SUBCENTRIC LINKING SYSTEMS

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Abstract. We propose a definition of a linking system associated to a saturated fusion system which is more general than the one currently in the literature and thus allows a more flexible choice of objects of linking systems. More precisely, we define subcentric subgroups of fusion systems in a way that every quasicentric subgroup of a saturated fusion system is subcentric. Whereas the objects of linking systems in the current definition are always quasicentric, the objects of our linking systems only need to be subcentric. We prove that, associated to each saturated fusion system $\mathcal{F}$, there is a unique linking system whose objects are the subcentric subgroups of $\mathcal{F}$. Furthermore, the nerve of such a subcentric linking system is homotopy equivalent to the nerve of the centric linking system associated to $\mathcal{F}$. We believe that the existence of subcentric linking systems opens a new way for a classification of fusion systems of characteristic p-type. The various results we prove about subcentric subgroups give furthermore some evidence that the concept is of interest for studying extensions of linking systems and fusion systems.

1. Introduction

Centric linking systems associated to fusion systems were introduced by Broto, Levi and Oliver [9] to be able to study p-completed classifying spaces of fusion systems. The existence and uniqueness of a centric linking system associated to each saturated fusion system was however a conjecture for many years until it was proved by Chermak [10] using the classification of finite simple groups. Chermak's proof was reformulated by Oliver [19] and, building on this reformulation, a recent result of Glauberman and Lynd [12] removes the dependence of the proof on the classification. It is an advantage in many contexts to work with linking systems rather than with fusion systems, but it often presents a problem that centric linking systems do not form a category in a meaningful way. Different notions of linking systems were introduced to allow a more flexible choice of objects making it at least in special cases possible to study extensions of linking systems. So Broto, Castellana, Grodal, Levi and Oliver [7] introduced *quasicentric linking systems* and, much later, Oliver [18] introduced a general notion of a *linking system* providing an axiomatic setup for the full subcategories of quasicentric linking systems studied before. *Transporter systems* as defined by Oliver and Ventura [20] give an even more general framework. The main purpose of this paper is to suggest a new notion of a linking system, allowing a more flexible choice of objects than in the existing notion. We prove furthermore some results indicating the usefulness of this new definition.

We write our functions usually on the right hand side. Accordingly, we always compose morphisms in categories from the left to the right.

Throughout, p is a prime, S is a finite p-group, and $\mathcal{F}$ is a fusion system over S.

We refer the reader to [6, Part I] for an introduction to fusion systems. Recall that a subgroup $Q \leq S$ is called *quasicentric* in $\mathcal{F}$ if, for any fully centralized $\mathcal{F}$-conjugate P of Q, $C_{\mathcal{F}}(P) = \mathcal{F}_{C_S(P)}(C_S(P))$. The set of quasicentric subgroups is denoted by $\mathcal{F}^q$. The objects of a linking system associated to $\mathcal{F}$ in the sense of Oliver are always quasicentric subgroups. The objects of linking systems in our new definition only need to satisfy a weaker condition. Namely, they are *subcentric* subgroups as defined next.

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Definition 1. A subgroup $Q \leq S$ is said to be *subcentric* in $\mathcal{F}$ if, for any fully normalized $\mathcal{F}$-conjugate P of Q, $O_p(N_{\mathcal{F}}(P))$ is centric in $\mathcal{F}$. Write $\mathcal{F}^s$ for the set of subcentric subgroups of $\mathcal{F}$.

Recall that $\mathcal{F}$ is called constrained if $\mathcal{F}$ is saturated and $C_S(O_p(\mathcal{F})) \leq O_p(\mathcal{F})$. As we show in detail in Lemma 3.1, assuming $\mathcal{F}$ is saturated, a subgroup $Q \leq S$ is subcentric if and only if for some (and thus for any) fully normalized $\mathcal{F}$-conjugate P of Q, $N_{\mathcal{F}}(P)$ is constrained. Similarly, Q is subcentric if and only if for some (and thus for any) fully centralized $\mathcal{F}$-conjugate P of Q, $C_{\mathcal{F}}(P)$ is constrained. It follows that every quasicentric subgroup is subcentric. Thus, provided $\mathcal{F}$ is saturated, we have the following inclusions:

$$\mathcal{F}^{cr} \subseteq \mathcal{F}^c \subseteq \mathcal{F}^q \subseteq \mathcal{F}^s$$

Even though the original motivation for the definition of linking systems came from homotopy theory, there is some evidence that linking systems are also useful from an algebraic point of view. Chermak [10] introduced with *localities* a concept which in a certain sense is equivalent to the concept of a transporter systems, but has a more group-like flavor. Chermak defines a partial group to be a set $\mathcal{L}$ together with a product which is only defined on certain words in $\mathcal{L}$, and with an inversion map $\mathcal{L} \to \mathcal{L}$ which is an involutory bijection, subject to certain axioms. So the product of a partial group is a map $\Pi \colon \mathbf{D} \to \mathcal{L}$ where $\mathbf{D}$ is a set of words in $\mathcal{L}$. A locality is a triple $(\mathcal{L}, \Delta, S)$ such that $\mathcal{L}$ is a partial group which is finite as a set, S is a p-subgroup of $\mathcal{L}$, and Δ is a set of subgroups of S, again subject to certain axioms. The set Δ is called the set of objects of the locality $(\mathcal{L}, \Delta, S)$. While Chermak defined localities first in the context of his proof of the existence and uniqueness of centric linking systems, he is currently developing a local theory of localities; see [11]. We refer the reader to Section 5 for a brief introduction to localities and to [10] and [11] for a detailed treatment of the subject.

Let $(\mathcal{L}, \Delta, S)$ be a locality. Given the group-like nature of $\mathcal{L}$, there is a natural notion of conjugation in $\mathcal{L}$, even though conjugation is not always defined, since products in partial groups are only defined on certain words in $\mathcal{L}$. For $P \subseteq \mathcal{L}$, the normalizer $N_{\mathcal{L}}(P)$ consists of all $f \in \mathcal{L}$ such that P^f is defined and equals P. It turns out that, for any $P \in \Delta$, the normalizer $N_{\mathcal{L}}(P)$ is a subgroup of $\mathcal{L}$ and thus forms a finite group. The fusion system $\mathcal{F}_S(\mathcal{L})$ is the fusion system over S generated by the conjugation maps between the subgroups of S. We say that a locality $(\mathcal{L}, \Delta, S)$ is a locality over $\mathcal{F}$ if $\mathcal{F} = \mathcal{F}_S(\mathcal{L})$. One can always construct a transporter system $\mathcal{T}(\mathcal{L}, \Delta)$ associated to $\mathcal{F}_S(\mathcal{L})$ whose set of objects is Δ. Moreover, every transporter system associated to $\mathcal{F}$ is isomorphic to a transporter system which comes in this way from a locality over $\mathcal{F}$. It follows from the construction of $\mathcal{T}(\mathcal{L}, \Delta)$ that $\mathrm{Aut}_{\mathcal{T}(\mathcal{L},\Delta)}(P) \cong N_{\mathcal{L}}(P)$ for every $P \in \Delta$. For more details on the connection between transporter systems and localities we refer the reader to Subsection 5.4.

Definition 2.

- A finite group G is said to be of *characteristic* p if $C_G(O_p(G)) \leq O_p(G)$.
- Define a locality $(\mathcal{L}, \Delta, S)$ to be of *objective characteristic* p if, for any $P \in \Delta$, the group $N_{\mathcal{L}}(P)$ is of characteristic p. A locality $(\mathcal{L}, \Delta, S)$ is called a *linking locality*, if $\mathcal{F}_S(\mathcal{L})^{cr} \subseteq \Delta$ and $(\mathcal{L}, \Delta, S)$ is of objective characteristic p.
- Let $\mathcal{T}$ be a transporter system associated to $\mathcal{F}$. Then $\mathcal{T}$ is said to be of *objective characteristic* p if $\mathrm{Aut}_{\mathcal{T}}(P)$ is a group of characteristic p for every object P of $\mathcal{T}$. Moreover, $\mathcal{T}$ is called a *linking system*, if $\mathcal{F}^{cr} \subseteq \mathrm{ob}(\mathcal{T})$ and $\mathcal{T}$ is of objective characteristic p.
- A *subcentric linking locality* over $\mathcal{F}$ is a linking locality $(\mathcal{L}, \mathcal{F}^s, S)$ over $\mathcal{F}$. Similarly, a *centric linking locality* over $\mathcal{F}$ is a linking locality $(\mathcal{L}, \mathcal{F}^c, S)$ over $\mathcal{F}$, and a *quasicentric linking locality* over $\mathcal{F}$ is a linking locality $(\mathcal{L}, \mathcal{F}^q, S)$ over $\mathcal{F}$.
- A linking system $\mathcal{T}$ associated to $\mathcal{F}$ is called a *subcentric linking system* if $\mathrm{ob}(\mathcal{T}) = \mathcal{F}^s$.

In the following remark, we summarize some basic but important properties of linking systems and linking localities. Moreover, we explain the connection between our notion of a linking system and the one currently in the literature. By a *model* for the fusion system $\mathcal{F}$ we always mean a finite group G of characteristic p such that $S \in \mathrm{Syl}_p(G)$ and $\mathcal{F}_S(G) = \mathcal{F}$. As shown in [7], there exists a model for $\mathcal{F}$ if and only if $\mathcal{F}$ is constrained. Moreover, if a model exists, then it is unique up to isomorphism.

Remark 1. Let $(\mathcal{L}, \Delta, S)$ be a locality over $\mathcal{F}$, and let $\mathcal{T}$ be a transporter system associated to $\mathcal{F}$. Then the following hold.

(a) $\mathcal{T}(\mathcal{L}, \Delta)$ is a linking system if and only if $(\mathcal{L}, \Delta, S)$ is a linking locality.
(b) If $(\mathcal{L}, \Delta, S)$ of objective characteristic p then $\Delta \subseteq \mathcal{F}^s$ and, for any $P \in \Delta \cap \mathcal{F}^f$, $N_\mathcal{L}(P)$ is a model for $N_\mathcal{F}(P)$. Similarly, if $\mathcal{T}$ is of objective characteristic p then $\mathrm{ob}(\mathcal{T}) \subseteq \mathcal{F}^s$ and, for any $P \in \mathrm{ob}(\mathcal{T}) \cap \mathcal{F}^f$, the group $\mathrm{Aut}_\mathcal{T}(P)$ is isomorphic to a model for $N_\mathcal{F}(P)$.
(c) Let $\Delta \subseteq \mathcal{F}^q$. Then $C_\mathcal{L}(P) = C_S(P)O_{p'}(C_\mathcal{L}(P))$ for every $P \in \Delta \cap \mathcal{F}^f$. As a consequence, $(\mathcal{L}, \Delta, S)$ is of objective characteristic p if and only if $C_\mathcal{L}(P)$ is a p-group for every $P \in \Delta$. If $\mathrm{ob}(\mathcal{T}) \subseteq \mathcal{F}^q$, then $\mathcal{T}$ is a linking system in the sense defined above if and only if it is a linking system in the sense of Oliver [18, Definition 3]. In particular, every linking system in Oliver's definition is a linking system in our definition.
(d) If $\Delta \subseteq \mathcal{F}^c$ then $(\mathcal{L}, \Delta, S)$ is of objective characteristic p if and only if $(\mathcal{L}, \Delta, S)$ is a Δ-linking system in the sense of Chermak [10], i.e. if and only if $C_\mathcal{L}(P) \leq P$ for every $P \in \Delta$. If $\Delta = \mathcal{F}^c$ then $(\mathcal{L}, \Delta, S)$ is a linking locality in our definition if and only if $(\mathcal{L}, \Delta, S)$ is a centric linking system in the sense of Chermak [10].
If $\mathrm{ob}(\mathcal{T}) = \mathcal{F}^c$, then $\mathcal{T}$ is a linking system in the sense defined above if and only if it is a centric linking system in the sense of [9, Definition 1.7].

Assume now that $\mathcal{F}$ is saturated. Suppose we are given a set Δ of subgroups such that $\mathcal{F}^{cr} \subseteq \Delta \subseteq \mathcal{F}^q$ and such that Δ is closed under $\mathcal{F}$-conjugation and with respect to overgroups. It follows from the existence and uniqueness of centric linking systems combined with [7, Theorem A, Proposition 3.12] that there is a linking system with object set Δ associated to $\mathcal{F}$, and that such a linking system is unique up to isomorphism. Moreover, the nerve of the linking system does not depend on the object set Δ. In particular, quasicentric linking systems exist and are unique up to isomorphism, and the nerve of a quasicentric linking system is homotopy equivalent to the nerve of a centric linking system. Except for the statement about nerves, a formulation of these results and an algebraic proof using the methods in [10] was given by Chermak in unpublished notes before the idea to define subcentric subgroups arose. We similarly give a version for subcentric linking systems. We also include a statement about nerves, which follows from a result of Oliver and Ventura [20, Proposition 4.7] generalizing the arguments in [7]. The crucial property here is that the radical objects of a linking system $\mathcal{T}$ (i.e. the objects P of $\mathcal{T}$ with $O_p(\mathrm{Aut}_\mathcal{T}(P)) \cong P$) are precisely the elements of $\mathcal{F}^{cr}$.

Theorem A. *Let $\mathcal{F}$ be saturated.*

(a) *Let $\mathcal{F}^{cr} \subseteq \Delta \subseteq \mathcal{F}^s$ such that Δ is closed under $\mathcal{F}$-conjugation and with respect to overgroups. Then there exists a linking locality over $\mathcal{F}$ with object set Δ, and such a linking locality is unique up to a rigid isomorphism. Similarly, there exists a linking system $\mathcal{T}$ associated to $\mathcal{F}$ whose set of objects is Δ, and such a linking system is unique up to an isomorphism of transporter systems. Moreover, the nerve $|\mathcal{T}|$ is homotopy equivalent to the nerve of a centric linking system associated to $\mathcal{F}$.*
(b) *The set $\mathcal{F}^s$ is closed under $\mathcal{F}$-conjugation and with respect to overgroups. In particular, there exists a subcentric linking locality over $\mathcal{F}$ which is unique up to a rigid isomorphism, and there exists a subcentric linking system associated to $\mathcal{F}$ which is unique up to an isomorphism of transporter systems.*

Recall here from [10] that a *rigid isomorphism* between localities $(\mathcal{L}, \Delta, S)$ and $(\mathcal{L}^*, \Delta, S)$ with the same set of objects is an isomorphism $\mathcal{L} \to \mathcal{L}^*$ of partial groups which restricts to the identity on S.

As we will explain next, the existence of subcentric linking localities seems to be important because it leads to a useful setup for a classification of fusion systems of characteristic p-type. Recall that a finite group G is said to be of *characteristic* p*-type* (or of *local characteristic* p), if every p-local subgroup (i.e. every normalizer of a non-trivial p-subgroup) is of characteristic p. Similarly, if $\mathcal{F}$ is saturated, then $\mathcal{F}$ is said to be of characteristic p-type if, for every non-trivial fully $\mathcal{F}$-normalized subgroup $P \leq S$, $N_{\mathcal{F}}(P)$ is constrained. The main examples of groups of characteristic p-type are the finite groups of Lie type in defining characteristic p. Moreover, if a finite group is of characteristic p-type then its fusion system turns out to be of characteristic p-type whereas the converse is not true in general.

Since a subgroup is subcentric if and only if the normalizer of every fully $\mathcal{F}$-normalized $\mathcal{F}$-conjugate is constrained, $\mathcal{F}$ is of characteristic p-type if and only if every non-trivial subgroup of S is subcentric. So supposing that $\mathcal{F}$ is of characteristic p-type and $(\mathcal{L}, \Delta, S)$ is a subcentric linking locality over $\mathcal{F}$, the set Δ is the set of non-trivial subgroups of S, and the normalizer $N_{\mathcal{L}}(P)$ of any non-trivial subgroup P of S is a finite group of characteristic p. Hence, "locally" the partial group $\mathcal{L}$ looks very much like a finite group of characteristic p-type. On the other hand, every group of characteristic p-type leads in an elementary way to a linking locality of this kind:

Example 1. Let G be a group of characteristic p-type and let $S \in \mathrm{Syl}_p(G)$. Let Δ be the set of non-trivial subgroups of S. Let $\mathcal{L}_\Delta(G)$ be the set of all elements $g \in G$ with $S \cap S^g \neq 1$. Moreover, define a partial product on $\mathcal{L}_\Delta(G)$ by taking the restriction of the (multivariable) product on G to the set $\mathbf{D}$ of all words $(g_1, \ldots, g_n)$ such that $g_i \in G$ and there exist elements $P_0, \ldots, P_n \in \Delta$ with $P_{i-1}^{g_i} = P_i$ for $i = 1, \ldots, n$. Define an inversion map on $\mathcal{L}_\Delta(G)$ by taking the restriction of the inversion map on G to the set $\mathcal{L}_\Delta(G)$. Then $(\mathcal{L}_\Delta(G), \Delta, S)$ is a locality by [10, Example/Lemma 2.10]. Moreover, $N_{\mathcal{L}_\Delta(G)}(P) = N_G(P)$ is of characteristic p for all $P \in \Delta$. Hence, $(\mathcal{L}_\Delta(G), \Delta, S)$ is a subcentric linking locality for $\mathcal{F}_S(G)$

Previous treatments of fusion systems of characteristic p-type (as for example in [2], [4], [5] and [13]) have used the existence of models for normalizers of fully normalized subgroups. Supposing that $\mathcal{F}$ is a fusion system of characteristic p-type, this involves moving from an arbitrary non-trivial subgroup of S to a fully normalized $\mathcal{F}$-conjugate whose normalizer can then be realized by a model. This process of moving between different $\mathcal{F}$-conjugates often complicates the arguments. Such technical difficulties can be avoided when working with a subcentric linking locality $(\mathcal{L}, \Delta, S)$ over $\mathcal{F}$, because then, for any non-trivial subgroup P of S, the normalizer $N_{\mathcal{L}}(P)$ is a finite group of characteristic p-type. We thus believe that subcentric linking localities allow a much more canonical translation of the arguments used to classify groups of characteristic p-type. Building on the ongoing program of Meierfrankenfeld, Stellmacher, Stroth to classify groups of local characteristic p, one can hope to achieve a classification of fusion systems of characteristic p-type once this program is complete.

It might be possible to give a unifying approach to the classification of fusion systems of characteristic p-type and of groups of characteristic p-type whilst avoiding to use Theorem A and the theory of fusion systems to prove classification theorems for groups of characteristic p-type. We suggest to proceed as follows: In a first step one proves a classification theorem for a linking locality $(\mathcal{L}, \Delta, S)$ where Δ is the set of non-trivial subgroups of S. Then in a second step one separately deduces from that a corresponding classification theorem for fusion systems of characteristic p-type (using the existence of subcentric linking systems), and for groups of characteristic p-type (working with the locality $(\mathcal{L}_\Delta(G), \Delta, S)$ introduced in Example 1). A similar approach should be possible for groups and fusion systems which are not of characteristic p-type, but satisfy a weaker condition like for example being of *parabolic characteristic* p. In Remark 10.8 we outline a possible approach after constructing linking localities and localities of objective characteristic p coming from arbitrary finite groups in Section 10.

We think that the existence of linking localities and linking systems with subcentric objects is also important for another reason. Namely, it seems that the more flexible choice of objects facilitates the study of extensions and of "maps" between linking systems in the spirit of [8], [18], [20], [1]. We continue by stating some results which point into this direction. In particular, in the next two propositions, we state some relations between the subcentric subgroups of $\mathcal{F}$ and the subcentric subgroups of local and certain normal subsystems. The proof of these propositions can be found in Section 3.

Proposition 1. *If $\mathcal{F}$ is saturated then the following hold:*

(a) *Let $R \trianglelefteq \mathcal{F}$ and $P \leq S$. Then $PR \in \mathcal{F}^s$ if and only if $P \in \mathcal{F}^s$.*
(b) *Let $Z \leq Z(\mathcal{F})$ and $P \leq S$. Then $P \in \mathcal{F}^s$ if and only if PZ/Z is subcentric in $\mathcal{F}/Z$.*
(c) *If $Q \in \mathcal{F}^f$ and $P \in N_{\mathcal{F}}(Q)^s$ then $PQ \in \mathcal{F}^s$. More generally, if $Q \leq S$ and $K \leq \mathrm{Aut}(Q)$ such that Q is fully K-normalized, then $PQ \in \mathcal{F}^s$ for every $P \in N_{\mathcal{F}}^K(Q)^s$.*
(d) *For any $Q \in \mathcal{F}^f$, we have $\{P \in \mathcal{F}^s \colon P \leq N_S(Q)\} \subseteq N_{\mathcal{F}}(Q)^s$. More generally, for any $Q \leq S$ and any $K \trianglelefteq \mathrm{Aut}_{\mathcal{F}}(Q)$ such that Q is fully K-normalized, $\{P \in \mathcal{F}^s \colon P \leq N_S^K(Q)\} \subseteq N_{\mathcal{F}}^K(Q)^s$.*

Property (b) holds accordingly for quasicentric subgroups as proved by Broto, Castellana, Grodal, Levi and Oliver in [8, Lemma 6.4(b)]. Building on this result, the authors show that a quasicentric linking system for $\mathcal{F}/Z$ ($Z \leq Z(\mathcal{F})$) can be constructed as a "quotient" of a quasicentric linking system associated to $\mathcal{F}$. A similar construction can be carried out in the world of localities. We prove this in Proposition 9.2 not only for quasicentric linking localities, but also correspondingly for subcentric linking localities and for arbitrary linking localities. Results corresponding to (c) and (d) are also true for centric and quasicentric subgroups; see Lemma 3.14. As we explain in more detail in Section 9.3, property (c) implies that a subcentric linking locality over $N_{\mathcal{F}}^K(Q)$ is contained in a subcentric linking locality over $\mathcal{F}$ such that the inclusion map is a homomorphism of partial groups. This leads also to a functor from the subcentric linking system of $N_{\mathcal{F}}^K(Q)$ to the subcentric linking system of $\mathcal{F}$. Similar results hold for centric and quasicentric linking systems and linking localities. We now turn attention to weakly normal subsystems.

Proposition 2. *Let $\mathcal{F}$ be saturated and let $\mathcal{E}$ be a weakly normal subsystem of $\mathcal{F}$ over T. Then the following hold:*

(a) *The set $\mathcal{E}^s$ is invariant under $\mathcal{F}$-conjugation.*
(b) *For every $P \in \mathcal{F}^s$ with $P \leq T$, $P \in \mathcal{E}^s$.*
(c) *If $\mathcal{E}$ is normal in $\mathcal{F}$ of index prime to p, then $\mathcal{E}^s = \mathcal{F}^s$.*
(d) *If $\mathcal{E}$ is normal in $\mathcal{F}$ of p-power index, then $\mathcal{E}^s = \{P \in \mathcal{F}^s \colon P \leq T\}$.*
(e) *If $R \trianglelefteq \mathcal{F}$ and $K \trianglelefteq \mathrm{Aut}_{\mathcal{F}}(R)$ then $N_{\mathcal{F}}^K(R)^s = \{P \in \mathcal{F}^s \colon P \leq N_S^K(R)\}$. In particular, $C_{\mathcal{F}}(R)^s = \{P \in \mathcal{F}^s \colon P \leq C_S(R)\}$.*

Corresponding statements to (a) and (b) are also true for centric and quasicentric subgroups. Property (c) is clearly also true if one considers centric subgroups rather than subcentric subgroups, and a statement corresponding to (d) is true for quasicentric subgroups by [8, Theorem 4.3]. It is shown in [8, Theorem 5.5] that, given a subsystem $\mathcal{E}$ of index prime to p, a centric linking system associated to $\mathcal{E}$ can be naturally constructed from the centric linking system associated to $\mathcal{F}$. Similarly, it is shown in [8, Theorem 4.4] that a quasicentric linking system of a subsystem of p-power index can be obtained from a quasicentric linking system associated to $\mathcal{F}$. Property (e) fails for centric and quasicentric subgroups as it is stated, but if $\mathrm{Inn}(R) \leq K$, it is true that every centric or quasicentric subgroup of $N_{\mathcal{F}}^K(R)$ which contains R is $\mathcal{F}$-centric or

$\mathcal{F}$-quasicentric respectively, and this is enough for many purposes. In [1, Definition 1.27], Andersen, Oliver and Ventura define normal linking systems. The results we summarized enable them to associate normal pairs of linking systems to $(\mathcal{E},\mathcal{F})$ if $\mathcal{E}$ is a weakly normal subsystem of $\mathcal{F}$ of index prime to p, or of p-power index, or if $\mathcal{E} = N_{\mathcal{F}}^K(R)$ for some normal subgroup $R \trianglelefteq \mathcal{F}$ and $\mathrm{Inn}(R) \leq K \trianglelefteq \mathrm{Aut}(Q)$; see [1, Proposition 1.31]. Andersen, Oliver and Ventura [1] define also the reduction of a fusion system $\mathcal{F}$ by starting with $\mathcal{F}_0 := C_{\mathcal{F}}(O_p(\mathcal{F}))/Z(O_p(\mathcal{F}))$ and then alternately taking $\mathcal{F}_i = O^p(\mathcal{F}_{i-1})$ and $\mathcal{F}_i = O^{p'}(\mathcal{F}_{i-1})$ for any positive integer i until the process terminates. Note that Proposition 1(b) together with Proposition 2(c),(d),(e) gives a very clean connection between the subcentric subgroups of $\mathcal{F}$ and the subcentric subgroups of the reduction of $\mathcal{F}$. Therefore it could be an advantage to work with subcentric linking systems rather than with centric and quasicentric linking systems in this context.

We now turn attention to normal subsystems which do not fulfill any additional properties, and we prove that its subcentric subgroups are still closely related to subcentric subgroups of the entire fusion system.

Theorem B. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S, and let $\mathcal{E}$ be a normal subsystem of $\mathcal{F}$. Then for every subcentric subgroup P of $\mathcal{E}$, $PC_S(\mathcal{E})$ is subcentric in $\mathcal{F}$.*

Here $C_S(\mathcal{E})$ is the subgroup introduced by Aschbacher [3, Chapter 6]. It is the largest subgroup X of S with $\mathcal{E} \subseteq C_{\mathcal{F}}(X)$. If $\mathcal{E}$ is realized by a partial normal subgroup then we prove that $C_S(\mathcal{E})$ is indeed easy to describe in the locality:

Proposition 3. *Let $\mathcal{E}$ be a normal subsystem of $\mathcal{F}$ over T and let $(\mathcal{L},\Delta,S)$ be a linking locality over $\mathcal{F}$. Suppose there exists a partial normal subgroup $\mathcal{N}$ of $\mathcal{L}$ such that $S\cap\mathcal{N} = T$ and $\mathcal{E} = \mathcal{F}_T(\mathcal{N})$. Then $C_S(\mathcal{E}) = C_S(\mathcal{N})$.*

Here $\mathcal{F}_T(\mathcal{N})$ is the smallest fusion system on T containing all conjugation maps by elements of $\mathcal{N}$ between subgroups of T. Supposing that $\mathcal{F}$ is saturated and $(\mathcal{L},\Delta,S)$ is a linking locality over $\mathcal{F}$, it is indeed work in progress of Chermak and the author of this paper to prove that every normal subsystem is of the form $\mathcal{F}_{S\cap\mathcal{N}}(\mathcal{N})$ for a unique partial normal subgroup $\mathcal{N}$ of $\mathcal{L}$, and that this leads to a one-to-one correspondence between the normal subsystems of $\mathcal{F}$ and the partial normal subgroups of $\mathcal{L}$. In particular, if $(\mathcal{L},\Delta,S)$ is a subcentric linking locality over $\mathcal{F}$, then a normal subsystem $\mathcal{E}$ of $\mathcal{F}$ is realized by a partial normal subgroup $\mathcal{N}$ of $\mathcal{L}$. This situation is explored further in Subsection 9.4. Using Theorem B and Proposition 3 we show that a subcentric linking locality for $\mathcal{E}$ is contained in $\mathcal{L}$, and that the inclusion map is a homomorphism of partial groups. This leads to a functor from the subcentric linking system of $\mathcal{E}$ to the subcentric linking system of $\mathcal{F}$. This functor maps every object $P \in \mathcal{E}^s$ to $PC_S(\mathcal{E}) \in \mathcal{F}^s$.

The following Proposition is needed in the proof of Theorem A. If $\mathcal{F}$ is saturated and $(\mathcal{L},\Delta,S)$ is a linking locality over $\mathcal{F}$ then the statement can be considered as a particular case of the correspondence between the normal subsystems of $\mathcal{F}$ and the partial normal subgroups of $\mathcal{L}$.

Proposition 4. *Let $(\mathcal{L},\Delta,S)$ be a locality over $\mathcal{F}$ of objective characteristic p. Then a subgroup $Q \leq S$ is normal in $\mathcal{F}$ if and only if $\mathcal{L} = N_{\mathcal{L}}(Q)$. Similarly, $Q \leq Z(\mathcal{F})$ if and only if $\mathcal{L} = C_{\mathcal{L}}(Q)$.*

Finally, a word about our proofs: Since there is some hope that the theory of fusion systems can be revisited using linking localities, we seek to keep the proofs of the results on subcentric subgroups of fusion systems as elementary as possible. In particular, we reprove some known results on constrained systems in Section 2 without using the theory of components of fusion systems. However, it should be pointed out that we require this theory and Aschbacher's version of the L-balance theorem for fusion systems for the proof of Theorem B.

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that subcentric linking systems should exist. He also pointed out that the nerve of a subcentric linking system would be homotopy equivalent to the nerve of a centric linking system. It was Andrew Chermak who suggested using the iterative procedure introduced in [10] to construct subcentric linking systems.

Throughout, this text, we continue to assume that $\mathcal{F}$ is a fusion system on a finite p-group S. Given a subsystem $\mathcal{E}$ of $\mathcal{F}$ we write $T = \mathcal{E} \cap S$ to express that $\mathcal{E}$ is a subsystem over $T \leq S$.

2. Groups of characteristic p and constrained fusion systems

Throughout this section, $\mathcal{F}$ is assumed to be saturated.

Recall that $\mathcal{F}$ is called *constrained* if $C_S(O_p(\mathcal{F})) \leq O_p(\mathcal{F})$, and a finite group G is said to be of *characteristic* p if $C_G(O_p(G)) \leq O_p(G)$ (or equivalently, $C_G(O_p(G)) = Z(O_p(G))$). If G is of characteristic p then $O_{p'}(G) = 1$ as $[O_p(G), O_{p'}(G)] \leq O_p(G) \cap O_{p'}(G) = 1$ and $C_G(O_p(G)) = Z(O_p(G))$ does not contain any non-trivial p'-elements. A finite group G is called a *model* for $\mathcal{F}$ if $S \in \mathrm{Syl}_p(G)$, $\mathcal{F} = \mathcal{F}_S(G)$ and G is of characteristic p. The following lemma summarizes the connection between constrained fusion systems and groups of characteristic p which was (except for some detail) established in [7].

