CONTROL OF FUSION BY ABELIAN SUBGROUPS OF THE HYPERFOCAL SUBGROUP

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Abstract. We prove that an isomorphism between saturated fusion systems over the same finite p-group is detected on the elementary abelian subgroups of the hyperfocal subgroup if p is odd, and on the abelian subgroups of the hyperfocal subgroup of exponent at most 4 if $p = 2$. For odd p, this has implications for mod p group cohomology.

1. Introduction

In 1971, Quillen [18] published two articles relating properties of the mod p cohomology ring of a group G to the elementary abelian p-subgroups of G. The results hold for any prime p and any group G which is a compact Lie group (e.g. a finite group). Quillen studied in particular varieties of mod p cohomology rings and proved a stratification theorem stating that the variety of the mod p cohomology ring of G can be broken up into pieces corresponding to the G-conjugacy classes of elementary abelian p-subgroups of G.[1] Therefore, it is of interest to study conjugacy relations between elementary abelian p-subgroups. From now on we assume that G is finite and H is a subgroup of G of index prime to p. For any two subgroups A and B of G, we write $\mathrm{Hom}_G(A, B)$ for the set of group homomorphisms from A to B that are obtained via conjugation by an element of G. As a consequence of Quillen's stratification theorem, H controls fusion of elementary abelian subgroups in G, if the inclusion map from H to G induces an isomorphism between the varieties of the mod p cohomology rings of H and G. Here we say that the subgroup H controls fusion of elementary abelian subgroups in G if $\mathrm{Hom}_H(A, B) = \mathrm{Hom}_G(A, B)$ for all elementary abelian subgroups A and B of H. Similarly we say that H controls p-fusion in G if $\mathrm{Hom}_H(A, B) = \mathrm{Hom}_G(A, B)$ for all p-subgroups A and B of H. By the Cartan–Eilenberg stable elements formula [9, XII.10.1], the inclusion map from H to G induces an isomorphism in mod p group cohomology if H controls fusion in G. Together with Quillen's fundamental results, this motivates the study of connections between control of fusion of elementary abelian subgroups and control of p-fusion.

If $H = S$ is a Sylow p-subgroup of G and p is odd, Quillen [17] proved as a first illustration of his theory that G is p-nilpotent if the inclusion map from S to G induces an isomorphism between the corresponding varieties. We recall that, by a classical theorem of Frobenius, G is p-nilpotent if and only if S controls fusion in G. So Quillen showed that S controls fusion in G if S controls fusion of elementary abelian subgroups. Variations of this theorem were proved in [12, 7, 10, 13, 8, 2], but all maintaining the hypothesis that $H = S$ is a Sylow p-subgroup. Only relatively recently, Benson, Grodal and the first author of this

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[1]More precisely, Quillen studied the variety of the commutative subring of $H^*(G; \mathbb{F}_p)$ of elements of even degree. However, his stratification theorem holds similarly for the variety of $H^*(G; \mathbb{F}_p)$; see Remark 3.2.

paper proved a result that holds more generally for any subgroup H of index prime to p; see [5]. More precisely, it is shown that H controls fusion in G (and thus the inclusion map from H to G induces an isomorphism in mod p group cohomology), if the inclusion map induces an isomorphism between the corresponding varieties, i.e. if H controls fusion of elementary abelian subgroups of G. This is obtained as a consequence of a theorem that is stated and proved for saturated fusion systems; see [5, Theorem B]. In this short note, we point out that actually a slightly stronger version of this theorem holds. We refer the reader to [1, Part I] for an introduction to fusion systems.

