CENTRALIZERS OF NORMAL SUBGROUPS AND THE Z*-THEOREM

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Abstract. Glauberman’s Z*-theorem and analogous statements for odd primes show that, for any prime $p$ and any finite group $G$ with Sylow $p$-subgroup $S$, the centre of $G/O_p(G)$ is determined by the fusion system $F_S(G)$. Building on these results we show a statement that seems a priori more general: For any normal subgroup $H$ of $G$ with $O_p(H) = 1$, the centralizer $C_S(H)$ is expressed in terms of the fusion system $F_S(G)$ and its normal subsystem induced by $H$.

Keywords: Finite groups; fusion systems; Glauberman’s Z*-theorem.

Throughout $p$ is a prime. Glauberman’s Z*-theorem [3] and its generalization to odd primes, which is shown using the classification of finite simple groups (see [7] and [4]), can be reformulated as follows:

Theorem A. Let $G$ be a finite group with $O_p(G) = 1$, and $S \in \text{Syl}_p(G)$. Then $Z(G) = Z(F_S(G))$.

We refer the reader here to [2] for basic definitions and results regarding fusion systems; see in particular Definitions I.4.1 and I.4.3 for the definition of central subgroups and the centre $Z(F)$. A more common formulation of the Z*-theorem states that, assuming the hypothesis of Theorem A, we have $t \in Z(G)$ if and only if $t^G \cap S = \{t\}$ for every element $t \in S$ of order $p$. Given a normal subgroup $H$ of a finite group $G$, a Sylow $p$-subgroup $S \in \text{Syl}_p(G)$, and an element $t \in S$ of order $p$, one can apply the Z*-theorem with $H(t)$ in place of $G$ to obtain the following corollary: Provided $O_p(H) = 1$, we have $t^H \cap S = \{t\}$ if and only if $t \in C_S(H)$. In this short note, we use Theorem A to give a less obvious characterization of $C_S(H)$.

Given a saturated fusion system $F$ on a finite $p$-group $S$ and a normal subsystem $\mathcal{E}$ of $F$ on $T \leq S$, Aschbacher [1] (6.7)(1)] showed that the set of subgroups $X$ of $C_S(T)$ with $\mathcal{E} \subseteq C_F(X)$ has a largest member $C_S(\mathcal{E})$. He furthermore constructed a normal subsystem $C_F(\mathcal{E})$ on $C_S(\mathcal{E})$, the centralizer of $\mathcal{E}$ in $F$; see [1] Chapter 6]. Note that $C_S(\mathcal{E})$ depends not only on $S$ and $\mathcal{E}$ but also on the fusion system $F$ in which both $S$ and $\mathcal{E}$ are contained.

The definition of $C_S(\mathcal{E})$ generalizes the definition of $Z(F)$ since $C_S(F) = Z(F)$. Moreover, for every normal subgroup $H$ of a finite group $G$ with Sylow $p$-subgroup $S$, $F_{S\cap H}(H)$ is a normal subsystem of $F_S(G)$ by [2] I.6.2. Thus, the following theorem, which we prove later on, can be seen as a generalization of Theorem A.

Theorem B. Let $G$ be a finite group and let $S$ be a Sylow $p$-subgroup of $G$. Let $H \leq G$ with $O_p(H) = 1$. Then $C_S(F_{S\cap H}(H)) = C_S(H)$.

In the statement of Theorem B it is understood that $C_S(F_{S\cap H}(H))$ is formed inside of $F_S(G)$. The result says in other words that, under the hypothesis of Theorem B, for any $X \leq S$ with $F_{S\cap H}(H) \subseteq C_{F_S(G)}(X)$, we have $X \leq C_S(H)$. 

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This is not true if one drops the assumption that \( H \) is normal in \( G \) as the following example shows: Let \( G := G_1 \times G_2 \) with \( G_1 \cong G_2 \cong S_3 \). Set \( p = 3 \), \( S = O_3(G) \), \( S_i := O_3(G_i) \) and let \( R \) be a subgroup of \( G \) of order \( 2 \) which acts fixed point freely on \( S \). Set \( H := S_1 \times R \). Then \( S_1 \cap H \in \text{Syl}_3(H) \) and \( F_{S_1}(H) = F_{S_1}(G_1) \subseteq C_{F_{S_1}(G)}(S_2) \) as \( S_2 = C_S(G_1) \). However, \( S_2 \not\subseteq C_S(H) \) by the choice of \( R \).

Theorem B was conjectured by the second author of this paper in [6]. Our proof of Theorem B builds on Theorem A and the reduction uses only elementary group theoretical results. Essential is the following lemma, whose proof is self-contained apart from using the conjugacy of Hall-subgroups in solvable groups.

**Lemma 1.** Let \( G \) be a finite group with Sylow \( p \)-subgroup \( S \) and a normal subgroup \( H \). Let \( P \leq S \) such that \( P \cap H \) is centric in \( F_{S \cap H}(H) \). Then for every \( p' \)-element \( \varphi \in \text{Aut}_G(P) \) with \([P, \varphi] \leq P \cap H \) and \( \varphi|_{P \cap H} \in \text{Aut}_H(P \cap H) \), we have \( \varphi \in \text{Aut}_H(P) \).