Theorem 2.1. (a) *$\mathcal{F}$ is constrained if and only if there exists a model for $\mathcal{F}$. In this case, a model is unique up to an isomorphism which is the identity on S.*
(b) *If $\mathcal{F}$ is constrained and G is a model for $\mathcal{F}$ then a subgroup of S is normal in $\mathcal{F}$ if and only if it is normal in G. If $Q \leq S$ is normal and centric in $\mathcal{F}$, then in addition $C_G(Q) \leq Q$.*

Proof. If G is a model for $\mathcal{F}$ then clearly any normal subgroup of G is normal in $\mathcal{F}$, so in particular, $\mathcal{F}$ is constrained. Thus, (a) follows from [6, Theorem 5.10]. Let now $\mathcal{F}$ be constrained and G a model for $\mathcal{F}$. If Q is a normal centric subgroup of $\mathcal{F}$ then it follows again from [6, Theorem 5.10] that $Q \trianglelefteq G$ and $C_G(Q) \leq Q$. In particular, $O_p(\mathcal{F}) \trianglelefteq G$. So if $g \in G$ then $c_g|_{O_p(\mathcal{F})} \in \mathrm{Aut}_{\mathcal{F}}(O_p(\mathcal{F}))$ and thus $P^g = P$ for any normal subgroup P of $\mathcal{F}$. This shows that any normal subgroup of $\mathcal{F}$ is normal in G completing the proof. □

We continue by listing some properties of groups of characteristic p.

Lemma 2.2. *Let G be a finite group of characteristic p. Then the following hold:*
(a) *$N_G(P)$ and $C_G(P)$ are of characteristic p for all non-trivial p-subgroups P of G.*
(b) *Every subnormal subgroup of G is of characteristic p.*

Proof. By Part (c) of [17, Lemma 1.2], $N_G(P)$ is of characteristic p and by Part (a) of the same lemma, (b) holds. As $C_G(P) \trianglelefteq N_G(P)$, it follows now that $C_G(P)$ is of characteristic p. □

Lemma 2.3. *Let G be a finite group of characteristic p and $Z \leq Z(G)$. Then $Z \leq O_p(G)$ and G/Z is of characteristic p.*

Proof. Note that $O_{p'}(Z) \leq O_{p'}(G) = 1$. As Z is abelian, it follows that Z is a p-group and thus $Z \leq O_p(G)$. Set $C := C_G(O_p(G)Z/Z)$. It is sufficient to show that $C \leq O_p(G)$. Note that $[O_p(G), C] \leq Z$ and $[Z, C] = 1$ as $Z \leq Z(G)$. Hence, $[O_p(G), O^p(C)] = 1$. As $C_G(O_p(G)) = Z(O_p(G))$ is a p-group, it follows $O^p(C) = 1$. So C is a p-group and thus $C \leq O_p(G)$. □

Lemma 2.4. *Let G be a finite group and $Z \leq Z(G) \cap O_p(G)$. Then G is of characteristic p if and only if G/Z is of characteristic p.*

Proof. Note that $O_p(G/Z) = O_p(G)/Z$ as $Z \leq O_p(G)$. Suppose first that G is of characteristic p and set $C := C_G(O_p(G)/Z)$. To prove that G/Z is of characteristic p it is sufficient to show that $C \leq O_p(G)$. Note that $[O_p(G), C] \leq Z$ and $[Z, C] = 1$ as $Z \leq Z(G)$. Hence, $[O_p(G), O^p(C)] = 1$.

As $C_G(O_p(G)) = Z(O_p(G))$ is a p-group, it follows $O^p(C) = 1$. So C is a normal p-subgroup of G and thus $C \leq O_p(G)$.

Assume now that G/Z is of characteristic p. As $O_p(G/Z) = O_p(G)/Z$, it follows $C_G(O_p(G)/Z) \leq O_p(G)$. In particular, $C_G(O_p(G)) \leq O_p(G)$ and G is of characteristic p. □

Lemma 2.5. *Let G be a finite group with a normal p-subgroup P such that, for $S \in \mathrm{Syl}_p(G)$, we have $\mathcal{F}_{C_S(P)}(C_G(P)) = \mathcal{F}_{C_S(P)}(C_S(P))$. Then $G = C_S(P)O_{p'}(C_G(P))$, and G is of characteristic p if and only if $C_G(P)$ is a p-group.*

Proof. By the Theorem of Frobenius [16, Theorem 1.4], $C_G(P) = C_S(P)O_{p'}(C_G(P))$. If G is of characteristic p, then $O_{p'}(C_G(P)) \leq O_{p'}(G) = 1$ and thus $C_G(P) = C_S(P)$ is a p-group. On the other hand, if $C_G(P)$ is a p-group then $C_G(P) \leq O_p(G)$ as $C_G(P) \trianglelefteq G$. Hence, as $P \leq O_p(G)$, $C_G(O_p(G)) \leq C_G(P) \leq O_p(G)$ and G is of characteristic p. □

Definition 2.6. Set $\Theta(G) = O_{p'}(G)$ for any finite group G. We say that a finite group G is *almost of characteristic* p if $G/\Theta(G)$ is of characteristic p.[1]

Remark 2.7. Let G be a finite group and P be a p-subgroup of G.

(a) We have $\Theta(N_G(P)) = \Theta(C_G(P))$.
(b) For $\overline{G} = G/\Theta(G)$, we have $N_{\overline{G}}(\overline{P}) = \overline{N_G(P)}$ and $C_{\overline{G}}(\overline{P}) = \overline{C_G(P)}$.

Proof. Since $C_G(P) \trianglelefteq N_G(P)$ and $\Theta(C_G(P))$ is characteristic in $C_G(P)$, we have $\Theta(C_G(P)) \leq \Theta(N_G(P))$. As $[\Theta(N_G(P)), P] \leq \Theta(N_G(P)) \cap P = 1$, (a) follows. By a Frattini argument, $N_{\overline{G}}(\overline{P}) = \overline{N_G(P)}$. If C is the preimage of $C_{\overline{G}}(\overline{P})$ in $N_G(P)$ then $[P, C] \leq \Theta(G) \cap P = 1$ and thus $C = C_G(P)$. Hence, (b) holds. □

Lemma 2.8. *Let G be a finite group which is almost of characteristic p and P a p-subgroup of G. Set $\overline{G} = G/\Theta(G)$.*

(a) *We have $\Theta(C_G(P)) = \Theta(N_G(P)) = \Theta(G) \cap N_G(P) = \Theta(G) \cap C_G(P)$.*
(b) *$N_G(P)$ and $C_G(P)$ are almost characteristic p.*

Proof. By [15, 8.2.12], $\Theta(N_G(P)) = \Theta(G) \cap N_G(P)$. Now (a) follows from Remark 2.7(a). By (a) and Remark 2.7(a), $N_G(P)/\Theta(N_G(P)) \cong \overline{N_G(P)} = N_{\overline{G}}(\overline{P})$ and $C_G(P)/\Theta(C_G(P)) \cong \overline{C_G(P)} = C_{\overline{G}}(\overline{P})$. So the assertion follows from Lemma 2.2(a). □

Lemma 2.9. *Let G be a finite group and let P be a p-subgroup of G. Then $N_G(P)$ is of characteristic p if and only if $C_G(P)$ is of characteristic p. Similarly, $C_G(P)$ is almost of characteristic p if and only if $N_G(P)$ is almost of characteristic p.*

Proof. By Lemma 2.2(a), $C_G(P)$ is of characteristic p if $N_G(P)$ is of characteristic p and, by Lemma 2.8(b), $C_G(P)$ is almost of characteristic p if $N_G(P)$ is almost of characteristic p. Note that $O_p(C_G(P)) \trianglelefteq N_G(P)$ as $C_G(P) \trianglelefteq N_G(P)$. Hence, if $C_G(P)$ is of characteristic p, we have $C_{N_G(P)}(O_p(N_G(P))) \leq C_{N_G(P)}(O_p(C_G(P))P) = C_{C_G(P)}(O_p(C_G(P))) \leq O_p(C_G(P)) \leq O_p(N_G(P))$ and $N_G(P)$ is of characteristic p. Set $H := N_G(P)$ and $\overline{H} := H/\Theta(H)$. Note $C_G(P) = C_H(P)$. By Remark 2.7, $\Theta(H) = \Theta(C_G(P))$ and $C_{\overline{H}}(\overline{P}) = \overline{C_G(P)}$. By what we have just shown, $\overline{H}$ is of characteristic p if $C_{\overline{H}}(\overline{P})$ is of characteristic p. So if $C_G(P)$ is almost of characteristic p then $H = N_G(P)$ is almost of characteristic p. □

In the remainder of this section is devoted to exploring some connections between $\mathcal{F}$ being constrained and certain subsystems or factor systems of $\mathcal{F}$ being constrained.

Lemma 2.10. *Let $Z \leq Z(\mathcal{F})$. Then $\mathcal{F}$ is constrained if and only if $\mathcal{F}/Z$ is constrained. Moreover, if G is a model for $\mathcal{F}$, then $Z \leq Z(G)$ and G/Z is a model for $\mathcal{F}/Z$.*

[1] In the literature, groups which are almost of characteristic p are usually called constrained, but we find our definition more intuitive in this context.

Proof. Suppose first that $\mathcal{F}$ is constrained and that G is a model for $\mathcal{F}$. Note that, by Theorem 2.1(a), a model G always exists if $\mathcal{F}$ is constrained. By Theorem 2.1(b), Z is normal in G. So every $g \in G$ induces an $\mathcal{F}$-automorphism of Z which then has to be the identity, as $Z \leq Z(\mathcal{F})$. Hence, $Z \leq Z(G)$ and G/Z is of characteristic p by Lemma 2.3. By [6, Example II.5.6], $\mathcal{F}/Z = \mathcal{F}_{S/Z}(G/Z)$ and so G/Z is a model for $\mathcal{F}/Z$. Hence, by Theorem 2.1(a), $\mathcal{F}/Z$ is constrained. Assume now that $\mathcal{F}/Z$ is constrained and let $Z \leq Q \leq S$ with $Q/Z = O_p(\mathcal{F}/Z)$. Then $C_S(Q) \leq Q$ as $C_{S/Z}(Q/Z) \leq Q/Z$. So it is sufficient to show that Q is normal in $\mathcal{F}$. Observe that Q is strongly closed in $\mathcal{F}$, since Q/Z is strongly closed in $\mathcal{F}/Z$ and every morphism in $\mathcal{F}$ induces a morphism in $\mathcal{F}/Z$. By [6, Proposition I.4.5], a subgroup of a fusion system is normal if and only if it is strongly closed and contained in every centric radical subgroup. So Q/Z is contained in every element of $(\mathcal{F}/Z)^{cr}$ and it is sufficient to show that Q is contained in every element of $\mathcal{F}^{cr}$. As shown in [14, Proposition 3.1], we have $R/Z \in (\mathcal{F}/Z)^{cr}$ for every $R \in \mathcal{F}^{cr}$. So Q is contained in every element of $\mathcal{F}^{cr}$ as required. □

We now turn attention to subsystems of $\mathcal{F}$, in particular to p-local subsystems and (weakly) normal subsystems.

Lemma 2.11. *Let $\mathcal{F}$ be constrained and $P \in \mathcal{F}^f$. Then $N_\mathcal{F}(P)$ and $C_\mathcal{F}(P)$ are constrained. Moreover, if G is a model for $\mathcal{F}$, then $N_G(P)$ is a model for $N_\mathcal{F}(P)$ and $C_G(P)$ is a model for $C_\mathcal{F}(P)$.*

Proof. Let $\mathcal{F}$ be a constrained fusion system on a finite p-group S and G a model for $\mathcal{F}$. Note that G always exists by Theorem 2.1(a). By [6, Proposition I.5.4], $N_S(P) \in \mathrm{Syl}_p(N_G(P))$, $C_S(P) \in \mathrm{Syl}_p(C_G(P))$, $N_\mathcal{F}(P) = \mathcal{F}_{N_S(P)}(N_G(P))$ and $C_\mathcal{F}(P) = \mathcal{F}_{C_S(P)}(C_G(P))$. By Lemma 2.2, $N_G(P)$ and $C_G(P)$ are of characteristic p, so $N_G(P)$ is a model for $N_\mathcal{F}(P)$ and $C_G(P)$ is a model for $C_\mathcal{F}(P)$. In particular, by Theorem 2.1(a), $N_\mathcal{F}(P)$ and $C_\mathcal{F}(P)$ are constrained. □

We continue with a general lemma needed afterwards to prove results about constrained fusion systems. It could be obtained as a consequence of [3, (7.4)] and the fact that for any $P \in \mathcal{F}$, $P \trianglelefteq \mathcal{F}$ if and only if $\mathcal{F}_P(P) \trianglelefteq \mathcal{F}$. We give however an elementary direct proof.

Lemma 2.12. *Let $\mathcal{E}$ be a weakly normal subsystem of $\mathcal{F}$. Then $O_p(\mathcal{E})$ is normal in $\mathcal{F}$.*

Proof. Let $T = \mathcal{E} \cap S$. As $\mathcal{E}$ is weakly normal in $\mathcal{F}$, every element of $\mathrm{Aut}_\mathcal{F}(T)$ induces an automorphism of $\mathcal{E}$. Thus $O_p(\mathcal{E})$ is $\mathrm{Aut}_\mathcal{F}(T)$-invariant. Since $O_p(\mathcal{E})$ is normal and thus strongly closed in $\mathcal{E}$, it follows now from the Frattini condition as stated in [6, Definition I.6.1] that $O_p(\mathcal{E})$ is strongly closed in $\mathcal{F}$. Hence, by [6, Theorem I.4.5], it is sufficient to prove that $O_p(\mathcal{E})$ is contained in any element of $\mathcal{F}^{cr}$. Let $R \in \mathcal{F}^{cr}$ and set $R_0 := R \cap T$. Recall that T is strongly closed and so R_0 is $\mathrm{Aut}_\mathcal{F}(R)$ invariant. As $O_p(\mathcal{E})$ is normal in $\mathcal{E}$, $\mathrm{Aut}_{O_p(\mathcal{E})}(R_0) \trianglelefteq \mathrm{Aut}_\mathcal{E}(R_0)$. Thus, $\mathrm{Aut}_{O_p(\mathcal{E})}(R_0) \leq O_p(\mathrm{Aut}_\mathcal{E}(R_0)) \leq O_p(\mathrm{Aut}_\mathcal{F}(R_0))$ since $\mathrm{Aut}_\mathcal{E}(R_0) \trianglelefteq \mathrm{Aut}_\mathcal{F}(R_0)$. It follows that the restriction of every element of $X := \langle \mathrm{Aut}_{O_p(\mathcal{E})}(R)^{\mathrm{Aut}_\mathcal{F}(R)} \rangle$ to R_0 lies in $O_p(\mathrm{Aut}_\mathcal{F}(R_0))$. Hence, $[R_0, O^p(X)] = 1$. Since $[R, N_{O_p(\mathcal{E})}(R)] \leq [R, N_T(R)] \leq T \cap R = R_0$, we have $[R, X] \leq R_0$. Thus, $O^p(X) = 1$ meaning that X is a normal p-subgroup of $\mathrm{Aut}_\mathcal{F}(R)$. Consequently, as R is centric radical, $\mathrm{Aut}_{O_p(\mathcal{E})}(R) \leq X \leq \mathrm{Inn}(R)$ and $O_p(\mathcal{E}) \leq R$. □

The following Lemma can be seen as a fusion system version of Lemma 2.9.

Lemma 2.13. *Let $Q \in \mathcal{F}^f$. Then $N_\mathcal{F}(Q)$ is constrained if and only if $C_\mathcal{F}(Q)$ is constrained.*

Proof. If $N_\mathcal{F}(Q)$ is constrained, then it follows from Lemma 2.11 applied to $N_\mathcal{F}(Q)$ in place of $\mathcal{F}$ that $C_\mathcal{F}(Q)$ is constrained. Assume now $C_\mathcal{F}(Q)$ is constrained. By [1, 1.25], $C_\mathcal{F}(Q)$ is weakly normal in $N_\mathcal{F}(Q)$. It follows now from Lemma 2.12 that $R := QO_p(C_\mathcal{F}(Q)) \trianglelefteq N_\mathcal{F}(Q)$. Moreover, $C_{N_S(Q)}(R) = C_{C_S(Q)}(O_p(C_\mathcal{F}(Q))) \leq O_p(C_\mathcal{F}(Q)) \leq R$ as $C_\mathcal{F}(Q)$ is constrained. Thus, $N_\mathcal{F}(Q)$ is constrained. □

The reader is referred to [6, Section I.7] for definitions and properties of subsystems of index prime to p and of subsystems of p-power index.

Lemma 2.14. *Let $\mathcal{E}$ be a normal subsystem of $\mathcal{F}$ of index prime to p. Then $\mathcal{E}$ is constrained if and only if $\mathcal{F}$ is constrained.*

Proof. Clearly, $O_p(\mathcal{F})$ is normal in $\mathcal{E}$, so $O_p(\mathcal{E}) = O_p(\mathcal{F})$ by Lemma 2.12. As $\mathcal{E} \cap S = S$, it follows that $\mathcal{E}$ is a constrained if and only if $C_S(O_p(\mathcal{F})) \leq O_p(\mathcal{F})$, which is the case if and only if $\mathcal{F}$ is constrained. □

Lemma 2.15. *Let $\mathcal{F}$ be constrained and let $\mathcal{E}$ be a weakly normal subsystem of $\mathcal{F}$. Then $\mathcal{E}$ is constrained.*

Proof. By Lemma 2.14, $\mathcal{E}$ is constrained if and only if $O^{p'}(\mathcal{E})$ is constrained. By a theorem of Craven [6, Theorem I.7.8], $O^{p'}(\mathcal{E})$ is normal in $\mathcal{F}$. So replacing $\mathcal{E}$ by $O^{p'}(\mathcal{E})$, we may assume that $\mathcal{E}$ is normal in $\mathcal{F}$. Let G be a model for $\mathcal{F}$, which exist by Theorem 2.1(a). By [6, Lemma II.7.4], there exists a normal subgroup N of G such that $T := \mathcal{E} \cap S = N \cap S \in \mathrm{Syl}_p(N)$ and $\mathcal{E} = \mathcal{F}_T(N)$. By Lemma 2.2(b), N is of characteristic p and thus $\mathcal{E}$ is constrained by Theorem 2.1(a). □

The following lemma is a version of [17, Lemma 1.3] for fusion systems, except that we do not require the subsystem $\mathcal{E}$ to be normal in $\mathcal{F}$. A different proof could be given using the theory of components of fusion systems as developed in [2], but we prefer to keep the proof as elementary as possible.

Lemma 2.16. *Let $\mathcal{E}$ be a subsystem of $\mathcal{F}$ of p-power index. Then $\mathcal{E}$ is constrained if and only if $\mathcal{F}$ is constrained.*

Proof. Let $T = \mathcal{E} \cap S$. Let $T = T_0 \trianglelefteq T_1 \trianglelefteq \cdots \trianglelefteq T_n = S$ be a chain of subgroups such that $|T_i/T_{i-1}| = p$ for $i = 1, \ldots, n$. By [6, Theorem I.7.4], there is a unique subsystem $\mathcal{F}_{T_i} = \langle \mathrm{Inn}(T_i), O^p(\mathrm{Aut}_{\mathcal{F}}(P))\colon P \leq T_i \rangle$ of $\mathcal{F}$ of p-power index over T_i for every $i = 1, \ldots, n$. In particular, $\mathcal{F}_T = \mathcal{F}_{T_0} = \mathcal{E}$. Again by [6, Theorem I.7.4], $\mathcal{F}_{T_{i-1}}$ is a normal subsystem of $\mathcal{F}_{T_i}$ of p-power index for $i = 1, \ldots, n$. Hence, we can reduce to the case that $|S : T| = p$ and $\mathcal{E}$ is normal in $\mathcal{F}$. By Lemma 2.12, $Q := O_p(\mathcal{E})$ is normal in $\mathcal{F}$. It is sufficient to show that $P := QC_S(Q)$ is normal in $\mathcal{F}$. As $\mathcal{E}$ is constrained, $C_T(Q) \leq Q$ and thus $|P : Q| \leq |S : T| = p$. As Q is normal in $\mathcal{F}$, P is weakly closed in $\mathcal{F}$. We prove now that P is strongly closed. Let $X \leq P$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(X, S)$. If $X \leq Q$ then $X\varphi \leq Q \leq P$. If $X \not\leq Q$ then $P = QX$ as $|P : Q| \leq p$. Since $Q \trianglelefteq \mathcal{F}$, φ extends in this case to an element $\mathrm{Hom}_{\mathcal{F}}(P, S)$. As P is weakly closed in $\mathcal{F}$, it follows $X\varphi \leq P$. So P is strongly closed. By [6, Proposition I.4.6], there exists a series $1 = P_0 \leq P_1 \leq \ldots P_n = Q$ of subgroups strongly closed in $\mathcal{F}$ such that $[P_i, Q] \leq P_{i-1}$ for $i = 1, \ldots, n$. Since $P = QC_S(Q) \leq QC_S(P_i)$, it follows $[P_i, P] \leq P_{i-1}$ for $i = 1, \ldots, n$. As $|P : Q| \leq p$, we have $[P, P] \leq Q$. Hence, $P \trianglelefteq \mathcal{F}$ by [6, Proposition I.4.6]. As $C_S(P) \leq C_S(Q) \leq P$, it follows that $\mathcal{F}$ is constrained. □

The reader is referred to [6, Section I.5] for the definitions of K-normalizers. We will need the following elementary lemma:

Lemma 2.17. *Let $Q \leq S$, $K \leq \mathrm{Aut}(Q)$, $P \leq N_S^K(Q)$ and $\alpha \in \mathrm{Hom}_{\mathcal{F}}(PQ, S)$. Then $P\alpha \leq N_S^{K^\alpha}(Q\alpha)$.*

Proof. Notice that for any $s \in P$ and any $x \in Q$, $(x\alpha)^{s\alpha} = (x^s)\alpha = (x\alpha)\alpha^{-1}c_s\alpha = (x\alpha)(c_s|_Q)^\alpha$. So as $c_s|_Q \in K$, $c_{s\alpha}|_{Q\alpha} = (c_s|_Q)^\alpha \in K^\alpha$ and thus $s\alpha \in N_S^{K^\alpha}(Q\alpha)$. □

Lemma 2.18. *Let $Q \leq S$ and $K \leq \mathrm{Aut}(Q)$ such that $N_{\mathcal{F}}^K(Q)$ is saturated.*

(a) *Let $K_0 \trianglelefteq K$ such that $N_{\mathcal{F}}^{K_0}(Q)$ is saturated. Then $N_{\mathcal{F}}^{K_0}(Q)$ is weakly normal in $N_{\mathcal{F}}^K(Q)$.*
(b) *Suppose $C_{\mathcal{F}}(Q)$ is saturated. Then $C_{\mathcal{F}}(Q)$ is weakly normal in $N_{\mathcal{F}}^K(Q)$. In particular, $C_{\mathcal{F}}(Q)$ is constrained if $N_{\mathcal{F}}^K(Q)$ is constrained.*

Proof. As $C_{\mathcal{F}}(Q) = N_{\mathcal{F}}^{\{\mathrm{id}_Q\}}(Q)$, part (b) follows from part (a) and Lemma 2.15. So it remains to prove (a). As $N_{\mathcal{F}}^{K_0}(Q)$ and $N_{\mathcal{F}}^{K}(Q)$ are by assumption both saturated it remains to prove that $N_{\mathcal{F}}^{K_0}(Q)$ is $N_{\mathcal{F}}^{K}(Q)$-invariant. We show first that $N_S^{K_0}(Q)$ is strongly closed in $N_{\mathcal{F}}^{K}(Q)$. Let $R \leq N_S^{K_0}(Q)$ and $\alpha \in \mathrm{Hom}_{N_{\mathcal{F}}^K(Q)}(R, N_S^K(Q))$. Then it follows from the definition of $N_{\mathcal{F}}^K(Q)$ that α extends to $\hat{\alpha} \in \mathrm{Hom}_{\mathcal{F}}(RQ, S)$ with $\hat{\alpha}|_Q = K$. As $K_0 \trianglelefteq K$ and $\hat{\alpha}|_Q \in K$, we have $K_0^{\hat{\alpha}} = K_0$. Hence, by Lemma 2.17, $R\alpha \leq N_S^{K_0^{\hat{\alpha}}}(Q\hat{\alpha}) = N_S^{K_0}(Q)$. This shows that $N_S^{K_0}(Q)$ is strongly closed in $N_{\mathcal{F}}^K(Q)$.

We show now that the strong invariance condition as given in [6, Proposition I.6.4] holds. For that let $A \leq B \leq N_S^{K_0}(Q)$, $\varphi \in \mathrm{Hom}_{N_{\mathcal{F}}^{K_0}(Q)}(A, B)$ and $\psi \in \mathrm{Hom}_{N_{\mathcal{F}}^K(Q)}(B, N_S^{K_0}(Q))$. We need to prove that $\varphi^\psi := (\psi|_A)^{-1}\varphi\psi \in \mathrm{Hom}_{N_{\mathcal{F}}^{K_0}(Q)}(A\psi, B\psi)$. By the definitions of $N_{\mathcal{F}}^{K_0}(Q)$ and $N_{\mathcal{F}}^K(Q)$, φ extends to $\hat{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(AQ, BQ)$ with $\hat{\varphi}|_Q \in K_0$, and ψ extends to $\hat{\psi} \in \mathrm{Hom}_{\mathcal{F}}(BQ, S)$ with $\hat{\psi}|_Q \in K$. Then $\hat{\varphi}^{\hat{\psi}} := (\hat{\psi}|_{AQ})^{-1}\hat{\varphi}\hat{\psi} \in \mathrm{Hom}_{\mathcal{F}}((A\psi)Q, (B\psi)Q)$ extends φ^ψ. Moreover, $(\hat{\varphi}^{\hat{\psi}})|_Q = (\hat{\psi}|_Q)^{-1}(\hat{\varphi}|_Q)(\hat{\psi}|_Q) \in K_0$ as $\hat{\varphi}|_Q \in K_0$, $\hat{\psi}|_Q \in K$ and $K_0 \trianglelefteq K$. This shows that φ^ψ is a morphism in $N_{\mathcal{F}}^{K_0}(Q)$ as required. □

3. Properties of subcentric subgroups

Throughout this section, $\mathcal{F}$ is assumed to be saturated.

Lemma 3.1. *For any $Q \in \mathcal{F}$, the following conditions are equivalent:*

- (a1) *The subgroup Q is subcentric in $\mathcal{F}$.*
- (a2) *For some fully normalized $\mathcal{F}$-conjugate P of Q, $O_p(N_{\mathcal{F}}(P))$ is centric in $\mathcal{F}$.*
- (b1) *For any fully normalized $\mathcal{F}$-conjugate P of Q, $N_{\mathcal{F}}(P)$ is constrained.*
- (b2) *For some fully normalized $\mathcal{F}$-conjugate P of Q, $N_{\mathcal{F}}(P)$ is constrained.*
- (c1) *For any fully centralized $\mathcal{F}$-conjugate P of Q, $C_{\mathcal{F}}(P)$ is constrained.*
- (c2) *For some fully centralized $\mathcal{F}$-conjugate P of Q, $C_{\mathcal{F}}(P)$ is constrained.*

Proof. If $P, P^* \in Q^{\mathcal{F}}$ are both fully normalized, then it follows from [6, Lemma I.2.6(c)] that there exists an isomorphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(N_S(P), N_S(P^*))$ such that $P\varphi = P^*$. It is straightforward to check that any such φ induces an isomorphism from $N_{\mathcal{F}}(P)$ to $N_{\mathcal{F}}(P^*)$ and thus $N_{\mathcal{F}}(P)$ is constrained if and only if $N_{\mathcal{F}}(P^*)$ is constrained. Moreover, $O_p(N_{\mathcal{F}}(P))\varphi = O_p(N_{\mathcal{F}}(P^*))$. Thus, conditions (b1) and (b2) are equivalent, and conditions (a1) and (a2) are equivalent. Similarly, if $P, P^* \in Q^{\mathcal{F}}$ are both fully centralized in $\mathcal{F}$, then by the extension axiom, there exists $\varphi \in \mathrm{Hom}_{\mathcal{F}}(C_S(P)P, C_S(P^*)P^*)$ with $P\varphi = P^*$ and $\varphi|_{C_S(P)}$ induces and isomorphism from $C_{\mathcal{F}}(P)$ to $C_{\mathcal{F}}(P^*)$. This proves that conditions (c1) and (c2) are equivalent. Let now $P \in Q^{\mathcal{F}}$ be fully normalized. By Lemma 2.13, $N_{\mathcal{F}}(P)$ is constrained if and only if $C_{\mathcal{F}}(P)$ is constrained. Since every fully normalized subgroup is fully centralized, this shows that (b2) implies (c2) and that (c1) implies (b1). Set now $R := O_p(N_{\mathcal{F}}(P))$. If Q is subcentric, then $C_{N_S(P)}(R) = C_S(R) \leq R$ and so $N_{\mathcal{F}}(P)$ is constrained. Hence, (a1) implies (b1). Assume now $N_{\mathcal{F}}(P)$ is constrained. By [6, Lemma I.2.6(c)], there exists $\varphi \in \mathrm{Hom}_{\mathcal{F}}(N_S(R), S)$ such that $R\varphi \in \mathcal{F}^f$. As $N_S(P) \leq N_S(R)$ and P is fully normalized, it follows $N_S(P)\varphi = N_S(P\varphi)$ and $P\varphi \in \mathcal{F}^f$. Again, $\varphi|_{N_S(P)}$ induces an isomorphism from $N_{\mathcal{F}}(P)$ to $N_{\mathcal{F}}(P\varphi)$ and thus $R\varphi = O_p(N_{\mathcal{F}}(P\varphi))$ and $N_{\mathcal{F}}(P\varphi)$ is constrained. Hence, $C_S(R\varphi) = C_{N_S(P\varphi)}(R\varphi) \leq R\varphi$. So $R\varphi$ and thus R is centric as $R\varphi$ is fully normalized. Hence, (b2) implies (a2). □

Looking more generally at K-normalizers rather than at centralizers and normalizers of subgroups of S, we get the following sufficient condition for a subgroup to be subcentric:

Lemma 3.2. *Let $Q \in \mathcal{F}$ and $K \leq \mathrm{Aut}(Q)$ such that Q is fully K-normalized. If $N_{\mathcal{F}}^K(Q)$ is constrained then Q is subcentric.*

Proof. Since Q is fully K-normalized, Q is fully centralized by [6, Proposition I.5.2]. Now by [6, Theorem I.5.5], $C_{\mathcal{F}}(Q)$ and $N_{\mathcal{F}}^K(Q)$ are saturated. If $N_{\mathcal{F}}^K(Q)$ is constrained, it follows therefore from Lemma 2.18 that $C_{\mathcal{F}}(Q)$ is constrained. So Q is subcentric by Lemma 3.1. □

Proposition 3.3. *The set $\mathcal{F}^s$ of subcentric subgroups of $\mathcal{F}$ is closed under taking $\mathcal{F}$-conjugates and overgroups.*

