, \quad m \mapsto xm.$$

This will be used in this subsection to define isomorphisms between certain tensor products of representation. Consider the representation $K^{i,j} \otimes M$. Identify it with M , as a vector space, via $1 \otimes m \mapsto m$. Under this identification, v_a and v_b act via

$\zeta^i v_a$ and $\zeta^j v_b$ respectively. Since $v_a v_b = \zeta v_b v_a$, we see that this is the same as ${}_x M$ for $x = v_a^{-j} v_b^i$. Then we have an isomorphism of representations

$$(3.3) \quad l_{i,j} : K^{i,j} \otimes M \rightarrow M, \quad 1 \otimes m \mapsto (v_a^{-j} v_b^i) m.$$

In a similar fashion, $M \otimes K^{i,j}$ is isomorphic to ${}_x M$ for $x = v_a^j v_b^i$ via

$$(3.4) \quad r_{i,j} : M \otimes K^{i,j} \rightarrow M, \quad m \otimes 1 \mapsto (v_a^j v_b^i) m.$$

We discuss separately the four different cases that occur in the description of $\theta_{V,W}$:

3.4.2. *Two representations of A_0 .* We begin by considering the case $V = K^{i,j}$ and $W = K^{k,l}$. We have $V \otimes W \simeq K^{i+k,j+l}$. We must give an isomorphism between $F(V \otimes W) \simeq F(K^{i+k,j+l}) \simeq K^{j+l,i+k}$ and $F(V) \otimes F(W) \simeq K^{j,i} \otimes K^{l,k} \simeq K^{j+l,i+k}$. It will be determined by a nonzero scalar $\mu((i,j), (k,l))$. Then:

$$\theta_{V,W} : F(V \otimes W) \rightarrow F(V) \otimes F(W), \quad 1 \otimes 1 \mapsto \mu((i,j), (k,l)) 1 \otimes 1.$$

The associativity constraints yield that $\mu : (C_p \times C_p)^2 \rightarrow K^\times$ is a 2-cocycle. We shall compute μ explicitly in the sequel. We will see that:

$$(3.5) \quad \theta_{K^{i,j}, K^{k,l}} : F(K^{i,j} \otimes K^{k,l}) \rightarrow F(K^{i,j}) \otimes F(K^{k,l}), \quad 1 \otimes 1 \mapsto \zeta^{il-jk} 1 \otimes 1.$$

3.4.3. *One representation of A_0 and one representation of A_1 .* We next consider the case $V = K^{i,j}$ and $W = M$ (and $V = M$ and $W = K^{i,j}$). We first deal with the values $(i,j) = (0,1), (1,0)$ and then we will deduce a formula for an arbitrary pair (i,j) .

We need to find an isomorphism between $F(K^{1,0} \otimes M)$ and $F(K^{1,0}) \otimes F(M)$. Both representations are isomorphic to M . Thus, up to a nonzero scalar, there is only one possible choice. Using (3.3), we see that such an isomorphism must be given by

$$\theta_{K^{1,0}, M} : F(K^{1,0} \otimes M) \rightarrow F(K^{1,0}) \otimes F(M), \quad 1 \otimes m \mapsto \alpha_{1,0} \otimes (v_a v_b) m,$$

for some $\alpha_{1,0} \in K$ (that will be determined later). In a similar fashion:

$$\begin{aligned} \theta_{K^{0,1}, M} &: F(K^{0,1} \otimes M) \rightarrow F(K^{0,1}) \otimes F(M), \quad 1 \otimes m \mapsto \alpha_{0,1} \otimes (v_b^{-1} v_a^{-1}) m, \\ \theta_{M, K^{1,0}} &: F(M \otimes K^{1,0}) \rightarrow F(M) \otimes F(K^{1,0}), \quad m \otimes 1 \mapsto \beta_{1,0} (v_a^{-1} v_b) m \otimes 1, \\ \theta_{M, K^{0,1}} &: F(M \otimes K^{0,1}) \rightarrow F(M) \otimes F(K^{0,1}), \quad m \otimes 1 \mapsto \beta_{0,1} (v_b^{-1} v_a) m \otimes 1, \end{aligned}$$

for $\alpha_{0,1}, \beta_{1,0}, \beta_{0,1} \in K$.

The tensor structure on F will depend on $\alpha_{1,0}, \alpha_{0,1}, \beta_{1,0}, \beta_{0,1}$, and μ . The compatibility of F with associativity constraints will impose some restrictions on the possible values of them.

We show by induction that the following formula holds for $(i,0)$ with $i \geq 2$:

$$\theta_{K^{i,0}, M} : F(K^{i,0} \otimes M) \rightarrow F(K^{i,0}) \otimes F(M), \quad 1 \otimes m \mapsto \alpha_{1,0}^i \otimes (v_a^i v_b^i) m.$$

Using naturality and compatibility of F with the associativity constraint we have the following commutative diagram:

$$\begin{array}{ccccc}
F(K^{i,0} \otimes M) & \longrightarrow & F(K^{i-1,0} \otimes K^{1,0} \otimes M) & \xrightarrow{F(id \otimes l_{1,0})} & F(K^{i-1,0} \otimes M) \\
\theta_{K^{i,0},M} \downarrow & & & & \downarrow \theta_{K^{i-1,0},M} \\
F(K^{i,0}) \otimes F(M) & & & & F(K^{i-1,0}) \otimes F(M) \\
\uparrow & & & & \downarrow id \otimes F(l_{1,0}^{-1}) \\
F(K^{i-1,0} \otimes K^{1,0}) \otimes F(M) & & & & F(K^{i-1,0}) \otimes F(K^{1,0} \otimes M) \\
\theta_{K^{i-1,0},K^{1,0}}^{-1} \otimes id \swarrow & & & \swarrow id \otimes \theta_{K^{1,0},M} & \\
& & F(K^{i-1,0}) \otimes F(K^{1,0}) \otimes F(M) & &
\end{array}$$

One can check that $1 \otimes m$ is mapped to

$$\alpha_{1,0}^i \mu((i-1,0), (1,0))^{-1} \otimes (v_a^i v_b^i) m.$$

Without loss of generality, we can assume that $\mu((i,0), (j,0)) = \mu((0,i), (0,j)) = 1$, and then we arrive at the desired formula.

By a similar calculation we also obtain:

$$\theta_{K^{0,j},M} : F(K^{0,j} \otimes M) \rightarrow F(K^{0,j}) \otimes F(M), \quad 1 \otimes m \mapsto \alpha_{0,1}^j \otimes (v_b^{-j} v_a^{-j}) m.$$

We can combine these two isomorphisms with the associativity constraint to get the following general formula:

$$(3.6) \quad \theta_{K^{i,j},M} : F(K^{i,j} \otimes M) \rightarrow F(K^{i,j}) \otimes F(M), \quad 1 \otimes m \mapsto \alpha_{1,0}^i \alpha_{0,1}^j \zeta^{i(i-j)} \otimes v_b^{i-j} v_a^{i-j} m.$$

This is done as follows. Using naturality and compatibility of F with the associativity constraint we can construct the following commutative diagram:

$$\begin{array}{ccccc}
F(K^{i,j} \otimes M) & \longrightarrow & F(K^{i,0} \otimes K^{0,j} \otimes M) & \xrightarrow{F(id \otimes l_{0,j})} & F(K^{i,0} \otimes M) \\
\theta_{K^{i,j},M} \downarrow & & & & \downarrow \theta_{K^{i,0},M} \\
F(K^{i,j}) \otimes F(M) & & & & F(K^{i,0}) \otimes F(M) \\
\uparrow & & & & \downarrow id \otimes F(l_{0,j}^{-1}) \\
F(K^{i,0} \otimes K^{0,j}) \otimes F(M) & & & & F(K^{i,0}) \otimes F(K^{0,j} \otimes M) \\
\theta_{K^{i,0},K^{0,j}}^{-1} \otimes id \swarrow & & & \swarrow id \otimes \theta_{K^{0,j},M} & \\
& & F(K^{i,0}) \otimes F(K^{0,j}) \otimes F(M) & &
\end{array}$$

Following the longest path, we obtain:

$$(3.7) \quad \theta_{K^{i,j},M}(1 \otimes m) = \frac{\alpha_{1,0}^i \alpha_{0,1}^j \zeta^{i^2}}{\mu((i,0), (0,j))} \otimes v_b^{i-j} v_a^{i-j} m.$$

We can write a similar diagram with $K^{0,j} \otimes K^{i,0} \otimes M$ in the upper central term and proceeding accordingly we get:

$$\theta_{K^{i,j},M}(1 \otimes m) = \frac{\alpha_{1,0}^i \alpha_{0,1}^j \zeta^{i^2-2ij}}{\mu((0,j), (i,0))} \otimes v_b^{i-j} v_a^{i-j} m.$$

These two equalities yield the following formula for μ :

$$\frac{\mu((0,j), (i,0))}{\mu((i,0), (0,j))} = \zeta^{-2ij}.$$

Since $C_p \times C_p$ is abelian and K is assumed to be algebraically closed of characteristic zero, this completely determines the cohomology class of μ . We choose the following representative from this cohomology class:

$$\mu((i,j), (k,l)) = \zeta^{il-jk}.$$

Substituting this in (3.7) we arrive at the desired formula for $\theta_{K^{i,j},M}$. By making this choice we also assure that $F^2 = Id$ on the subcategory of representations of A_0 .

By a similar calculation, we obtain:

$$(3.8) \quad \theta_{M,K^{i,j}} : F(M \otimes K^{i,j}) \rightarrow F(M) \otimes F(K^{i,j}), m \otimes 1 \mapsto \beta_{1,0}^i \beta_{0,1}^j \zeta^{j(j-i)} v_a^{j-i} v_b^{i-j} m \otimes 1.$$

We have described so the tensor structure on F for the tensor product of representations of A_0 with representations of A_1 . One can verify that *this structure is indeed compatible with all the associativity constraints involving two irreducible representations of A_0 if and only if $\alpha_{1,0}, \alpha_{0,1}, \beta_{1,0}$, and $\beta_{0,1}$ are p -th roots of unity. Moreover, $F^2 = Id$ on $K^{i,j} \otimes M$ and $M \otimes K^{i,j}$ if and only if*

$$(3.9) \quad \alpha_{1,0} \alpha_{0,1} = \beta_{1,0} \beta_{0,1} = 1.$$

We shall assume that this holds henceforth.