Theorem A (Small exponent abelian subgroups of the hyperfocal subgroup control fusion)**.** *Let $\mathcal{G} \subseteq \mathcal{F}$ be two saturated fusion systems over the same finite p-group S. Suppose that* $\mathrm{Hom}_{\mathcal{G}}(A,B) = \mathrm{Hom}_{\mathcal{F}}(A,B)$ *for all $A, B \leq \mathfrak{hyp}(\mathcal{F})$ with A, B elementary abelian if p is odd, and abelian of exponent at most 4 if $p = 2$. Then $\mathcal{G} = \mathcal{F}$.*

If one replaces $\mathfrak{hyp}(\mathcal{F})$ by S, then the above theorem coincides with [5, Theorem B]. We recall that the hyperfocal subgroup $\mathfrak{hyp}(\mathcal{F})$ is the subgroup of S generated by all elements of the form $x^{-1}\varphi(x)$ where $x \in Q$ and $\varphi \in O^p(\mathrm{Aut}_{\mathcal{F}}(Q))$ for some subgroup Q of S. If $\mathcal{F} = \mathcal{F}_S(G)$ is the fusion system of a finite group G with Sylow p-subgroup S, then Puig's hyperfocal subgroup theorem [16, §1.1] states that $\mathfrak{hyp}(\mathcal{F}) = O^p(G) \cap S$. In the situation of Theorem A, Quillen's example $Q_8 \leq Q_8 : C_3$ shows that it is indeed not enough to consider only elementary abelian subgroups for $p = 2$.

A fusion system $\mathcal{F}$ on S is called nilpotent if $\mathcal{F} = \mathcal{F}_S(S)$. Restricting attention to subgroups of the hyperfocal subgroup is motivated by a theorem of the second author of this paper together with Zhang, which characterizes nilpotency of a saturated fusion system $\mathcal{F}$ by the fusion on certain subgroups of the hyperfocal subgroup of $\mathcal{F}$; see [14]. Another motivation comes from work of Ballester-Bolinches, Ezquerro, Su and Wang [2] showing that, in certain special cases, fusion is detected on the subgroups of the focal subgroup of $\mathcal{F}$ which are cyclic of order p or 4. We show here that in Theorem A and C of [2], the focal subgroup can actually be replaced by the hyperfocal subgroup. More precisely, we prove the following theorem which gives in particular a new characterization of nilpotent fusion systems:

Theorem B. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S, and let $\mathcal{G} = N_{\mathcal{F}}(S)$ or $\mathcal{G} = \mathcal{F}_S(S)$. Suppose that* $\mathrm{Hom}_{\mathcal{G}}(A,B) = \mathrm{Hom}_{\mathcal{F}}(A,B)$ *for all $A, B \leq \mathfrak{hyp}(\mathcal{F})$ which are cyclic subgroups of order p or 4. Then $\mathcal{G} = \mathcal{F}$.*

We remark that, in general, it is not the case that the subgroup H controls fusion in G if it controls fusion on cyclic subgroups of order p for odd p, or on subgroups of order at most 4 for $p = 2$. This is not even the case if G has a normal Sylow p-subgroup as the following example shows: Let n be an integer such that $n \geq 2$ and p does not divide n. Let S be the field of order p^n, so that S under addition forms in particular an elementary abelian group of order p^n. Note that every non-zero element of S induces a group automorphism of S via multiplication. Let D be the group of all these automorphisms. Then D is a subgroup of $\mathrm{Aut}(S)$ of order $p^n - 1$ acting freely and transitively on the non-trivial elements of S. Let σ be the Frobenius automorphism of the field S. Then σ has order n and is also a group automorphism of S. Moreover, σ normalizes D, as conjugation by σ takes every element of D to its pth power. Hence, $\hat{D} = D \rtimes \langle \sigma \rangle$ is a group of order $(p^n - 1)n$. Since p does not divide n, it follows that S is a normal Sylow p-subgroup of $G := S \rtimes \hat{D}$. Moreover, $H := S \rtimes D$