Proof. Note first that the set of subcentric subgroups is by definition closed under $\mathcal{F}$-conjugation. Let $Q \in \mathcal{F}^s$ and R an overgroup of Q. We need to show that R is subcentric. By induction on the length of a subnormal series of Q in R, we reduce to the case that $Q \trianglelefteq R$. Since every $\mathcal{F}$-conjugate of Q is subcentric, and any $\mathcal{F}$-conjugate of R contains an $\mathcal{F}$-conjugate of Q, we can and will furthermore assume from now on that $R \in \mathcal{F}^f$. Replacing Q be a suitable conjugate of Q in $N_{\mathcal{F}}(R)$ we will also assume that $Q \in N_{\mathcal{F}}(R)^f$. By [6, I.2.6(c)], there exists $\alpha \in \mathrm{Hom}_{\mathcal{F}}(N_S(Q), S)$ such that $Q\alpha \in \mathcal{F}^f$. Then by [2, (2.2)(1),(2)], $(N_S(Q) \cap N_S(R))\alpha = N_S(Q\alpha) \cap N_S(R\alpha)$, $R\alpha \in N_{\mathcal{F}}(Q\alpha)^f$, and α induces an isomorphism from $\mathcal{N}_1 := N_{N_{\mathcal{F}}(R)}(Q)$ to $\mathcal{N}_2 := N_{N_{\mathcal{F}}(Q\alpha)}(R\alpha)$. As Q is subcentric and $Q\alpha \in \mathcal{F}^f$, $N_{\mathcal{F}}(Q\alpha)$ is constrained by Lemma 3.1. Therefore, by Lemma 2.11 applied with $N_{\mathcal{F}}(Q\alpha)$ in place of $\mathcal{F}$, $C_{\mathcal{N}_2}(R\alpha) = C_{N_{\mathcal{F}}(Q\alpha)}(R\alpha)$ is constrained. So since α induces an isomorphism $\mathcal{N}_1 \to \mathcal{N}_2$, $C_{\mathcal{F}}(R) = C_{\mathcal{N}_1}(R)$ is constrained. Now by Lemma 3.1, R is subcentric. □

Lemma 3.4. *Let $R \trianglelefteq \mathcal{F}$ and $P \in \mathcal{F}$. Then $PR \in \mathcal{F}^s$ if and only if $P \in \mathcal{F}^s$.*

Proof. If $P \in \mathcal{F}^s$ then by Proposition 3.3, $PR \in \mathcal{F}^s$. From now on we assume that $PR \in \mathcal{F}^s$ and want to show that $P \in \mathcal{F}^s$. Since $\mathcal{F}^s$ is closed under $\mathcal{F}$-conjugation, we can assume without loss of generality that $PR \in \mathcal{F}^f$. As $PR \in \mathcal{F}^s$ this means that $N_{\mathcal{F}}(PR)$ is a constrained fusion system. If Q is a fully normalized $\mathcal{F}$-conjugate of P then an isomorphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, P)$ extends to $\hat{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(QR, S)$ with $R\hat{\varphi} = R$. Hence, as $(QR)\hat{\varphi} = PR \in \mathcal{F}^f$, there exists by [6, I.2.6(c)] a morphism $\alpha \in \mathrm{Hom}_{\mathcal{F}}(N_S(QR), S)$ such that $(Q\alpha)R = (QR)\alpha = PR$. As $N_S(Q) \leq N_S(QR)$ and Q is fully normalized, it follows that $Q\alpha$ is fully normalized. So replacing P by $Q\alpha$, we may assume that P is fully normalized in $\mathcal{F}$. Then P is also fully normalized in $N_{\mathcal{F}}(PR)$ and thus $N_{N_{\mathcal{F}}(PR)}(P)$ is constrained by Lemma 2.11. One easily observes that $N_{\mathcal{F}}(P) = N_{N_{\mathcal{F}}(PR)}(P)$, as R is normal in $\mathcal{F}$. So $N_{\mathcal{F}}(P)$ is constrained and P is subcentric by Lemma 3.1. □

Lemma 3.5. *Let $Z \leq Z(\mathcal{F})$ and $P \leq S$. Then $P \in \mathcal{F}^s$ if and only if PZ/Z is subcentric in $\mathcal{F}/Z$.*

Proof. By Lemma 3.4, we may assume that $Z \leq P$. Moreover, we can assume $P \in \mathcal{F}^f$. Since $Z \leq P$, we have $Z \leq Q$ for every $Q \in P^{\mathcal{F}}$. Moreover, the $\mathcal{F}/Z$-conjugates of P/Z are precisely the subgroups of the form Q/Z with $Q \in P^{\mathcal{F}}$. Observe also that $N_{S/Z}(Q/Z) = N_S(Q)/Z$ for every $Q \in P^{\mathcal{F}}$. Hence, P/Z is fully normalized in $\mathcal{F}/Z$. Clearly, $N_{\mathcal{F}}(P)/Z = N_{\mathcal{F}/Z}(P/Z)$ and $Z \leq Z(N_{\mathcal{F}}(P))$. Therefore, by Lemma 2.10, $N_{\mathcal{F}}(P)$ is constrained if and only if $N_{\mathcal{F}/Z}(P/Z)$ is constrained. The assertion follows now from Lemma 3.1. □

Suppose $\tilde{\mathcal{F}}$ is a fusion system on a p-group $\tilde{S}$ and $\alpha\colon S \to \tilde{S}$ is an isomorphism of groups. We say that α induces an isomorphism of fusion systems from $\mathcal{F}$ to $\tilde{\mathcal{F}}$ if, for all $P, Q \leq S$, the map $\alpha_{P,Q}\colon \mathrm{Hom}_{\mathcal{F}}(P, Q) \to \mathrm{Hom}_{\tilde{\mathcal{F}}}(P\alpha, Q\alpha)$ with $\varphi \mapsto \varphi^\alpha := \alpha^{-1}\varphi\alpha$ is a bijection. The maps $\alpha_{P,Q}$ $(P, Q \leq S)$ together with the map $P \mapsto P\alpha$ from the set of objects of $\mathcal{F}$ to the set of objects of $\tilde{\mathcal{F}}$ give us an invertible functor from $\mathcal{F}$ to $\tilde{\mathcal{F}}$. Moreover, α together with the maps $\alpha_{P,Q}$ $(P, Q \leq S)$ is a morphism in the sense of [6, Definition II.2.2].

Lemma 3.6. *Let $\tilde{\mathcal{F}}$ be a saturated fusion system on a p-group $\tilde{S}$ and $\alpha\colon S \to \tilde{S}$ a group isomorphism which induces an isomorphism of fusion systems $\mathcal{F} \to \tilde{\mathcal{F}}$. Then $\tilde{\mathcal{F}}^s = \{P\alpha\colon P \in \mathcal{F}^s\}$.*

Proof. Note that $N_S(Q)\alpha = N_{\tilde{S}}(Q\alpha)$ for any $Q \leq S$. Moreover, for $P \leq S$ and $\psi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$, $P\psi\alpha = P\alpha(\alpha^{-1}\psi\alpha) \in (P\alpha)^{\tilde{\mathcal{F}}}$, since $\psi^\alpha = \alpha^{-1}\psi\alpha$ is a morphism in $\tilde{\mathcal{F}}$ as α induces an isomorphism of fusion systems. Hence, $\{Q\alpha : Q \in P^{\mathcal{F}}\} = (P\alpha)^{\tilde{\mathcal{F}}}$ and $Q \in P^{\mathcal{F}}$ is fully $\mathcal{F}$-normalized if and only if $Q\alpha$ is fully $\tilde{\mathcal{F}}$-normalized. Let now $Q \in P^{\mathcal{F}}$ be fully $\mathcal{F}$-normalized. Then $\alpha|_{N_S(Q)} : N_S(Q) \to N_{\tilde{S}}(Q\alpha)$ induces an isomorphism from $N_{\mathcal{F}}(Q)$ to $N_{\tilde{\mathcal{F}}}(Q\alpha)$. In particular, $N_{\mathcal{F}}(Q)$ is constrained if and only if $N_{\tilde{\mathcal{F}}}(Q\alpha)$ is constrained. Hence, by Lemma 3.1 $P \in \mathcal{F}^s$ if and only if $P\alpha \in \tilde{\mathcal{F}}^s$. □

Lemma 3.7. *Let $\mathcal{E}$ be weakly normal in $\mathcal{F}$, $P \in \mathcal{E}^s$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$. Then $P\varphi \in \mathcal{E}^s$.*

Proof. Let $T = \mathcal{E} \cap S$. Note that $P\varphi \leq T$ as T is strongly closed. By the Frattini condition [6, Definition I.6.1], there are $\alpha \in \mathrm{Aut}_{\mathcal{F}}(T)$ and $\varphi_0 \in \mathrm{Hom}_{\mathcal{E}}(P,T)$ such that $\varphi = \varphi_0\alpha$. As φ_0 is a morphism in $\mathcal{E}$, $P\varphi_0 \in \mathcal{E}^s$. As $\mathcal{E}$ is normal in $\mathcal{F}$, α induces and automorphism of $\mathcal{E}$. Hence, by Lemma 3.6 applied with $\mathcal{E}$ in the role of $\mathcal{F}$ and $\tilde{\mathcal{F}}$, $P\varphi = (P\varphi_0)\alpha \in \mathcal{E}^s$. □

Before we continue proving properties of subcentric subgroups we need two general lemmas.

Lemma 3.8. *Let $\mathcal{E}$ be an $\mathcal{F}$-invariant subsystem of $\mathcal{F}$ over $T \leq S$. Let $P \in \mathcal{E}^f$ and $\alpha \in \mathrm{Hom}_{\mathcal{F}}(N_T(P),S)$. Then $P\alpha \in \mathcal{E}^f$, $N_T(P)\alpha = N_T(P\alpha)$ and α induces an isomorphism from $N_{\mathcal{E}}(P)$ to $N_{\mathcal{E}}(P\alpha)$.*

Proof. By the Frattini condition [6, Definition I.6.1] there are $\alpha_0 \in \mathrm{Hom}_{\mathcal{E}}(N_T(P),T)$ and $\beta \in \mathrm{Aut}_{\mathcal{F}}(T)$ such that $\alpha = \alpha_0\beta$. Clearly, $N_T(P)\alpha_0 \leq N_T(P\alpha_0)$ because T is strongly closed in $\mathcal{F}$. As $P \in \mathcal{E}^f$, it follows $N_T(P)\alpha_0 = N_T(P\alpha_0)$. Since β is an automorphism of T, $N_T(P\alpha_0)\beta = N_T(P\alpha_0\beta) = N_T(P\alpha)$. Hence, $N_T(P)\alpha = N_T(P)\alpha_0\beta = N_T(P\alpha_0)\beta = N_T(P\alpha)$. Since $\mathcal{E}$ is $\mathcal{F}$-invariant, it is now straightforward to check that α induces an isomorphism from $N_{\mathcal{E}}(P)$ to $N_{\mathcal{E}}(P\alpha)$. □

Lemma 3.9. *Let $\mathcal{E}$ be an $\mathcal{F}$-invariant subsystem of $\mathcal{F}$ over $T \leq S$, and $P \leq T$. If $P \in \mathcal{F}^f$ then $P \in \mathcal{E}^f$.*

Proof. Suppose $P \in \mathcal{F}^f$ and choose a fully $\mathcal{E}$-normalized $\mathcal{E}$-conjugate Q of P. By [6, I.2.6(c)], there exists $\alpha \in \mathrm{Hom}_{\mathcal{F}}(N_S(Q),S)$ such that $Q\alpha = P$. Applying Lemma 3.8 with Q in place of P yields then $|N_T(Q)| = |N_T(P)|$ and thus $P \in \mathcal{E}^f$. □

Lemma 3.10. *Let $\mathcal{E}$ be a weakly normal subsystem of $\mathcal{F}$ over $T \leq S$. Then $P \in \mathcal{E}^s$ for any $P \in \mathcal{F}^s$ with $P \leq T$.*

Proof. By Lemma 3.7, we may replace P by any $\mathcal{F}$-conjugate of P and can thus assume that $P \in \mathcal{F}^f$. Then by Lemma 3.9, $P \in \mathcal{E}^f$. So $N_{\mathcal{F}}(P)$ and $N_{\mathcal{E}}(P)$ are saturated. It is now easy to check that $N_{\mathcal{E}}(P)$ is weakly normal in $N_{\mathcal{F}}(P)$. Since $P \in \mathcal{F}^s$, $N_{\mathcal{F}}(P)$ is constrained by Lemma 3.1. Hence, by Lemma 2.15, $N_{\mathcal{E}}(P)$ is constrained and $P \in \mathcal{E}^s$ again by Lemma 3.1. □

Lemma 3.11. *Let $\mathcal{E}$ be a normal subsystem of $\mathcal{F}$ of index prime to p. Then $\mathcal{E}^s = \mathcal{F}^s$.*

Proof. By Lemma 3.10, we only need to prove that $\mathcal{E}^s \subseteq \mathcal{F}^s$. By Lemma 3.7, it is sufficient to prove $\mathcal{E}^s \cap \mathcal{F}^f \subseteq \mathcal{F}^s$. Let $P \in \mathcal{E}^s \cap \mathcal{F}^f$. By Lemma 3.9, $P \in \mathcal{E}^f$. Thus $N_{\mathcal{F}}(P)$ and $N_{\mathcal{E}}(P)$ are saturated subsystems and one sees easily that $N_{\mathcal{E}}(P)$ is a weakly normal subsystem of $N_{\mathcal{F}}(P)$. As they are both fusion systems over $N_S(P)$, it follows that $N_{\mathcal{E}}(P)$ actually normal in $N_{\mathcal{F}}(P)$ and a subsystem of index prime to p. As $P \in \mathcal{E}^s$, $N_{\mathcal{E}}(P)$ is constrained by Lemma 3.1. Hence, by Lemma 2.14, $N_{\mathcal{F}}(P)$ is constrained and $P \in \mathcal{F}^s$ again by Lemma 3.1. □

Lemma 3.12. *Let $\mathcal{E}$ be a normal subsystem of $\mathcal{F}$ of p-power index and $T = \mathcal{E} \cap S$. Then $\mathcal{E}^s = \{P \in \mathcal{F}^s \colon P \leq T\}$.*

Proof. By Lemma 3.10, it remains only to prove that $\mathcal{E}^s \subseteq \mathcal{F}^s$. By Lemma 3.7, it is sufficient to prove $\mathcal{E}^s \cap \mathcal{F}^f \subseteq \mathcal{F}^s$. Let $P \in \mathcal{E}^s \cap \mathcal{F}^f$. By Lemma 3.9, $P \in \mathcal{E}^f$. Hence, $N_{\mathcal{F}}(P)$ and $N_{\mathcal{E}}(P)$ are saturated. It follows from the definition of the hyperfocal subgroup that $\mathfrak{hyp}(N_{\mathcal{F}}(P)) \leq \mathfrak{hyp}(\mathcal{F}) \leq T$ and thus $\mathfrak{hyp}(N_{\mathcal{F}}(P)) \leq N_T(P)$. For any $R \leq N_T(P)$, a p'-element $\alpha \in \mathrm{Aut}_{N_{\mathcal{F}}(P)}(R)$ extends to a p'-element $\hat{\alpha} \in \mathrm{Aut}_{\mathcal{F}}(PR)$ normalizing P. As $\mathcal{E}$ is a subsystem of $\mathcal{F}$ of p-power index, $\hat{\alpha} \in O^p(\mathrm{Aut}_{\mathcal{F}}(PR)) \leq \mathrm{Aut}_{\mathcal{E}}(PR)$. Hence, α extends to an element of $\mathrm{Aut}_{\mathcal{E}}(PR)$ normalizing

R, which means $\alpha \in \mathrm{Aut}_{N_{\mathcal{E}}(P)}(R)$. This shows that $N_{\mathcal{E}}(P)$ is a subsystem of $N_{\mathcal{F}}(P)$ of p-power index. As $P \in \mathcal{E}^s$, $N_{\mathcal{E}}(P)$ is constrained by Lemma 3.1. Hence, by Lemma 2.16, it follows that $N_{\mathcal{F}}(P)$ is constrained and $P \in \mathcal{F}^s$ by Lemma 3.1. □

Lemma 3.13. *Let $Q \leq S$ and $K \leq \mathrm{Aut}(Q)$ such that Q is fully K-normalized. Let $P \leq N_S^K(Q)$ such that P is fully centralized in $N_{\mathcal{F}}^K(Q)$. Then PQ is fully centralized in $\mathcal{F}$.*

Proof. We note first that $N_{\mathcal{F}}^K(Q)$ is saturated by [6, Theorem I.5.5] as Q is fully K-normalized.

Step 1: Let $P_0 \in P^{N_{\mathcal{F}}^K(Q)}$. Then we show that $|C_S(P_0Q)| \leq |C_S(PQ)|$. Observe first that, by the extension axiom, an $\mathcal{F}$-isomorphism from P_0 to P extends to a morphism

$$\alpha \in \mathrm{Hom}_{N_{\mathcal{F}}^K(Q)}(P_0C_{N_S^K(Q)}(P_0), PC_{N_S^K(Q)}(P)).$$

So we can fix $\alpha \in \mathrm{Hom}_{N_{\mathcal{F}}^K(Q)}(P_0C_{N_S^K(Q)}(P_0), PC_{N_S^K(Q)}(P))$ with $P_0\alpha = P$. It follows from the definition of $N_{\mathcal{F}}^K(Q)$ that α extends to $\hat{\alpha} \in \mathrm{Hom}_{\mathcal{F}}(P_0C_{N_S^K(Q)}(P_0)Q, PC_{N_S^K(Q)}(P)Q)$ with $Q\hat{\alpha} = Q$. Note $C_S(P_0Q) \leq C_{N_S^K(Q)}(P_0)$. As $P_0\hat{\alpha} = P_0\alpha = P$ we have $(P_0Q)\hat{\alpha} = PQ$. Hence $C_S(P_0Q)\alpha = C_S(P_0Q)\hat{\alpha} \subseteq C_S((P_0Q)\hat{\alpha}) = C_S(PQ)$. Therefore, as α is injective, $|C_S(P_0Q)| = |C_S(P_0Q)\alpha| \leq |C_S(PQ)$. This finishes Step 1.

Step 2: We are now in a position to complete the proof. Let $\varphi \in \mathrm{Hom}_{\mathcal{F}}(PQ, S)$ such that $(PQ)\varphi$ is fully centralized. Our goal will be to show that $|C_S((PQ)\varphi)| \leq |C_S(PQ)|$. Note that φ^{-1} restricts to an $\mathcal{F}$-isomorphism from $Q\varphi$ to Q. As Q is fully K-normalized it follows from [6, Proposition I.5.2] that there exists $\chi \in \mathrm{Aut}_{\mathcal{F}}^K(Q)$ and $\psi \in \mathrm{Hom}_{\mathcal{F}}((Q\varphi)N_S^{K^\varphi}(Q\varphi), S)$ such that $\psi|_{Q\varphi} = (\varphi^{-1}|_{Q\varphi})\chi$. As $P \leq N_S^K(Q)$, we have $P\varphi \leq N_S^{K^\varphi}(Q\varphi)$ by Lemma 2.17. In particular, ψ is defined on $(PQ)\varphi = (P\varphi)(Q\varphi)$ and $\varphi\psi$ is defined on PQ. Note that $(\varphi\psi)|_Q = \chi \in K$ and so $(\varphi\psi)|_P$ is a morphism $N_{\mathcal{F}}^K(Q)$. Therefore, by Step 1, we have $|C_S((P\varphi\psi)Q)| \leq |C_S(PQ)|$. Note that $Q\varphi\psi = Q$ and thus $(PQ)\varphi\psi = (P\varphi\psi)Q$. Moreover, ψ is defined on $C_S((PQ)\varphi) \leq C_S(Q\varphi) \leq N_S^{K^\varphi}(Q\varphi)$ and $C_S((PQ)\varphi)\psi \leq C_S((PQ)\varphi\psi) = C_S((P\varphi\psi)Q)$. Putting these properties together, we obtain $|C_S((PQ)\varphi)| = |C_S((PQ)\varphi)\psi| \leq |C_S((P\varphi\psi)Q)| \leq |C_S(PQ)|$. As $(PQ)\varphi$ is fully centralized, it follows that PQ is fully centralized as well. □

Lemma 3.14. *Let $Q \in \mathcal{F}$ and $K \leq \mathrm{Aut}(Q)$ such that Q is fully K-normalized. Then $PQ \in \mathcal{F}^s$ for every $P \in N_{\mathcal{F}}^K(Q)^s$.*

A similar result holds for centric and quasicentric subgroups: For every $P \in N_{\mathcal{F}}^K(Q)^c$ we have $PQ \in \mathcal{F}^c$, and for every $P \in N_{\mathcal{F}}^K(Q)^q$ we have $PQ \in \mathcal{F}^q$.

Proof. Let $P \leq N_S^K(Q)$. We want to show that $PQ \in \mathcal{F}^s$ if $P \in N_{\mathcal{F}}^K(Q)^s$, that $PQ \in \mathcal{F}^c$ if $P \in N_{\mathcal{F}}^K(Q)^c$ and that $PQ \in \mathcal{F}^q$ if $P \in N_{\mathcal{F}}^K(Q)^q$. Since the collections of centric, quasicentric and subcentric subgroups are closed under taking conjugates in the respective fusion system, we can replace P by a suitable $N_{\mathcal{F}}^K(Q)$-conjugate and will assume without loss of generality that P is fully centralized in $N_{\mathcal{F}}^K(Q)$. Then PQ is fully centralized in $\mathcal{F}$ by Lemma 3.13.

Assume first $P \in N_{\mathcal{F}}^K(Q)^s$. Then by Lemma 3.1, $\mathcal{C} := C_{N_{\mathcal{F}}^K(Q)}(P)$ is a constrained (saturated) subsystem. Note that $\mathcal{C} = N_{\mathcal{F}}^{\tilde{K}}(PQ)$ where $\tilde{K} := \{\alpha \in \mathrm{Aut}(PQ)\colon \alpha|_Q \in K,\ \alpha|_P = \mathrm{id}_P\}$. Moreover, as PQ is fully $\mathcal{F}$-centralized, $C_{\mathcal{F}}(PQ)$ is saturated. Hence, by Lemma 2.18(b), $C_{\mathcal{F}}(PQ)$ is constrained. Since PQ is fully $\mathcal{F}$-centralized, Lemma 3.1 implies $PQ \in \mathcal{F}^s$ as required.

If $P \in N_{\mathcal{F}}^K(Q)^c$ then $C_S(PQ) = C_{C_S(Q)}(P) \leq C_{N_S^K(Q)}(P) \leq P \leq PQ$ and so PQ is centric since PQ is fully centralized. Suppose now $P \in N_{\mathcal{F}}^K(Q)^q$. Then $\mathcal{C} := C_{N_{\mathcal{F}}^K(Q)}(P)$ is the fusion system of the p-group $C_{N_S^K(Q)}(P)$. As $C_{\mathcal{F}}(PQ)$ is a subsystem of $\mathcal{C}$, it follows that $\mathrm{Aut}_{C_{\mathcal{F}}(PQ)}(R)$ is a p-group for every $R \leq C_S(PQ)$. So if $R \in C_{\mathcal{F}}(PQ)^f$ then $\mathrm{Aut}_{C_{\mathcal{F}}(PQ)}(R) = \mathrm{Aut}_{C_S(PQ)}(R)$. Since PQ is fully centralized, $C_{\mathcal{F}}(PQ)$ is saturated. So it follows from Alperin's fusion theorem (see [6, Theorem I.3.6]) that $C_{\mathcal{F}}(PQ) = \mathcal{F}_{C_S(PQ)}(C_S(PQ))$. As PQ is fully centralized, this implies $PQ \in \mathcal{F}^q$ which completes the proof. □

Lemma 3.15. *Let R be a subgroup of S normal in $\mathcal{F}$ and $K \trianglelefteq \mathrm{Aut}_{\mathcal{F}}(R)$. Then $N_{\mathcal{F}}^K(R)$ is weakly normal in $\mathcal{F}$ and $N_{\mathcal{F}}^K(R)^s = \{P \in \mathcal{F}^s \colon P \leq N_S^K(R)\}$. In particular, $C_{\mathcal{F}}(R)^s = \{P \in \mathcal{F}^s \colon P \leq C_S(R)\}$.*

Proof. By Lemma 2.18(a), $N_{\mathcal{F}}^K(R)$ is weakly normal in $\mathcal{F} = N_{\mathcal{F}}^{\mathrm{Aut}_{\mathcal{F}}(R)}(R)$. Hence, by Lemma 3.10, every $P \in \mathcal{F}^s$ with $P \leq N_S^K(R)$ is a member of $N_{\mathcal{F}}^K(R)^s$. Let now $P \in N_{\mathcal{F}}^K(R)^s$. By Lemma 3.14, $PR \in \mathcal{F}^s$. So by Lemma 3.4, $P \in \mathcal{F}^s$. □

Lemma 3.16. *Let $Q \in \mathcal{F}^f$ and $P \in \mathcal{F}^s$ with $P \leq N_S(Q)$. Then $P \in N_{\mathcal{F}}(Q)^s$.*

Proof. By Lemma 3.3, $PQ \in \mathcal{F}^s$. Moreover, by Lemma 3.4, $P \in N_{\mathcal{F}}(Q)^s$ if $PQ \in N_{\mathcal{F}}(Q)^s$. Hence, replacing P by PQ, we may assume $Q \leq P$. Moreover, replacing P by a $N_{\mathcal{F}}(Q)$-conjugate, we may assume that P is fully centralized in $N_{\mathcal{F}}(Q)$. Then $P = PQ$ is fully centralized in $\mathcal{F}$ by Lemma 3.13. So by Lemma 3.1, $C_{\mathcal{F}}(P)$ is constrained. As $C_{\mathcal{F}}(P) = C_{N_{\mathcal{F}}(Q)}(P)$, Lemma 3.1 applied with $N_{\mathcal{F}}(Q)$ in place of $\mathcal{F}$ gives $P \in N_{\mathcal{F}}(Q)^s$. □

Lemma 3.17. *Let $Q \in \mathcal{F}$ and $K \trianglelefteq \mathrm{Aut}_{\mathcal{F}}(Q)$ such that Q is fully K-normalized. Then $\{P \in \mathcal{F}^s \colon P \leq N_S^K(Q)\} \subseteq N_{\mathcal{F}}^K(Q)^s$.*

Proof. By [6, I.2.6(c)], there exists a morphism $\alpha \in \mathrm{Hom}_{\mathcal{F}}(N_S(Q), S)$ such that $Q\alpha \in \mathcal{F}^f$. By Lemma 2.17, $N_S^K(Q)\alpha \leq N_S^{K^\alpha}(Q\alpha)$. As Q is fully K-normalized, it follows that $Q\alpha$ is fully K^α-normalized and $N_S^K(Q)\alpha = N_S^{K^\alpha}(Q\alpha)$. It is straightforward to check that α induces an isomorphism from $N_{\mathcal{F}}^K(Q)$ to $N_{\mathcal{F}}^{K^\alpha}(Q\alpha)$. Thus, by Lemma 3.6, for any $P \leq N_S^K(Q)$, we have $P\alpha \in N_{\mathcal{F}}^{K^\alpha}(Q\alpha)^s$ if and only if $P \in N_{\mathcal{F}}^K(Q)^s$. Hence, as $\mathcal{F}^s$ is invariant under $\mathcal{F}$-conjugation, replacing Q by $Q\alpha$, we may assume that $Q \in \mathcal{F}^f$. Then $N_{\mathcal{F}}(Q)$ is saturated and, as $N_{\mathcal{F}}^K(Q) = N_{N_{\mathcal{F}}(Q)}^K(Q)$, it follows from Lemma 3.15 that $N_{\mathcal{F}}^K(Q)^s = \{P \in N_{\mathcal{F}}(Q)^s \colon P \leq N_S^K(Q)\}$. If $P \in \mathcal{F}^s$ with $P \leq N_S^K(Q)$ then $P \in N_{\mathcal{F}}(Q)^s$ by Lemma 3.16, and therefore $P \in N_{\mathcal{F}}^K(Q)^s$. This proves the assertion. □

Lemma 3.18. *Let $Q \in \mathcal{F}^s \cap \mathcal{F}^f$ such that $Q = O_p(N_{\mathcal{F}}(Q))$. Then $Q \in \mathcal{F}^{frc}$.*

Proof. As $Q \in \mathcal{F}^s \cap \mathcal{F}^f$, we have $Q = O_p(N_{\mathcal{F}}(Q)) \in \mathcal{F}^c$ by definition of subcentric subgroups. Moreover Theorem 3.1 yields that $N_{\mathcal{F}}(Q)$ is constrained. So by Theorem 2.1, there exists a model G for $N_{\mathcal{F}}(Q)$ and $O_p(G) = O_p(N_{\mathcal{F}}(Q)) = Q$. Note $\mathrm{Aut}_{\mathcal{F}}(Q) = \mathrm{Aut}_{N_{\mathcal{F}}(Q)}(Q) \cong G/C_G(Q) = G/Z(Q)$. Then $O_p(\mathrm{Aut}_{\mathcal{F}}(Q)) \cong O_p(G/Z(Q)) = Q/Z(Q) \cong \mathrm{Inn}(Q)$ and so Q is radical. □

Proof of Proposition 1. This follows from Lemma 3.4, Lemma 3.5, Lemma 3.14 and Lemma 3.17. Compare also Lemma 3.16. □

Proof of Proposition 2. The proposition follows from Lemma 3.7, Lemma 3.10, Lemma 3.11, Lemma 3.12 and Lemma 3.15. □

4. The proof of Theorem B

Throughout this section, $\mathcal{F}$ is assumed to be saturated. Moreover, $\mathcal{E}$ is always a normal subsystem of $\mathcal{F}$ over $T \leq S$.

The subgroup $C_S(\mathcal{E})$ was introduced in [3, Chapter 6]. We will use throughout the following characterization: The subgroup $C_S(\mathcal{E})$ is the largest subgroup X of $C_S(T)$ such that $\mathcal{E} \subseteq C_{\mathcal{F}}(X)$.

Lemma 4.1. *The subsystem $\mathcal{E}$ is also a normal subsystem of $N_{\mathcal{F}}(C_S(\mathcal{E}))$.*

Proof. Recall $\mathcal{E} \subseteq C_{\mathcal{F}}(C_S(\mathcal{E})) \subseteq N_{\mathcal{F}}(C_S(\mathcal{E}))$. Clearly $\mathcal{E}$ is weakly normal in $N_{\mathcal{F}}(C_S(\mathcal{E}))$. Set $T = \mathcal{E} \cap S$. As $\mathcal{E} \trianglelefteq \mathcal{F}$, every element $\varphi \in \mathrm{Aut}_{\mathcal{E}}(T)$ extends to $\overline{\varphi} \in \mathrm{Aut}_{\mathcal{F}}(TC_S(T))$ such that $[C_S(T), \overline{\varphi}] \leq T$. Since $C_S(\mathcal{E}) \leq C_S(T)$ and $C_S(\mathcal{E})$ is strongly closed in $\mathcal{F}$ by [3, (6.7)(2)], we have $C_S(\mathcal{E})\overline{\varphi} = C_S(\mathcal{E})$. Hence, $\overline{\varphi} \in \mathrm{Aut}_{N_{\mathcal{F}}(C_S(\mathcal{E}))}(TC_S(T))$. This shows the assertion. □

Lemma 4.2. *Let $Q \in \mathcal{F}^f$ such that $Q = (Q \cap T)C_S(\mathcal{E})$. Then $Q \cap T \in \mathcal{E}^f$.*

Proof. Set $P := Q \cap T$. Let $\alpha_0 \in \mathrm{Hom}_{\mathcal{E}}(P, T)$ such that $P\alpha_0$ is fully $\mathcal{E}$-normalized. By the characterization of $C_S(\mathcal{E})$ above, α_0 extends to $\alpha \in \mathrm{Hom}_{\mathcal{F}}(Q, S)$ such that α fixes every element of $C_S(\mathcal{E})$. In particular, $C_S(\mathcal{E})\alpha = C_S(\mathcal{E})$ and $Q\alpha = (P\alpha)C_S(\mathcal{E})$. Moreover, $P\alpha = (Q\cap T)\alpha \leq Q\alpha \cap T$ and $(Q\alpha \cap T)(\alpha|_Q)^{-1} \leq Q \cap T$. So $Q\alpha \cap T = (Q \cap T)\alpha = P\alpha$.