3.4.4. *Two representations of A_1 .* Lastly, we compute the isomorphism between $F(M \otimes M)$ and $F(M) \otimes F(M)$. We know that $M \otimes M \simeq \bigoplus_{i,j=0}^{p-1} K^{i,j}$. One can easily check that the element $q_{i,j} \in M \otimes M$ spanning the 1-dimensional representation isomorphic to $K^{i,j}$ must be of the form

$$q_{i,j} = \lambda_{i,j} \sum_t \zeta^{-tj} m_t \otimes m_{i-t}, \text{ with } \lambda_{i,j} \in K.$$

(Unless otherwise specified, throughout the limits in the sums are understood to run from 0 to $p-1$.) We take $\lambda_{i,j} = 1$ for every i, j . The isomorphism is given by:

$$(3.10) \quad \theta_{M,M} : F(M \otimes M) \rightarrow F(M) \otimes F(M), q_{i,j} \mapsto \gamma_{i,j} q_{j,i},$$

for some $\gamma_{i,j} \in K$. Using naturality and compatibility of F with the associativity constraint at $K^{i,j} \otimes M \otimes M$ we obtain the following commutative diagram:

$$\begin{array}{ccc}
\bigoplus_{s,t} F(K^{i,j} \otimes K^{s,t}) & \longrightarrow & F(K^{i,j} \otimes M \otimes M) \xrightarrow{F(l_{i,j} \otimes id)} F(M \otimes M) \\
\bigoplus_{s,t} \theta_{K^{i,j}, K^{s,t}} \downarrow & & \downarrow \theta_{M,M} \\
\bigoplus_{s,t} F(K^{i,j}) \otimes F(K^{s,t}) & & F(M) \otimes F(M) \\
\downarrow & & \downarrow F(l_{i,j}^{-1}) \otimes id \\
F(K^{i,j}) \otimes F(M \otimes M) & & F(K^{i,j} \otimes M) \otimes F(M) \\
\searrow id \otimes \theta_{M,M} & & \swarrow \theta_{K^{i,j}, M} \otimes id \\
& & F(K^{i,j}) \otimes F(M) \otimes F(M)
\end{array}$$

Through the isomorphism on the upper right side, $1 \otimes 1 \in K^{i,j} \otimes K^{s,t}$ is mapped to $\gamma_{s+i,t+j} \alpha_{1,0}^i \alpha_{0,1}^j \zeta^{it-jt} \otimes q_{t,s}$. Through the isomorphism on the left side, $1 \otimes 1$ is mapped to $\gamma_{s,t} \zeta^{it-jt} \otimes q_{t,s}$. From here,

$$(3.11) \quad \gamma_{i,j} = \alpha_{1,0}^{j-i} \gamma_{0,0}.$$

By considering the associativity constraint for $M \otimes M \otimes K^{i,j}$ and writing the analogous diagram we get $\gamma_{i,j} = \beta_{1,0}^{j-i} \gamma_{0,0}$. This implies

$$(3.12) \quad \alpha_{1,0} = \beta_{1,0}.$$

The tensor structure of F on $M \otimes M$ depends therefore on $\alpha_{1,0}$ (which is a p -th root of unity) and $\gamma_{0,0}$ (which equals ± 1 since $F^2 = Id$ on $M \otimes M$).

By checking compatibility with all associativity constraints we see that the isomorphism we have constructed does furnish a tensor structure on F . It can be shown directly that no matter what choice we make for $\gamma_{0,0}$ and $\alpha_{1,0}$, we will always end up with an isomorphic functor. *We can thus assume, without loss of generality, that*

$$\gamma_{0,0} = \alpha_{1,0} = 1.$$

Then, the scalars $\alpha_{0,1}, \beta_{1,0}, \beta_{0,1}$, and $\gamma_{i,j}$ equal 1 by equations (3.9), (3.11), and (3.12). This finishes the description of the tensor structure on F .

We summarize our discussion in the following result.

Proposition 3.2. *Let A be the Hopf algebra defined in Subsection 3.1. Consider its irreducible representations $K^{i,j}$, with $0 \leq i, j < p$, and M defined in Subsection 3.3. There exists (up to isomorphism) only one tensor functor $F : \text{Rep}(A) \rightarrow \text{Rep}(A)$ such that $F(K^{i,j}) \simeq K^{j,i}$ and $F(M) \simeq M$. It is given by the equations (3.5), (3.6), (3.8), and (3.10), where the scalars $\alpha_{1,0}, \alpha_{0,1}, \beta_{1,0}, \beta_{0,1}$, and $\gamma_{i,j}$ equal 1.*

3.5. The comultiplication of g . The category $\text{Rep}(H)$ can be identified with that of F -equivariant representations of A as follows: if $V \in \text{Rep}(H)$, then $V \in \text{Rep}(A)$ by restriction, and $\tilde{g} : V \rightarrow V, v \mapsto gv$ establishes an isomorphism between V and $F(V)$.

We now consider the regular representation of H . The following diagram should be commutative:

$$\begin{array}{ccc} H \otimes H & \xrightarrow{\Delta(g)} & F(H \otimes H) \\ & \searrow (g \otimes g) & \downarrow \Omega \\ & & F(H) \otimes F(H) \end{array}$$

where Ω comes from the tensor structure of F . Since $g = g^{-1}$, we have:

$$\Delta(g) = (g \otimes g)\Omega.$$

For $V, W \in \text{Rep}(A)$ the isomorphism $\theta_{V,W} : F(V \otimes W) \rightarrow F(V) \otimes F(W)$ is given by multiplication by $\Omega \in A \otimes A$. The reason for this is the following: the isomorphism $\theta_{A,A} : A \otimes A \simeq F(A \otimes A) \rightarrow F(A) \otimes F(A) \simeq A \otimes A$ is natural, and hence it must commute with multiplication from the right by elements of $A \otimes A$. So, it must be given by multiplication from the left by some element $\Omega \in A \otimes A$. The same holds for $V, W \in \text{Rep}(A)$ by the naturality of θ again with respect to any morphisms $A \rightarrow V$ and $A \rightarrow W$. Then, the computation of Ω can be derived from our knowledge of these isomorphisms for any two irreducible representations of A . To do this, we first need the decomposition of the regular representation of A as a direct sum of irreducible representations. For $i, j = 0, \dots, p-1$ let $f_{ij} \in A_0$ denote the idempotent upon which u_a acts by ζ^i and u_b by ζ^j . It is:

$$(3.13) \quad f_{ij} = \frac{1}{p^2} \sum_{k,l} \zeta^{-(ik+jl)} u_a^k u_b^l.$$

Let $V_{ij} = A_0 f_{ij}$. Then $V_{ij} \simeq K^{i,j}$. Consider in A_1 the element

$$h_i = \frac{1}{p} \sum_k \zeta^{-ik} v_a^k.$$

Let W_i be the subspace spanned by $v_b^l h_i$ for $l = 0, \dots, p-1$. Then $W_i \simeq M$ by mapping $v_b^{l-i} h_i$ to m_l . Thus we have:

$$A = \left(\bigoplus_{i,j} V_{ij} \right) \oplus \left(\bigoplus_i W_i \right).$$

We claim that:

$$(3.14) \quad \begin{aligned} \Omega = & \frac{1}{p^2} \sum_{i,j,k,l} \zeta^{kj-il} u_a^i u_b^j \otimes u_a^k u_b^l + \frac{1}{p} \sum_{k,l} \zeta^{-(k+l)k} u_a^k u_b^l \otimes v_a^{k+l} v_b^{k+l} \\ & + \frac{1}{p} \sum_{k,l} \zeta^{k(k+l)} v_a^{k+l} v_b^{-(k+l)} \otimes u_a^k u_b^l + \frac{1}{p} \sum_{k,l} v_a^k v_b^l \otimes v_a^{-l} v_b^k. \end{aligned}$$

Using (3.5), (3.8), (3.6), and (3.10), this formula for Ω is proved by checking directly the following equalities, which we leave to the reader:

$$\begin{aligned} \theta_{V_{ij}, V_{kl}}(f_{ij} \otimes f_{kl}) &= \zeta^{il-jk} f_{ij} \otimes f_{kl} = \Omega(f_{ij} \otimes f_{kl}), \\ \theta_{V_{ij}, M}(f_{ij} \otimes v_b^{k-l} h_l) &= \zeta^{(i-j)(k+i)} f_{ij} \otimes v_b^{k+i-j-l} h_l = \Omega(f_{ij} \otimes v_b^{k-l} h_l), \\ \theta_{M, V_{ij}}(v_b^{k-l} h_l \otimes f_{ij}) &= \zeta^{(j-i)(k+i)} v_b^{k+i-j-l} h_l \otimes f_{ij} = \Omega(v_b^{k-l} h_l \otimes f_{ij}), \end{aligned}$$

$$\begin{aligned} \theta_{M,M} \left(\sum_k \zeta^{-jk} v_b^{k-l} h_l \otimes v_b^{i-k-l} h_l \right) &= \sum_k \zeta^{-ik} v_b^{k-l} h_l \otimes v_b^{j-k-l} h_l \\ &= \Omega \left(\sum_k \zeta^{-jk} v_b^{k-l} h_l \otimes v_b^{i-k-l} h_l \right). \end{aligned}$$

A careful calculation reveals that $S(g) = g$. This finishes the description of the Hopf algebra structure of H and hence the proof of Theorem 3.1.

Remark 3.3. Although we used that K is algebraically closed to reconstruct H , a posteriori we see from Theorem 3.1 that H is defined over $\mathbb{Q}(\zeta)$.

4. DUALITY, (CO)CHARACTERS, AND HOPF AUTOMORPHISMS

In this section we study further the structure of H : we describe its irreducible (co)representations and (co)characters, its Hopf automorphisms and we show that it is self-dual. The description of the (co)characters is one of the essential points in the proof of our main result since they provide elements in any Hopf order in view of Proposition 1.2. We keep the notation of the previous section.

4.1. Dual Hopf algebra. We present here the Hopf algebra structure of H^* . As a vector space, $H = A_0 \oplus A_1 \oplus gA_0 \oplus gA_1$. We consider the following basis of H :

$$(4.1) \quad \mathcal{B} := \{u_a^i u_b^j\} \cup \{v_a^i v_b^j\} \cup \{gu_a^i u_b^j\} \cup \{gv_a^i v_b^j\}.$$

We denote the dual basis by:

$$(4.2) \quad \mathcal{B}^* := \{s_{ij}\} \cup \{t_{ij}\} \cup \{\alpha_{ij}\} \cup \{\beta_{ij}\}.$$

From (3.1) and (3.2), we easily see that $H = A \oplus gA$ as a coalgebra. Then

$$(4.3) \quad H^* = A^* \oplus (gA)^*$$

as an algebra. We denote by ε_A and ε_{gA} the counit of H restricted to A and gA respectively. Then, ε_A and ε_{gA} are the central idempotents of H^* giving the previous decomposition. The following result provides the full description of H^* .