**Proof.** This is [5, Proposition 3.1].

**Proof of Theorem B.** We assume the hypothesis of Theorem B. Furthermore, we set \( F := F_S(G) \), \( T := S \cap H \) and \( E := F_T(H) \). If a homomorphism \( \varphi \) between subgroups \( A \) and \( B \) of \( T \) is induced by conjugation with an element \( h \in H \), then \( \varphi \) extends to \( c_h : AC_S(H) \to BC_S(H) \) and \( c_h \) restricts to the identity on \( C_S(H) \).

Thus \( E \subseteq C_T(C_S(H)) \), so by the definition of \( C_S(E) \), we have \( C_S(H) \leq C_S(E) \). To prove the converse inclusion, choose \( t \in C_S(E) \). Define:

\[
G_0 = H^t \quad \text{and} \quad S_0 := T^t,
\]
so that plainly \( S_0 \) is a Sylow \( p \)-subgroup of \( G_0 \) and \( F_0 := F_{S_0}(G_0) \) is a saturated fusion system on \( S_0 \). Note also that \( O_{p'}(G_0) = 1 \) as \( O_p(G_0) = O_p(H) \) and \( O_{p'}(H) = 1 \) by assumption.

By Theorem A, \( Z(F_0) = Z(G_0) \leq C_S(H) \). It thus suffices to prove \( t \in Z(F_0) \). As \( t \in C_S(E) \leq C_S(T) \), \( t \in Z(S_0) \). Let \( P \) be a subgroup of \( S_0 \) which is centric radical and fully normalized in \( F_0 \). Then \( t \in Z(S_0) \leq C_{S_0}(P) \leq P \). It is sufficient to prove \([t, \text{Aut}_{F_0}(P)] = 1\). For as \( P \) is arbitrary, Alperin’s fusion theorem [2, Theorem 3.6] implies then \( t \in Z(F_0) \). As \( F_0 \)-normal, \( \text{Aut}_{S_0}(P) \in \text{Syl}_p(\text{Aut}_{F_0}(P)) \) and thus \( \text{Aut}_{F_0}(P) = \text{Aut}_{S_0}(P) \text{Op}(\text{Aut}_{F_0}(P)) \). Note that \([t, \text{Aut}_{S_0}(P)] = 1\) as \( t \in Z(S_0) \). Hence, it is enough to prove

\[
[t, \text{Op}(\text{Aut}_{F_0}(P))] = 1.
\]

Let \( \varphi \in \text{Aut}_{F_0}(P) \) be a \( p' \)-element. Since \( \text{Op}(H) = \text{Op}(G_0) \), we have \( \text{Op}(\text{Aut}_{F_0}(P)) = \text{Op}(\text{Aut}_H(P)) \). In particular, \( \varphi \in \text{Aut}_H(P) \) and thus \( \varphi|_{P \cap T} \in \text{Aut}_H(P \cap T) = \text{Aut}_{E}(P \cap T) \). As \( t \in P \leq S_0 = T^t \), we have \( P = (P \cap T)^t \). Moreover, \( t \in C_S(E) \) implies that \( E \subseteq C_{F}(t) \). Hence, \( \varphi|_{P \cap T} \) extends to \( \psi \in \text{Aut}_E(P) \) with the property that \( t \psi = t \). Note that \( o(\psi) = o(\varphi|_{P \cap T}) \) and thus \( \psi \) is a \( p' \)-element as \( \varphi \) has order prime to \( p \). Moreover, plainly \([P, \psi] \leq P \cap T \) and \( \psi|_{P \cap T} = \varphi|_{P \cap T} \in \text{Aut}_H(P \cap T) \).

Since \( E \subseteq F_0 \), \( P \cap T \) is \( E \)-centric by [1] 7.18. Now it follows from Lemma 1 that \( \psi \in \text{Aut}_H(P) \). Thus, \( \chi := \varphi \circ \psi^{-1} \in \text{Aut}_H(P) \leq \text{Aut}_{F_0}(P) \). Clearly \( \chi|_{P \cap T} = \text{Id} \) as \( \psi \) extends \( \varphi|_{P \cap T} \). Moreover, using that \( H \) is normal in \( G \), we obtain

\[
[P, \chi] \leq \text{Aut}_H(P) \leq P \cap H = P \cap T.
\]

Hence, by [2] Lemma A.2, \( \chi \in C_{\text{Aut}_{F_0}(P)}(P/(P \cap T)) \) and \( C_{\text{Aut}_{F_0}(P)}(P \cap T) = \text{Op}(\text{Aut}_{F_0}(P)) = \text{Inn}(P) \) as \( P \) is radical in \( F_0 \). As \( \text{Inn}(P) \leq \text{Aut}_{S_0}(P) \) and \([t, \text{Aut}_{S_0}(P)] = 1\), it follows \( t \chi = t \). By the choice of \( \psi \), also \( t \psi = t \) and consequently \( t \varphi = t \). Since \( \varphi \) was chosen to be an
arbitrary $p'$-element in $\text{Aut}_{F_0}(P)$ and $O^p(\text{Aut}_{F_0}(P))$ is the subgroup generated by these elements, it follows that $[t, O^p(\text{Aut}_{F_0}(P))] = 1$. As argued above, this yields the assertion. □

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