As $Q = PC_S(\mathcal{E})$ and $C_S(\mathcal{E}) \leq C_S(T)$, we have $N_T(P) \leq N_T(Q)$. As $P = Q \cap T$, $N_T(Q) \leq N_T(P)$. Hence, $N_T(P) = N_T(Q)$. Similarly, $N_T(Q\alpha) = N_T(P\alpha)$. By [6, Lemma I.2.6(c)], there exists $\beta \in \mathrm{Hom}_{\mathcal{F}}(N_S(Q\alpha), S)$ such that $Q\alpha\beta = Q$. For such β, we have $N_T(Q\alpha)\beta \leq N_T(Q)$ and thus $|N_T(P\alpha_0)| = |N_T(P\alpha)| = |N_T(Q\alpha)| \leq |N_T(Q)| = |N_T(P)|$. Hence, $P \in \mathcal{E}^f$ as $P\alpha_0 \in \mathcal{E}^f$. □

Lemma 4.3. *Let $\mathcal{E}$ be a normal subsystem of $\mathcal{F}$ over T. Let $Q \in \mathcal{F}^f$ such that $Q = (Q\cap T)C_S(\mathcal{E})$. Then $N_{\mathcal{E}}(Q \cap T)$ is weakly normal in $N_{\mathcal{F}}(Q)$.*

Proof. Set $P := Q \cap T$. By Lemma 4.2, $P \in \mathcal{E}^f$. By assumption $Q \in \mathcal{F}^f$, so both $N_{\mathcal{E}}(P)$ and $N_{\mathcal{F}}(Q)$ are saturated. Every morphism $\alpha \in \mathrm{Hom}_{N_{\mathcal{E}}(P)}(A, B)$ $(A, B \leq N_T(P))$ extends to an element of $\mathrm{Hom}_{\mathcal{E}}(AP, BP)$ normalizing P, which then by definition of $C_S(\mathcal{E})$ extends to $\overline{\alpha} \in \mathrm{Hom}_{\mathcal{F}}(APC_S(\mathcal{E}), BPC_S(\mathcal{E}))$ centralizing $C_S(\mathcal{E})$. As $Q = PC_S(\mathcal{E})$, it follows $Q\overline{\alpha} = Q$ and so α is a morphism in $N_{\mathcal{F}}(Q)$. This shows that $N_{\mathcal{E}}(P)$ is a subsystem of $N_{\mathcal{F}}(Q)$. Hence, it remains to prove only that $N_{\mathcal{E}}(P)$ is invariant in $N_{\mathcal{F}}(P)$. We prove the strong invariance condition as stated in [6, Proposition 6.4(d)]. Let $A \leq B \leq N_T(P)$, $\varphi \in \mathrm{Hom}_{N_{\mathcal{E}}(P)}(A, B)$ and $\psi \in \mathrm{Hom}_{N_{\mathcal{F}}(Q)}(B, N_T(P))$. We need to prove that $(\psi|_A)^{-1}\varphi\psi \in \mathrm{Hom}_{N_{\mathcal{E}}(P)}(A\psi, B\psi)$. By definition of the normalizer subsystems, φ extends to $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{E}}(AP, BP)$ with $P\overline{\varphi} = P$, and ψ extends to $\overline{\psi} \in \mathrm{Hom}_{\mathcal{F}}(BQ, N_T(P)Q)$ with $Q\overline{\psi} = Q$. As T is strongly closed and, by assumption, $P = Q \cap T$, we have $P\overline{\psi} = P$ and thus $\hat{\psi} := \overline{\psi}|_{BP} \in \mathrm{Hom}_{\mathcal{F}}(BP, N_T(P))$. Since the strong invariance condition holds for $(\mathcal{E}, \mathcal{F})$, we have that $(\hat{\psi}|_{AP})^{-1}\overline{\varphi}\hat{\psi}$ is a morphism in $\mathcal{E}$. Moreover, $P(\hat{\psi}|_{AP})^{-1}\overline{\varphi}\hat{\psi} = P$ and $(\hat{\psi}|_{AP})^{-1}\overline{\varphi}\hat{\psi}$ extends $(\psi|_A)^{-1}\varphi\psi$. So $(\psi|_A)^{-1}\varphi\psi$ is a morphism in $N_{\mathcal{E}}(P)$ as required. □

Lemma 4.4. *Let $\mathcal{E}$ be a normal subsystem of $\mathcal{F}$ and $\mathcal{C}$ a component of $\mathcal{F}$. Then $\mathcal{C} \subseteq \mathcal{E}$ or $\mathcal{C} \cap S \leq C_S(\mathcal{E})$.*

Proof. Suppose $\mathcal{C}$ is not contained in $\mathcal{E}$. Then $\mathcal{C}$ is in the set J of components of $\mathcal{F}$ which are not components of $\mathcal{E}$. Then $\mathcal{D} := \prod_{\mathcal{C}' \in J} \mathcal{C}'$ is a well-defined subsystem of $\mathcal{F}$ containing $\mathcal{C}$ by [3, (9.8)(2)]. It is furthermore shown in [3, (9.13)] that $\mathcal{ED}$ is well-defined and a central product of $\mathcal{E}$ and $\mathcal{D}$. If $\mathcal{F}$ is the central product of two subsystems $\mathcal{F}_1$ and $\mathcal{F}_2$ then, by the construction of central products in [3, Chapter 2], $\mathcal{F}_2 \cap S \leq C_S(\mathcal{F}_1)$. So $\mathcal{C} \cap S \leq \mathcal{D} \cap S \leq C_S(\mathcal{E})$. □

Proof of Theorem B. Let $\mathcal{E}$ be a normal subsystem of $\mathcal{F}$ over $T \leq S$. Let $P \in \mathcal{E}^s$ and set $Q := PC_S(\mathcal{E})$.

Step 1: We show that it is enough to prove the assertion in the case that $Q \in \mathcal{F}^f$ and $P = Q \cap T$. For that take $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, S)$ such that $Q\varphi$ is fully $\mathcal{F}$-normalized. Then by Lemma 3.7, $P\varphi \in \mathcal{E}^s$. Moreover, as $C_S(\mathcal{E})$ is strongly closed by [3, (6.7)(2)], $C_S(\mathcal{E})\varphi = C_S(\mathcal{E})$ and thus $Q\varphi = (P\varphi)C_S(\mathcal{E})$. So replacing (P, Q) by $(P\varphi, Q\varphi)$, we may assume that Q is fully $\mathcal{F}$-normalized. Note also that $P \leq Q \cap T$, so by Proposition 3.3, $Q \cap T$ is subcentric in $\mathcal{E}$. Moreover, $Q = (Q \cap T)C_S(\mathcal{E})$. Hence, replacing P by $Q \cap T$, we may assume that $P = Q \cap T$.

From now on we assume that $Q \in \mathcal{F}^f$ and $P = Q \cap T$.

Step 2: We show that $E(N_{\mathcal{F}}(C_S(\mathcal{E}))) \subseteq \mathcal{E}$. Let $\mathcal{C}$ be a component of $N_{\mathcal{F}}(C_S(\mathcal{E}))$. By [3, (9.9)(1)], a component of a saturated fusion system centralizes every normal subgroup of the same fusion system. Hence, as $C_S(\mathcal{E})$ is normal in $N_{\mathcal{F}}(C_S(\mathcal{E}))$, we have $\mathcal{C} \subseteq C_{\mathcal{F}}(C_S(\mathcal{E}))$. By Lemma 4.1, $\mathcal{E}$ is normal in $N_{\mathcal{F}}(C_S(\mathcal{E}))$. Therefore, by Lemma 4.4, $\mathcal{C} \subseteq \mathcal{E}$ or $C := \mathcal{C} \cap S \leq C_S(\mathcal{E})$. Assume

$C \leq C_S(\mathcal{E})$. As $\mathcal{C} \subseteq C_{\mathcal{F}}(C_S(\mathcal{E}))$ this means that C is abelian, contradicting [3, (9.1)(2)] and the fact that $\mathcal{C}$ is quasisimple. This proves $\mathcal{C} \subseteq \mathcal{E}$ and, as $\mathcal{C}$ was arbitrary, $E(N_{\mathcal{F}}(C_S(\mathcal{E}))) \subseteq \mathcal{E}$.

Step 3: We complete the proof by showing that Q is subcentric in $\mathcal{F}$. Suppose this is not true. As we assume that Q is fully normalized, this means by Lemma 3.1 that $N_{\mathcal{F}}(Q)$ is not constrained. Thus, by [3, (14.2)], $E(N_{\mathcal{F}}(Q)) \neq 1$. By [3, (6.7)(2)], $C_S(\mathcal{E})$ is strongly closed in $\mathcal{F}$. So as $C_S(\mathcal{E}) \leq Q$, we have $N_{\mathcal{F}}(Q) = N_{N_{\mathcal{F}}(C_S(\mathcal{E}))}(Q)$. Since Q is fully normalized in $\mathcal{F}$ and $C_S(\mathcal{E}) \trianglelefteq S$, $N_{\mathcal{F}}(C_S(\mathcal{E}))$ is saturated and Q is fully normalized in $N_{\mathcal{F}}(C_S(\mathcal{E}))$. Thus, by Aschbacher's version of the L-Balance Theorem for fusion systems [3, Theorem 7], $E(N_{\mathcal{F}}(Q)) = E(N_{N_{\mathcal{F}}(C_S(\mathcal{E}))}(Q)) \subseteq E(N_{\mathcal{F}}(C_S(\mathcal{E})))$. So by Step 2, $E(N_{\mathcal{F}}(Q)) \subseteq \mathcal{E}$. Let $\mathcal{D}$ be a component of $N_{\mathcal{F}}(Q)$ and $D = S \cap \mathcal{D}$. By assumption, $P = Q \cap T$ and thus $Q = (Q \cap T)C_S(\mathcal{E})$. Hence, by Lemma 4.2, P is fully $\mathcal{E}$-normalized. Moreover, by Lemma 4.3, $N_{\mathcal{E}}(P)$ is weakly normal in $N_{\mathcal{F}}(Q)$. It follows from the latter fact and Lemma 2.12 that $O_p(N_{\mathcal{E}}(P))$ is normal in $N_{\mathcal{F}}(Q)$. By [3, (9.6)] the component $\mathcal{D}$ of $N_{\mathcal{F}}(Q)$ centralizes every normal subgroup of $N_{\mathcal{F}}(Q)$. So in particular, $\mathcal{D} \subseteq C_{N_{\mathcal{F}}(Q)}(O_p(N_{\mathcal{E}}(P)))$ and thus $[D, O_p(N_{\mathcal{E}}(P))] = 1$. As $E(N_{\mathcal{F}}(Q)) \subseteq \mathcal{E}$, we have $D \leq T$. Hence, $D \leq C_T(O_p(N_{\mathcal{E}}(P))) = Z(O_p(N_{\mathcal{E}}(P)))$, because P is subcentric and fully normalized in $\mathcal{E}$ and thus $O_p(N_{\mathcal{E}}(P))$ is centric in $\mathcal{E}$. Thus, D is abelian, again contradicting [3, (9.1)(2)] and the fact that $\mathcal{D}$ is quasisimple. □

5. Partial groups, localities and transporter systems

5.1. **Partial groups.** Adapting the notation from [10] and [11], we denote the set of words in a set $\mathcal{L}$ by $\mathbf{W}(\mathcal{L})$. Moreover, we write $\emptyset$ for the empty word, and $v_1 \circ v_2 \circ \cdots \circ v_n$ for the concatenation of words $v_1, \dots, v_n \in \mathbf{W}(\mathcal{L})$. Roughly speaking, a partial group is a set $\mathcal{L}$ together with a product which is only defined on certain words in $\mathcal{L}$, and an inversion map $\mathcal{L} \to \mathcal{L}$ which is an involutory bijection, subject to certain axioms. We refer the reader to [10, Definition 2.1] or [11, Definition 1.1] for the precise definition of a partial group, and to the elementary properties of partial groups stated in [10, Lemma 2.2] or [11, Lemma 1.4].

For the remainder of this section let $\mathcal{L}$ be a partial group with product $\Pi\colon \mathbf{D} \to \mathcal{L}$ defined on the domain $\mathbf{D} \subseteq \mathbf{W}(\mathcal{L})$.

It follows from the axioms of a partial group that $\emptyset \in \mathbf{D}$. We set $\mathbf{1} = \Pi(\emptyset)$. Moreover, given a word $v = (f_1, \dots, f_n) \in \mathbf{D}$, we write sometimes $f_1 f_2 \dots f_n$ for the product $\Pi(v)$.

A *partial subgroup* of $\mathcal{L}$ is a subset $\mathcal{H}$ of $\mathcal{L}$ such that $f^{-1} \in \mathcal{H}$ for all $f \in \mathcal{H}$ and $\Pi(w) \in \mathcal{H}$ for all $w \in \mathbf{W}(\mathcal{H}) \cap \mathbf{D}$. Note that $\emptyset \in \mathbf{W}(\mathcal{H}) \cap \mathbf{D}$ and thus $\mathbf{1} = \Pi(\emptyset) \in \mathcal{H}$ if $\mathcal{H}$ is a partial subgroup of $\mathcal{L}$. It is easy to see that a partial subgroup of $\mathcal{L}$ is always a partial group itself whose product is the restriction of the product Π to $\mathbf{W}(\mathcal{H}) \cap \mathbf{D}$. Observe furthermore that $\mathcal{L}$ forms a group in the usual sense if $\mathbf{W}(\mathcal{L}) = \mathbf{D}$; see [11, Lemma 1.3]. So it makes sense to call a partial subgroup $\mathcal{H}$ of $\mathcal{L}$ a *subgroup of* $\mathcal{L}$ if $\mathbf{W}(\mathcal{H}) \subseteq \mathbf{D}$. In particular, we can talk about *p-subgroups of* $\mathcal{L}$ meaning subgroups of $\mathcal{L}$ which are finite and whose order is a power of p.

For any $g \in \mathcal{L}$, $\mathbf{D}(g)$ denotes the set of $x \in \mathcal{L}$ with $(g^{-1}, x, g) \in \mathbf{D}$. Thus, $\mathbf{D}(g)$ denotes the set of elements $x \in \mathcal{L}$ for which the conjugation $x^g := \Pi(g^{-1}, x, g)$ is defined.

If $g \in \mathcal{L}$ and $X \subseteq \mathbf{D}(g)$ we set $X^g := \{x^g\colon x \in X\}$. If we write X^g for some $g \in \mathcal{L}$ and some subset $X \subseteq \mathcal{L}$, we will always implicitly mean that $X \subseteq \mathbf{D}(g)$. Similarly, if we write x^g for $x, g \in \mathcal{L}$, we always mean that $x \in \mathbf{D}(g)$.

If X is a subsets of $\mathcal{L}$ then we set

$$N_{\mathcal{L}}(X) = \{g \in \mathcal{L}\colon X^g = X\} \text{ and } C_{\mathcal{L}}(X) = \{g \in \mathcal{L}\colon x^g = x \text{ for all } x \in X\}.$$

Note that $C_{\mathcal{L}}(X) \subseteq N_{\mathcal{L}}(X)$. It follows easily from the axioms of a partial group that $\mathbf{1}$ is contained in the centralizer of any subset of $\mathcal{L}$; see [10, Lemma 2.5(d)].

If X and Y are subsets of $\mathcal{L}$ then set $N_Y(X) = N_{\mathcal{L}}(X) \cap Y$ and $C_Y(X) = C_{\mathcal{L}}(X) \cap Y$. Moreover, define

$$Z(\mathcal{L}) := C_{\mathcal{L}}(\mathcal{L}).$$

Generalizing the notion of the normalizer, we set $N_{\mathcal{L}}(X, Y) = \{f \in \mathcal{L}\colon X^f \subseteq Y\}$.

Since there is a natural notion of conjugation, there is also a natural notion of partial normal subgroups of partial groups. Namely, a partial subgroup $\mathcal{N}$ of $\mathcal{L}$ is called a *partial normal subgroup* of $\mathcal{L}$ if $n^f \in \mathcal{N}$ for all $f \in \mathcal{L}$ and all $n \in \mathcal{N} \cap \mathbf{D}(f)$.

Let $\mathcal{L}'$ be a partial group with domain $\mathbf{D}'$ and product $\Pi'\colon \mathbf{D}' \to \mathcal{L}'$. Let $\mathbf{1}' = \Pi'(\emptyset)$ be the identity in $\mathcal{L}'$. Let $\beta\colon \mathcal{L} \to \mathcal{L}'$ be a map and let $\beta^*\colon \mathbf{W}(L) \to \mathbf{W}(\mathcal{L}')$ be the induced map. Recall from [11, Definition 1.11] that β is called a homomorphism of partial groups if $\mathbf{D}\beta^* \subseteq \mathbf{D}'$ and $\Pi'(v\beta^*) = (\Pi(v))\beta$ for all $v \in \mathbf{D}$. If β is a homomorphism of partial groups, define the kernel of β via

$$\ker(\beta) = \{f \in \mathcal{L}\colon \beta(f) = \mathbf{1}'\}.$$

By [11, Lemma 1.14], the kernel of a homomorphism of partial groups forms always a partial normal subgroup.

A homomorphism $\beta\colon \mathcal{L} \to \mathcal{L}'$ of partial groups is called an *isomorphism* of partial groups if $\mathbf{D}\beta^* = \mathbf{D}'$ and β is injective. As every word in $\mathcal{L}'$ of length one is in $\mathbf{D}'$, the condition $\mathbf{D}\beta^* = \mathbf{D}'$ implies that β is surjective. Thus, every isomorphism of partial groups is bijective.

5.2. **Localities.**

Definition 5.1. Let S be a p-subgroup of $\mathcal{L}$ and let Δ be a non-empty set of subgroups of S.

The set Δ is said to be *closed under taking $\mathcal{L}$-conjugates and overgroups in S* if for any $P \in \Delta$ the following holds: For every $g \in \mathcal{L}$ with $P \subseteq \mathbf{D}(g)$ and $P^g \subseteq S$ we have $P^g \in \Delta$ (so in particular, P^g is a subgroup of S), and for every subgroup Q of S containing P we have $Q \in \Delta$.

We say that $(\mathcal{L}, \Delta, S)$ is a *locality* if the partial group $\mathcal{L}$ is finite as a set and the following conditions hold:

- (L1) S is maximal with respect to inclusion among the p-subgroups of $\mathcal{L}$.
- (L2) $\mathbf{D}$ is the set of words $(f_1, \dots, f_n) \in \mathbf{W}(\mathcal{L})$ such that there exist $P_0, \dots, P_n \in \Delta$ with
  (*) $P_{i-1} \subseteq \mathbf{D}(f_i)$ and $P_{i-1}^{f_i} = P_i$ for all $i = 1, \dots, n$.
- (L3) The set Δ is closed under taking $\mathcal{L}$-conjugates and overgroups in S.

If $(\mathcal{L}, \Delta, S)$ is a locality and $v = (f_1, \dots, f_n) \in \mathbf{W}(\mathcal{L})$, then we say that $v \in \mathbf{D}$ via $P_0, \dots, P_n$ (or $v \in \mathbf{D}$ via P_0), if $P_0, \dots, P_n$ are elements of Δ such that (*) holds.

Remark 5.2. Our definition of a locality differs slightly from the one given by Chermak in [10] and [11], but can be shown to be equivalent. It can be easily seen that a locality as defined by Chermak is a triple $(\mathcal{L}, \Delta, S)$ such that $\mathcal{L}$ is a finite partial group, S is a p-subgroup of $\mathcal{L}$, Δ is a set of subgroups of S, and such that the conditions (L1) and (L2) together with the following condition hold:

- (L3') For any subgroup Q of S, for which there exist $P \in \Delta$ and $g \in \mathcal{L}$ with $P \subseteq \mathbf{D}(g)$ and $P^g \leq Q$, we have $Q \in \Delta$.

Clearly (L3) implies (L3'). If $(\mathcal{L}, \Delta, S)$ is a locality in Chermak's definition, then it is shown in [11, Proposition 2.6] that P^g is a subgroup of S and thus an element of Δ if $g \in \mathcal{L}$ with $P \subseteq \mathbf{D}(g)$ and $P^g \subseteq S$. Moreover, $P \subseteq \mathbf{D}(\mathbf{1})$ and $P^{\mathbf{1}} = P$. Therefore, if $(\mathcal{L}, \Delta, S)$ is a locality in Chermak's definition then (L3) holds and $(\mathcal{L}, \Delta, S)$ is indeed also a locality in our definition.

If $(\mathcal{L}, \Delta, S)$ is a locality and $g \in \mathcal{L}$, we set

$$S_g := \{s \in \mathbf{D}(g)\colon s^g \in S\}.$$

More generally, if $v = (g_1, \dots, g_n) \in \mathbf{W}(\mathcal{L})$, we write S_w for the set of $s \in S$ such that there exist elements $s = s_0, s_1, \dots, s_n$ of S such that, for $i = 1, \dots, n$, $s_{i-1} \in \mathbf{D}(g_i)$ and $s_{i-1}^{g_i} = s_i$.

Lemma 5.3 (Important properties of localities)**.** *Let $(\mathcal{L}, \Delta, S)$ be a locality. Then the following hold:*

(a) *$N_{\mathcal{L}}(P)$ is a subgroup of $\mathcal{L}$ for each $P \in \Delta$.*
(b) *Let $P \in \Delta$ and $g \in \mathcal{L}$ with $P \subseteq S_g$. Then $Q := P^g \in \Delta$, $N_{\mathcal{L}}(P) \subseteq \mathbf{D}(g)$ and*
$$c_g \colon N_{\mathcal{L}}(P) \to N_{\mathcal{L}}(Q)$$
is an isomorphism of groups.
(c) *Let $w = (g_1, \ldots, g_n) \in \mathbf{D}$ via $(X_0, \ldots, X_n)$. Then*
$$c_{g_1} \circ \cdots \circ c_{g_n} = c_{\Pi(w)}$$
is a group isomorphism $N_{\mathcal{L}}(X_0) \to N_{\mathcal{L}}(X_n)$.
(d) *For every $g \in \mathcal{L}$, $S_g \in \Delta$. In particular, S_g is a subgroup of S. Moreover, $S_g^g = S_{g^{-1}}$.*
(e) *For every $g \in \mathcal{L}$, $c_g \colon \mathbf{D}(g) \to \mathbf{D}(g^{-1})$ is a bijection with inverse map $c_{g^{-1}}$.*
(f) *For any $w \in \mathbf{W}(\mathcal{L})$, S_w is a subgroup of $S_{\Pi(w)}$, and $S_w \in \Delta$ if and only if $w \in \mathbf{D}$.*
(g) *Let $P \in \Delta$. Then P is fully-normalized in $\mathcal{F}_S(\mathcal{L})$ if and only if $N_S(P) \in \mathrm{Syl}_p(N_{\mathcal{L}}(P))$. If so then $N_{\mathcal{F}}(P) = \mathcal{F}_{N_S(P)}(N_{\mathcal{L}}(P))$ and $C_{\mathcal{F}}(P) = \mathcal{F}_{C_S(P)}(C_{\mathcal{L}}(P))$. Moreover, for any $P \in \Delta$ there exists $f \in \mathcal{L}$ such that $N_S(P) \leq S_f$ and $N_S(P^f) \in \mathrm{Syl}_p(N_{\mathcal{L}}(P^f))$.*

Proof. Properties (a),(b) and (c) correspond to the statements (a),(b) and (c) in [11, Lemma 2.3] except for the fact stated in (b) that $Q \in \Delta$, which is however clearly true if one uses our definition of a locality. Property (d) holds by [11, Proposition 2.6(a),(b)] and property (e) is stated in [10, Lemma 2.5(c)]. Property (f) is [11, Corollary 2.7]. Property (g) combines Lemma 2.10 and Lemma 2.13(b) in [11]. □

We will use the properties stated in Lemma 5.3 most of the time without reference. Note that, by parts (b) and (d), $c_g \colon S_g \to S$ is a homomorphism of groups, which by part (e) is injective. If $(\mathcal{L}, \Delta, S)$ is a locality, then $\mathcal{F}_S(\mathcal{L})$ is the fusion system generated by the conjugation maps $c_g \colon S_g \to S$ with $g \in \mathcal{L}$. Equivalently, $\mathcal{F}_S(\mathcal{L})$ is generated by the conjugation maps between subgroups of Δ, or by the conjugation maps between subgroups of S.

Lemma 5.4. *Let $(\mathcal{L}, \Delta, S)$ be a locality and R a subgroup of S. Then $N_{\mathcal{L}}(R)$ and $C_{\mathcal{L}}(R)$ are partial subgroups of $\mathcal{L}$.*

Proof. By [10, Lemma 2.19(a)], $N_{\mathcal{L}}(R)$ is a partial subgroup of $\mathcal{L}$. So it remains to prove that $C_{\mathcal{L}}(R)$ is a partial subgroup. It follows from Lemma 5.3(e) that $C_{\mathcal{L}}(R)$ is closed under inversion. Let now $w = (f_1, \ldots, f_n) \in \mathbf{D} \cap \mathbf{W}(C_{\mathcal{L}}(R))$. Then $w \in \mathbf{D}$ via a sequence $P_0, \ldots, P_n$ of elements of Δ. Since Δ is closed under taking overgroups in S, $Q_i := \langle P_i, R\rangle \in \Delta$ for $i = 0, 1, \ldots, n$. Moreover, for $i = 1, \ldots, n$, $Q_{i-1} \leq S_{f_i}$, $c_{f_i}|_R = \mathrm{id}_R$, and $Q_{i-1}^{f_i} = Q_i$ as $c_{f_i} \colon S_{f_i} \to S$ is a homomorphism of groups. So $w \in \mathbf{D}$ via $Q_0, \ldots, Q_n$ and it follows from Lemma 5.3(c) that $R \leq Q_0 \leq S_{\Pi(w)}$ and $c_{\Pi(w)}|_R = \mathrm{id}_R$, So $\Pi(w) \in C_{\mathcal{L}}(R)$. □

The following remark is used throughout, usually without reference:

Remark 5.5. Let $P \in \Delta$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}_S(\mathcal{L})}(P, S)$. Then there exists $g \in \mathcal{L}$ with $P \leq S_g$ and $\varphi = c_g|_P$.

Proof. By definition of $\mathcal{F}_S(\mathcal{L})$, φ is the composition of suitable restrictions of conjugation maps $c_{g_1}, c_{g_2}, \ldots, c_{g_n}$ with $g_1, g_2, \ldots, g_n \in \mathcal{L}$. Then $(g_1, \ldots, g_n) \in \mathbf{D}$ via P, Moreover, setting $g = \Pi(g_1, g_2, \ldots, g_n)$, Lemma 5.3(c) implies $P \leq S_g$ and $\varphi = c_g|_P$. □

5.3. **Projections of localities.** There is a theory of morphisms and factor systems of fusion systems, where factor systems are formed modulo strongly closed subgroups and similarly the kernels of morphisms are strongly closed. We refer the reader to [6, Section II.5] for details.

Let $\mathcal{F}$ and $\mathcal{F}'$ be fusion systems over S and S' respectively. Then we say that a group homomorphism $\alpha\colon S \to S'$ induces a morphism from $\mathcal{F}$ to $\mathcal{F}'$ if for each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$ there exists $\psi \in \mathrm{Hom}_{\mathcal{F}'}(P\alpha, Q\alpha)$ such that $(\alpha|_P)\psi = \varphi(\alpha|_Q)$. Such ψ is then uniquely determined, so α induces a map $\alpha_{P,Q}\colon \mathrm{Hom}_{\mathcal{F}}(P,Q) \to \mathrm{Hom}_{\mathcal{F}'}(P\alpha,Q\alpha)$. Together with the map $P \mapsto P\alpha$ from the set of objects of $\mathcal{F}$ to the set of objects of $\mathcal{F}'$ this gives a functor from $\mathcal{F}$ to $\mathcal{F}'$. Moreover, α together with the maps $\alpha_{P,Q}$ $(P,Q \leq S)$ is a morphism of fusion systems in the sense of [6, Definition II.2.2]. We say that α induces an epimorphism from $\mathcal{F}$ to $\mathcal{F}'$ if $(\alpha, \alpha_{P,Q}\colon P,Q \leq S)$ is a surjective morphism of fusion systems. This means that α is surjective as a map $S \to S'$ and, for every $P,Q \leq S$ with $\ker(\alpha) \leq P \cap Q$ the map $\alpha_{P,Q}$ is surjective, i.e. for each $\psi \in \mathrm{Hom}_{\mathcal{F}'}(P\alpha,Q\alpha)$, there exists $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$ with $(\alpha|_P)\psi = \varphi(\alpha|_Q)$. If α is in addition injective then we say that α induces an isomorphism from $\mathcal{F}$ to $\mathcal{F}'$. Note that this fits with the definition we gave earlier in Section 3.

If α induces an epimorphism from $\mathcal{F}$ to $\mathcal{F}'$ then notice that the induced map

$$S/\ker(\alpha) \to S'$$

is a fusion preserving isomorphism from $\mathcal{F}/\ker(\alpha)$ to $\mathcal{F}'$.

In the remainder of this subsection we will summarize the theory of projections and quotients of localities, and relate this theory to the theory of morphisms and quotients of fusion systems.

From now on let $(\mathcal{L},\Delta,S)$ be a locality.

Theorem 5.6. *Let $\mathcal{L}'$ be a partial group with product defined on the domain $\mathbf{D}'$. Let $\beta\colon \mathcal{L} \to \mathcal{L}'$ be a homomorphism of partial groups such that $\mathbf{D}\beta^* = \mathbf{D}'$, where $\beta^*\colon \mathbf{W}(\mathcal{L}) \to \mathbf{W}(\mathcal{L}')$ is the map induced by β. Set $T = \ker(\beta) \cap S$, $S' = S\beta$ and $\Delta' = \{P\beta\colon P \in \Delta\}$.*

(a) *$(\mathcal{L}',\Delta',S')$ is a locality.*
(b) *The restriction $\beta|_S\colon S \to S'$ of β to S induces an epimorphism from $\mathcal{F}_S(\mathcal{L})$ to $\mathcal{F}_{S'}(\mathcal{L}')$ with kernel T. In particular, the group isomorphism $S/T \to S'$, $sT \mapsto s\beta$ induces an isomorphism from $\mathcal{F}_S(\mathcal{L})/T$ to $\mathcal{F}_{S'}(\mathcal{L}')$.*
(c) *Let $P,Q \in \Delta$ with $T \leq P \cap Q$. Then β restricts to a surjection $N_{\mathcal{L}}(P,Q) \to N_{\mathcal{L}'}(P\beta,Q\beta)$, and to a surjective homomorphism of groups if $P = Q$.*

Proof. For properties (a) and (c) see [10, Theorem 4.4]. Recall that $\mathcal{F} := \mathcal{F}_S(\mathcal{L})$ is generated by the conjugation maps between elements of Δ, and similarly $\mathcal{F}' := \mathcal{F}_{S'}(\mathcal{L}')$ is generated by the conjugation maps between the elements of Δ'. Thus, it is sufficient to prove the following two properties for $P,Q \in \Delta$:

(1) For every conjugation map $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$ there exists $\psi \in \mathrm{Hom}_{\mathcal{F}'}(P\beta,Q\beta)$ such that $\varphi\beta|_Q = \beta|_P\psi$.
(2) If $T \leq P \cap Q$ then, for any conjugation map $\psi \in \mathrm{Hom}_{\mathcal{F}'}(P\beta,Q\beta)$ there exists $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$ such that $\varphi\beta|_Q = \beta|_P\psi$.