Proposition 4.1. *As an algebra, H^* is the direct sum of the algebras A^* and $(gA)^*$. The algebra A^* is spanned by the elements s_{ij} and t_{ij} and its multiplication is given by:*

$$(4.4) \quad \begin{aligned} s_{ij} s_{kl} &= \delta_{i,k} \delta_{j,l} s_{ij}, & t_{kl} s_{ij} &= \delta_{i,k} \delta_{j,-l} t_{kl}, \\ s_{ij} t_{kl} &= \delta_{i,k} \delta_{j,l} t_{kl}, & t_{ij} t_{kl} &= \delta_{i,k} \delta_{j,-l} s_{ij}. \end{aligned}$$

The algebra $(gA)^*$ is generated by the elements γ_{ij} and B subject to the following relations:

$$(4.5) \quad B^2 = \varepsilon_{gA}, \quad \gamma_{ij} \gamma_{kl} = \zeta^{il-jk} \gamma_{i+k, j+l}, \quad \text{and} \quad B \gamma_{ij} = \gamma_{ij} B.$$

The comultiplication, counit, and antipode of H^* are given by:

$$\begin{aligned}
\Delta(s_{ij}) &= \sum_{k,l} s_{kl} \otimes s_{i-k,j-l} + \frac{1}{p^2} \zeta^{-(il+jk)} \gamma_{kl} \otimes \gamma_{lk}, \\
\varepsilon(s_{ij}) &= \delta_{i,0} \delta_{j,0}, & S(s_{ij}) &= s_{-i,-j}, \\
\Delta(t_{ij}) &= \sum_{k,l} \zeta^{l(k-i)} t_{kl} \otimes t_{i-k,j-l} + \frac{1}{p^2} \zeta^{-il} \gamma_{kl} B \otimes \gamma_{l-j,k} B, \\
\varepsilon(t_{ij}) &= \delta_{i,0} \delta_{j,0}, & S(t_{ij}) &= \zeta^{-ij} t_{-ij}, \\
\Delta(\gamma_{ij}) &= \sum_{k,l} \zeta^{li+kj} s_{kl} \otimes \gamma_{ij} + \zeta^{ki+lj} \gamma_{ij} \otimes s_{kl}, \\
\varepsilon(\gamma_{ij}) &= 0, & S(\gamma_{ij}) &= \gamma_{-j,-i}, \\
\Delta(B) &= \sum_{k,l} \zeta^{kl} \gamma_{l0} B \otimes t_{kl} + t_{kl} \otimes \gamma_{0-l} B, \\
\varepsilon(B) &= 0, & S(B) &= B.
\end{aligned}
\tag{4.6}$$

(The operations in the indices are all done modulo p .)

Proof. From the dual basis \mathcal{B}^* in (4.2), we are going to construct a new basis of H^* which is more convenient to express the multiplication. In $(gA_0)^*$, instead of $\{\alpha_{ij}\}$ we take the dual basis of $\{gf_{ij}\}$, where $\{f_{ij}\}$ are the idempotents in (3.13). We denote this basis by $\{\gamma_{ij}\}$. Then:

$$\gamma_{ij}(g u_a^k u_b^l) = \zeta^{ik+jl}.$$

The s_{ij} 's and t_{ij} 's form a basis of A^* and the β_{ij} 's and γ_{ij} 's form one of $(gA)^*$. A direct and tedious calculation yields the following formulas:

$$\begin{aligned}
s_{ij} s_{kl} &= \delta_{i,k} \delta_{j,l} s_{ij}, & t_{kl} s_{ij} &= \delta_{i,k} \delta_{j,-l} t_{kl}, \\
s_{ij} t_{kl} &= \delta_{i,k} \delta_{j,l} t_{kl}, & t_{ij} t_{kl} &= \delta_{i,k} \delta_{j,-l} s_{ij}, \\
\gamma_{ij} \beta_{kl} &= \zeta^{j(l+j+k-i)} \beta_{k-i+j, l-i+j}, & \beta_{kl} \gamma_{ij} &= \zeta^{j(i+k-j-l)} \beta_{k+i-j, l+j-i}, \\
\gamma_{ij} \gamma_{kl} &= \zeta^{il-jk} \gamma_{i+k, j+l}.
\end{aligned}$$

This gives the statement for the multiplication in A^* . For the one in $(gA)^*$ we proceed as follows: consider the element

$$B = \sqrt{p} \sum_k \beta_{k0}.$$

It commutes with the γ_{ij} 's in view of the above formulas. A simple computation shows that $B^2 = \varepsilon_{gA}$. Each β_{ij} can be expressed as

$$\beta_{ij} = \frac{1}{p\sqrt{p}} \sum_k \zeta^{-ki} \gamma_{k-j,k} B.$$

This can be verified directly by using the equality:

$$(\gamma_{ij} B)(g v_a^k v_b^l) = \sqrt{p} \zeta^{jk} \delta_{l, j-i}.$$

Then $\{\gamma_{ij}\} \cup \{\gamma_{ij}B\}$ is a basis of $(gA)^*$. We change our basis of H^* again to

$$(4.9) \quad \mathcal{L} := \{s_{ij}\} \cup \{t_{ij}\} \cup \{\gamma_{ij}\} \cup \{\gamma_{ij}B\}.$$

The multiplication of H^* is then fully described on \mathcal{L} by (4.4) and (4.5).

We next compute the formulas for the comultiplication of H^* given in (4.6). These formulas follow from direct calculations, just using the multiplication in H . The calculations do not present any special difficulty. We briefly indicate how to proceed for s_{ij} and leave the details and the other cases to the reader. The element s_{ij} vanishes on A_1, gA_0 and gA_1 . Since $A_0A_0 = (gA_0)(gA_0) = A_0$ and $A_0A_1 = A_1A_0 = 0$ no other kind of summands can occur in the right-hand side. Hence it suffices to evaluate $\Delta(s_{ij})$ at $u_a^k u_b^l \otimes u_a^m u_b^n$ and $gf_{kl} \otimes gf_{mn}$. The coefficients of $s_{kl} \otimes s_{mn}$ and $\gamma_{kl} \otimes \gamma_{mn}$ must be respectively:

$$\langle s_{ij}, (u_a^k u_b^l)(u_a^m u_b^n) \rangle = \delta_{i,k+m} \delta_{j,l+n} \quad \text{and} \quad \langle s_{ij}, (gf_{kl})(gf_{mn}) \rangle = \frac{1}{p^2} \zeta^{-(il+jk)} \delta_{k,n} \delta_{l,m}.$$

Finally, one can check with no effort that the counit and antipode are the ones given in (4.6). \square

4.2. Self-duality. Nikshych proved in [11, Proposition 5.2] that H and H^* are isomorphic as algebras. In this subsection we strengthen this result by the following proposition:

Proposition 4.2. *The Hopf algebras H and H^* are isomorphic.*

Proof. Let us begin by finding inside H^* a Hopf subalgebra isomorphic to A . Set $d = \frac{p+1}{2}$. Consider the elements:

$$(4.10) \quad \bar{u}_a = \sum_{k,l} \zeta^{(k+l)d} s_{kl}, \quad \bar{u}_b = \sum_{k,l} \zeta^{(k-l)d} s_{kl}, \quad \bar{v}_a = \gamma_{dd}, \quad \bar{v}_b = \gamma_{-dd}.$$

Let \bar{A} be the subalgebra generated by $\bar{u}_a, \bar{u}_b, \bar{v}_a$, and \bar{v}_b . Using the multiplication rules (4.4) and (4.5) one easily checks that the assignment $u_x \mapsto \bar{u}_x, v_x \mapsto \bar{v}_x$ for $x \in \{a, b\}$ establishes an algebra isomorphism Ψ between A and \bar{A} . The elements corresponding to the central idempotents e_0 and e_1 in Subsection 3.1 are

$$(4.11) \quad \varepsilon_A = \sum_{k,l} s_{kl} \quad \text{and} \quad \varepsilon_{gA} = \gamma_{00}.$$

Notice that $\varepsilon_A + \varepsilon_{gA} = \varepsilon_H = 1_{H^*}$. Using formulas (4.6) one can verify with a long but direct computation that the above isomorphism is actually an isomorphism of Hopf algebras.

Consider finally the element

$$\bar{g} = B + \sum_{k,l} \zeta^{dkl} t_{kl}.$$

It can be shown that $\bar{g}^2 = 1_{H^*}$, conjugation by \bar{g} stabilizes \bar{A} , and, by the above isomorphism, \bar{g} acts on \bar{A} as g acts on A . Moreover, one can show that Ψ extends to a Hopf algebra isomorphism from H to H^* by defining $g \mapsto \bar{g}$. This finishes the proof. \square

Remark 4.3. If $p = 1 \pmod{4}$, then $\sqrt{p} \in \mathbb{Q}(\zeta)$ and the above isomorphism is defined over $\mathbb{Q}(\zeta)$. Otherwise, it is not defined over $\mathbb{Q}(\zeta)$ but over $\mathbb{Q}(\zeta, \omega)$, with ω a primitive fourth root of unity, and maps B to ωB . Consider H as defined over $\mathbb{Q}(\zeta)$. Then B belongs to $H \otimes_{\mathbb{Q}(\zeta)} K$ but not to H because $\sqrt{p} \notin \mathbb{Q}(\zeta)$ in this case. In fact, since the orbit of B under the group of Hopf automorphisms of H is $\{B, -B\}$, see Subsection 4.5, it follows that an isomorphism between H and H^* cannot be defined over $\mathbb{Q}(\zeta)$. The Hopf algebra H^* will be a form of H but not isomorphic to it over $\mathbb{Q}(\zeta)$.

In the next two subsections we describe the irreducible representations of H and H^* and their characters, see [11, Proposition 5.2], which will be used to find the possible Hopf orders of H .