Notice that for $f \in N_{\mathcal{L}}(P,Q)$ and $x \in P$, we have $x^f\beta = (x\beta)^{f\beta}$ as β is a homomorphism of partial groups. Hence, $x(c_f|_P)\beta = x^f\beta = (x\beta)^{f\beta} = x\beta(c_{f\beta}|_{P\beta})$. This proves

$$(*)\quad (c_f|_P)\beta|_Q = \beta|_P(c_{f\beta}|_{P\beta}) \text{ for any } f \in N_{\mathcal{L}}(P,Q).$$

Since any conjugation homomorphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$ is of the form $c_f|_P$ with $f \in N_{\mathcal{L}}(P,Q)$, and as $c_{f\beta}|_{P\beta} \in \mathrm{Hom}_{\mathcal{F}'}(P\beta,Q\beta)$, this shows (1). Assume now $T \leq P \cap Q$. Every conjugation homomorphism $\psi \in \mathrm{Hom}_{\mathcal{F}'}(P\beta,Q\beta)$ is of the form $c_g|_{P\beta}$ with $g \in N_{\mathcal{L}'}(P\beta,Q\beta)$. By (c), there exists $f \in N_{\mathcal{L}}(P,Q)$ with $f\beta = g$. So (2) follows also from (*) as $c_f|_P \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$. □

Definition 5.7. Let $(\mathcal{L}', \Delta', S')$ be a locality with the partial product defined on a domain $\mathbf{D}'$. Then a homomorphism $\beta\colon \mathcal{L} \to \mathcal{L}'$ of partial groups is called a *projection* (of localities) from $(\mathcal{L}, \Delta, S)$ to $(\mathcal{L}', \Delta', S')$ if $\mathbf{D}\beta^* = \mathbf{D}'$ and $\Delta' = \{P\beta\colon P \in \Delta\}$.

If β is a projection of localities as in the above definition then note that $S\beta = S'$ as $\Delta' = \{P\beta\colon P \in \Delta\}$.

As we mentioned before, the kernels of homomorphisms of partial groups form partial normal subgroups. On the other hand, given a partial normal subgroup $\mathcal{N}$ of $\mathcal{L}$, one can form a quotient locality $\mathcal{L}/\mathcal{N}$ such that there is a natural homomorphism from $\mathcal{L}$ onto $\mathcal{L}/\mathcal{N}$. This uses however that $(\mathcal{L}, \Delta, S)$ is a locality. The quotient locality $\mathcal{L}/\mathcal{N}$ is more precisely defined as follows: Call a subset of $\mathcal{L}$ of the form $\mathcal{N}f := \{\Pi(n, f)\colon n \in \mathcal{N},\ (n, f) \in \mathbf{D}\}$ a right coset of $\mathcal{N}$ in $\mathcal{L}$. A maximal right coset of $\mathcal{N}$ is a right coset which is maximal with respect to inclusion among the right cosets of $\mathcal{N}$. By [10, Proposition 3.14(a)], the maximal right cosets of $\mathcal{N}$ form a partition of $\mathcal{L}$. As a set, $\mathcal{L}/\mathcal{N}$ is the set of maximal right cosets of $\mathcal{N}$ in $\mathcal{L}$. Then there is a natural map $\rho\colon \mathcal{L} \to \mathcal{L}/\mathcal{N}$ which sends every element $g \in \mathcal{L}$ to the (unique) maximal right coset of $\mathcal{N}$ containing g. Writing ρ^* for the map $\mathbf{W}(\mathcal{L}) \to \mathbf{W}(\mathcal{L}/\mathcal{N})$ induced by ρ and setting $\overline{\mathbf{D}} = \mathbf{D}\rho^*$, $\overline{\mathcal{L}} = \mathcal{L}/\mathcal{N}$ forms a partial groups with product $\overline{\Pi}\colon \overline{\mathbf{D}} \to \overline{\mathcal{L}}$ defined by $\overline{\Pi}(v\rho^*) = \Pi(v)\rho$ for all $v \in \mathbf{D}$. By construction, ρ is then a homomorphism of partial groups; see [10, Lemma 3.16]. The identity element of $\overline{\mathcal{L}}$ is $\mathcal{N}\mathbf{1} = \mathcal{N}$. So $\ker(\rho) = \mathcal{N}$. Setting $\overline{S} = S\rho$ and $\overline{\Delta} = \{P\rho\colon P \in \Delta\}$, $(\overline{\mathcal{L}}, \overline{\Delta}, \overline{S})$ is a locality by Lemma 5.6. We call this locality the quotient locality of $\mathcal{L}$ modulo $\mathcal{N}$.

Corollary 5.8. *Let $(\mathcal{L}, \Delta, S)$ be a locality over $\mathcal{F}$ and $R \leq S$ such that R forms a partial normal subgroup of $\mathcal{L}$. Then R is strongly closed in $\mathcal{F}$. Furthermore, setting $\overline{\mathcal{L}} = \mathcal{L}/R$, $\overline{S} = S/R$ and $\overline{\Delta} = \{PR/R\colon P \in \Delta\}$, the triple $(\overline{\mathcal{L}}, \overline{\Delta}, \overline{S})$ is a locality over $\mathcal{F}/R$.*

Proof. As R is a partial normal subgroup of $\mathcal{L}$, R is strongly closed in $\mathcal{F} = \mathcal{F}_S(\mathcal{L})$. Let $\beta\colon \mathcal{L} \to \mathcal{L}/R$ be the natural projection. Then $\ker(\beta) = R = \ker(\beta) \cap S$, $S\beta = S/R$ and the induced map $S/R \to S\beta$, $sR \mapsto s\beta$ is just the identity on S/R. Hence, the claim follows from Theorem 5.6. □

5.4. **Transporter systems coming from localities.** If G is a group and Δ a set of subgroups of G then $\mathcal{T}_\Delta(G)$ denotes the *transporter category of* G with object set Δ. That is, for $P, Q \in \Delta$, the set of morphism from P to Q is given by $\mathrm{Hom}_{\mathcal{T}_\Delta(G)}(P, Q) = \{(g, P, Q)\colon g \in G \text{ with } P^g \leq Q\}$.

Our use of the term "transporter system associated to a fusion system $\mathcal{F}$" has been slightly sloppy so far. A transporter system is not just a category $\mathcal{T}$, but it comes always together with "structural maps", namely a pair of functors

$$\mathcal{T}_{\mathrm{ob}(\mathcal{T})}(S) \xrightarrow{\varepsilon} \mathcal{T} \xrightarrow{\rho} \mathcal{F}$$

subject to certain axioms. In particular, ε is the identity on objects, $\mathrm{ob}(\mathcal{T}) \subseteq \mathrm{ob}(\mathcal{F})$, and ρ is the inclusion on objects. So a transporter system should be thought of more correctly as a triple $(\mathcal{T}, \varepsilon, \rho)$ with ε and ρ as above. Given such a transporter system $(\mathcal{T}, \varepsilon, \rho)$, the map $\rho_{P,P}\colon \mathrm{Aut}_\mathcal{T}(P) \to \mathrm{Aut}_\mathcal{F}(P)$ is a group homomorphism for any $P \in \mathrm{ob}(\mathcal{T})$. Its kernel is denoted by $E(P)$. We refer the reader to [20, Definition 3.1] for the precise definition of a transporter system. Comparing this definition with the definitions of linking systems in [9, Definition 1.7] and [18, Definition 3] one observes:

Remark 5.9. Let $(\mathcal{T}, \varepsilon, \rho)$ be a transporter system associated to $\mathcal{F}$ and $E(P) = \ker(\rho_{P,P})$ for any $P \in \mathrm{ob}(\mathcal{T})$. Then $(\mathcal{T}, \varepsilon, \rho)$ is a centric linking system as defined in [9, Definition 1.7] if and only if $\mathrm{ob}(\mathcal{T}) = \mathcal{F}^c$ and $E(P) = Z(P\varepsilon_{P,P})$. Moreover, $(\mathcal{T}, \varepsilon, \rho)$ is a linking system associated to $\mathcal{F}$ in the sense of Oliver [18, Definition 3] if and only if $\mathcal{F}^{cr} \subseteq \mathrm{ob}(\mathcal{T})$ and $E(P)$ is a p-group for every object P of $\mathcal{T}$.

Recall that a transporter system is a linking system in our sense if and only if $\mathcal{F}^{cr} \subseteq \mathrm{ob}(\mathcal{T})$ and $\mathrm{Aut}_{\mathcal{T}}(P)$ is of characteristic p for all $P \in \mathrm{ob}(\mathcal{T})$.

Two transporter systems $(\mathcal{T}, \varepsilon, \rho)$ and $(\mathcal{T}', \varepsilon', \rho')$ associated to $\mathcal{F}$ are called isomorphic if there exists an isomorphism between them, i.e. an invertible functor $\alpha\colon \mathcal{T} \to \mathcal{T}'$ such that (in right handed notation) $\varepsilon \circ \alpha = \varepsilon'$ and $\alpha \circ \rho' = \rho$.

Remark 5.10. Let $(\mathcal{T}, \varepsilon, \rho)$ and $(\mathcal{T}', \varepsilon', \rho')$ be transporter systems associated to $\mathcal{F}$ with $\mathrm{ob}(\mathcal{T}) = \mathrm{ob}(\mathcal{T}')$, and let $\alpha\colon \mathcal{T} \to \mathcal{T}'$ be an isomorphism between them which is the identity on objects. Then for any $P \in \mathrm{ob}(\mathcal{T})$, the map $\alpha_{P,P}\colon \mathrm{Aut}_{\mathcal{T}}(P) \to \mathrm{Aut}_{\mathcal{T}'}(P)$ is an isomorphism of groups. In particular, $(\mathcal{T}, \varepsilon, \rho)$ is a linking system if and only if $(\mathcal{T}', \varepsilon', \rho')$ is a linking system.

Moreover, for every $P \in \mathrm{ob}(\mathcal{T})$, the isomorphism $\alpha_{P,P}$ maps $E(P) = \ker(\rho_{P,P})$ to $E'(P) = \ker(\rho'_{P,P})$ and $P\varepsilon_{P,P}$ to $P\varepsilon'_{P,P}$. So $(\mathcal{T}, \varepsilon, \rho)$ is a centric linking system if and only if $(\mathcal{T}', \varepsilon', \rho')$ is a centric linking system. Similarly $(\mathcal{T}, \varepsilon, \rho)$ is a linking locality in the sense of Oliver [18, Definition 3] if and only if the same holds for $(\mathcal{T}', \varepsilon', \rho')$.

Proof. The first part is clear. Let $P \in \Delta$ and $g \in \mathrm{Aut}_{\mathcal{T}}(P)$. As $\alpha \circ \rho' = \rho$, we have $g \in E(P)$ if and only if $g\alpha_{P,P}\rho'_{P,P} = g\rho_{P,P} = \mathrm{id}_P$, i.e. if and only if $g\alpha_{P,P} \in E'(P)$. Hence, $E(P)\alpha_{P,P} = E'(P)$. As $\varepsilon \circ \alpha = \varepsilon'$, we have $(P\varepsilon_{P,P})\alpha_{P,P} = P\varepsilon'_{P,P}$. The last part follows now from Remark 5.9. □

Suppose now we are given a locality $(\mathcal{L}, \Delta, S)$. Then we can construct a transporter system associated to $\mathcal{F} = \mathcal{F}_S(\mathcal{L})$ as follows: The objects of $\mathcal{T}(\mathcal{L}, \Delta)$ are the elements of Δ, and a morphism between objects $P, Q \in \Delta$ is a triple (f, P, Q) with $f \in \mathcal{L}$ such that $P \subseteq \mathbf{D}(f)$ and $P^f \leq Q$. Composition of morphisms is given by multiplication in the locality $\mathcal{L}$, i.e. $(f, P, Q) \circ (g, Q, R) = (fg, P, R)$ for all morphisms (f, P, Q) and (g, Q, R) in $\mathcal{T}(\mathcal{L}, \Delta)$.

Note that $\mathrm{Hom}_{\mathcal{T}_\Delta(S)}(P, Q) \subseteq \mathrm{Hom}_{\mathcal{T}(\mathcal{L},\Delta)}(P, Q)$ for all $P, Q \in \Delta$. Let $\varepsilon = \varepsilon_{\mathcal{L},\Delta}\colon \mathcal{T}_\Delta(S) \to \mathcal{T}(\mathcal{L}, \Delta)$ be the functor which is the identity on objects and the inclusion map on morphism sets. Let $\rho = \rho_{\mathcal{L},\Delta}\colon \mathcal{T}(\mathcal{L}, \Delta) \to \mathcal{F}$ be the functor which is the inclusion on objects, and for $P, Q \in \Delta$, $\rho_{P,Q}\colon \mathrm{Hom}_{\mathcal{T}(\mathcal{L},\Delta)} \to \mathrm{Hom}_{\mathcal{F}}(P, Q)$ is defined by $(f, P, Q) \mapsto c_f|_P$.

Theorem 5.11. *Let $(\mathcal{L}, \Delta, S)$ be a locality over $\mathcal{F}$. Let $\varepsilon = \varepsilon_{\mathcal{L},\Delta}$ and $\rho = \rho_{\mathcal{L},\Delta}$.*

(a) *The triple $(\mathcal{T}(\mathcal{L}, \Delta), \varepsilon, \rho)$ forms a transporter system associated to $\mathcal{F}$.*

(b) *For any $P \in \Delta$, we have $\mathrm{Aut}_{\mathcal{T}(\mathcal{L},\Delta)}(P) \cong N_{\mathcal{L}}(P)$ and $E(P) := \ker(\rho_{P,P}) = \{(f, P, P)\colon f \in C_{\mathcal{L}}(P)\} \cong C_{\mathcal{L}}(P)$.*

(c) *The locality $(\mathcal{L}, \Delta, S)$ is a linking locality if and only if $(\mathcal{T}(\mathcal{L}, \Delta), \varepsilon, \rho)$ is a linking system.*

(d) *The transporter system $(\mathcal{T}(\mathcal{L}, \Delta), \varepsilon, \rho)$ is a linking system in the sense of Oliver* [18, Definition 3] *if and only if $\mathcal{F}^{cr} \subseteq \Delta$ and $C_{\mathcal{L}}(P)$ is a p-group for every $P \in \Delta$. Moreover, $(\mathcal{T}(\mathcal{L}, \Delta), \varepsilon, \rho)$ is a centric linking system if and only if $\Delta = \mathcal{F}^c$ and $C_{\mathcal{L}}(P) \leq P$ for every $P \in \Delta$.*

Proof. Property (a) is shown in [10, Proposition A.3(a)]. Clearly, for any $P \in \Delta$, the map $N_{\mathcal{L}}(P) \to \mathrm{Aut}_{\mathcal{T}(\mathcal{L},\Delta)}(P)$ with $f \mapsto (f, P, P)$ is an isomorphism of groups. Moreover, any element $(f, P, P) \in \mathrm{Aut}_{\mathcal{T}(\mathcal{L},\Delta)}(P)$ lies in $E(P)$ if and only if $c_f|_P = \mathrm{id}_P$, i.e. if and only if $f \in C_{\mathcal{L}}(P)$. This shows (b). Property (c) follows now from (b), and (d) follows from (b) and Remark 5.9. □

Theorem 5.12. *Let $(\mathcal{T}, \varepsilon, \rho)$ be a transporter system associated to $\mathcal{F}$. Then there exists a locality $(\mathcal{L}, \Delta, S)$ over $\mathcal{F}$ with $\Delta = \mathrm{ob}(\mathcal{T})$ and an isomorphism $\eta\colon \mathcal{T} \to \mathcal{T}(\mathcal{L}, \Delta)$ between $(\mathcal{T}, \varepsilon, \rho)$ and $(\mathcal{T}(\mathcal{L}, \Delta), \varepsilon_{\mathcal{L},\Delta}, \rho_{\mathcal{L},\Delta})$ which is the identity on objects.*

Proof. Chermak [10, Appendix A] constructs a locality $(\mathcal{L}, \Delta, S)$ with $\Delta = \mathrm{ob}(\mathcal{T})$; see in particular [10, Proposition A.13]. It is then shown in Lemma A.14 and Lemma A.15 of [10] that there exists an invertible functor $\eta\colon \mathcal{T} \to \mathcal{T}(\mathcal{L}, \Delta)$ with certain properties. These properties imply that $\mathcal{F} = \mathcal{F}_S(\mathcal{L})$ and η is an isomorphism of transporter systems. The argument is exactly the same as the argument in the proof of Theorem A in [10] that the two left hand squares in the diagram on p.137 commute. □

6. Localities of objective characteristic p

In this section $\mathcal{F}$ is not necessarily assumed to be saturated.

Lemma 6.1. *Let $(\mathcal{L}, \Delta, S)$ be a locality over $\mathcal{F}$ of objective characteristic p. If $P \in \Delta \cap \mathcal{F}^f$ then $N_{\mathcal{L}}(P)$ is a model for $N_{\mathcal{F}}(P)$. In particular, $\Delta \subseteq \mathcal{F}^s$.*

Proof. Let $P \in \Delta \cap \mathcal{F}^f$. By Lemma 5.3(a),(g), $N_{\mathcal{L}}(P)$ is a subgroup of $\mathcal{L}$, $N_S(P) \in \mathrm{Syl}_p(N_{\mathcal{L}}(P))$ and $N_{\mathcal{F}}(P) = \mathcal{F}_{N_S(P)}(N_{\mathcal{L}}(P))$. As $N_{\mathcal{L}}(P)$ is of characteristic p, it follows that $N_{\mathcal{L}}(P)$ is a model for $N_{\mathcal{F}}(P)$. So by Lemma 2.1(b), $R := O_p(N_{\mathcal{F}}(P)) = O_p(N_{\mathcal{L}}(P))$. By Lemma 5.3(g), there exists $h \in \mathcal{L}$ such that $N_S(R) \leq S_h$, $R^h \in \mathcal{F}^f$ and $N_S(R^h) \in \mathrm{Syl}_p(N_{\mathcal{L}}(R^h))$. Note that $N_S(P) \leq N_S(R) \leq S_h$. By Lemma 5.3(b), $P^h \in \Delta$, $N_{\mathcal{L}}(P) \subseteq \mathbf{D}(h)$ and $c_h \colon N_{\mathcal{L}}(P) \to N_{\mathcal{L}}(P^h)$ is an isomorphism of groups. In particular, $R^h = O_p(N_{\mathcal{L}}(P^h))$. Moreover, $N_S(P) \cong N_S(P)^h \leq N_S(P^h)$. As P is fully normalized, it follow $N_S(P)^h = N_S(P^h)$ and P^h is fully normalized. Since P was arbitrary, what we proved before gives us $O_p(N_{\mathcal{F}}(P^h)) = O_p(N_{\mathcal{L}}(P^h)) = R^h$ and $N_{\mathcal{L}}(P^h)$ is a model for $N_{\mathcal{F}}(P^h)$. So, as $P^h \leq R^h$, we have $C_S(R^h) = C_{N_S(P^h)}(R^h) \leq R^h$. This implies that R is centric, because R^h is fully normalized. As P was arbitrary, this shows that, for any $Q \in \Delta$ and any fully normalized $\mathcal{F}$-conjugate P of Q, $O_p(N_{\mathcal{F}}(P))$ is $\mathcal{F}$-centric. So every $Q \in \Delta$ is subcentric by definition showing $\Delta \subseteq \mathcal{F}^s$. □

If $(\mathcal{L}, \Delta, S)$ is a locality, define $P \in \Delta$ to be *$\mathcal{L}$-radical* if $O_p(N_{\mathcal{L}}(P)) = P$.

Lemma 6.2. *Let $(\mathcal{L}, \Delta, S)$ be a locality over $\mathcal{F}$ of objective characteristic p and $P \in \Delta$. Then P is $\mathcal{L}$-radical if and only if $P \in \mathcal{F}^{cr}$.*

Proof. It follows from Lemma 5.3(b) that the set of $\mathcal{L}$-radical subgroups is closed under $\mathcal{F}$-conjugation. The set $\mathcal{F}^{cr}$ is closed under $\mathcal{F}$-conjugation as well. Hence, we may assume that $P \in \mathcal{F}^f$. Then by Lemma 6.1, $G := N_{\mathcal{L}}(P)$ is a model for $N_{\mathcal{F}}(P)$. Note $G/C_G(P) \cong \mathrm{Aut}_{\mathcal{F}}(P)$ and $P/Z(P) \cong \mathrm{Inn}(P)$. Hence, if $C_G(P) = Z(P)$ then $O_p(\mathrm{Aut}_{\mathcal{F}}(P)) = \mathrm{Inn}(P)$ if and only if $P = O_p(G)$. If $P \in \mathcal{F}^{cr}$ then $P \in N_{\mathcal{F}}(P)^c$ and so by Theorem 2.1, $C_G(P) = Z(P)$. Hence, by what we just stated, $P = O_p(G)$ and P is $\mathcal{L}$-radical. On the other hand, assuming that P is $\mathcal{L}$-radical, $C_G(P) = Z(P)$ as G is of characteristic p. So, again by what we stated before, $P \in \mathcal{F}^r$. Moreover, $C_S(P) = C_{N_S(P)}(P) \leq C_G(P) \leq P$. So $P \in \mathcal{F}^c$ as $P \in \mathcal{F}^f$. This proves the assertion. □

Proof of Proposition 4. Clearly, $Q \trianglelefteq \mathcal{F}$ if $\mathcal{L} = N_{\mathcal{L}}(Q)$ and $Q \leq Z(\mathcal{F})$ if $\mathcal{L} = C_{\mathcal{L}}(Q)$. Moreover, if $Q \leq Z(\mathcal{F})$ and $\mathcal{L} = N_{\mathcal{L}}(Q)$ then clearly, $\mathcal{L} = C_{\mathcal{L}}(Q)$. Hence, it is sufficient to prove that $\mathcal{L} = N_{\mathcal{L}}(Q)$ if $Q \trianglelefteq \mathcal{F}$. So assume $Q \trianglelefteq \mathcal{F}$ and $\mathcal{L} \neq N_{\mathcal{L}}(Q)$. Choose $f \in \mathcal{L} \backslash N_{\mathcal{L}}(Q)$ such that $|S_f|$ is maximal. Since $Q \trianglelefteq \mathcal{F}$ it follows $Q \not\leq S_f$. In particular, $S_f < S$ and thus $S_f^f < N_S(S_f^f)$. By [11, Lemma 2.10], there exists $h \in \mathcal{L}$ such that $N_S(S_f^f) \leq S_h$ and $N_S(S_f^{fh}) \in \mathrm{Syl}_p(N_{\mathcal{L}}(S_f^{fh}))$. Then $(f, h, h^{-1}) \in \mathbf{D}$ via S_f. By the maximality of $|S_f|$, $h \in N_{\mathcal{L}}(Q)$. So if $fh \in N_{\mathcal{L}}(Q)$ then $f = (fh)h^{-1} \in N_{\mathcal{L}}(Q)$ as $N_{\mathcal{L}}(Q)$ is a partial subgroup of $\mathcal{L}$ by Lemma 5.4. Hence, $fh \notin N_{\mathcal{L}}(Q)$ and by the maximality of $|S_f|$, $S_f = S_{fh}$. So replacing f by fh, we may assume that $N_S(S_f^f) \in \mathrm{Syl}_p(N_{\mathcal{L}}(S_f^f))$.

Since $c_f : S_f \to S_f^f$ is a morphism in $\mathcal{F}$ and $Q \trianglelefteq \mathcal{F}$, there exists $g \in \mathcal{L}$ such that $S_f Q \leq S_g$, $c_g|_{S_f} = c_f$ and $g \in N_{\mathcal{L}}(Q)$. Then $(f^{-1}, g) \in \mathbf{D}$ via S_f^f and $f^{-1}g \in C_{\mathcal{L}}(S_f^f) \subseteq N_{\mathcal{L}}(S_f^f)$. Since $Q \trianglelefteq \mathcal{F}$, we have $\mathrm{Aut}_{QP}(P) \leq O_p(\mathrm{Aut}_{\mathcal{F}}(P))$ for any $P \leq S$. Hence, $Q \leq P$ for every $P \in \mathcal{F}^{cr}$. In particular, $S_f \notin \mathcal{F}^{cr}$ as $Q \not\leq S_f$. So $S_f^f \notin \mathcal{F}^{cr}$ and thus, by Lemma 6.2, $S_f^f < R := O_p(N_{\mathcal{L}}(S_f^f))$. As $N_S(S_f^f) \in \mathrm{Syl}_p(N_{\mathcal{L}}(S_f^f))$ we have $R \leq S$. So the maximality of $|S_f| = |S_f^f|$ yields $f^{-1}g \in N_{\mathcal{L}}(Q)$. As $(f^{-1}, g, g^{-1}) \in \mathbf{D}$ via S_f^f, it follows $f^{-1} = (f^{-1}g)g \in N_{\mathcal{L}}(Q)$. This yields a contradiction to $f \notin N_{\mathcal{L}}(Q)$. □

Lemma 6.3. *Let $(\mathcal{L}, \Delta, S)$ be a locality over $\mathcal{F}$.*

(a) *If $P \in \Delta \cap \mathcal{F}^{fc}$, then $N_{\mathcal{L}}(P)$ is of characteristic p (and thus a model for $N_{\mathcal{F}}(P)$) if and only if $C_{\mathcal{L}}(Q) \leq Q$ for any $Q \in P^{\mathcal{F}}$.*

(b) *If $P \in \Delta \cap \mathcal{F}^{fq}$, then $C_{\mathcal{L}}(P) = C_S(P)O_{p'}(C_{\mathcal{L}}(P))$. Moreover, $N_{\mathcal{L}}(P)$ is of characteristic p (and thus a model for $N_{\mathcal{F}}(P)$) if and only if $C_{\mathcal{L}}(Q)$ is a p-group for any $Q \in P^{\mathcal{F}}$.*

Proof. Let $P \in \Delta \cap \mathcal{F}^f$. Then by Lemma 5.3(a),(g), $G := N_{\mathcal{L}}(P)$ is a finite group with $N_S(P) \in \mathrm{Syl}_p(G)$, $N_{\mathcal{F}}(P) = \mathcal{F}_{N_S(P)}(G)$ and $C_{\mathcal{F}}(P) = \mathcal{F}_{C_S(P)}(C_G(P))$. In particular, G is a model for $N_{\mathcal{F}}(P)$ if and only if G is of characteristic p.

By Remark 5.5, every $\mathcal{F}$-morphism between P and an $\mathcal{F}$-conjugate Q of P can be realized as a conjugation map by an element of $f \in \mathcal{L}$ and then, by Lemma 5.3(b), $c_f \colon N_{\mathcal{L}}(P) \to N_{\mathcal{L}}(Q)$ is an isomorphism of groups. In particular, $C_{\mathcal{L}}(P) \cong C_{\mathcal{L}}(Q)$ for any $Q \in P^{\mathcal{F}}$.

Suppose now that P is $\mathcal{F}$-centric. Then P is also centric in $N_{\mathcal{F}}(P)$. As $P \trianglelefteq G$, G is of characteristic p if $C_G(P) = C_{\mathcal{L}}(P) \leq P$. On the other hand, if G is a model, then Theorem 2.1(b) yields that $C_{\mathcal{L}}(P) = C_G(P) \leq P$. So G is a model for $N_{\mathcal{F}}(P)$ if and only if $C_{\mathcal{L}}(P) \leq P$. As $C_{\mathcal{L}}(P) \cong C_{\mathcal{L}}(Q)$ for any $Q \in P^{\mathcal{F}}$, we have $C_{\mathcal{L}}(P) \leq P$ if and only if $C_{\mathcal{L}}(Q) \leq Q$. This proves (a).

Assume now that P is quasicentric, i.e. $\mathcal{F}_{C_S(P)}(C_S(P)) = C_{\mathcal{F}}(P) = \mathcal{F}_{C_S(P)}(C_G(P))$. So by Lemma 2.5, $C_{\mathcal{L}}(P) = C_G(P) = C_S(P)O_{p'}(C_{\mathcal{L}}(P))$ and G is a model for $N_{\mathcal{F}}(P)$ if and only if $C_{\mathcal{L}}(P) = C_G(P)$ is a p-group. As $C_{\mathcal{L}}(P) \cong C_{\mathcal{L}}(Q)$ for any $Q \in P^{\mathcal{F}}$, this yields (b). □

Proof of Remark 1. Property (a) is stated in its more precise form in Proposition 5.11(c). The statement in (b) about $(\mathcal{L}, \Delta, S)$ is proved in Lemma 6.1. The statements in (c) and (d) about $(\mathcal{L}, \Delta, S)$ follow from Lemma 6.3.

We argue now that the statements in (b),(c),(d) about $(\mathcal{L}, \Delta, S)$ imply the statements about $\mathcal{T}$, where $(\mathcal{T}, \varepsilon, \rho)$ is a transporter system associated to $\mathcal{F}$: By Remark 5.10, we can replace $(\mathcal{T}, \Delta, S)$ by any transporter system which is isomorphic via an isomorphism which is the identity on objects. So by Lemma 5.12, we can assume $(\mathcal{T}, \varepsilon, \rho) = (\mathcal{T}(\mathcal{L}, \Delta), \varepsilon_{\mathcal{L},\Delta}, \rho_{\mathcal{L},\Delta})$. However, then the statements about $\mathcal{T}$ follow from Theorem 5.11(b),(c),(d). So it remains only to prove the statement in (c) that every linking system in Oliver's definition is a linking system in our definition. This follows however from what we have shown and the following fact: If $(\mathcal{T}, \varepsilon, \rho)$ is a linking system associated to $\mathcal{F}$ in the sense of Oliver [18, Definition 3], then the objects of $\mathcal{T}$ are by [18, Proposition 4(g)] quasicentric in $\mathcal{F}$. □

We close this section by giving a method to produce localities of objective characteristic p in certain circumstances. The result we state is a slight generalization of [11, Theorem 4.8]. It builds on the notion of a quotient locality modulo a partial normal subgroup as introduced in [11, Section 4]. In particular, we adapt the definition of a canonical projection modulo a partial normal subgroup from there.

Proposition 6.4. *Let $(\mathcal{L}, \Delta, S)$ be a locality such that $N_{\mathcal{L}}(P)$ is almost of characteristic p for every $P \in \Delta$. Set $\Theta(P) := \Theta(N_{\mathcal{L}}(P))$ for every $P \in \Delta$, and $\Theta := \bigcup\{\Theta(P) \colon P \in \Delta\}$.*

Then Θ is a partial normal subgroup of $\mathcal{L}$ with $\Theta \cap S = 1$. The canonical projection $\rho \colon \mathcal{L} \to \mathcal{L}/\Theta$ restricts to an isomorphism $S \to S\rho$. Upon identifying S with $S\rho$, the following properties hold:

(a) *$(\mathcal{L}/\Theta, \Delta, S)$ is a locality of objective characteristic p.*

(b) *$\mathcal{F}_S(\mathcal{L}/\Theta) = \mathcal{F}_S(\mathcal{L})$.*

(c) *For every $P \in \Delta$, the restriction $N_{\mathcal{L}}(P) \to N_{\mathcal{L}/\Theta}(P)$ of ρ has kernel $\Theta(P)$ and induces an isomorphism $N_{\mathcal{L}}(P)/\Theta(P) \cong N_{\mathcal{L}/\Theta}(P)$.*

Proof. Step 1: We show that $\Theta(Q) = \Theta(P) \cap C_{\mathcal{L}}(Q)$ for any $P, Q \in \Delta$ with $P \leq Q$. For the proof note that P is subnormal in Q, and by induction on the subnormal length, we may assume that $P \trianglelefteq Q$. Then $Q \leq N_{\mathcal{L}}(P)$ and $C_{N_{\mathcal{L}}(Q)}(Q) = C_{\mathcal{L}}(Q) = C_{N_{\mathcal{L}}(P)}(Q)$. Hence, by Lemma 2.8(a),

$\Theta(Q) = \Theta(C_{\mathcal{L}}(Q)) = \Theta(C_{N_{\mathcal{L}}(P)}(Q)) = \Theta(N_{\mathcal{L}}(P)) \cap C_{N_{\mathcal{L}}(P)}(Q) = \Theta(P) \cap C_{\mathcal{L}}(Q)$. This completes Step 1.