4.3. Characters of H . We have the following irreducible representations of H and corresponding characters:

4.3.1. Dimension 1. There are $2p$ irreducible representations of H of dimension 1. They arise from the elements in A_0 that are g -invariant. For $i = 0, \dots, p-1$ we have the representation V_i^+ (resp. V_i^-), upon which A_1 acts trivially, $u_a^k u_b^l$ acts through the scalar $\zeta^{(k+l)i}$, and g acts as 1 (resp. -1). By using the previously chosen basis \mathcal{L} of H^* (see Equation 4.9) we can write the characters of these representations as:

$$(4.12) \quad \chi_{V_i^\pm} = \pm \gamma_{ii} + \sum_{k,l} \zeta^{(k+l)i} s_{kl}.$$

4.3.2. Dimension 2. The irreducible representations of H of dimension 2 come from the 1-dimensional representations of A_0 which are not g -invariant. Therefore, their orbits have two elements: $K^{i,j}$ and $K^{j,i}$ for $i \neq j$. Such representations are parameterized by pairs (i, j) with $i < j$. We denote them by W_{ij} . There are $\frac{p(p-1)}{2}$ such representations. The elements g and $u_a^k u_b^l$ act on W_{ij} as the matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \zeta^{ik+jl} & 0 \\ 0 & \zeta^{il+jk} \end{pmatrix}$$

respectively, and A_1 acts trivially. The associated characters with respect to the basis \mathcal{L} of H^* are:

$$(4.13) \quad \chi_{W_{ij}} = \sum_{k,l} (\zeta^{ik+jl} + \zeta^{il+jk}) s_{kl}.$$

4.3.3. Dimension p . Finally, there are two irreducible representations of H of dimension p . They arise from the p -dimensional representation M of A_1 , see Subsection 3.3. We denote them by M^+ and M^- . They have basis $\{m_0, \dots, m_{p-1}\}$, the elements in A_1 act as $v_a m_i = \zeta^i m_i$, $v_b m_i = m_{i+1}$ and g acts as ± 1 . The elements of A_0 act trivially. The corresponding characters in the basis \mathcal{L} of H^* are:

$$(4.14) \quad \chi_{M^\pm} = pt_{00} \pm \frac{1}{\sqrt{p}} \sum_i \gamma_{ii} B.$$

4.4. **Characters of H^* .** To describe the irreducible representations of H^* we will use the decomposition (4.3) expressing H^* as the direct sum of algebras $H^* = A^* \oplus (gA)^*$. We start with the irreducible representations of A^* . By the multiplication rules (4.4), A^* is the direct sum of algebras

$$A^* = \left(\bigoplus_i R_i \right) \oplus \left(\bigoplus_{i,j} R_{i,j} \right),$$

where R_i is spanned by s_{i0} and t_{i0} and R_{ij} by $s_{ij}, s_{i-j}, t_{ij}, t_{i-j}$. The index i runs from 0 to $p-1$ and j from 1 to $\frac{p-1}{2}$ to avoid repetitions.

4.4.1. *Dimension 1.* The algebra R_i has two 1-dimensional representations, on both of which s_{i0} acts as 1 whereas t_{i0} acts as ± 1 . We denote them by L_i^+ and L_i^- respectively. The characters of these representations, expressed in the basis \mathcal{B} of H , see Equation 4.1, are:

$$(4.15) \quad \psi_{L_i^\pm} = u_a^i \pm v_a^i.$$

4.4.2. *Dimension 2.* The algebra R_{ij} is isomorphic to $M_2(K)$. Therefore, it has one irreducible 2-dimensional representation, which we denote by P_{ij} . This representation is given by the following map:

$$s_{ij} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad s_{i-j} \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad t_{ij} \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad t_{i-j} \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

In the basis \mathcal{B} of H the characters of these representations are expressed as:

$$(4.16) \quad \psi_{P_{ij}} = u_a^i u_b^j + u_a^i u_b^{-j}.$$

4.4.3. *Dimension p .* Lastly, we discuss the irreducible representations of $(gA)^*$. Since $B^2 = \varepsilon|_{gA} = 1_{(gA)^*}$, we have the following two central idempotents:

$$\kappa = \frac{1}{2}(\varepsilon|_{gA} + B) \quad \text{and} \quad \kappa' = \frac{1}{2}(\varepsilon|_{gA} - B).$$

They induce the algebra decomposition $(gA)^* = (gA)^* \kappa \oplus (gA)^* \kappa'$. From (4.5) we obtain $\gamma_{10}^p = \gamma_{01}^p = \varepsilon_{gA}$ and $\gamma_{10}\gamma_{01} = \zeta^2\gamma_{01}\gamma_{10}$. Then $(gA)^* \kappa$ and $(gA)^* \kappa'$ are isomorphic to $M_p(K)$. Hence $(gA)^*$ has two p -dimensional irreducible representations, which we denote by N^+ and N^- . Both have a basis $\{n_0, \dots, n_{p-1}\}$ with actions

$$\gamma_{ij} n_l = \zeta^{ij+2il} n_{l+j}, \quad B n_l = \pm n_l.$$

The characters of the above representations are given by:

$$(4.17) \quad \psi_{N^\pm} = \frac{1}{p} \sum_{i,j} g u_a^i u_b^j \pm \frac{1}{\sqrt{p}} \sum_i g v_a^i.$$

4.5. Hopf automorphisms. The group of Hopf automorphisms of H is described by the following result:

Proposition 4.4. *The group $Aut_{Hopf}(H)$ is isomorphic to $C_2 \times (C_2 \times C_p)$. Writing C_2 as $\{\pm 1\}$, the Hopf automorphism ϕ of H corresponding to the triple $(\epsilon_1, \epsilon_2, t)$ is:*

$$\begin{aligned}\phi(u_a) &= u_a^{\epsilon_2}, & \phi(u_b) &= u_b^{\epsilon_2}, \\ \phi(v_a) &= v_a^{\epsilon_2}, & \phi(v_b) &= \zeta^t v_b^{\epsilon_2}, & \phi(g) &= g(e_0 + \epsilon_1 e_1).\end{aligned}$$

Proof. We know from (3.1) and (3.2) that $H = A \oplus gA$ as coalgebras and hence $H^* = A^* \oplus (gA)^*$ as algebras. The algebra A^* splits as a direct sum of matrix algebras over K of dimension 1 or 4 (Subsections 4.4.1 and 4.4.2). On the other hand, the algebra $(gA)^*$ is the direct sum of two matrix algebras of dimension p^2 (Subsection 4.4.3). Let $\sigma \in Aut_{Hopf}(H)$. Since σ must preserve the Wedderburn decomposition of H^* , it must hold that $\sigma(A) \subseteq A$. Thus $\sigma|_A$ is a Hopf automorphism of A . We are so led to compute $Aut_{Hopf}(A)$. This gives a group morphism

$$\Theta : Aut_{Hopf}(H) \rightarrow Aut_{Hopf}(A), \quad \sigma \mapsto \sigma|_A.$$

Using this morphism, we are going to compute $Aut_{Hopf}(H)$ in two steps:

Step 1. Hopf automorphisms of A . We know from Subsection 3.1 that A has an algebra decomposition $A = A_0 \oplus A_1$, where $A_0 = K(C_p \times C_p)$ and $A_1 = K^c(C_p \times C_p)$. Considering, as before, the dimensions of the simple components of the Wedderburn decomposition of A_0 and A_1 we get $\sigma(A_0) = A_0$ and $\sigma(A_1) = A_1$. The group-like elements of A are $u_a^i \pm v_a^i$ with $0 \leq i < p$. Since σ preserves group-like elements and the relations $u_a^p = e_0$ and $v_a^p = e_1$, we must have $\sigma(u_a + v_a) = u_a^r + v_a^r$ for some $r \neq 0$. As $\sigma(u_a) \in A_0$ and $\sigma(v_a) \in A_1$, we obtain

$$(4.18) \quad \sigma(u_a) = u_a^r \quad \text{and} \quad \sigma(v_a) = v_a^r.$$

On the other hand, $\sigma(u_b) = u_a^k u_b^s$ for some $k, s \neq 0$ because σ induces a Hopf automorphism on the quotient Hopf algebra A_0 of A . We derive that $k = 0$ from the equality $\mu \Delta \sigma(u_b) = \sigma \mu \Delta(u_b)$. Here μ stands for the multiplication of H . So $\sigma(u_b) = u_b^s$. Using the equality $\Delta \sigma(u_b) = (\sigma \otimes \sigma) \Delta(u_b)$ we arrive to $\sigma(v_b) = \lambda v_b^s$ for some $\lambda \in K^\times$. Moreover, $\lambda^p = 1$ because $\sigma(v_b)^p = e_1$. Put $\lambda = \zeta^t$ with $0 \leq t < p$. Applying σ to the relation $v_a v_b = \zeta v_b v_a$ we get $sr = 1 \pmod{p}$. Then

$$(4.19) \quad \sigma(u_b) = u_b^s \quad \text{and} \quad \sigma(v_b) = \zeta^t v_b^s, \quad \text{with } s = r^{-1} \pmod{p}.$$

Thus σ determines a pair $(r, t) \in C_p^\times \times C_p$. Conversely, one can check that any such a pair together with (4.18) and (4.19) defines a Hopf automorphism of A . Finally, by composing two automorphisms one sees that $Aut_{Hopf}(A) \simeq C_p^\times \times C_p$.

Step 2. Computing the kernel and image of Θ . We claim that $\text{Ker } \Theta \simeq C_2$. Let $\nu \in \text{Ker } \Theta$. We know that H has a coalgebra decomposition $H = A \oplus gA$, that ν must preserve. Then $\nu(g) = gz$ for some $z \in A$. Since $\nu|_A = id_A$, we have for every $x \in A$:

$$gxg^{-1} = \nu(gxg^{-1}) = gzxz^{-1}g^{-1}.$$

From this it follows that $z \in Z(A)$. Recall that $\Delta(g) = (g \otimes g)\Omega$, where Ω is given in Equation 3.14. Using this and that ν is a coalgebra map we get:

$$(g \otimes g)\Omega\Delta(z) = \Delta(gz) = \Delta\nu(g) = (\nu \otimes \nu)\Delta(g) = (gz \otimes gz)\Omega.$$

We also used here that $\Omega \in A \otimes A$ and $\nu|_A = id_A$. Since $z \in Z(A)$ and Ω and g are invertible, the above equality implies that z is a group-like element of A . As $1 = \nu(g)^2 = gzgz$, the only nontrivial option is $z = e_0 - e_1$. Conversely, one can easily check that a map of this form defines an element of order 2 in $\text{Ker } \Theta$.

We claim now that $\text{Im } \Theta \simeq C_2 \times C_p$. Let $\sigma \in \text{Im } \Theta$. Assume that σ is given by $(r, t) \in C_p^\times \times C_p$ and equations (4.18) and (4.19). Then, arguing as before, $\sigma(g) = gz$ for some $z \in A$. We have:

$$u_b^{r^{-1}} = \sigma(u_b) = \sigma(gu_a g^{-1}) = gz u_a^r z^{-1} g^{-1} = u_b^r.$$

From this, $r^2 = 1 \pmod{p}$ and so $r = \pm 1$. Conversely, the Hopf automorphism τ of A corresponding to $(1, t)$ is given by conjugation by the group-like element $u_a^t + v_a^t$. Conjugation by the same element defines $\bar{\tau} \in \text{Aut}_{\text{Hopf}}(H)$ such that $\Theta(\bar{\tau}) = \tau$. Let $\varphi \in \text{Aut}_{\text{Hopf}}(A)$ be corresponding to $(-1, 0)$. One can check effortlessly that $\varphi \in \text{Im } \Theta$ with preimage $\bar{\varphi}$ defined by $\bar{\varphi}|_A = \varphi$ and $\bar{\varphi}(g) = g$.

Thus we have a short exact sequence

$$1 \rightarrow C_2 \rightarrow \text{Aut}_{\text{Hopf}}(H) \rightarrow C_2 \times C_p \rightarrow 1.$$

This sequence splits because $\bar{\varphi}$ has order 2. The action on C_2 is trivial (this is the only possible action), and then

$$\text{Aut}_{\text{Hopf}}(H) \simeq C_2 \times (C_2 \times C_p).$$

□

5. ORDERS OF NIKSHYCH'S HOPF ALGEBRA

In this section we will use the results of the previous sections to classify the orders of Nikshych's Hopf algebra. We will see that Nikshych's Hopf algebra admits at most one order over any number field.

We keep the conventions and notations of Section 3: ζ is a primitive p -th root of unity; K is a number field containing ζ ; $R = \mathcal{O}_K$ is the ring of integers of K ; H denotes Nikshych's Hopf algebra of dimension $4p^2$, and A stands for Masuoka's Hopf algebra of dimension $2p^2$, both defined over K .

Recall from Remark 3.3 that H is defined over $\mathbb{Q}(\zeta)$. However, we will prove here that H does not have orders over $\mathcal{O}_{\mathbb{Q}(\zeta)}$, but only over the ring of integers of some extension of $\mathbb{Q}(\zeta)$. Set $K = \mathbb{Q}(\zeta, \omega)$, where ω is a primitive fourth root of unity. The field $\mathbb{Q}(\zeta)$ contains either \sqrt{p} or $\sqrt{-p}$, depending on the value of $p \pmod{4}$. The existence of ω allows us to assume that $\sqrt{p} \in K$ and treat our computations in a unified way avoiding the distinction of cases.

The proof of Theorem 2 is quite involved. We will divide it into several parts.

5.1. Elements that must be in any Hopf order. Suppose that X is a Hopf order of H over R . Our goal in this first part is to prove that several elements of H , arising from (co)characters, must belong to X . This will be used later to show that all basis elements of H , given in (4.1), must be in X .

We retain the notation of Section 3: e_0, e_1 are the units of A_0 and A_1 and $\varepsilon_A, \varepsilon_{gA}$ denote the counits of A and gA respectively. We start with the following:

Lemma 5.1. *The elements e_0, e_1 are in X and $\varepsilon_A, \varepsilon_{gA}$ are in X^* .*

Proof. We first show that $e_0, e_1 \in X$. The subalgebra H_b of H generated by u_b and v_b is a Hopf subalgebra. Consider the algebra maps $\sigma : H_b \rightarrow K, u_b \mapsto \zeta, v_b \mapsto 0$ and $\tau : H_b \rightarrow K, u_b \mapsto 0, v_b \mapsto \zeta$. They are group-like elements of H_b^* and $\sigma^2 = \tau^p = 1$ and $\sigma\tau = \tau^{p-1}\sigma$. Then $H_b^* \simeq K(C_2 \times C_p)$ as Hopf algebras and $X \cap H_b$ may be viewed as a Hopf order of $K(C_2 \times C_p)^*$ by Proposition 1.1(iii). According to the proof of [1, Proposition 2.1], $X \cap H_b$ contains the idempotents t_0, t_1 (notation as there). Let $\{\nu_{\sigma^i \tau^j}\}_{i,j} \subset K(C_2 \times C_p)^*$ be the dual basis of $\{\sigma^i \tau^j\}_{i,j}$. Recall that $t_0 = \sum_j \nu_{\tau^j}$ and $t_1 = \sum_j \nu_{\sigma \tau^j}$. One can verify directly that $\nu_{\tau^j} = \frac{1}{p} \sum_k \zeta^{-jk} u_b^k$ and $\nu_{\sigma \tau^j} = \frac{1}{p} \sum_k \zeta^{(j-1)k} v_b^k$. Then $t_0 = e_0$ and $t_1 = e_1$.

For the second statement, take into account that H is self-dual by Proposition 4.2. The isomorphism between H and H^* established there maps e_0, e_1 to $\varepsilon_A, \varepsilon_{gA}$ respectively, see (4.11). We now get that $\varepsilon_A, \varepsilon_{gA} \in X^*$ from self-duality of H , the above fact, and the first statement applied to X^* and H^* . \square

Recall from (4.7) the element B used in describing H^* .

Lemma 5.2. *The elements ge_1 and B belong to X and X^* respectively.*

Proof. We first prove that $ge_1 \in X$. We know from Proposition 1.2 that characters of H^* are in X and characters of H are in X^* . Using the previous lemma, (4.17) and (4.14) we obtain that

$$(5.1) \quad \begin{aligned} \Gamma_1 &:= e_0 \psi_{N^+} = \frac{1}{p} \sum_{i,j} g u_a^i u_b^j \in X, \\ \Gamma_2 &:= \varepsilon_{gA} \chi_{M^+} = \frac{1}{\sqrt{p}} \sum_k \gamma_{kk} B \in X^*. \end{aligned}$$

Then $(\Gamma_2 \otimes_R id_X) \Delta(\Gamma_1) \in X$. We check that $(\Gamma_2 \otimes_R id_X) \Delta(\Gamma_1) = ge_1$. Recall that Γ_2 vanishes on $A_0 \oplus A_1 \oplus gA_0$, so we only need to compute the part of $\Delta(\Gamma_1)$ in

$gA_1 \otimes gA_1$. It is:

$$\begin{aligned}
& \frac{1}{p^2} \sum_{i,j,k,l} (gv_a^k v_b^l \otimes gv_a^{-l} v_b^k) (v_a^i v_b^j \otimes v_a^i v_b^{-j}) \\
&= \frac{1}{p^2} \sum_{i,j,k,l} \zeta^{-i(k+l)} gv_a^{k+i} v_b^{l+j} \otimes gv_a^{i-l} v_b^{k-j} \\
&= \frac{1}{p^2} \sum_{i',j',k,l'} \zeta^{(l'-i')(i'-k)} gv_a^{i'} v_b^{j'} \otimes gv_a^{l'} v_b^{i'-j'-l'} \\
& \hspace{15em} \text{putting } i' = k+i, j' = l+j, \text{ and } l' = i-l, \\
&= \frac{1}{p} \sum_{i,j,l} \zeta^{(l-i)i} \left(\frac{1}{p} \sum_k \zeta^{(i-l)k} \right) gv_a^i v_b^j \otimes gv_a^l v_b^{i-j-l} \\
& \hspace{15em} \text{putting } i = i', j = j', \text{ and } l = l', \\
&= \frac{1}{p} \sum_{i,j} gv_a^i v_b^j \otimes gv_a^i v_b^{-j}.
\end{aligned}$$

Applying $\Gamma_2 \otimes_R id_X$ to this expression we get

$$\begin{aligned}
& \frac{1}{p\sqrt{p}} \sum_{i,j,k} (\gamma_{kk} B) (gv_a^i v_b^j) gv_a^i v_b^{-j} \stackrel{(4.8)}{=} \frac{1}{p} \sum_{i,k} \zeta^{ik} gv_a^i \\
&= \sum_i \left(\frac{1}{p} \sum_k \zeta^{ik} \right) gv_a^i \\
&= ge_1.
\end{aligned}$$

Therefore $ge_1 \in X$.

We next show that $B \in X^*$. From (4.14) and Proposition 1.2, we know that $\chi_{M^+} = pt_{00} + \frac{1}{\sqrt{p}} \sum_i \gamma_{ii} B \in X^*$. Using Lemma 5.1, we obtain $\varepsilon_{A\chi_{M^+}} = pt_{00} \in X^*$. Now,

$$(5.2) \quad (\varepsilon_{gA} \otimes \varepsilon_{gA}) \Delta(pt_{00}) \stackrel{(4.6)}{=} \frac{1}{p} \sum_{k,l} \gamma_{kl} B \otimes \gamma_{lk} B \in X^* \otimes_R X^*.$$

On the other hand, by (4.17) and Proposition 1.2, we have

$$\psi_{N^+} = \frac{1}{p} \sum_{i,j} gv_a^i v_b^j + \frac{1}{\sqrt{p}} \sum_i gv_a^i \in X.$$

Using again Lemma 5.1, we get

$$e_1 \psi_{N^+} = \frac{1}{\sqrt{p}} \sum_i gv_a^i \in X.$$

Finally, applying $e_1 \psi_{N^+} \otimes_R id_{X^*}$ to (5.2) we obtain

$$\frac{1}{p\sqrt{p}} \sum_{i,k,l} (\gamma_{kl} B) (gv_a^i) \gamma_{lk} B = \frac{1}{p} \sum_{i,k,l} \zeta^{il} \delta_{l-k,0} \gamma_{lk} B = \gamma_{00} B = B.$$

So, $B \in X^*$. □

Lemma 5.3. *The elements $u_a, v_a, \frac{1}{\sqrt{p}} \sum_i u_a^i$, and $\frac{1}{\sqrt{p}} \sum_i v_a^i$ belong to X .*

Proof. By (4.15) and Proposition 1.2, $u_a + v_a \in X$. Then $e_1(u_a + v_a) = v_a \in X$ and $u_a = (u_a + v_a) - v_a \in X$. We have just seen in the above proof that $\frac{1}{\sqrt{p}} \sum_i g v_a^i \in X$. Multiplying by $g e_1$, we have $\frac{1}{\sqrt{p}} \sum_i v_a^i \in X$. Let H_a be the Hopf subalgebra of H generated by u_a and v_a . Proposition 1.1(iii) entails that $X \cap H_a$ is a Hopf order of H_a . Then

$$\Delta\left(\frac{1}{\sqrt{p}} \sum_i v_a^i\right) \stackrel{(3.1)}{=} \frac{1}{\sqrt{p}} \sum_i u_a^i \otimes v_a^i + v_a^i \otimes u_a^i \in (X \cap H_a) \otimes_R (X \cap H_a).$$

Consider the character φ of H_a given by $\varphi(u_a) = 0$ and $\varphi(v_a) = 1$. By Proposition 1.2, $\varphi \in (X \cap H_a)^*$. Applying $\varphi \otimes_R id_{X \cap H_a}$ to the above equality we conclude that $\frac{1}{\sqrt{p}} \sum_i u_a^i \in X$. \square

5.2. A special case. If we show that $g e_0 \in X$, then it will follow from Lemmas 5.1, 5.2, and 5.3, that all elements of the basis \mathcal{B} in (4.1) of H will be in any Hopf order X . Unlike for other elements, this can not be shown directly. The strategy will be to adjoin to K an element π such that $\pi^2 = \zeta - 1$, prove the statement in this case and then derive it for K . So, in this subsection we assume that K contains such an element π . The proof requires some preparations.