Step 2: We show $x \in \Theta(S_x)$ for any $x \in \Theta$. Let $x \in \Theta$. Then by definition of Θ, the element x lies in $\Theta(P)$ for some $P \in \Delta$. Choose such P maximal with respect to inclusion. We have $P \leq S_x$ and $[N_{S_x}(P), x] \leq \Theta(P) \cap N_S(P) = 1$. Hence, using Step 1, $x \in \Theta(P) \cap C_{\mathcal{L}}(N_{S_x}(P)) = \Theta(N_{S_x}(P))$. So the maximality of P yields $P = N_{S_x}(P)$ and thus $P = S_x$. Hence, $x \in \Theta(S_x)$ as required.

Step 3: We show that Θ is a partial normal subgroup of $\mathcal{L}$. Note that $1 \in \Theta$ as $1 \in \Theta(P)$ for any $P \in \Delta$. Moreover, clearly Θ is closed under inversion, since $\Theta(P)$ is a group for any $P \in \Delta$. Let now $(x_1, \dots, x_n) \in \mathbf{D}$ with $x_i \in \Theta$ for $i = 1. \dots, n$. Then $R := S_{(x_1,\dots,x_n)} \in \Delta$ by Lemma 5.3(f). Induction on i together with Step 1 and Step 2 shows $R \leq S_{x_i}$ and $x_i \in \Theta(R) \leq C_{\mathcal{L}}(R)$ for every $i = 1, \dots, n$. Hence, $\Pi(x_1, x_2, \dots, x_n) \in \Theta(R) \subseteq \Theta$. Thus, Θ is a partial subgroup of $\mathcal{L}$. Let $x \in \Theta$ and $f \in \mathcal{L}$ with $(f^{-1}, x, f) \in \mathbf{D}$. By Lemma 5.3(f), $X := S_{(f^{-1},x,f)} \in \Delta$. Moreover, $X^{f^{-1}} \leq S_x$. By Step 2, we have $x \in \Theta(S_x)$, and then by Step 1, $x \in \Theta(X^{f^{-1}})$. It follows now from Lemma 5.3(b) that $x^f \in \Theta(X^{f^{-1}})^f = \Theta(X) \subseteq \Theta$. Hence, Θ is a partial normal subgroup of $\mathcal{L}$.

Step 4: We are now in a position to complete the proof. Notice first that $\Theta \cap S = 1$ as $\Theta(P) \cap S = \Theta(P) \cap N_S(P) = 1$ for every $P \in \Delta$. The quotient map $\rho\colon \mathcal{L} \to \mathcal{L}/\Theta$ is a homomorphism of partial groups with $\ker(\rho) = \Theta$; see Section 5.3. Therefore, $\rho|_S\colon S \to S\rho$ is a homomorphism of groups with kernel $S \cap \Theta = 1$ and thus an isomorphism of groups. Upon identifying S with $S\rho$, it follows now from Theorem 5.6(a),(b) that $(\mathcal{L}/\Theta, \Delta, S)$ is a locality and $\mathcal{F}_S(\mathcal{L}) = \mathcal{F}_S(\mathcal{L}/\Theta)$. So (b) holds. Let $P \in \Delta$. By Theorem 5.6(c), the restriction of ρ to a map $N_{\mathcal{L}}(P) \to N_{\mathcal{L}/\Theta}(P)$ is an epimorphism with kernel $N_{\mathcal{L}}(P) \cap \Theta$. For any $x \in N_{\mathcal{L}}(P) \cap \Theta$, we have $P \leq S_x$ and then $x \in \Theta(S_x) \leq \Theta(P)$ by Step 1 and Step 2. This shows $N_{\mathcal{L}}(P) \cap \Theta = \Theta(P)$ and so (c) holds. In particular, our assumption yields that $N_{\mathcal{L}/\Theta}(P)$ is a group of characteristic p and therefore (a) holds. □

7. Construction of linking localities

Lemma 7.1. *Suppose $(\mathcal{L}, \Delta, S)$ is a locality of objective characteristic p over $\mathcal{F}$. Let $T \in \mathcal{F}^f$ such that any proper overgroup of T is in Δ and $O_p(N_{\mathcal{F}}(T)) \in \Delta$. Then $N_{\mathcal{L}}(T)$ is a subgroup of $\mathcal{L}$ which is a model for $N_{\mathcal{F}}(T)$.*

Proof. As every proper overgroup of T is in Δ,

$$\Delta_T := \{N_P(T)\colon T \leq P \in \Delta\} = \{P \in \Delta\colon T \leq P \leq N_S(T)\} \subseteq \Delta.$$

Set

$$R := O_p(N_{\mathcal{F}}(T)).$$

Step 1: We show that $(N_{\mathcal{L}}(T), \Delta_T, N_S(T))$ is a linking locality of objective characteristic p. First of all, by [10, Lemma 2.19(c)], $(N_{\mathcal{L}}(T), \Delta_T, N_S(T))$ is a locality. If $P \in \Delta_T$ then $N_{\mathcal{L}}(P)$ is a group of characteristic p, as $P \in \Delta$ and $(\mathcal{L}, \Delta, S)$ is of objective characteristic p. In particular, $N_{N_{\mathcal{L}}(T)}(P) = N_{N_{\mathcal{L}}(P)}(T)$ is a group of characteristic p by Lemma 2.2(a). Hence, $(N_{\mathcal{L}}(T), \Delta_T, N_S(T))$ is of objective characteristic p.

Step 2: We show that $N_{\mathcal{F}}(T) = \mathcal{F}_{N_S(T)}(N_{\mathcal{L}}(T))$. Clearly, $\mathcal{F}_{N_S(T)}(N_{\mathcal{L}}(T)) \subseteq N_{\mathcal{F}}(T)$. Let now $A, B \leq N_S(T)$ and $\varphi \in \mathrm{Hom}_{N_{\mathcal{F}}(T)}(A, B)$. As R is normal in $N_{\mathcal{F}}(T)$, φ extends to $\hat{\varphi} \in \mathrm{Hom}_{N_{\mathcal{F}}(T)}(AR, BR)$ with $R\hat{\varphi} = R$. By assumption, $R \in \Delta$ and thus $AR \in \Delta$. So by Remark 5.5, there exists $f \in \mathcal{L}$ with $RA \leq S_f$ and $\hat{\varphi} = c_f|_{AR}$. As $\hat{\varphi}$ is a morphism in $N_{\mathcal{F}}(T)$ and $T \leq R \leq RA \leq S_f$, it follows $T^f = T\hat{\varphi} = T$ and thus $f \in N_{\mathcal{L}}(T)$. Hence, $\varphi = c_f|_A$ is a morphism in $\mathcal{F}_{N_S(T)}(N_{\mathcal{L}}(T))$. This completes Step 2.

Step 3: We complete the proof. By Step 1 and Step 2, $(N_{\mathcal{L}}(T), N_S(T), \Delta_T)$ is a locality of objective characteristic p over $N_{\mathcal{F}}(T)$. Hence, by Proposition 4, we have $N_{\mathcal{L}}(T) = N_{N_{\mathcal{L}}(T)}(R)$. As $R \in \Delta$ by assumption, $R \in \Delta_T$. Moreover, as R is normal in $N_{\mathcal{F}}(T)$, R is fully normalized in $N_{\mathcal{F}}(T)$. So it follows from Lemma 6.1 applied with $N_{\mathcal{L}}(T)$ and R in place of $\mathcal{L}$ and P that $N_{\mathcal{L}}(T) = N_{N_{\mathcal{L}(T)}}(R)$ is a model for $N_{\mathcal{F}}(T) = N_{N_{\mathcal{F}(T)}}(R)$. In particular, $N_{\mathcal{L}}(T)$ is a subgroup of $\mathcal{L}$. $\square$

Suppose $(\mathcal{L}^+, \Delta^+, S)$ is a locality with partial product $\Pi^+ \colon \mathbf{D}^+ \to \mathcal{L}^+$. Suppose Δ is a non-empty subset of Δ^+ which is closed under taking $\mathcal{L}^+$-conjugates and overgroups in S. Set

$$\mathcal{L}^+|_\Delta := \{f \in \mathcal{L}^+ \colon \exists P \in \Delta \text{ such that } P \subseteq \mathbf{D}^+(f) \text{ and } P^f \leq S\}$$

and write $\mathbf{D}$ for the set of words $w = (f_1, \ldots, f_n)$ such that $w \in \mathbf{D}^+$ via $P_0, \ldots, P_n$ for some $P_0, \ldots, P_n \in \Delta$. Observe that $\mathbf{D} \subseteq \mathbf{W}(\mathcal{L}^+|_\Delta)$. It is easy to check that $\mathcal{L}^+|_\Delta$ forms a partial group with partial product $\Pi^+|_{\mathbf{D}} \colon \mathbf{D} \to \mathcal{L}^+|_\Delta$, and that $(\mathcal{L}^+|_\Delta, \Delta, S)$ forms a locality. We call $\mathcal{L}^+|_\Delta$ the *restriction* of $\mathcal{L}^+$ to Δ.

Note that $\mathcal{T}(\mathcal{L}^+|_\Delta, \Delta)$ is the full subcategory of $\mathcal{T}(\mathcal{L}^+, \Delta^+)$ with object set Δ.

Theorem 7.2. *Suppose $\mathcal{F}$ is saturated. Let Δ and Δ^+ be collections of subgroups of S which are both closed under $\mathcal{F}$-conjugation and with respect to overgroups. Suppose that $\mathcal{F}^{cr} \subseteq \Delta \subseteq \Delta^+ \subseteq \mathcal{F}^s$, and let $(\mathcal{L}, \Delta, S)$ be a linking locality over $\mathcal{F}$.*

- (a) *There exists a linking locality $(\mathcal{L}^+, \Delta^+, S)$ such that $\mathcal{L}$ is the restriction $\mathcal{L}^+|_\Delta$ of $\mathcal{L}^+$ to Δ and $\mathcal{F}_S(\mathcal{L}^+) = \mathcal{F}$. The inclusion of nerves $|\mathcal{T}(\mathcal{L}, \Delta)| \subseteq |\mathcal{T}(\mathcal{L}^+, \Delta^+)|$ is a homotopy equivalence.*
- (b) *If $(\widetilde{\mathcal{L}}^+, \Delta^+, S)$ is another linking locality over $\mathcal{F}$ with object set Δ^+ and $\beta : \mathcal{L} \to \widetilde{\mathcal{L}}^+|_\Delta$ is a rigid isomorphism, then β extends to a rigid isomorphism $\mathcal{L}^+ \to \widetilde{\mathcal{L}}^+$. So in particular, $\mathcal{L}^+$ is unique up to an isomorphism that restricts to the identity on $\mathcal{L}$.*
- (c) *If $\Delta^+ \backslash \Delta$ is a single $\mathcal{F}$-conjugacy class then $N_{\mathcal{L}}(R) = N_{\mathcal{L}^+}(R)$ for every $R \in \Delta^+ \backslash \Delta$ which is fully $\mathcal{F}$-normalized.*

Proof. We may assume $\Delta \neq \Delta^+$. Choose $T \in \Delta^+ \backslash \Delta$ such that T is maximal with respect to inclusion. Since Δ^+ is closed under taking overgroups, it follows that every proper overgroup of T is in Δ. Therefore, as Δ is closed under $\mathcal{F}$-conjugation, every proper overgroup of an $\mathcal{F}$-conjugate of T is in Δ. Hence, $\Delta \cup T^{\mathcal{F}}$ is closed under taking overgroups. By construction, this set is closed under taking $\mathcal{F}$-conjugates. Furthermore, $\Delta \cup T^{\mathcal{F}} \subseteq \Delta^+$, as Δ^+ is closed under taking $\mathcal{F}$-conjugates. Now by induction on $|\Delta^+ \backslash \Delta|$, we may assume $\Delta^+ = \Delta \cup T^{\mathcal{F}}$. Replacing T by a suitable $\mathcal{F}$-conjugate, we may assume $T \in \mathcal{F}^f$. As $\mathcal{F}^{cr} \subseteq \Delta$ and $T \notin \Delta$, we have $T \notin \mathcal{F}^{cr}$. Then by Lemma 3.18, $T < O_p(N_{\mathcal{F}}(T))$ and thus $O_p(N_{\mathcal{F}}(T)) \in \Delta$, as every proper overgroup of T is in Δ. Hence, by Lemma 7.1, $M := N_{\mathcal{L}}(T)$ is a subgroup of $\mathcal{L}$ which is a model for $N_{\mathcal{F}}(T)$. Now clearly properties (1)-(4) of [10, Hypothesis 5.3] hold. By Theorem 2.1(b), $O_p(M) = O_p(N_{\mathcal{F}}(T)) \in \Delta$. So setting $\Delta_T := \{P \in \Delta \colon T \trianglelefteq P\}$, the locality $\mathcal{L}_{\Delta_T}(M)$ introduced in [10, Example/Lemma 2.10] is just the group M and $\lambda = \mathrm{id}_M$ can be considered as a rigid isomorphism $N_{\mathcal{L}}(T) \to \mathcal{L}_{\Delta_T}(M)$. So Hypothesis 5.3 in [10] is fulfilled. So by [10, Theorem 5.14], there exists a locality $(\mathcal{L}^+, \Delta^+, S)$ such that $\mathcal{L}$ is the restriction $\mathcal{L}^+|_\Delta$ of $\mathcal{L}^+$ to Δ and $\mathcal{F}_S(\mathcal{L}^+) = \mathcal{F}$. Furthermore, $\mathcal{L}^+$ can be taken to be the locality $\mathcal{L}^+(\lambda)$ constructed in [10, Section 5]. So the first part of (a) holds. To prove (b) let $(\widetilde{\mathcal{L}}^+, \Delta^+, S)$ be another linking locality over $\mathcal{F}$ with object set Δ^+ and let $\beta : \mathcal{L} \to \widetilde{\mathcal{L}}^+|_\Delta$ be a rigid isomorphism. Then $\widetilde{\mathcal{L}} := \widetilde{\mathcal{L}}^+|_\Delta$ is a linking locality as well and has thus the same properties we proved above for $\mathcal{L}$. In particular, $N_{\widetilde{\mathcal{L}}}(T)$ is a subgroup of $\widetilde{\mathcal{L}}$ which is a model for $N_{\mathcal{F}}(T)$. Then $\beta_T = \beta|_M \colon M \to N_{\widetilde{\mathcal{L}}}(T)$ will be an isomorphism of groups which restricts to the identity on $N_S(T)$, as β is a rigid isomorphism. As $(\widetilde{\mathcal{L}}^+, \Delta^+, S)$ is a linking locality and $T \in \Delta^+ \cap \mathcal{F}^f$, $N_{\widetilde{\mathcal{L}}^+}(T)$ is a model for $N_{\mathcal{F}}(T)$ by Lemma 6.1. Clearly, $N_{\widetilde{\mathcal{L}}}(T) \subseteq N_{\widetilde{\mathcal{L}}^+}(T)$ and thus $N_{\widetilde{\mathcal{L}}}(T) = N_{\widetilde{\mathcal{L}}^+}(T)$ by Theorem 2.1(a). Hence, β_T is also a group isomorphism $M \to N_{\widetilde{\mathcal{L}}^+}(T)$ which restrict to the identity on $N_S(T)$. So by [10, Theorem 5.15(a)] applied with $\widetilde{\mathcal{L}}^+$ in place of $\mathcal{L}^*$ and β_T in place of μ, there exists a rigid isomorphism $\beta^+ \colon \mathcal{L}^+(\lambda) \to \widetilde{\mathcal{L}}^+$ which restricts to the identity on $\mathcal{L}$. This proves (b). Since our choice of T was arbitrary, the arguments above give that $N_{\mathcal{L}}(R)$ is a model for $N_{\mathcal{F}}(R)$ for any $R \in T^{\mathcal{F}} \cap \mathcal{F}^f$ and thus $N_{\mathcal{L}^+}(T) = N_{\mathcal{L}}(R)$. This proves (c).

It remains to prove the statement in part (a) about the nerves of the transporter systems. Note that $\mathcal{T}(\mathcal{L}, \Delta)$ is the full subcategory of $\mathcal{T}(\mathcal{L}^+, \Delta^+)$ with object set Δ. If $(\mathcal{T}, \varepsilon, \rho)$ is a transporter system then $P \in \mathrm{ob}(\mathcal{T})$ is $\mathcal{T}$-radical in the sense defined in [20, p. 1015] if $O_p(\mathrm{Aut}_{\mathcal{T}}(P)) = P\varepsilon_{P,P}$. As $\mathrm{Aut}_{\mathcal{T}(\mathcal{L}^+,\Delta^+)}(P) \cong N_{\mathcal{L}^+}(P)$ for every $P \in \Delta^+$, it follows that $P \in \Delta^+$ is $\mathcal{T}(\mathcal{L}^+, \Delta^+)$-radical if and only if P is $\mathcal{L}^+$-radical in the sense defined above. Hence, by Lemma 6.2, the $\mathcal{T}(\mathcal{L}^+, \Delta)$-radical elements of Δ^+ are precisely the elements of $\mathcal{F}^{cr}$. As by assumption $\mathcal{F}^{cr} \subseteq \Delta$, it follows $\mathcal{T}(\mathcal{L}^+, \Delta^+)^r \subseteq \mathcal{T}(\mathcal{L}, \Delta)$, where $\mathcal{T}(\mathcal{L}^+, \Delta^+)^r$ denotes the full subcategory of $\mathcal{T}(\mathcal{L}^+, \Delta^+)$ with object set the $\mathcal{T}(\mathcal{L}^+, \Delta^+)$-radical subgroups. Hence, by [20, Proposition 4.7], the inclusion of nerves $|\mathcal{T}(\mathcal{L}, \Delta)| \subseteq |\mathcal{T}(\mathcal{L}^+, \Delta^+)|$ is a homotopy equivalence. □

Theorem A is now easy to deduce from Theorem 7.2 using the existence and uniqueness of centric linking systems which we state here in the formulation in which we will apply it:

Theorem 7.3 (Chermak, Oliver, Glauberman–Lynd)**.** *Let $\mathcal{F}$ be a saturated fusion system over S. Then there exists a linking locality $(\mathcal{L}, \Delta, S)$ over $\mathcal{F}$ with object set $\Delta = \mathcal{F}^c$, and such a linking locality is unique up to a rigid isomorphism.*

By Remark 1, $(\mathcal{L}, \Delta, S)$ is a linking locality with object set $\Delta = \mathcal{F}^c$ if and only if it is a centric linking system in the sense of Chermak [10]. Hence, Theorem 7.3 is a restatement of the main theorem in [10]. The proof in [10] uses the classification of finite simple groups. However, by Theorem 5.11 and Theorem 5.12, the statement of Theorem 7.3 is equivalent to the existence and uniqueness of centric linking systems which can be proved without the classification of finite simple groups if combining [19] and [12].

Proof of Theorem A. Suppose $\mathcal{F}$ is saturated. By Lemma 3.3, the set $\mathcal{F}^s$ is closed under taking $\mathcal{F}$-conjugates and overgroups. Hence, it is sufficient to prove (a). Let Δ_0 be the set of overgroups of the elements of $\mathcal{F}^{cr}$ in S. Then Δ_0 is closed under taking $\mathcal{F}$-conjugates, as $\mathcal{F}^{cr}$ is closed under taking $\mathcal{F}$-conjugates.

Step 1: We show that, up to a rigid isomorphism, there exists a unique linking locality $(\mathcal{L}_0, \Delta_0, S)$ over $\mathcal{F}$ and the nerve of $\mathcal{T}(\mathcal{L}_0, \Delta)$ is homotopy equivalent to the nerve of a centric linking system. By Theorem 7.3, a linking locality $(\mathcal{L}^*, \mathcal{F}^c, S)$ over $\mathcal{F}$ exists and is unique up to a rigid isomorphism. Then clearly, setting $\mathcal{L}_0 := \mathcal{L}^*|_{\Delta_0}$, the triple $(\mathcal{L}_0, \Delta_0, S)$ is a linking locality. Suppose we are given another linking locality $(\widetilde{\mathcal{L}}_0, \Delta_0, S)$ over $\mathcal{F}$. Then by Theorem 7.2, there exists a linking locality $(\widetilde{\mathcal{L}}^*, \mathcal{F}^s, S)$ over $\mathcal{F}$ with $\widetilde{\mathcal{L}}^*|_{\Delta_0} = \widetilde{\mathcal{L}}_0$. Moreover, $|\mathcal{T}(\widetilde{\mathcal{L}}^*, \Delta)| \simeq |\mathcal{T}(\widetilde{\mathcal{L}}_0, \Delta_0)|$. Since $(\mathcal{L}^*, \mathcal{F}^c, S)$ is unique up to a rigid isomorphism, there exists then a rigid isomorphism $\lambda : \mathcal{L}^* \to \widetilde{\mathcal{L}}^*$. Clearly, λ restricts to a rigid isomorphism $\mathcal{L}_0 \to \widetilde{\mathcal{L}}_0$. By [10, Proposition A.3(b)], every rigid isomorphism of localities leads to an isomorphism between the corresponding transporter systems. In particular, $|\mathcal{T}(\mathcal{L}^*, \Delta^*)| \simeq |\mathcal{T}(\widetilde{\mathcal{L}}^*, \Delta)| \simeq |\mathcal{T}(\widetilde{\mathcal{L}}_0, \Delta_0)| \simeq |\mathcal{T}(\mathcal{L}_0, \Delta_0)|$.

Step 2: We complete the proof by showing that, up to a rigid isomorphism, there exists a unique linking locality $(\mathcal{L}, \Delta, S)$ and $|\mathcal{T}(\mathcal{L}, \Delta)|$ is homotopy equivalent to the nerve of a centric linking system. Note that $\mathcal{F}^{cr} \subseteq \Delta_0 \subseteq \Delta \subseteq \mathcal{F}^s$. By Step 1 there is a linking locality $(\mathcal{L}_0, \Delta_0, S)$ which is unique up to rigid isomorphism and $|\mathcal{T}(\mathcal{L}_0, \Delta_0)|$ is homotopy equivalent to the nerve of a centric linking system. By Theorem 7.2, there exists a linking locality $(\mathcal{L}, \Delta, S)$ over $\mathcal{F}$ such that $\mathcal{L}|_{\Delta_0} = \mathcal{L}_0$ and $|\mathcal{T}(\mathcal{L}, \Delta)| \simeq |\mathcal{T}(\mathcal{L}_0, \Delta_0)|$ is homotopy equivalent to the nerve of a centric linking system. Moreover, for every linking locality $(\widetilde{\mathcal{L}}, \Delta, S)$, any rigid isomorphism $\mathcal{L}_0 \to \widetilde{\mathcal{L}}|_{\Delta_0}$ extends to a rigid isomorphism $\mathcal{L} \to \widetilde{\mathcal{L}}$. Let $(\widetilde{\mathcal{L}}, \Delta, S)$ be a linking locality. Note that $(\widetilde{\mathcal{L}}|_{\Delta_0}, \Delta_0, S)$ is a linking locality. So by the uniqueness of $\mathcal{L}_0$, there exists a rigid isomorphism $\gamma : \mathcal{L}_0 \to \widetilde{\mathcal{L}}|_{\Delta_0}$. This extends to a rigid isomorphism $\mathcal{L} \to \widetilde{\mathcal{L}}$ proving that $\mathcal{L}$ is unique up to a rigid isomorphism. □

8. Centralizers of partial normal subgroups

Lemma 8.1. *Let $(\mathcal{L},\Delta,S)$ be a locality and $\mathcal{N}$ a partial normal subgroup of $\mathcal{L}$. Let $Q\in\Delta$. Then there exists $x\in\mathcal{N}$ such that $N_S(Q)\leq S_x$ and $N_T(Q^x)\in\mathrm{Syl}_p(N_{\mathcal{N}}(Q^x))$.*

Proof. By Lemma 5.3(g) there exists $g\in\mathcal{L}$ such that $N_S(Q)\leq S_g$ and $N_S(Q^g)\in\mathrm{Syl}_p(N_{\mathcal{L}}(Q^g))$. As $N_{\mathcal{L}}(Q^g)$ is a subgroup of $\mathcal{L}$ with normal subgroup $N_{\mathcal{N}}(Q^g)$, it follows $N_T(Q^g)=N_S(Q^g)\cap N_{\mathcal{N}}(Q^g)\in\mathrm{Syl}_p(N_{\mathcal{N}}(Q^g))$. Chermak [10, Definition 4.3] defines a reflexive and transitive relation $\uparrow$ on the set $\mathcal{L}\circ\Delta$ of pairs $(f,P)\in\mathcal{L}\times\Delta$ such that $P\leq S_f$. Furthermore, he calls a pair (f,P) maximal in $\mathcal{L}\circ\Delta$ if $(f,P)\uparrow(f',P')$ implies $|P|=|P'|$. The definition of $\uparrow$ yields that $P=S_f$ if (f,P) is $\uparrow$-maximal. As S is finite, we can take $f\in\mathcal{L}$ and $R\in\Delta$ such that $(g,S_g)\uparrow(f,R)$ and (f,R) is $\uparrow$-maximal. Then $R=S_f$, so it follows from [10, Proposition 4.5] that $T\leq S_f=R$. Then by [10, Lemma 4.6], there exists $x\in\mathcal{N}$ such that $g=xf$, $S_g\leq S_{(x,f)}$ and $N_T(Q^x)\in\mathrm{Syl}_p(N_{\mathcal{N}}(Q^x))$. Then $N_S(Q)\leq S_g=S_{(x,f)}\leq S_x$ and the assertion holds. □

Proposition 8.2. *Suppose $(\mathcal{L},\Delta,S)$ is a linking locality over $\mathcal{F}$. Let $\mathcal{N}$ be a partial normal subgroup of $\mathcal{L}$ and $T=\mathcal{N}\cap S$. Assume that R is a subgroup of $C_S(T)$ which is weakly closed in $\mathcal{F}$. Then the following are equivalent:*

(1) *$N_{\mathcal{N}}(T)\subseteq C_{\mathcal{L}}(R\cap T)$.*
(2) *$N_{\mathcal{N}}(T)\subseteq C_{\mathcal{L}}(R)$.*
(3) *$\mathcal{N}\subseteq C_{\mathcal{L}}(R)$.*

Proof. Assume first that (1) holds. To prove (2) let $x\in N_{\mathcal{N}}(T)$. We want to show that $x\in C_{\mathcal{L}}(R)$. Set $P:=S_x$. As $x\in N_{\mathcal{N}}(T)$, we have $T\leq S_x$. Then by [11, Lemma 3.1(b)], $x\in N_{\mathcal{L}}(P)$ and thus $x\in N:=N_{N_{\mathcal{N}}(P)}(T)$. So it is sufficient to show that $N\subseteq C_{\mathcal{L}}(R)$. Recall that $P\in\Delta$ and thus $N_{\mathcal{N}}(P)\leq N_{\mathcal{L}}(P)$ is a subgroup of $\mathcal{L}$. In particular, N is a subgroup of $\mathcal{L}$. By [11, Lemma 3.1(c)], T is maximal in the poset of p-subgroups of $\mathcal{N}$. Thus, as $N\subseteq\mathcal{N}$, it follows that $T\in\mathrm{Syl}_p(N)$. As $(\mathcal{L},\Delta,S)$ is of objective characteristic p it follows from [11, Lemma 3.5] that $N_{\mathcal{N}}(T)\subseteq N_{\mathcal{L}}(TC_S(T))$. Since $R\leq C_S(T)$ and R is weakly closed in $\mathcal{F}$, this implies $N\subseteq N_{\mathcal{N}}(T)\subseteq N_{\mathcal{L}}(R)$. Note that the commutator group $[R,N]$ is defined the group $N_{\mathcal{L}}(P)$. By (1), $[R\cap T,N]=1$. Moreover, $[R,N]\leq R\cap\mathcal{N}=R\cap T$. Thus, $[R,O^p(N)]=[R,O^p(N),O^p(N)]=1$ and so $O^p(N)\subseteq C_{\mathcal{L}}(R)$. As $T\in\mathrm{Syl}_p(N)$ and $R\leq C_S(T)$, it follows $N=TO^p(N)\subseteq C_{\mathcal{L}}(R)$ as required. So we have shown that (1) implies (2). Clearly, (3) implies (1), so it remains only to prove that (2) implies (3).

Suppose (2) holds and that $\mathcal{N}\not\subseteq C_{\mathcal{L}}(R)$. Choose $n\in\mathcal{N}$ such that $n\notin C_{\mathcal{L}}(R)$ and $P:=S_n$ is of maximal order subject to this property. We proceed in two steps.

Step 1: We show that $N_{\mathcal{N}}(Q)\subseteq C_{\mathcal{L}}(R)$ for all $Q\in\Delta$ with $|Q|\geq|P|$ and $N_T(Q)\in\mathrm{Syl}_p(N_{\mathcal{N}}(Q))$. Assuming this is wrong we choose a counterexample Q. Then $|Q|=|P|$ because of the maximality of P. Set $G:=N_{\mathcal{L}}(Q)$ and notice that $N:=N_{\mathcal{N}}(Q)$ is a normal subgroup of G. As $N_T(Q)\in\mathrm{Syl}_p(N)$, we have $O_p(N)\leq N_T(Q)$. As $N\leq N_{\mathcal{N}}(QO_p(N))$, the maximality of $|Q|=|P|$ yields $O_p(N)\leq Q$. As $Q_0:=Q\cap T=Q\cap\mathcal{N}\trianglelefteq N$, it follows $Q_0=O_p(N)$. Since $(\mathcal{L},\Delta,S)$ is a linking locality, $G=N_{\mathcal{L}}(Q)$ is of characteristic p. So by Lemma 2.2(b), N is of characteristic p and thus $C_N(Q_0)\leq Q_0$. Hence, $[N_{C_S(Q_0)}(Q),N]\leq C_N(Q_0)\leq Q_0$ and $QN_{C_S(Q_0)}(Q)$ is normalized by N. The maximality of $|Q|=|P|$ yields now $N_{C_S(Q_0)}(Q)\leq Q$. As $QC_S(Q_0)$ is a p-group, this implies $C_S(Q_0)\leq Q$. In particular, $R\leq C_S(T)\leq C_S(Q_0)\leq Q$. As R is weakly closed in $\mathcal{F}$, it follows that $R\trianglelefteq G$. By assumption $[R,T]=1$ and $N_T(Q)\in\mathrm{Syl}_p(N)$ yielding $[R,O^{p'}(N)]=[R,\langle N_T(Q)^N\rangle]=1$. If $T\leq Q$ then, as T is strongly closed, $N\leq N_{\mathcal{N}}(T)$. So (2) would yield that $N\leq N_{\mathcal{N}}(T)\subseteq C_{\mathcal{L}}(R)$, contradicting the choice of Q. Thus $T\not\leq Q$ and, as TQ is a p-group, we have $N_T(Q)\not\leq Q$. Thus, by the maximality of $|Q|=|P|$, $N_N(N_T(Q))\subseteq N_{\mathcal{N}}(N_T(Q)Q)\subseteq C_{\mathcal{L}}(R)$. By a Frattini argument, $N=O^{p'}(N)N_N(N_T(Q))\leq C_G(R)\subseteq C_{\mathcal{L}}(R)$. This contradicts our assumption and thus completes Step 1.

Step 2: We derive the final contradiction. By Lemma 8.1, there exists $x \in \mathcal{N}$ such that $N_S(P^n) \leq S_x$ and $N_T(P^{nx}) \in \mathrm{Syl}_p(N_{\mathcal{N}}(P^{nx}))$. If $T \leq P$ then, as T is strongly closed, $n \in N_{\mathcal{N}}(T) \subseteq C_{\mathcal{L}}(R)$ contradicting the choice of n. Hence, $T \not\leq P$ and $T \not\leq P^n$. In particular, $N_S(P^n) \not\leq P^n$ and the maximality of $|P| = |P^n|$ yields that $x \in C_{\mathcal{L}}(R)$. By Lemma 5.3(b), conjugation with nx induces a group isomorphism from $N_{\mathcal{L}}(P)$ to $N_{\mathcal{L}}(P^{nx})$ and so $N_T(P)^{nx}$ is a p-subgroup of $N_{\mathcal{N}}(P^{nx})$. As $N_T(P^{nx}) \in \mathrm{Syl}_p(N_{\mathcal{N}}(P^{nx}))$, there exists $y \in N_{\mathcal{N}}(P^{nx})$ such that $(N_T(P)^{nx})^y \leq N_T(P^{nx})$. Then by Lemma 5.3(c), $N_T(P)^{nxy} = (N_T(P)^{nx})^y \leq N_T(P^{nx})$. As $T \not\leq P$ and TP is a p-group, we have $N_T(P) \not\leq P$. Moreover, $N_T(P)P \leq S_{nxy}$. Hence, the maximality of $|P|$ yields $nxy \in C_{\mathcal{L}}(R)$. By Step 1, $y \in N_{\mathcal{N}}(P^{nx}) \subseteq C_{\mathcal{L}}(R)$. By Lemma 5.4, $C_{\mathcal{L}}(R)$ is a partial subgroup of $\mathcal{L}$. As $(n, x, y, y^{-1}) \in \mathbf{D}$ via P, it follows that $nx = (nx)(yy^{-1}) = (nxy)y^{-1} \in C_{\mathcal{L}}(R)$. Similarly, as $x \in C_{\mathcal{L}}(R)$ and $(n, x, x^{-1}) \in \mathbf{D}$ via P, $n = n(xx^{-1}) = (nx)x^{-1} \in C_{\mathcal{L}}(R)$. This contradicts the choice of n and gives thus the final contradiction. □

Proof of Proposition 3. Clearly, $C_S(\mathcal{N}) \subseteq C_S(\mathcal{E})$. So it is sufficient to show that $R := C_S(\mathcal{E})$ is contained in $C_S(\mathcal{N})$, or equivalently, $\mathcal{N} \subseteq C_{\mathcal{L}}(R)$. By [3, (6.7)(1)], R is strongly closed in $\mathcal{F}$ and thus weakly closed in $\mathcal{F}$. Furthermore, $R \leq C_S(T)$. As $\mathcal{E} \subseteq C_{\mathcal{F}}(R)$, $c_n|_{R\cap T}$ is the identity for every $n \in N_{\mathcal{N}}(T)$, i.e. $N_{\mathcal{N}}(T) \subseteq C_{\mathcal{L}}(R \cap T)$. Hence, by Lemma 8.2, $R \leq C_S(\mathcal{N})$. □

Remark 8.3. Our arguments show actually that in the situation of Proposition 3, the subgroup $C_S(\mathcal{E}) = C_S(\mathcal{N})$ is the largest subgroup of $C_S(T)$ weakly closed in $\mathcal{F}$ such that $N_{\mathcal{N}}(T) \subseteq C_{\mathcal{L}}(R \cap T)$. Similarly, it is the largest subgroup of $C_S(T)$ strongly closed in $\mathcal{F}$ such that $N_{\mathcal{N}}(T) \subseteq C_{\mathcal{L}}(R \cap T)$.

9. Maps between linking systems

9.1. **Quotients modulo central subgroups.** We will study quotients of localities modulo central subgroups contained in S. We start by summarizing some crucial facts about quotients of fusion systems modulo central subgroups:

Lemma 9.1. *Let $\mathcal{F}$ be a saturated fusion system on S and $Z \leq Z(\mathcal{F})$. Then the following hold for every subgroup $P \leq S$:*

(a) *We have $PZ/Z \in (\mathcal{F}/Z)^s$ if and only if $P \in \mathcal{F}^s$.*
(b) *We have $PZ/Z \in (\mathcal{F}/Z)^q$ if and only if $P \in \mathcal{F}^q$.*
(c) *We have $P \leq \mathcal{F}^{cr}$ if and only if $Z \leq P$ and $P/Z \in (\mathcal{F}/Z)^{cr}$.*

Proof. Part (a) was shown in Lemma 3.5. If $Z \leq P$ then it is shown in [8, Lemma 6.4(b)] that $P \in \mathcal{F}^q$ if and only if $P/Z \in (\mathcal{F}/Z)^q$. So for (b) it remains to show that $P \in \mathcal{F}^q$ if and only if $PZ \in \mathcal{F}^q$. As $\mathcal{F}^q$ is closed under taking overgroups, $PZ \in \mathcal{F}^q$ if $P \in \mathcal{F}^q$. Assume now $PZ \in \mathcal{F}^q$. Since $\mathcal{F}^q$ is closed under taking $\mathcal{F}$-conjugates, we may assume that PZ is fully centralized. As $PZ \in \mathcal{F}^q$ it follows that $C_{\mathcal{F}}(PZ) = \mathcal{F}_{C_S(PZ)}(C_S(PZ))$. Since $Z \leq Z(\mathcal{F})$, we have $C_S(P) = C_S(PZ)$ and $C_{\mathcal{F}}(P) = C_{\mathcal{F}}(PZ) = \mathcal{F}_{C_S(P)}(C_S(P))$. Suppose Q is an $\mathcal{F}$-conjugate of P. Then an $\mathcal{F}$-isomorphism $P \to Q$ extends to an $\mathcal{F}$-isomorphism $PZ \to QZ$ as $Z \leq Z(\mathcal{F})$. Hence, QZ is $\mathcal{F}$-conjugate to PZ. So, as PZ is fully centralized, $|C_S(Q)| = |C_S(QZ)| \leq |C_S(PZ)| = |C_S(P)|$. Therefore, P is fully centralized in $\mathcal{F}$ and thus quasicentric as $C_{\mathcal{F}}(P) = \mathcal{F}_{C_S(P)}(C_S(P))$. This shows (b).

If $P \in \mathcal{F}^{cr}$ then $Z \leq C_S(P) \leq P$ as P is centric. Moreover, $P/Z \in (\mathcal{F}/Z)^{cr}$ by [14, Proposition 3.1]. Assume now $Z \leq P$ and $P \in (\mathcal{F}/Z)^{cr}$. We need to show that P is centric radical in $\mathcal{F}$. Note that P is centric radical in $\mathcal{F}$ if and only if some $\mathcal{F}$-conjugate of P is centric radical in $\mathcal{F}$. Moreover, the $\mathcal{F}/Z$-conjugates of P/Z are precisely the subgroups of the form Q/Z with $Q \in P^{\mathcal{F}}$, and every $\mathcal{F}/Z$-conjugate of P/Z is centric radical in $\mathcal{F}/Z$. Hence, we can replace P by any $\mathcal{F}$-conjugate of P and will assume without loss of generality that P is fully normalized in $\mathcal{F}$. Then $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$. Set

$$C := C_{\mathrm{Aut}_{\mathcal{F}}(P)}(P/Z).$$

It follows from the definition of $\mathcal{F}/Z$ that

$$\mathrm{Aut}_{\mathcal{F}/Z}(P/Z) \cong \mathrm{Aut}_{\mathcal{F}}(P)/C.$$

As Z is central in $\mathcal{F}$, we have $[Z, \mathrm{Aut}_{\mathcal{F}}(P)] = 1$. In particular, Z is $\mathrm{Aut}_{\mathcal{F}}(P)$-invariant and thus C is normal in $\mathrm{Aut}_{\mathcal{F}}(P)$. Moreover, $[P, O^p(C)] = [P, O^p(C), O^p(C)] \leq [Z, O^p(C)] = 1$. Thus $O^p(C) = 1$ and C is a normal p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$. This implies $C \leq O_p(\mathrm{Aut}_{\mathcal{F}}(P))$ and $O_p(\mathrm{Aut}_{\mathcal{F}}(P)/C) = O_p(\mathrm{Aut}_{\mathcal{F}}(P))/C$. Moreover, as $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$, it follows $C \leq \mathrm{Aut}_S(P)$ and thus $C = C_{\mathrm{Aut}_S(P)}(P/Z) = \mathrm{Aut}_{C_S(P/Z)}(P)$. Since P/Z is centric in $\mathcal{F}/Z$, we have $C_S(P/Z) \leq P$. Thus, $C \leq \mathrm{Inn}(P)$ and $\mathrm{Inn}(P/Z) \cong \mathrm{Inn}(P)/C$. Since P/Z is radical in $\mathcal{F}/Z$, we obtain

$$O_p(\mathrm{Aut}_{\mathcal{F}}(P))/C = O_p(\mathrm{Aut}_{\mathcal{F}}(P)/C) \cong O_p(\mathrm{Aut}_{\mathcal{F}/Z}(P/Z)) = \mathrm{Inn}(P/Z) \cong \mathrm{Inn}(P)/C.$$

As $\mathrm{Inn}(P) \leq O_p(\mathrm{Aut}_{\mathcal{F}}(P))$, this implies $\mathrm{Inn}(P) = O_p(\mathrm{Aut}_{\mathcal{F}}(P))$ and P is radical in $\mathcal{F}$. Since $C_S(P) \leq C_S(P/Z) \leq P$ and P is fully normalized in $\mathcal{F}$, P is also centric in $\mathcal{F}$. This shows (c). □

If $(\mathcal{L}, \Delta, S)$ is a locality then every central subgroup $Z \leq Z(\mathcal{L})$ is a partial normal subgroup of $\mathcal{L}$. Suppose now $(\mathcal{L}, \Delta, S)$ is a linking locality. Then $Z(\mathcal{L}) \leq C_{\mathcal{L}}(S) \leq S$, so every central subgroup if $\mathcal{L}$ is contained in S and then normal in $\mathcal{F}_S(\mathcal{L})$. On the other hand, if $Z \leq Z(\mathcal{F}_S(\mathcal{L}))$ then $Z \leq Z(\mathcal{L})$ by Proposition 4. We will now consider quotients modulo subgroups $Z \leq Z(\mathcal{L}) \cap S$.

Proposition 9.2. *Let $(\mathcal{L}, \Delta, S)$ be a locality over $\mathcal{F}$ and $Z \leq Z(\mathcal{L}) \cap S$. Set $\overline{\mathcal{L}} = \mathcal{L}/Z$, $\overline{S} = S/Z$ and $\overline{\Delta} = \{\overline{P} \colon P \in \Delta\}$.*

- (a) *The triple $(\overline{\mathcal{L}}, \overline{\Delta}, \overline{S})$ is a locality over $\mathcal{F}/Z$.*
- (b) *The locality $(\overline{\mathcal{L}}, \overline{\Delta}, \overline{S})$ is of objective characteristic p if and only if $(\mathcal{L}, \Delta, S)$ is of objective characteristic p.*
- (c) *We have $\mathcal{F}^{cr} \subseteq \Delta$ if and only if $(\mathcal{F}/Z)^{cr} \subseteq \overline{\Delta}$, we have $\Delta = \mathcal{F}^s$ if and only $\overline{\Delta} = (\mathcal{F}/Z)^s$, and we have $\Delta = \mathcal{F}^q$ if and only if $\overline{\Delta} = \mathcal{F}^q$.*
- (d) *The locality $(\overline{\mathcal{L}}, \overline{\Delta}, \overline{S})$ is a linking locality if and only if $(\mathcal{L}, \Delta, S)$ is a linking locality. Similarly, $(\overline{\mathcal{L}}, \overline{\Delta}, \overline{S})$ is a subcentric linking locality if and only if $(\mathcal{L}, \Delta, S)$ is a subcentric linking locality, and $(\overline{\mathcal{L}}, \overline{\Delta}, \overline{S})$ is a quasicentric linking locality if and only if $(\mathcal{L}, \Delta, S)$ is a quasicentric linking locality.*

Proof. It follows from Corollary 5.8 that $(\overline{\mathcal{L}}, \overline{\Delta}, \overline{S})$ is a locality over $\mathcal{F}/Z$. Let $P \in \Delta$. As $Z \leq Z(\mathcal{L})$, we have $N_{\mathcal{L}}(P) \subseteq N_{\mathcal{L}}(PZ)$ and then $N_{\mathcal{L}}(P) = N_{N_{\mathcal{L}}(PZ)}(P)$. So by Lemma 2.2(a), $N_{\mathcal{L}}(P)$ is of characteristic p if $N_{\mathcal{L}}(PZ)$ is of characteristic p. So for the proof of (b), we can and will assume $Z \leq P$, and we need to show that $N_{\mathcal{L}}(P)$ is of characteristic p if and only if $N_{\mathcal{L}}(P)/Z$ is of characteristic p. The latter statement is however true by Lemma 2.4 since $Z \leq Z(\mathcal{L}) \cap P \leq Z(N_{\mathcal{L}}(P)) \cap O_p(N_{\mathcal{L}}(P))$. This proves (b). As $Z \leq Z(\mathcal{F})$, (c) follows from Lemma 9.1. Parts (b) and (c) imply (d). □

Proposition 9.3. *Let $(\mathcal{L}, \Delta, S)$ and $(\mathcal{L}', \Delta', S')$ be localities. Let β be a projection of localities from $(\mathcal{L}, \Delta, S)$ to $(\mathcal{L}', \Delta', S')$ with $\ker(\beta) \leq Z(\mathcal{L})$.*

Then $(\mathcal{L}, \Delta, S)$ is of objective characteristic p if and only if $\ker(\beta) \leq S$ and $(\mathcal{L}', \Delta', S')$ is of objective characteristic p.

Similarly, $(\mathcal{L}, \Delta, S)$ is a linking locality if and only if $\ker(\beta) \leq S$ and $(\mathcal{L}', \Delta', S')$ is a linking locality; $(\mathcal{L}, \Delta, S)$ is a quasicentric linking locality if and only if $\ker(\beta) \leq S$ and $(\mathcal{L}', \Delta', S')$ is a quasicentric linking locality; and $(\mathcal{L}, \Delta, S)$ is a subcentric linking locality if and only if $\ker(\beta) \leq S$ and $(\mathcal{L}', \Delta', S')$ is a subcentric linking locality.

Proof. Set $Z := \ker(\beta)$. If $(\mathcal{L}, \Delta, S)$ is of objective characteristic p then $Z \leq C_{\mathcal{L}}(S) \leq S$. In particular, this is the case if $(\mathcal{L}, \Delta, S)$ is a linking locality. So assume now $Z \leq S$. Set $\overline{\mathcal{L}} = \mathcal{L}/Z$, $\overline{S} = S/Z$ and $\overline{\Delta} = \{\overline{P} \colon P \in \Delta\}$. Recall that $(\overline{\mathcal{L}}, \overline{\Delta}, \overline{S})$ is a locality. By [11, Theorem 4.7], the induced map $\gamma\colon \overline{\mathcal{L}} \to \mathcal{L}', Zf \mapsto f\beta$ is an isomorphism of partial groups and a projection from $(\overline{\mathcal{L}}, \overline{\Delta}, \overline{S})$ to $(\mathcal{L}', \Delta', S')$. Hence, $(\overline{\mathcal{L}}, \overline{\Delta}, \overline{S})$ is of objective characteristic p if and only $(\mathcal{L}', \Delta', S')$ is of objective characteristic p. By Theorem 5.6(b), $\beta|_{\overline{S}}\colon \overline{S} \to S'$ induces an isomorphism from $\mathcal{F}_{\overline{S}}(\overline{\mathcal{L}})$ to $\mathcal{F}_{S'}(\mathcal{L}')$, and induces thus a bijection from $\mathcal{F}_{\overline{S}}(\overline{\mathcal{L}})^{cr}$ to $\mathcal{F}_{S'}(\mathcal{L}')^{cr}$, from $\mathcal{F}_{\overline{S}}(\overline{\mathcal{L}})^{q}$ to $\mathcal{F}_{S'}(\mathcal{L}')^{q}$, and from $\mathcal{F}_{\overline{S}}(\overline{\mathcal{L}})^{s}$ to $\mathcal{F}_{S'}(\mathcal{L}')^{s}$. Hence, $(\overline{\mathcal{L}}, \overline{\Delta}, \overline{S})$ is a linking locality if and only if $(\mathcal{L}', \Delta', S')$ is a linking locality; $(\overline{\mathcal{L}}, \overline{\Delta}, \overline{S})$ is a quasicentric linking locality if and only if $(\mathcal{L}', \Delta', S')$ is a quasicentric linking locality; and $(\overline{\mathcal{L}}, \overline{\Delta}, \overline{S})$ is a subcentric linking locality if and only if $(\mathcal{L}', \Delta', S')$ is a subcentric linking locality. So the assertion follows from Proposition 9.2(b),(d). □

9.2. **Partial subgroups leading to localities.** Let $(\mathcal{L}, \Delta, S)$ be a locality. The partial product on $\mathcal{L}$ will be denoted by $\Pi\colon \mathbf{D} \to \mathcal{L}$. Let $\mathcal{H}$ be a partial subgroup of $\mathcal{L}$ and set $T := \mathcal{H} \cap S$.

Note that, for any $f \in \mathcal{H}$, $(S_f \cap T)^f \in S \cap \mathcal{H} = T$. By $\mathcal{F}_T(\mathcal{H})$ we denote the fusion system on T generated by the conjugation maps $c_f\colon S_f \cap T \to T, x \mapsto x^f$.

Let Γ be a non-empty set of subgroups of T closed under taking $\mathcal{H}$-conjugates and overgroups in T. Let $Q \le S$ and assume that the following properties hold:

(R1) For all $P \in \Gamma$, we have $\langle P, Q\rangle \in \Delta$.
(R2) For all $P_1, P_2 \in \Gamma$, we have $N_{\mathcal{H}}(P_1, P_2) \subseteq N_{\mathcal{L}}(\langle P_1, Q\rangle, \langle P_2, Q\rangle)$.

Note that (R2) holds in particular if $\mathcal{H} \subseteq N_{\mathcal{L}}(Q)$. Set

$$\mathcal{H}|_\Gamma := \{f \in \mathcal{H}\colon S_f \cap T \in \Gamma\}.$$

Let $\mathbf{D}_0$ be the set of words $(f_1, \dots, f_n)$ in $\mathcal{H}$ such that there exists $P_0, P_1, \dots, P_n \in \Gamma$ with $P_{i-1}^{f_i} = P_i$ for all $i = 1, \dots, n$.[2]

Remark 9.4. (a) We have $\mathbf{D}_0 \subseteq \mathbf{D}$.

(b) If $(f_1, \dots, f_n) \in \mathbf{D}_0$ and $P_0, P_1, \dots, P_n \in \Gamma$ with $P_{i-1}^{f_i} = P_i$, then $P_0^{\Pi(f_1,\dots,f_n)} = P_n$. In particular, $\Pi(f_1, \dots, f_n) \in \mathcal{H}|_\Gamma$.

Proof. Let $(f_1, \dots, f_n) \in \mathbf{D}_0$ and $P_0, P_1, \dots, P_n \in \Gamma$ such that $P_{i-1}^{f_i} = P_i$ for $i = 1, \dots, n$. Then property (R1) implies $R_0 := \langle P_0, Q\rangle \in \Delta$. In particular, every $\mathcal{L}$-conjugate of R_0 in S is an element of Δ since Δ is closed under taking $\mathcal{L}$-conjugates in S. So defining $R_i := R_{i-1}^{f_i}$ for $i = 1, \dots, n$, property (R2) yields that R_i is well-defined, $R_i \le \langle P_i, Q\rangle$, and R_i is an element of Δ. Hence, $(f_1, \dots, f_n) \in \mathbf{D}$ via $R_0, R_1, \dots, R_n$. This shows $\mathbf{D}_0 \subseteq \mathbf{D}$, so (a) holds. Moreover, by Lemma 5.3(c), $c_{\Pi(f_1,\dots,f_n)} = c_{f_1} \circ \cdots \circ c_{f_n}$ as a map $N_{\mathcal{L}}(R_0) \to N_{\mathcal{L}}(R_n)$. In particular, $P_0 \subseteq \mathbf{D}(\Pi(f_1, \dots, f_n))$ and $P_0^{\Pi(f_1,\dots,f_n)} = P_n$. So $P_0 \le S_{\Pi(f_1,\dots,f_n)} \cap T$ and $S_{\Pi(f_1,\dots,f_n)} \cap T \in \Gamma$ as Γ is by assumption closed under taking overgroups in T. Therefore, $\Pi(f_1, \dots, f_n) \in \mathcal{H}|_\Gamma$. This shows (b). □

Lemma 9.5. *The set $\mathcal{H}|_\Gamma$ together with the restriction of the inversion on $\mathcal{L}$ to $\mathcal{H}|_\Gamma$ and with the product $\Pi_0 := \Pi|_{\mathbf{D}_0}\colon \mathbf{D}_0 \to \mathcal{H}|_\Gamma$ forms a partial group. The inclusion map $\mathcal{H}|_\Gamma \to \mathcal{L}$ is a homomorphism of partial groups.*

Proof. By Remark 9.4, we have $\mathbf{D}_0 \subseteq \mathbf{D}$ and $\Pi(v) \in \mathcal{H}|_\Gamma$ for every $v \in \mathbf{D}_0$. Hence, Π_0 is well-defined. Note that, for every element $f \in \mathcal{H}$, Lemma 5.3(d) implies $(S_f \cap T)^f \le S_{f^{-1}} \cap (\mathcal{H} \cap S) = S_{f^{-1}} \cap T$. As Γ is closed under taking $\mathcal{H}$-conjugates and overgroups in T, it follows that $S_{f^{-1}} \cap T \in \Gamma$ for every $f \in \mathcal{H}|_\Gamma$. Hence, $f^{-1} \in \mathcal{H}|_\Gamma$ for every $f \in \mathcal{H}|_\Gamma$ and the restriction of the inversion on $\mathcal{L}$ to $\mathcal{H}|_\Gamma$ gives us an involutory bijection $\mathcal{H}|_\Gamma \to \mathcal{H}|_\Gamma$.

It is immediate from the definition of $\mathbf{D}_0$ that $u, v \in \mathbf{D}_0$ for all $u, v \in \mathbf{W}(\mathcal{H}|_\Gamma)$ with $u \circ v \in \mathbf{D}_0$. Moreover, if $u \circ v \circ w \in \mathbf{D}_0$ then it follows from the definition of $\mathbf{D}_0$ and from Remark 9.4(b) that $u \circ (\Pi(v)) \circ w \in \mathbf{D}_0$. If $w \in \mathbf{D}_0$ then Lemma 5.3(e) yields $w^{-1} \circ w \in \mathbf{D}_0$. As $\mathcal{L}$ satisfies the axioms

[2] In particular, we mean here $\emptyset \in \mathbf{D}_0$.

of a partial group, it is now easy to observe that $\mathcal{H}|_\Gamma$ together with the partial product Π_0 and the restriction of the inversion map to $\mathcal{H}|_\Gamma$ satisfies the axioms of a partial group. Since $\mathbf{D}_0 \subseteq \mathbf{D}$ it follows moreover that the inclusion map $\mathcal{H}|_\Gamma \to \mathcal{L}$ is a homomorphism of partial groups. $\square$

From now on we consider the partial group structure on $\mathcal{H}|_\Gamma$ as defined in the previous lemma. We call this partial group the *restriction* of $\mathcal{H}$ to Γ.

Remark 9.6. Since $\mathbf{D}_0 \subseteq \mathbf{D}$, we have $\mathbf{D}_0(f) \subseteq \mathbf{D}(f)$ for every $f \in \mathcal{H}|_\Gamma$. The conjugate of an element of $\mathbf{D}_0(f)$ by f in the partial group $\mathcal{H}|_\Gamma$ coincides with the corresponding conjugate in $\mathcal{L}$ as $\Pi_0 = \Pi|_{\mathbf{D}_0}$.

The subgroup T of $\mathcal{H}$ is also a subgroup of the partial group $\mathcal{H}|_\Gamma$. Moreover, $S_f \cap T = T_f$ where $T_f := \{x \in T : x \in \mathbf{D}_0(f),\ x^f \in T\}$ for every $f \in \mathcal{H}|_\Gamma$.

Proof. The first part is clear and we will only argue that the second part holds. As Γ is non-empty and closed under taking overgroups in T, we have $T \in \Gamma$. If $f \in T$ then $S_f \cap T = S \cap T = T \in \Gamma$ and so $f \in \mathcal{H}|_\Gamma$. This shows $T \subseteq \mathcal{H}|_\Gamma$. Similarly, if $v = (f_1, \ldots, f_n) \in \mathbf{W}(T)$ then $T^{f_i} = T$ and thus $v \in \mathbf{D}_0$. Note that $\Pi_0(v) = \Pi(v) \in T$ as T is a subgroup of $\mathcal{L}$. Thus, T is a subgroup of $\mathcal{H}_0$.

Let now $f \in \mathcal{H}|_\Gamma$. As $\mathbf{D}_0(f) \subseteq \mathbf{D}(f)$, we have $T_f \leq S_f \cap T$. Let now $x \in S_f \cap T$. As $f \in \mathcal{H}|_\Gamma$, we have $P_1 := S_f \cap T \in \Gamma$. Thus also $P_0 := (S_f \cap T)^f \in \Gamma$. Moreover, $P_0^{f^{-1}} = P_1$, $P_1^x = P_1$ and $P_1^f = P_0$. Therefore, by definition of $\mathbf{D}_0$, $(f^{-1}, x, f) \in \mathbf{D}_0$ and so $x \in \mathbf{D}_0(f)$. Moreover, $x^f \in S \cap \mathcal{H} = T$ and thus $x \in T_f$. $\square$

Lemma 9.7. *Suppose T is a (with respect to inclusion) maximal p-subgroup of $\mathcal{H}$. Then $(\mathcal{H}|_\Gamma, \Gamma, T)$ is a locality. If $\mathcal{F}_T(\mathcal{H})$ is saturated and $\mathcal{F}_T(\mathcal{H})^{cr} \subseteq \Gamma$ then $(\mathcal{H}|_\Gamma, \Gamma, T)$ is a locality over $\mathcal{F}_T(\mathcal{H})$.*

Proof. As $\mathcal{L}$ is a finite set, the set $\mathcal{H}|_\Gamma$ is also finite. By Remark 9.6, T is a p-subgroup of $\mathcal{H}|_\Gamma$. Since $\mathbf{D}_0 \subseteq \mathbf{D}$ and $\Pi_0 = \Pi|_{\mathbf{D}_0}$, every subgroup of $\mathcal{H}_0$ is also a subgroup $\mathcal{H}$. Hence, our assumption yields that T is a maximal p-subgroup of the partial group $\mathcal{H}|_\Gamma$. It follows from the definition of $\mathbf{D}_0$ and Remark 9.6 that property (L2) in Definition 5.1 holds for the partial group $\mathcal{H}|_\Gamma$ and the set Γ in place of the partial group $\mathcal{L}$ and the set Δ. Using Remark 9.6 one sees also that Γ is closed under taking $\mathcal{H}|_\Gamma$-conjugates and overgroups in T, as Γ is closed under taking $\mathcal{H}$-conjugates and overgroups in T. Thus, $(\mathcal{H}|_\Gamma, \Gamma, T)$ is a locality.

Suppose now that $\mathcal{F}_T(\mathcal{H})$ is saturated and that $\mathcal{F}_T(\mathcal{H})^{cr} \subseteq \Gamma$. Clearly, by Remark 9.6, $\mathcal{F}_T(\mathcal{H}|_\Gamma) \subseteq \mathcal{F}_T(\mathcal{H})$. So it remains to prove that $\mathcal{F}_T(\mathcal{H}) \subseteq \mathcal{F}_T(\mathcal{H}|_\Gamma)$. Let $R \in \mathcal{F}_T(\mathcal{H}|_\Gamma)^{cr}$ and $\alpha \in \mathrm{Aut}_{\mathcal{F}_T(\mathcal{H})}(R)$. By Alperin's fusion theorem, it is sufficient to show that α is a morphism in $\mathcal{F}_T(\mathcal{H}|_\Gamma)$. Since $\mathcal{F}_T(\mathcal{H})$ is generated by the conjugation maps of the form $c_f : S_f \cap T \to T$ with $f \in \mathcal{H}$, it follows that there exist $f_1, \ldots, f_n \in \mathcal{H}$ and subgroups $P = P_0, P_1, \ldots, P_n = P$ of T such that $P_{i-1} \subseteq \mathbf{D}(f_i)$, $P_{i-1}^{f_i} = P_i$ and $\alpha = (c_{f_1}|_{P_0}) \circ \cdots \circ (c_{f_n}|_{P_{n-1}})$. As $P \in \Gamma$ and Γ is closed under taking $\mathcal{H}$-conjugates in T, it follows that $P_i \in \Gamma$ for $i = 0, \ldots, n$. Hence, $f_i \in \mathcal{H}|_\Gamma$ for $i = 1, \ldots, n$. By Remark 9.6, the conjugation map $c_{f_i} : P_{i-1} \to P_i$ is well-defined in $\mathcal{H}|_\Gamma$ and coincides with the corresponding conjugation map in $\mathcal{L}$. As $\mathcal{F}_T(\mathcal{H}|_\Gamma)$ is generated by conjugation maps between the elements of Γ, it follows that α is a morphism in $\mathcal{F}_T(\mathcal{H}|_\Gamma)$. $\square$

Remark 9.8. Suppose T is (with respect to inclusion) a maximal p-subgroup of $\mathcal{H}$. Consider the transporter systems attached to the localities $(\mathcal{H}|_\Gamma, \Gamma, T)$ and $(\mathcal{L}, \Delta, S)$. Then we can naturally define a functor $\mathcal{T}_\Gamma(\mathcal{H}|_\Gamma) \to \mathcal{T}_\Delta(\mathcal{L})$ by sending an object $P \in \Gamma$ to $\langle P, Q \rangle \in \Delta$, and a morphism $(f, P_0, P_1) \in \mathrm{Hom}_{\mathcal{T}_\Gamma(\mathcal{H}|_\Gamma)}(P_0, P_1)$ to $(f, \langle P_0, Q \rangle, \langle P_1, Q \rangle)$ for all $P_0, P_1 \in \Gamma$.

9.3. **Inclusions of linking systems associated to p-local subsystems.** Let $(\mathcal{L}, \Delta, S)$ be a linking locality over $\mathcal{F}$. The partial product on $\mathcal{L}$ will be denoted by $\Pi : \mathbf{D} \to \mathcal{L}$. Let Δ be one of the sets $\mathcal{F}^c$, $\mathcal{F}^q$ or $\mathcal{F}^s$ so that $(\mathcal{L}, \Delta, S)$ is a centric, quasicentric or subcentric linking locality.