Lemma 5.4. *The map $T : A_1 \rightarrow gA_0, v_a^i v_b^j \mapsto (B \otimes_R id_X) \Delta(g v_a^i v_b^j)$ can be expressed as*

$$T(v_a^i v_b^j) = \frac{1}{\sqrt{p}} \sum_k \zeta^{jk} g u_a^k u_b^{i-k}.$$

Moreover, $T(X \cap A_1) \subseteq X \cap (gA_0)$.

Proof. Since B vanishes on $A_0 \oplus A_1 \oplus gA_0$, only the part of $\Delta(g v_a^i v_b^j)$ in $gA_1 \otimes gA_0$ is relevant for the computation. We have:

$$\begin{aligned} T(v_a^i v_b^j) &\stackrel{\text{Th. 3.1}}{=} \frac{1}{p} \sum_{k,l} \zeta^{k(k+l)} B(g v_a^{k+l} v_b^{-(k+l)} v_a^i v_b^j) g u_a^{k+i} u_b^{l-j} \\ &= \frac{1}{p} \sum_{k,l} \zeta^{(k+i)(k+l)} B(g v_a^{k+l+i} v_b^{-(k+l)+j}) g u_a^{k+i} u_b^{l-j} \\ &\stackrel{(4.8)}{=} \frac{1}{\sqrt{p}} \sum_k \zeta^{(k+i)j} g u_a^{k+i} u_b^{-k} \\ &= \frac{1}{\sqrt{p}} \sum_k \zeta^{kj} g u_a^k u_b^{i-k}. \end{aligned}$$

Let now $x \in X \cap A_1$. By Lemma 5.2, we know that $g e_1 \in X$ and $B \in X^*$. Then $g x = g e_1 x \in X$ and $\Delta(g x) \in X \otimes_R X$. From here, $T(x) = (B \otimes_R id_X) \Delta(g x) \in X$. \square

Proposition 5.5. *Let Z be an R -algebra and $z, e \in Z$. Assume that $z e = e z = z$. Set $\tilde{z} = \frac{1}{\pi}(z - e)$. If $\tilde{z} \in Z$, then*

$$\frac{1}{\sqrt{p}} \sum_i z^i$$

is an R -linear combination of powers of \tilde{z} .

Proof. Set

$$\frac{(\pi\tilde{z} + e)^p - e}{\pi\tilde{z}} = \sum_{k=1}^p \binom{p}{k} (\pi\tilde{z})^{k-1}.$$

As in the proof of Lemma 2.3, the fractional expression is just symbolic. The left-hand side equals $\sum_{i=0}^{p-1} z^i$. We obtain the result by dividing this equation by \sqrt{p} , noticing that $\pi^{p-1} = \xi\sqrt{p}$ for some invertible $\xi \in R$, and $\binom{p}{k}$ is divisible by p for any $k = 1, \dots, p-1$. \square

We are now ready to tackle the difficult point.

Lemma 5.6. *The element ge_0 belongs to X .*

Proof. View A as a Hopf subalgebra of H and A_0 as a quotient Hopf algebra of A via projecting any element on its component in A_0 . Then $X \cap A_0$ is a Hopf order of A_0 in light of Proposition 1.1. Look now at the Hopf subalgebra of A_0 generated by u_a . Lemma 5.3 shows that $\frac{1}{\sqrt{p}} \sum_i u_a^i \in X$. Applying Corollary 2.9(ii), we have $\frac{1}{\pi}(u_a - e_0) \in X \cap A_0$.

On the other hand, Lemmas 5.1 and 5.4 yield that

$$T(e_1) = \frac{1}{\sqrt{p}} \sum_k g u_a^k u_b^{-k} \in X.$$

Put $e = \frac{1}{p} \sum_k u_a^k u_b^{-k}$. Observe that e is an idempotent and $T(e_1) = \sqrt{p}ge \in X$. Let G be the group generated by σ, τ subject to $\sigma^2 = \tau^p = 1, \sigma\tau = \tau\sigma$. The assignments $e_1 \mapsto 0; u_a, u_b \mapsto \tau; g \mapsto \sigma$ define a surjective algebra map $f : H \rightarrow KG$. It is easy to check that f is a Hopf algebra map and $\text{Ker } f$ equals the ideal generated by e_1 and $u_a u_b^{-1} - e_0$. By Proposition 1.1(iv), $f(X)$ is a Hopf order of KG . The element σ must be in $f(X)$ because it can be received from characters of $(KG)^*$. Take $x \in X \cap A_0$ such that $f(x) = \sigma$. Then $x - ge_0 = h(u_a u_b^{-1} - e_0)$ for some $h \in H$. Multiplying by $\sqrt{p}ge$ we arrive to $\sqrt{p}(xge - e) = 0$. Thus $\sqrt{p}e = x(\sqrt{p}ge) \in X \cap A_0$. Consider the Hopf subalgebra E of A_0 generated by $u_a u_b^{-1}$. As $\sqrt{p}e = \frac{1}{\sqrt{p}} \sum_k u_a^k u_b^{-k} \in X \cap E$, Corollary 2.9(i) implies $\frac{1}{\pi}(u_a u_b^{-1} - e_0) \in X$. Hence

$$\frac{1}{\pi}(u_b^{-1} - e_0) = u_a^{-1} \left(\frac{1}{\pi}(u_a u_b^{-1} - e_0) - \frac{1}{\pi}(u_a - e_0) \right) \in X.$$

By Proposition 5.5, $\frac{1}{\sqrt{p}} \sum_i u_b^i \in X$. Let H_b be the Hopf subalgebra of H generated by u_b and v_b . Arguing as we did for H_a in the proof of Lemma 5.3, we obtain that $\frac{1}{\sqrt{p}} \sum_i v_b^i \in X$. Applying Lemma 5.4, we have

$$T\left(\frac{1}{\sqrt{p}} \sum_i v_b^i\right) = ge_0 \in X$$

and we are done. \square

5.3. The necessary condition. We next derive that all basis elements of H must be in the Hopf order X . This will be key to establish the necessary condition of our main result and to prove later that a Hopf order of H , if exists, must be unique.

Proposition 5.7. *All elements of the basis \mathcal{B} in (4.1) of H belong to X .*

Proof. From Lemmas 5.1 and 5.3, we know that $e_0, e_1, u_a, v_a \in X$. We next see that $g \in X$. Take $\pi \in \mathbb{C}$ such that $\pi^2 = \zeta - 1$ and set $L = K(\pi), S = \mathcal{O}_L$. Then $X \otimes_R S$ is a Hopf order of $H_L := H \otimes_K L$. Lemma 5.2 combined with Lemma 5.6 yields that $g \in X \otimes_R S$. We can identify $H \otimes_R L$ with H_L via multiplication. Inside $H \otimes_R L$ we have $(X + Rg) \otimes_R S \subseteq X \otimes_R S + Rg \otimes_R S = X \otimes_R S \subseteq (X + Rg) \otimes_R S$. This equality holds indeed in $H \otimes_R S \subseteq H \otimes_R L$. Since S is faithfully flat as an R -module, we obtain $X = X + Rg$. Therefore $g \in X$.

It remains to prove that $u_b, v_b \in X$. We have that $u_b g = g u_a \in X$. Then $u_b = (u_b g) g \in X$ and consequently $\Delta(u_b) \in X \otimes_R X$. It follows from the latter that $v_b \in X$ arguing for H_b as we did for H_a in the proof of Lemma 5.3. \square

As a consequence of Lemma 5.3, we get

$$\frac{1}{\sqrt{p}} \sum_i u_a^i + v_a^i \in X.$$

Let E be the Hopf subalgebra of H generated by the group-like element $h := u_a + v_a$. Clearly, $E \simeq KC_p$ as Hopf algebras. Put $Z = E \cap X$ and denote by Λ the set of left integrals in the Hopf order Z of E .

Lemma 5.8. *We have $\Lambda = R(\frac{1}{\sqrt{p}} \sum_i h^i)$.*

Proof. Obviously, $R(\frac{1}{\sqrt{p}} \sum_i h^i) \subseteq \Lambda$. For the reverse inclusion, let $f \in \Lambda$. There is $\lambda \in K$ such that $f = \frac{\lambda}{\sqrt{p}} \sum_i h^i$. We will prove that $\lambda \in R$. Using Proposition 5.7, $\varpi := (f \otimes f) \Delta(g) \in X \otimes_R X$. Then $(\Gamma_2 \otimes_R \Gamma_2)(\varpi) \in R$, with Γ_2 being the element defined in (5.1). We next show that $(\Gamma_2 \otimes_R \Gamma_2)(\varpi) = \lambda^2$.

Taking into account that Γ_2 vanishes on $A_0 \oplus A_1 \oplus gA_0$, it suffices to compute the part of ϖ in $gA_1 \otimes gA_1$. We have:

$$\begin{aligned} (\Gamma_2 \otimes_R \Gamma_2)(\varpi) &\stackrel{(5.1)}{=} \frac{\lambda^2}{p^3} \sum_{i,j,k,l} \sum_{r,s} (\gamma_{rr} B)(g v_a^{i+k} v_b^l) (\gamma_{ss} B)(g v_a^{j-l} v_b^k) \\ &\stackrel{(4.8)}{=} \frac{\lambda^2}{p^2} \sum_{i,j,k,l} \sum_{r,s} \zeta^{r(i+k)} \delta_{l,0} \zeta^{s(j-l)} \delta_{k,0} \\ &= \lambda^2. \end{aligned}$$

So $\lambda^2 \in R$ and thus $\lambda \in R$. \square

We can now establish the necessary condition in our main result from the previous lemma and Corollary 2.9(i):

Proposition 5.9. *Suppose that H admits a Hopf order over R . Then there is an ideal I of R such that $I^{2(p-1)} = (p)$.*

5.4. The Hopf order. Assume that there is an ideal I of R such that $I^{2(p-1)} = (p)$. In this subsection we will construct from I a Hopf order of H which will turn out to be the only Hopf order. Consider the fractional ideal $J := I^{-1} = \{\alpha \in K : \alpha I \subseteq R\}$. By the unique factorization property in R , from $I^{2(p-1)} = (p) = (\zeta - 1)^{p-1} = (\sqrt{p})^2$, it follows that $I^2 = (\zeta - 1)$ and $I^{p-1} = (\sqrt{p})$. Then $J^2 = (\frac{1}{\zeta-1})$ and $J^{p-1} = (\frac{1}{\sqrt{p}})$.