Fix $Q \in \mathcal{F}$ and $K \leq \mathrm{Aut}_\mathcal{F}(Q)$ such that Q is fully K-normalized. Set

$$N_\mathcal{L}^K(Q) := \{f \in N_\mathcal{L}(Q) : c_f|_Q \in K\}.$$

Let $\Gamma = N_{\mathcal{F}}^K(Q)^c$ if $\Delta = \mathcal{F}^c$, $\Gamma = N_{\mathcal{F}}^K(Q)^q$ if $\Delta = \mathcal{F}^q$, and $\Gamma = N_{\mathcal{F}}^K(Q)^s$ if $\Delta = \mathcal{F}^s$.

Recall from [6, Theorem I.5.5] that $N_{\mathcal{F}}^K(Q)$ is saturated as Q is fully K-normalized. Our goal will be to show that a locality for $N_{\mathcal{F}}^K(Q)$ with object set Γ is contained in $\mathcal{L}$, and that this locality is a linking locality if $K \trianglelefteq \mathrm{Aut}_{\mathcal{F}}(Q)$. So if $K \trianglelefteq \mathrm{Aut}_{\mathcal{F}}(Q)$, we will show that a centric linking locality over $\mathcal{F}$ contains a centric linking locality over $N_{\mathcal{F}}^K(Q)$, a quasicentric linking locality over $\mathcal{F}$ contains a quasicentric linking locality over $N_{\mathcal{F}}^K(Q)$, and a subcentric linking locality over $\mathcal{F}$ contains a subcentric linking locality over $N_{\mathcal{F}}^K(Q)$. In particular, this is true if $K = \mathrm{Aut}_{\mathcal{F}}(Q)$ and $N_{\mathcal{F}}^K(Q) = N_{\mathcal{F}}(Q)$, or if $K = \{\mathrm{id}_Q\}$ and $N_{\mathcal{F}}^K(Q) = C_{\mathcal{F}}(Q)$.

Lemma 9.9. *The subset $N_{\mathcal{L}}^K(Q)$ is a partial subgroup of $\mathcal{L}$ and $N_S^K(Q)$ is (with respect to inclusion) a maximal p-subgroup of $N_{\mathcal{L}}^K(Q)$.*

Proof. It follows from Lemma 5.3(e) that $N_{\mathcal{L}}^K(Q)$ is closed under inversion. Let $v = (f_1, \dots, f_n) \in \mathbf{D} \cap \mathbf{W}(N_{\mathcal{L}}^K(Q))$. Then $v \in \mathbf{D}$ via some $P_0, \dots, P_n \in \Delta$. Replacing P_i by $\langle P_{i-1}, Q\rangle$ we may assume that $Q \leq P_i$. Then by Lemma 5.3(c), $c_{f_1} \circ \cdots \circ c_{f_n} = c_{\Pi(f_1,\dots,f_n)}$ as a map from $N_{\mathcal{L}}(P_0)$ to $N_{\mathcal{L}}(P_n)$ and thus in particular as a map from Q to Q. As $(c_{f_i})|_Q \in K$ for $i = 1, \dots, n$, it follows $c_{\Pi(f_1,\dots,f_n)}|_Q \in K$. Therefore $\Pi(f_1, \dots, f_n) \in N_{\mathcal{L}}^K(Q)$. This shows that $N_{\mathcal{L}}^K(Q)$ is a partial subgroup of $\mathcal{L}$.

Clearly, $N_S^K(Q)$ is a p-subgroup of $N_{\mathcal{L}}^K(Q)$. Let R be a p-subgroup of $N_{\mathcal{L}}^K(Q)$ such that $N_S^K(Q) \leq R$. By [11, Proposition 2.11(a)], every subgroup of $\mathcal{L}$ is contained in the normalizer of some element of Δ. So in particular, $R \leq N_{\mathcal{L}}(P)$ for some $P \in \Delta$. For such P, we have $R \subseteq N_{\mathcal{L}}(\langle P, Q\rangle)$ and $\langle P, Q\rangle \in \Delta$ as Δ is closed under taking overgroups in S. Hence, we can fix $P \in \Delta$ such that $Q \leq P$ and $R \leq N_{\mathcal{L}}(P)$. As R normalizes P, RP is a p-subgroup of the finite group $N_{\mathcal{L}}(P)$. Thus, RP is also a p-subgroup of the partial group $\mathcal{L}$. By [11, Proposition 2.11(b)], every p-subgroup of $\mathcal{L}$ is conjugate into S. So there exists $f \in \mathcal{L}$ such that $RP \subseteq \mathbf{D}(f)$ and $(RP)^f \leq S$. Note that $Q \leq P \leq S_f$.

Let $r \in R$ and $q \in Q$. We show next that $v := (f^{-1}, r^{-1}, f, f^{-1}, q, f, f^{-1}, r, f) \in \mathbf{D}$. As $R \subseteq \mathbf{D}(f)$, we have $(f^{-1}, r, f) \in \mathbf{D}$ via some $P_0, P_1, P_2, P_3 \in \Delta$. Recall that Δ is closed under taking overgroups in S. Using Lemma 5.3(d) and $R \subseteq N_{\mathcal{L}}^K(Q) \subseteq N_{\mathcal{L}}(Q)$, we conclude that $(f^{-1}, r, f) \in \mathbf{D}$ via $\langle P_0, Q^f\rangle, \langle P_1, Q\rangle, \langle P_2, Q\rangle, \langle P_3, Q^f\rangle$. So we may assume from now on that $Q^f \leq P_0$ and $Q \leq P_1 \cap P_2$. Then $q \in P_1$ and thus $P_1^q = P_1$. As $(f^{-1}, r, f) \in \mathbf{D}$ via P_0, P_1, P_2, P_3, we get the following series of conjugations:

$$P_0 \xrightarrow{f^{-1}} P_1 \xrightarrow{r} P_2 \xrightarrow{f} P_3 \text{ and } P_3 \xrightarrow{f^{-1}} P_2 \xrightarrow{r^{-1}} P_1 \xrightarrow{f} P_0.$$

Hence, we get the following series of conjugations:

$$P_3 \xrightarrow{f^{-1}} P_2 \xrightarrow{r^{-1}} P_1 \xrightarrow{f} P_0 \xrightarrow{f^{-1}} P_1 \xrightarrow{q} P_1 \xrightarrow{f} P_0 \xrightarrow{f^{-1}} P_1 \xrightarrow{r} P_2 \xrightarrow{f} P_3.$$

So $v \in \mathbf{D}$ via P_3. By [11, Lemma 1.6(b)], $(r^{-1})^f = (r^f)^{-1}$. So by the axioms of a partial group, $((r^f)^{-1}, q^f, r^f) = ((r^{-1})^f, q^f, r^f) \in \mathbf{D}$ and $(q^f)c_{r^f} = (q^f)^{r^f} = \Pi(v) = (q^f)c_{f^{-1}}c_r c_f$. Set $P := Q^f$ and let $\varphi = c_f|_Q \in \mathrm{Hom}_{\mathcal{F}}(Q, P)$. Then $(q^f)c_{r^f} = (q^f)\varphi^{-1}c_r\varphi$. As q is arbitrary, this shows $c_{r^f}|_P = \varphi^{-1}c_r|_Q\varphi$. Since $r \in R \subseteq N_{\mathcal{L}}^K(Q)$, we have $c_r|_Q \in K$. It follows $c_{r^f}|_P \in K^\varphi$ and thus $r^f \in N_{\mathcal{L}}^{K^\varphi}(P)$. Since $R^f \subseteq S$ and $r \in R$ was arbitrary, this shows $R^f \subseteq N_S^{K^\varphi}(P)$. By Lemma 5.3(e), we have $|R| = |R^f|$. Recall that $N_S^K(Q) \subseteq R$. As Q is fully K-normalized, it follows

$$|R| = |R^f| \leq |N_S^{K^\varphi}(P)| \leq |N_S^K(Q)| \leq |R|.$$

Hence, equality holds and $N_S^K(Q) = R$. This shows that $N_S^K(Q)$ is a maximal p-subgroup. □

Note that Γ is closed under taking $\mathcal{F}$-conjugates and overgroups in $N_S^K(Q)$; if $\Gamma = N_{\mathcal{F}}^K(Q)^s$ this follows from Proposition 3.3. By Lemma 3.14, we have

$$PQ \in \Delta \text{ for every } P \in \Gamma.$$

In particular, property (R1) holds. Moreover, clearly $N_{\mathcal{L}}^K(Q) \subseteq N_{\mathcal{L}}(Q)$ and so property (R2) holds with $N_{\mathcal{L}}^K(Q)$ in place of $\mathcal{H}$.

Lemma 9.10. *We have* $N_{\mathcal{F}}^K(Q) = \mathcal{F}_{N_S^K(Q)}(N_{\mathcal{L}}^K(Q))$.

Proof. Clearly $\mathcal{F}_{N_S^K(Q)}(N_{\mathcal{L}}^K(Q)) \subseteq N_{\mathcal{F}}^K(Q)$, so it remains to prove the converse inclusion. Let $P \in N_{\mathcal{F}}^K(Q)^{cr}$ and $\varphi \in \mathrm{Aut}_{N_{\mathcal{F}}^K(Q)}(P)$. As $N_{\mathcal{F}}^K(Q)$ is saturated, it is by Alperin's fusion theorem [6, Theorem I.3.6] sufficient to prove that φ is a morphism in $\mathcal{F}_{N_S^K(Q)}(N_{\mathcal{L}}^K(Q))$. By definition of $N_{\mathcal{F}}^K(Q)$, φ extends to $\hat{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(PQ)$ with $\hat{\varphi}|_Q \in K$. By Lemma 3.14, $PQ \in \Delta$. So by Remark 5.5, $\hat{\varphi} = c_f|_{PQ}$ for some $f \in \mathcal{L}$ with $PQ \leq S_f$. Then $c_f|_Q = \hat{\varphi}|_Q \in K$ and thus $f \in N_{\mathcal{L}}^K(Q)$. Hence, $\varphi = \hat{\varphi}|_P = c_f|_P$ is a morphism in $\mathcal{F}_{N_S^K(Q)}(N_{\mathcal{L}}^K(Q))$. □

Recall from Lemma 9.9 that $N_{\mathcal{L}}^K(Q)$ is a partial subgroup of $\mathcal{L}$. Set

$$\mathcal{L}_0 := N_{\mathcal{L}}^K(Q)|_\Gamma = \{f \in N_{\mathcal{L}}^K(Q) \colon S_f \cap N_S^K(Q) \in \Gamma\}.$$

Let $\mathbf{D}_0$ be the set of words $(f_1, \ldots, f_n) \in \mathbf{W}(\mathcal{L}_0)$ for which there exist $P_0, P_1, \ldots, P_n \in \Gamma$ with $P_{i-1}^{f_i} = P_i$ for all $i = 1, \ldots, n$.

Lemma 9.11. *The set $\mathcal{L}_0$ together with the restriction of the inversion on $\mathcal{L}$ to $\mathcal{L}_0$ and with the product $\Pi_0 := \Pi|_{\mathbf{D}_0} \colon \mathbf{D}_0 \to \mathcal{L}_0$ forms a partial group.*

Regarding $\mathcal{L}_0$ as a partial group in this way, the inclusion map $\mathcal{L}_0 \to \mathcal{L}$ is a homomorphism of partial groups and the triple $(\mathcal{L}_0, \Gamma, N_S^K(Q))$ is a locality over $N_{\mathcal{F}}^K(Q)$. If K is subnormal in $\mathrm{Aut}_{\mathcal{F}}(Q)$ then $(\mathcal{L}_0, \Gamma, N_S^K(Q))$ is a linking locality.

Proof. By Lemma 9.5, $\mathcal{L}_0 = N_{\mathcal{L}}^K(Q)|_\Gamma$ forms a partial group as described above, and the inclusion map $\mathcal{L}_0 \to \mathcal{L}$ is a homomorphism of partial groups. By Lemma 9.9, $N_S^K(Q) = N_{\mathcal{L}}^K(Q) \cap S$ is a maximal p-subgroup of $N_{\mathcal{L}}^K(Q)$, and by Lemma 9.10, $\mathcal{F}_{N_S^K(Q)}(N_{\mathcal{L}}^K(Q)) = N_{\mathcal{F}}^K(Q)$ is saturated. Moreover, by our choice of Γ, we have $N_{\mathcal{F}}^K(Q)^{cr} \subseteq \Gamma$. Hence, by Lemma 9.7, $(\mathcal{L}_0, \Gamma, N_S^K(Q))$ is a locality over $N_{\mathcal{F}}^K(Q)$.

Assume now that K is subnormal in $\mathrm{Aut}_{\mathcal{F}}(Q)$. Let $P \in \Gamma$. We need to show that the $N_{\mathcal{L}_0}(P)$ is of characteristic p. A priori, $N_{\mathcal{L}_0}(P)$ means here the normalizer of P formed in the partial group $\mathcal{L}_0$. However, by Remark 9.6, this normalizer coincides with the normalizer in $\mathcal{L}_0$ of P formed in the partial group $\mathcal{L}$. Moreover, $N_{\mathcal{L}_0}(P) = N_{N_{\mathcal{L}}^K(Q)}(P)$. Recall that $PQ \in \Delta$. So since $(\mathcal{L}, \Delta, S)$ is a linking locality, $G := N_{\mathcal{L}}(PQ)$ is a group of characteristic p. Observe that $N_{\mathcal{L}_0}(P) = N_{N_{\mathcal{L}}^K(Q)}(P) = N_{N_G(P)}^K(Q)$. As G is of characteristic p, it follows from Lemma 2.2(a) that $N_G(P)$ is of characteristic p, and thus $N_{N_G(P)}(Q)$ is of characteristic p. As K is subnormal in $\mathrm{Aut}_{\mathcal{F}}(Q)$, $K_0 := \mathrm{Aut}_G(Q) \cap K$ is subnormal in $\mathrm{Aut}_G(Q) \leq \mathrm{Aut}_{\mathcal{F}}(Q)$. Let $K_0 \trianglelefteq K_1 \trianglelefteq \cdots \trianglelefteq K_n = \mathrm{Aut}_G(Q)$ be a subnormal series. Then $N_{\mathcal{L}_0}(P) = N_{N_G(P)}^K(Q) = N_{N_G(P)}^{K_0}(Q) \trianglelefteq N_{N_G(P)}^{K_1}(Q) \trianglelefteq \cdots \trianglelefteq N_{N_G(P)}^{K_n}(Q) = N_{N_G(P)}(Q)$ and $N_{\mathcal{L}_0}(P)$ is a subnormal subgroup of $N_{N_G(P)}(Q)$. Thus, $N_{\mathcal{L}_0}(P)$ is of characteristic p by Lemma 2.2(b). □

Remark 9.12. As described more generally in Remark 9.8, we are naturally given a functor $\mathcal{T}(\mathcal{L}_0, \Delta_0) \to \mathcal{T}(\mathcal{L}, \Delta)$ which is injective on morphism sets. It sends an object $P \in \Gamma$ to $PQ \in \Delta$, and a morphism (f, P_1, P_2) to (f, P_1Q, P_2Q). So if K is subnormal in $\mathrm{Aut}_{\mathcal{F}}(Q)$ then we get functors from the centric linking system of $N_{\mathcal{F}}^K(Q)$ to the centric linking system of $\mathcal{F}$, from the quasicentric linking system of $N_{\mathcal{F}}^K(Q)$ to the quasicentric linking system of $\mathcal{F}$, and from the subcentric linking system of $N_{\mathcal{F}}^K(Q)$ to the subcentric linking system of $\mathcal{F}$.

If $N_{\mathcal{F}}^K(Q) = N_{\mathcal{F}}(Q)$, $\Delta_0 = N_{\mathcal{F}}(Q)^c$ and $\Delta = \mathcal{F}^c$ then note that $\mathcal{T}(\mathcal{L}_0, \Delta_0)$ is a centric linking system associated to $N_{\mathcal{F}}(Q)$. In [9, Definition 6.1] a centric linking system associated to $N_{\mathcal{F}}(Q)$ was directly constructed from a centric linking system associated to $\mathcal{F}$. So our construction of $(\mathcal{L}_0, \Delta_0, S_0)$ can be seen as a locality version of this construction.

9.4. **Inclusions of linking systems associated to normal subsystems.** Let $(\mathcal{L}, \Delta, S)$ be a subcentric linking locality over $\mathcal{F}$, i.e. $\Delta = \mathcal{F}^s$. Suppose $\mathcal{E}$ is a normal subsystem of $\mathcal{F}$ over $T \leq S$ and there exists a partial normal subgroup $\mathcal{N}$ of $\mathcal{L}$ such that $\mathcal{F}_T(\mathcal{N}) = \mathcal{E}$. Set $\Gamma := \mathcal{E}^s$ and $\mathcal{N}_0 := \mathcal{N}|_\Gamma = \{f \in \mathcal{N} \colon S_f \cap T \in \Delta_0\}$.

By Proposition 3, $Q := C_S(\mathcal{E}) = C_S(\mathcal{N})$. So by Theorem B we have $PQ = PC_S(\mathcal{N}) \in \Delta$ for all $P \in \Gamma$. Note moreover that $\mathcal{N} \subseteq N_\mathcal{L}(Q)$. Hence, properties (R1) and (R2) from Subsection 9.2 hold with $\mathcal{N}$ in place of $\mathcal{H}$. Let $\mathbf{D}_0$ be the set of words $(f_1, \dots, f_n) \in \mathbf{W}(\mathcal{N}_0)$ such that there exist $P_0, \dots, P_n \in \Gamma$ with $P_{i-1}^{f_i} = P_i$ for $i = 1, \dots, n$.

Lemma 9.13. *The set $\mathcal{N}_0$ together with the restriction of the inversion on $\mathcal{L}$ to $\mathcal{N}_0$ and with the product $\Pi_0 := \Pi|_{\mathbf{D}_0} \colon \mathbf{D}_0 \to \mathcal{N}_0$ forms a partial group.*

Regarding $\mathcal{N}_0$ as a partial group in this way, the inclusion map $\mathcal{N}_0 \to \mathcal{L}$ is a homomorphism of partial groups and the triple $(\mathcal{N}_0, \Gamma, T)$ is a subcentric linking locality over $\mathcal{E}$.

Proof. By Lemma 9.5, $\mathcal{N}_0$ forms a partial group in the way described, and the inclusion map $\mathcal{N}_0 \to \mathcal{L}$ is a homomorphism of partial groups. By [11, Lemma 3.1(c)], T is a maximal p-subgroup of $\mathcal{N}$. So as $\mathcal{F}_T(\mathcal{N}) = \mathcal{E}$ is saturated and $\mathcal{E}^{cr} \subseteq \mathcal{E}^s = \Gamma$, it follows from Lemma 9.7 that $(\mathcal{N}_0, \Gamma, T)$ is a locality over $\mathcal{E}$. Let $P \in \Gamma$. We need to show that $N_{\mathcal{N}_0}(P)$ is of characteristic p. A priori, $N_{\mathcal{N}_0}(P)$ denotes here the normalizer in the partial group $\mathcal{N}_0$, but by Remark 9.6 and the definition of the partial group $\mathcal{N}_0$, this normalizer coincides with the normalizer $N_{\mathcal{N}_0}(P)$ in $\mathcal{L}$ and $N_{\mathcal{N}_0}(P) = N_\mathcal{N}(P)$. Recall that $PQ \in \Delta$ for $Q := C_S(\mathcal{N})$. So $G := N_\mathcal{L}(PQ)$ is a finite group of characteristic p. Note that $N_\mathcal{N}(P) = N_G(P) \cap \mathcal{N}$ is a normal subgroup of $N_G(P)$. Hence, by Lemma 2.2, $N_\mathcal{N}(P)$ is of characteristic p. This implies the assertion. □

Remark 9.14. As described more generally in Remark 9.8 we get a functor $\mathcal{T}(\mathcal{N}_0, \Gamma) \to \mathcal{T}(\mathcal{L}, \Delta)$ which sends an object $P \in \Gamma$ to $PC_S(\mathcal{N}) \in \Delta$ and a morphism (f, P_1, P_2) to $(f, P_1C_S(\mathcal{N}), P_2C_S(\mathcal{N}))$. So there is a functor from the subcentric linking system of $\mathcal{E}$ to the subcentric linking system of $\mathcal{F}$ which is injective on the morphism sets.

10. Localities of objective characteristic p coming from finite groups

Throughout let G be a finite group and $S \in \mathrm{Syl}_p(G)$. Recall that $\Theta(H) = O_{p'}(H)$ for any group H.

Definition 10.1. Let Γ be a set of subgroups of S closed under taking G-conjugates and overgroups in S. Let $\mathcal{L}_\Gamma(G)$ be the set of all elements $g \in G$ with $S \cap S^g \in \Gamma$. Moreover, let $\mathbf{D}_\Gamma$ be the set of all words $(g_1, \dots, g_n)$ such that $g_i \in G$ and there exist elements $P_0, \dots, P_n \in \Gamma$ with $P_{i-1}^{g_i} = P_i$ for $i = 1, \dots, n$. Define a partial product $\Pi \colon \mathbf{D} \to \mathcal{L}_\Gamma(G)$ by mapping $(g_1, g_2, \dots, g_n) \in \mathbf{D}$ to the product $g_1 g_2 \dots g_n$ in G. Define an inversion map on $\mathcal{L}_\Gamma(G)$ by taking the restriction of the inversion map on G to the set $\mathcal{L}_\Gamma(G)$.

By [10, Example/Lemma 2.10], $(\mathcal{L}_\Gamma(G), \Gamma, S)$ is a locality.

In this section we are concerned with the case that Γ is one of the sets Δ and Δ^* defined as follows:

$$\Delta := \{P \leq S \colon N_G(P) \text{ is of characteristic } p\}$$

and

$$\Delta^* := \{P \leq S \colon N_G(P) \text{ is almost of characteristic } p\}.$$

By Lemma 2.9, we have

$$\Delta = \{P \leq S \colon C_G(P) \text{ is of characteristic } p\}$$

and

$$\Delta^* := \{P \leq S \colon C_G(P) \text{ is almost of characteristic } p\}.$$

Lemma 10.2. *The sets Δ and Δ^* are closed under $\mathcal{F}_S(G)$-conjugation and taking overgroups in S. In particular, if $\Delta \neq \emptyset$ then $S \in \Delta$.*

Proof. Clearly, $N_G(P^g) = N_G(P)^g \cong N_G(P)$ for any $g \in G$ and $P \leq S$. Hence, Δ and Δ^* are closed under $\mathcal{F}_S(G)$-conjugation. Let $P \leq R \leq S$. We want to show that $R \in \Delta$ if $P \in \Delta$, and $R \in \Delta^*$ if $P \in \Delta^*$. Since R is a p-group, P is subnormal in R. Thus, by induction on the subnormal length of P in R, we may assume that $P \trianglelefteq R$. Then $R \leq H := N_G(P)$ and $C_G(R) = C_H(R)$. If $P \in \Delta$ then H is of characteristic p, and if $P \in \Delta^*$ then H is almost of characteristic p. So if $P \in \Delta$ then, by Lemma 2.2(a), $C_G(R) = C_H(R)$ is of characteristic p and thus $R \in \Delta$ by Lemma 2.9. Similarly, if $P \in \Delta^*$ then $C_G(R) = C_H(R)$ is almost of characteristic p by Lemma 2.8(b), and thus $R \in \Delta^*$ by Lemma 2.9. □

Lemma 10.3. *We have $\Delta \subseteq \Delta^* \subseteq \mathcal{F}_S(G)^s$.*

Proof. Since every group of characteristic p is almost of characteristic p, we have $\Delta \subseteq \Delta^*$. If $P \in \Delta^* \cap \mathcal{F}_S(G)^f$ then $N_{\mathcal{F}_S(G)}(P) = \mathcal{F}_{N_S(P)}(N_G(P)) \cong \mathcal{F}_{\overline{N_S(P)}}(\overline{N_G(P)})$ for $\overline{N_G(P)} = N_G(P)/\Theta(N_G(P))$. Hence, $N_{\mathcal{F}_S(G)}(P)$ is constrained by Lemma 2.1(a) and $P \in \mathcal{F}_S(G)^s$ by Lemma 3.1. Since Δ^* and $\mathcal{F}_S(G)^s$ are closed under $\mathcal{F}_S(G)$-conjugation, it follows $\Delta^* \subseteq \mathcal{F}_S(G)^s$. □

Clearly, the sets Δ and Δ^* can be different in general, since there are groups which are almost of characteristic p, but not of characteristic p. The next example shows that Δ^* is not equal to $\mathcal{F}_S(G)^s$ in general.

Example 10.4. Let P be a P-group, $G = P \times A_5$ and $S \in \mathrm{Syl}_p(G)$. As $P = O_p(G)$, G is not of characteristic p. Since $O_{p'}(G) = 1$, it follows that $P \notin \Delta^*$. However, S is normal in $\mathcal{F}_S(G) = \mathcal{F}_S(P \times A_4)$ and thus $\mathcal{F}_S(G)$ is constrained. Hence, as P is normal in $\mathcal{F}_S(G)$, we have $P \in \mathcal{F}_S(G)^s$ by Lemma 3.1.

Lemma 10.5. *We have $\mathcal{F}_S(G)^q \subseteq \Delta^*$. In particular, $\mathcal{F}_S(G)^c \subseteq \Delta^*$ and $S \in \Delta^*$.*

Proof. Since both $\mathcal{F}_S(G)^q$ and Δ^* are closed under $\mathcal{F}_S(G)$-conjugation, it is sufficient to prove that $P \in \Delta^*$ for every $P \in \mathcal{F}_S(G)^{fq}$. Let $P \in \mathcal{F}_S(G)^{fq}$. Then $N_S(P) \in \mathrm{Syl}_p(N_G(P))$ and $\mathcal{F}_{C_S(P)}(C_G(P)) = C_{\mathcal{F}_S(G)}(P) = \mathcal{F}_{C_S(G)}(C_S(G))$ by [6, Proposition I.5.4]. Hence, Lemma 2.5 applied with $N_G(P)$ in place of G gives $C_G(P) = C_S(P)O_{p'}(C_G(P)) = C_S(P)\Theta(C_G(P))$. So $C_G(P)/\Theta(C_G(P))$ is a p-group and thus $C_G(P)$ is almost of characteristic p. Therefore, $P \in \Delta^*$ by Lemma 2.9. □

Example 10.6. Let P be an abelian p-group and $G = P \times S_4$. Then G is of characteristic p and thus $P \in \Delta \subseteq \Delta^*$. However, as $P \leq Z(G)$, P is fully $\mathcal{F}_S(G)$-centralized and $C_{\mathcal{F}_S(G)}(P) = \mathcal{F}_S(G) \neq \mathcal{F}_S(S)$. Therefore, $P \notin \mathcal{F}_S(G)^q$.

Set now $\Theta := \bigcup_{P \in \Delta^*} \Theta(N_G(P))$.

Lemma 10.7. *The locality $(\mathcal{L}_\Delta(G), \Delta, S)$ is of objective characteristic p. Moreover, Θ is a partial normal subgroup of $\mathcal{L}_{\Delta^*}(G)$ with $S \cap \Theta = 1$, the canonical map $\rho\colon \mathcal{L}_{\Delta^*}(G) \to \mathcal{L}_{\Delta^*}(\mathcal{L})/\Theta$ restricts to an isomorphism $S \to S\rho$ and, upon identifying S with $S\rho$, the locality $(\mathcal{L}_{\Delta^*}(G)/\Theta, \Delta^*, S)$ is a linking locality over $\mathcal{F}_S(G)$.*

Proof. Note that, for every $P \in \Delta$, $N_{\mathcal{L}_\Delta(G)}(P) = N_G(P)$ is of characteristic p. So the locality $(\mathcal{L}_\Delta(G), \Delta, S)$ is of objective characteristic p. Similarly, for every $P \in \Delta^*$, $N_G(P) = N_{\mathcal{L}_{\Delta^*}(G)}(P)$ is almost of characteristic p. Thus, it follows from Proposition 6.4 that Θ is a partial normal subgroup of $\mathcal{L}_{\Delta^*}(G)$ with $S \cap \Theta = 1$, the canonical map $\rho\colon \mathcal{L}_{\Delta^*}(G) \to \mathcal{L}_{\Delta^*}(\mathcal{L})/\Theta$ restricts to an isomorphism $S \to S\rho$ and, upon identifying S with $S\rho$, $(\mathcal{L}_{\Delta^*}(G)/\Theta, \Delta, S)$ is a locality over $\mathcal{F}_S(\mathcal{L}_{\Delta^*}(G))$ of objective characteristic p. By Lemma 10.5, $\mathcal{F}_S(G)^{cr} \subseteq \Delta^*$. Hence, by Alperin's fusion theorem, $\mathcal{F}_S(G) = \mathcal{F}_S(\mathcal{L}_{\Delta^*}(G))$. So $(\mathcal{L}_{\Delta^*}(G)/\Theta, \Delta, S)$ is a linking locality over $\mathcal{F}_S(G)$. □

To our knowledge there is no good way of constructing the subcentric linking system of $\mathcal{F}_S(G)$ directly from the group G. However, using Theorem 7.2, one can extend $\mathcal{L}_{\Delta^*}(G)/\Theta$ to a subcentric linking locality over $\mathcal{F}_S(G)$.

Remark 10.8. Many classification theorems in the program of Meierfrankenfeld, Stellmacher and Stroth are proved not only for groups of local characteristic p, but more generally for groups of *parabolic characteristic* p. These are finite groups where every p-local subgroup containing a Sylow p-subgroup is of characteristic p. We say similarly that $\mathcal{F}$ is of parabolic characteristic p if $\mathcal{F}$ is saturated and the normalizer of every normal subgroup of S is constrained. If G is of parabolic characteristic p, note that Δ and Δ^* contain every non-trivial normal subgroup of S. Similarly, if $\mathcal{F}$ is of parabolic characteristic p, then $\mathcal{F}^s$ contains every non-trivial normal subgroup of S.

As pointed out in the introduction, it might be possible to give a unifying approach to the classification of fusion systems of characteristic p-type and of groups of characteristic p-type whilst avoiding to use Theorem A and the theory of fusion systems to prove classification theorems for groups of characteristic p-type. Similarly, such an approach could presumably be implemented for groups and fusion systems of parabolic characteristic p if one proceeds roughly as follows: In a first step one proves a classification theorem for a locality $(\mathcal{L}, \Gamma, S)$ of objective characteristic p, where Γ contains every non-trivial normal subgroup of S. Then in a second step one separately deduces from that a corresponding classification theorem for fusion systems of parabolic characteristic p (using the existence of subcentric linking systems), and for groups of parabolic characteristic p (working with the locality $(\mathcal{L}_\Delta(G), \Delta, S)$ with the set Δ as above). If this approach turns out to be problematic, one could also in the first step prove a classification theorem for a linking locality $(\mathcal{L}, \Gamma, S)$ (over a saturated fusion system) where Γ includes every non-trivial normal subgroup of S. Then one would work with the locality $(\mathcal{L}_{\Delta^*}(G)/\Theta, \Delta^*, S)$ to deduce the corresponding classification theorem for groups of parabolic characteristic p. Working with $(\mathcal{L}_{\Delta^*}(G)/\Theta, \Delta^*, S)$ has here not only the advantage that $(\mathcal{L}_{\Delta^*}(G)/\Theta, \Delta^*, S)$ is a linking locality, but also that its fusion system is isomorphic to $\mathcal{F}_S(G)$.

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