We need the following version of Proposition 5.5:

Proposition 5.10. *Let Z be an R -algebra and $z, e \in Z$. Assume that $ze = ez = z$. If $J(z - e) \subset Z$, then*

$$\frac{1}{\sqrt{p}} \sum_i z^i \in Z.$$

Proof. Put $\tilde{z} = z - e$, proceed like in the other proof and use that $\frac{1}{\sqrt{p}} \tilde{z}^{p-1} \in Z$. \square

Theorem 5.11. *The R -subalgebra Y of H generated by $e_0, e_1, g, J(u_a - e_0), J(u_b - e_0), J(v_a - e_1),$ and $J(v_b - e_1)$ is a Hopf order of H .*

Proof. We will first prove that Y is finitely generated as an R -module. Observe that J is finitely generated. Write

$$x_a = u_a - e_0, \quad x_b = u_b - e_0, \quad y_a = v_a - e_1, \quad y_b = v_b - e_1.$$

We have that $x_a, x_b, y_a, y_b \in Y$ because $IJ = R$. Since $e_0, e_1 \in Y$, we also have $u_a, u_b, v_a, v_b \in Y$. We next check that $(Jx_a)^n \subset \sum_{i=1}^{p-1} J^i x_a^i$ for $n \geq p$. The element x_a satisfy $\sum_{i=1}^p \binom{p}{i} x_a^i = 0$. As $J^p J^{p-2} = J^{2(p-1)} = \left(\frac{1}{p}\right)$, we get $R = (J^p p) J^{p-2}$. Then $J^p p = I^{p-2} \subset R$. Hence

$$(Jx_a)^p = J^p x_a^p \subset \sum_{i=1}^{p-1} J^p \binom{p}{i} x_a^i \subset \sum_{i=1}^{p-1} R x_a^i \subset \sum_{i=1}^{p-1} J^i x_a^i.$$

The same holds for $x_b, y_a,$ and y_b . Consider now the equality:

$$\begin{aligned} y_a y_b &= v_a v_b - v_a - v_b + e_1 \\ &= \zeta v_b v_a - v_a - v_b + e_1 \\ &= \zeta y_b y_a + (\zeta - 1)(y_a + y_b + e_1). \end{aligned}$$

Then, for $\alpha_a, \alpha_b \in J$ the coefficient of e_1 in $(\alpha_a y_a)(\alpha_b y_b)$ belongs to R because $J^2 = \left(\frac{1}{\zeta-1}\right)$. Using the previous equality one can prove that any product of the form $(\beta_a y_a^k)(\beta_b y_b^l)$ with $\beta_a \in J^k, \beta_b \in J^l$ can be expressed as an R -linear combination of elements in $(J^i y_a^i)(J^j y_b^j)$ with $0 \leq i \leq l, 0 \leq j \leq k$. Notice that the coefficient of e_1 always belong to R . All these facts, together with the relations among $x_a, x_b, y_a, y_b,$ and g inside H , show that Y is finitely generated as an R -module. More precisely, using that J is finitely generated, the following elements generate Y over R :

$$e_0, e_1, g e_0, g e_1, J^{i+j}(x_b^i x_a^j), J^{i+j}(g x_b^i x_a^j), J^{i+j}(y_b^i y_a^j), J^{i+j}(g y_b^i y_a^j),$$

$$i, j = 0, \dots, p-1.$$

Removing the powers of J from these elements, we obtain a K -basis of H (we understand that i, j are not simultaneously zero). Hence Y is an order of H .

We next prove that Y is closed under comultiplication and antipode. It is easy to check that the comultiplication of the e 's, x 's and y 's lie in $Y \otimes_R Y$, the counits of them lies in R , and $S(Y) \subset Y$. For instance, for $\alpha \in J$ we have:

$$\begin{aligned} \Delta(\alpha x_a) &= \alpha x_a \otimes u_a + \alpha y_a \otimes v_a + e_0 \otimes \alpha x_a + e_1 \otimes \alpha y_a \in Y \otimes_R Y, \\ \Delta(\alpha x_b) &= \alpha x_b \otimes u_b + e_0 \otimes \alpha x_b + \alpha y_b \otimes v_b^{p-1} \\ &\quad + \sum_{k=1}^{p-1} e_1 \otimes \binom{p-1}{k} (\alpha y_b) y_b^{k-1} \in Y \otimes_R Y. \end{aligned}$$

It only remains to show that $\Delta(g) \in Y \otimes_R Y$. For, we need to rewrite $\Delta(g)$ as an R -linear combination of elements in $Y \otimes_R Y$. Recall from Equation 3.2 that $\Delta(g)$ consists of four summands. We treat each of them separately:

- Part in $A_0 \otimes A_0$. Consider the sum

$$\frac{1}{p^2} \sum_{i,j,k,l} \zeta^{jk-il} u_a^i u_b^j \otimes u_a^k u_b^l = \left(\frac{1}{p} \sum_{i,l} \zeta^{-il} u_a^i \otimes u_b^l \right) \left(\frac{1}{p} \sum_{j,k} \zeta^{jk} u_b^j \otimes u_a^k \right).$$

We argue on the first factor, the second one being similar. Replace u_a and u_b by $x_a + e_0$ and $x_b + e_0$ respectively and expand. The coefficient of $x_a^r \otimes x_b^s$ equals $\frac{\zeta^{-1}}{p}$ if $r = s = p - 1$. Then

$$\frac{\zeta^{-1}}{p} x_a^{p-1} \otimes x_b^{p-1} = \frac{\zeta^{-1}}{\sqrt{p}} x_a^{p-1} \otimes \frac{1}{\sqrt{p}} x_b^{p-1}$$

belongs to $Y \otimes_R Y$ because $\frac{1}{\sqrt{p}} \in J^{p-1}$. For either r or s different from $p - 1$ we use the following argument. The coefficient of $x_a^r \otimes x_b^s$ will be the same as the coefficient of $y_a^r \otimes y_b^s$ in the sum

$$\frac{1}{p} \sum_{i,l} \zeta^{-il} v_a^i \otimes v_b^l.$$

This in turn will be the same as the coefficient of $y_a^r y_b^s$ in the sum

$$\begin{aligned} \frac{1}{p} \sum_{i,l} \zeta^{-il} v_a^i v_b^l &= \frac{1}{p} \sum_{i,l} v_b^l v_a^i \\ &= \frac{1}{p} \left(\sum_l v_b^l \right) \left(\sum_i v_a^i \right) \\ (5.3) \qquad &= \frac{1}{p} \frac{(y_b + e_1)^p - e_1}{y_b} \frac{(y_a + e_1)^p - e_1}{y_a}. \end{aligned}$$

We are using here the convention in the proof of Proposition 2.3 for these fractional expressions. The coefficient of $y_a^r y_b^s$ in this sum contains the binomial coefficient $\binom{p}{k}$ for $k = 1, \dots, p - 1$. Therefore the first factor belong to $Y \otimes_R Y$.

- Part in $A_0 \otimes A_1$. We have the summand

$$\frac{1}{p} \sum_{k,l} \zeta^{-(k+l)k} u_a^k u_b^l \otimes v_a^{k+l} v_b^{k+l} = \left(\frac{1}{\sqrt{p}} \sum_k u_a^k \otimes v_b^k v_a^k \right) \left(\frac{1}{\sqrt{p}} \sum_l u_b^l \otimes v_a^l v_b^l \right).$$

We show that each of the sums belongs to $Y \otimes_R Y$. We do it only for the first one. For the second one proceed similarly. The coefficient of $y_a^r \otimes y_b^s y_a^t$ in this sum will be the same as the coefficient of $x_a^r \otimes x_b^s x_a^t$ in the sum

$$(5.4) \qquad \frac{1}{\sqrt{p}} \sum_k u_a^k \otimes u_b^k u_a^k = \frac{1}{\sqrt{p}} \sum_k (u_a \otimes u_b u_a)^k.$$

Observe that $u_a \otimes u_b u_a \in Y \otimes_R Y$ and

$$\begin{aligned} J(u_a \otimes u_b u_a - e_0 \otimes e_0) &= Jx_a \otimes x_b x_a + Jx_a \otimes x_b + Jx_a \otimes x_a + Jx_a \otimes e_0 \\ &\quad + e_0 \otimes (Jx_b)x_a + e_0 \otimes Jx_b + e_0 \otimes Jx_a \in Y \otimes_R Y. \end{aligned}$$

This together Proposition 5.10 yields that the sum belongs to $Y \otimes_R Y$.

- Part in $A_1 \otimes A_0$. We argue as before with the summand

$$\frac{1}{p} \sum_{k,l} \zeta^{(k+l)k} v_a^{k+l} v_b^{-(k+l)} \otimes u_a^k u_b^l = \left(\frac{1}{\sqrt{p}} \sum_k v_b^{(p-1)k} v_a^k \otimes u_a^k \right) \left(\frac{1}{\sqrt{p}} \sum_l v_a^l v_b^{(p-1)l} \otimes u_b^l \right).$$

but using the following variation: $v_b^{p-1} = \bar{y}_b + e_1$ with $\bar{y}_b = \sum_{j=1}^{p-1} \binom{p-1}{j} y_b^j$ and $J \subset J^j$ for $j = 1, \dots, p-1$.

- Part in $A_1 \otimes A_1$. Consider the summand

$$\frac{1}{p} \sum_{k,l} v_a^k v_b^l \otimes v_a^{-l} v_b^k = \frac{1}{p} \sum_{k,l} v_a^k v_b^l \otimes v_a^{(p-1)l} v_b^k.$$

Write it in $H \otimes H^{op}$ as

$$\left(\frac{1}{\sqrt{p}} \sum_k v_a^k \otimes v_b^k \right) \left(\frac{1}{\sqrt{p}} \sum_l v_b^l \otimes v_a^{(p-1)l} \right)$$

and proceed as before. This finishes the proof. \square

Proposition 5.12. *The Hopf order Y is unique.*

Proof. Let $\pi \in \mathbb{C}$ be such that $\pi^2 = \zeta - 1$ and set $L = K(\pi)$. We will first prove that H_L admits a unique Hopf order over $S = \mathcal{O}_L$ and derive the uniqueness for H arguing as we did in Proposition 5.7. Write $I = (\pi)$. Then $I^{2(p-1)} = (p)$. Let $J \subset L$ be the inverse of I , which is generated by $\frac{1}{\pi}$. We have seen in the precedent proof that the order Y (over S) is generated as an algebra by e_0, e_1, g and the elements

$$\tilde{x}_a := \frac{1}{\pi}(u_a - e_0), \quad \tilde{x}_b := \frac{1}{\pi}(u_b - e_0), \quad \tilde{y}_a := \frac{1}{\pi}(v_a - e_1), \quad \tilde{y}_b := \frac{1}{\pi}(v_b - e_1).$$

Let X be any Hopf order of H_L . By Lemma 5.3 and Corollary 2.9(ii), X must contain the element $\frac{1}{\pi}(u_a + v_a - 1)$. By Proposition 5.7, X contains all basis elements of H_L . Using multiplication by e_0 and e_1 , conjugation by g and translation by the character $\rho : H_b \rightarrow K, u_b \mapsto 0, v_b \mapsto 1$, we see that X must contain $\tilde{x}_a, \tilde{y}_a, \tilde{x}_b$, and \tilde{y}_b . Then $Y \subseteq X$ and thus Y is a minimal Hopf order.

We know that H_L is self-dual. Then H_L^* has also a minimal order, which we denote by Z . This implies that Z^* is a maximal Hopf order of H_L . Thus any Hopf order of H_L lies between Y and Z^* . We will prove that $Y = Z^*$. The R -submodule Λ_Y of left integrals in Y is spanned by $\frac{1}{p}(1+g) \sum_{i,j} u_a^i u_b^j$. Then $\varepsilon(\Lambda_Y) = (2p)$. Using self-duality of H_L , we also have $\varepsilon(\Lambda_Z) = (2p)$. Since $(\dim H) = (4p^2)$, by Proposition 1.3, $\varepsilon(\Lambda_{Z^*}) = (2p)$. Proposition 1.4 yields $Y = Z^*$.

Finally, let X, X' be two Hopf orders of H . The Hopf orders $X \otimes_R S$ and $X' \otimes_R S$ of H_L must be equal. Then $X \otimes_R S = (X + X') \otimes_R S = X' \otimes_R S$. As S is faithfully flat as an R -module, we obtain $X = X + X' = X'$ and we are done. \square

Remark 5.13. The precedent result shows that the behavior of orders for semisimple Hopf algebras can be quite different to that of group algebras. When we take larger number fields, the number of Hopf orders of the group algebra on C_p tends to infinity whereas the number of orders of H is constantly 1.

Remark 5.14. In [8, Theorem 1.8] the second author proved that every semisimple Hopf algebra over a number field only admits finitely many Hopf orders over its ring of integers.

5.5. Main result. We are finally in a position to prove our main result:

Theorem 5.15. *Let p be an odd prime number and K a number field containing a primitive p -th root of unity. Nikshych's Hopf algebra admits a Hopf order over \mathcal{O}_K , which must be unique, if and only if there is an ideal I of \mathcal{O}_K such that $I^{2(p-1)} = (p)$. In particular, K can not be neither a cyclotomic field nor an abelian extension of \mathbb{Q} if a Hopf orders exist.*

Proof. The necessary condition was established in Proposition 5.9. The sufficient condition and uniqueness were proved in Theorem 5.11 and Proposition 5.12.

We prove that K can not be a cyclotomic field if H admits a Hopf order over \mathcal{O}_K . Let $I \subset \mathcal{O}_K$ be the given ideal such that $I^{2(p-1)} = (p)$. Suppose that K is a cyclotomic field, say $K = \mathbb{Q}(\eta)$ with η a primitive m -th root of unity. Since (p) ramifies in \mathcal{O}_K , by [16, Proposition 2.3], p is a prime factor of m . Call n the exponent with which p occurs. By [10, Theorem 4.40], there is a prime ideal \mathfrak{P} of \mathcal{O}_K appearing in the factorization of (p) with exponent $e := (p-1)p^{n-1}$. The exponent of \mathfrak{P} in the factorization of $I^{2(p-1)}$ will be $2l(p-1)$ for some $l \in \mathbb{N}$. Then p should be divisible by 2, a contradiction.

That K can not be an abelian extension of \mathbb{Q} in this case follows from the Kronecker-Weber Theorem. \square

6. ON ORDERS OF FORMS

Let L/K be a Galois extension of fields with Galois group Γ . We have seen before that it could happen that Nikshych's Hopf algebra H over K does not admit an order over any cyclotomic ring of integers, but could a L/K -form of H do? Namely, could there be another Hopf algebra H' over K such that $H' \otimes_K L \simeq H \otimes_K L$ and H' admits an order over some cyclotomic ring of integers? We will show in this last section that the answer to this question is affirmative.

We first recall from [2, Proposition 1.1] and [13, Proposition 1] some basics about Galois descent in the Hopf algebra setting. Put $H_L = H \otimes_K L$. Given $\gamma \in \Gamma$, a *Hopf γ -automorphism* of H_L is a K -linear automorphism $f : H_L \rightarrow H_L$ which satisfies:

- (1) f is γ -semilinear, i.e., $f(\alpha h) = \gamma(\alpha)f(h)$ for all $\alpha \in L, h \in H_L$.
- (2) f is compatible with the multiplication, comultiplication, and antipode.
- (3) $f(1_{H_L}) = 1_{H_L}$.
- (4) $\varepsilon f = \gamma \varepsilon$.

According to Galois descent, L/K -forms of H correspond to group homomorphisms $\Phi : \Gamma \rightarrow \text{Aut}_K(H_L), \gamma \mapsto \Phi_\gamma$ such that Φ_γ is a Hopf γ -automorphism for all $\gamma \in \Gamma$. For such a Φ the set of invariants $(H_L)^\Gamma$ is a Hopf algebra over K and the natural map $(H_L)^\Gamma \otimes_K L \rightarrow H_L$ is an isomorphism of Hopf algebras.

Our goal is to prove the following:

Theorem 6.1. *Let $\zeta_n \in \mathbb{C}$ be a primitive n -th root of unity, with n divisible by p . Consider Nikshych's Hopf algebra H as defined over $\mathbb{Q}(\zeta_n)$. Let $w \in \mathbb{Z}[\zeta_n]$ and $t \in \mathbb{C}$*

be such that w is invertible and $t^2 = w(\zeta_p - 1)$. Assume that there is $d \in \mathbb{Z}[\zeta_n]$ such that $\frac{1}{2}(d + t) \in \mathcal{O}_{\mathbb{Q}(\zeta_n, t)}$. Then, H admits a $\mathbb{Q}(\zeta_n, t)/\mathbb{Q}(\zeta_n)$ -form H' which in turn admits an order over $\mathbb{Z}[\zeta_n]$.

Proof. Set $L = \mathbb{Q}(\zeta_n, t)$. We will construct H' and show that the unique order Y of H_L descends to an order of H' over $\mathbb{Z}[\zeta_n]$. The Galois group Γ of $L/\mathbb{Q}(\zeta_n)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. We denote the generator by γ . Bear in mind the Hopf automorphism σ of H of order two given by

$$\sigma(u_\theta) = u_\theta^{-1}, \sigma(v_\theta) = v_\theta^{-1} \text{ for } \theta = a, b, \text{ and } \sigma(g) = g.$$

We can define a Hopf γ -automorphism σ' of H_L by $\sigma'(h \otimes \alpha) = \sigma(h) \otimes \gamma(\alpha)$ for all $h \in H, \alpha \in L$. Let $\Phi : \Gamma \rightarrow \text{Aut}_K(H_L)$ be the group morphism mapping γ to σ' . Consider the form H' of H given by $H' = (H_L)^\Gamma$.

We claim that the order Y of H_L descends to an order $Y' := Y^\Gamma$ of H' over $\mathbb{Z}[\zeta_n]$. It is enough to check that the natural map $\rho : Y^\Gamma \otimes_{\mathbb{Z}[\zeta_n]} \mathcal{O}_L \rightarrow Y$ is an isomorphism (this will ensure us that Y^Γ is really a Hopf order). Since ρ is injective, it suffices to check the surjectivity. We have seen in Proposition 5.12 that Y is generated over \mathcal{O}_L by e_0, e_1, g , and

$$\tilde{x}_a := \frac{1}{t}(u_a - e_0), \quad \tilde{x}_b := \frac{1}{t}(u_b - e_0), \quad \tilde{y}_a := \frac{1}{t}(v_a - e_1), \quad \tilde{y}_b := \frac{1}{t}(v_b - e_1).$$

Clearly, $e_0, e_1, g \in \text{Im } \rho$ as they are invariants. We will show that $\text{Im } \rho$ contains the rest of the generators. Since $\text{Im } \rho$ is a subring of Y , this will give $\text{Im } \rho = Y$. Let us show that $\tilde{x}_a \in \text{Im } \rho$. The proof for the other generators is similar. The element $q := \frac{1}{t^2}(2e_0 - u_a - u_a^{-1}) = -\tilde{x}_a^2 u_a^{-1}$ belongs to Y^Γ . Since $\gamma(t) = -t$, a direct calculation reveals that $\sigma'(\tilde{x}_a) = \tilde{x}_a + tq$. Set $z = \tilde{x}_a + \frac{1}{2}(d + t)q$. One can easily check that $z \in Y^\Gamma$, and therefore $z \in \text{Im } \rho$. Finally, $\tilde{x}_a = z - \frac{1}{2}(d + t)q$, and $\frac{1}{2}(d + t)q \in \text{Im } \rho$, so $\tilde{x}_a \in \text{Im } \rho$ as well, as desired. \square

With the previous theorem in hand, we will describe an example in which an order of a form does exist.

Example. Consider the case $p = 7$ and $n = 28$. Let $\zeta := \zeta_{28}$ be a primitive 28-th root of unity. A computation done by Dror Speiser with the computer algebra system MAGMA showed that if w is the inverse of the element

$$\begin{aligned} & 21747826028152\zeta^{11} - 25061812676688\zeta^{10} + 5371269408312\zeta^9 - 2754700868376\zeta^8 \\ & + 21747826028152\zeta^7 - 22307111808312\zeta^6 + 4963799311635\zeta^4 + 12069132874072\zeta^3 \\ & - 1115355904156\zeta^2 - 12069132874072\zeta + 17343312496677 \end{aligned}$$

and $d = 1$, then the condition of the theorem holds. We take t such that $t^2 = w(1 - \zeta^4)$. We thus have an order over $\mathbb{Z}[\zeta]$ of a form of H_7 .

Then H_7 , as defined over the complex numbers, admits an order over a cyclotomic ring of integers.

The following questions remain open:

Questions. *Does there exist a value of p for which Nikshych's Hopf algebra H_p , as defined over the complex numbers, does not admit an order over any cyclotomic ring of integers? More generally, does there exist a complex semisimple Hopf algebra which admits an order over a number ring but not over any cyclotomic ring of integers?*

Acknowledgements. The first author is supported by grant MTM2014-54439-P from MICINN and FEDER and by the research group FQM0211 from Junta de Andalucía. The second author was supported by the Danish National Research Foundation (DNRF) through the Centre for Symmetry and Deformation. The authors are grateful to Dror Speiser for doing the previous computer calculation and to Bjorn Poonen for a conversation about the number theoretical condition in Theorem 6.1. The authors are finally indebted to the referee for his/her comments and suggestions, which helped to improve substantially the presentation of the results.

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