is a subgroup of G of index prime to p. Note also that $S = [S, D] = \mathfrak{hyp}(\mathcal{F}_S(G))$. Let $\mathcal{V}$ be the set of subgroups of S of order p. Then $\mathcal{V}$ has $\frac{p^n-1}{p-1}$ elements. As D acts freely and transitively on the non-trivial elements of S, it follows that D acts also transitively on $\mathcal{V}$, and that $C_D(A) = 1$ for all $A \in \mathcal{V}$. Thus $|\mathrm{Aut}_D(A)| = |N_D(A)| = \frac{|D|}{|\mathcal{V}|} = p - 1$ for every $A \in \mathcal{V}$. As any two elements of $\mathcal{V}$ are conjugate under D, it follows $|\mathrm{Hom}_D(A, B)| = p - 1$ for all $A, B \in \mathcal{V}$. Thus, $\mathrm{Hom}_H(A, B) = \mathrm{Hom}_D(A, B)$ is the set $\mathrm{Inj}(A, B)$ of injective group homomorphism from A to B. As $\mathrm{Hom}_H(A, B) \subseteq \mathrm{Hom}_G(A, B) \subseteq \mathrm{Inj}(A, B)$, this implies $\mathrm{Hom}_H(A, B) = \mathrm{Hom}_G(A, B)$ for all $A, B \in \mathcal{V}$. So H controls fusion in G of the cyclic subgroups of order p (and thus for $p = 2$ also of the cyclic subgroups of order at most 4). However, as $D \neq \hat{D}$, the subgroup H does not control fusion in G.

We conclude by stating a version of Theorem A in terms of varieties of cohomology rings. We continue to assume that G is a finite group and we fix moreover an algebraically closed field Ω of prime characteristic p. We either set $k = \Omega$ or $k = \mathbb{F}_p$. Moreover, set $H^*(G) := H^*(G, k)$ and define the variety V_G to be the variety $\mathrm{Hom}_k(H^*(G), \Omega)$ of k-algebra homomorphisms from $H^*(G)$ to Ω; see Remark 3.2 for alternative definitions of V_G. Then every k-algebra homomorphism $\alpha\colon H^*(G) \to H^*(H)$ induces a map of varieties $\alpha^*\colon V_H \to V_G$ by sending any homomorphism $\beta \in V_H = \mathrm{Hom}_k(H^*(H), \Omega)$ to $\beta \circ \alpha \in V_G = \mathrm{Hom}_k(H^*(G), \Omega)$. For an arbitrary subgroup H of G, we write $\mathrm{res}_{G,H} : H^*(G) \longrightarrow H^*(H)$ for the map induced by the inclusion map $H \to G$, and hence $\mathrm{res}^*_{G,H} : V_H \longrightarrow V_G$ for the corresponding map of varieties.

If H is a subgroup of G containing a Sylow p-subgroup S of G, then we have the inclusion maps $S \cap O^p(G) \hookrightarrow H \hookrightarrow G$ which induce the following maps of varieties:

$$V_{S\cap O^p(G)} \xrightarrow{\mathrm{res}^*_{H,S\cap O^p(G)}} V_H \xrightarrow{\mathrm{res}^*_{G,H}} V_G$$

So in particular, we can consider the restriction of the map $\mathrm{res}^*_{G,H}\colon V_H \to V_G$ to the subvariety $\mathrm{res}^*_{H,S\cap O^p(G)} V_{S\cap O^p(G)}$ of V_H. If p is an odd prime and H is a subgroup of G of index prime to p, then the results in [5] say basically that H controls fusion in G if $\mathrm{res}^*_{G,H}\colon V_H \to V_G$ is an isomorphism of varieties. Theorem A implies a slightly stronger statement which is stated in the next theorem. Notice that a subgroup H of G has index prime to p if and only if H contains a Sylow p-subgroup of G.

Theorem C. *Let G be a finite group, let p be an odd prime, and let H be a subgroup of G containing a Sylow p-subgroup S of G. Suppose the restriction of the map $\mathrm{res}^*_{G,H}$ to $\mathrm{res}^*_{H,S\cap O^p(G)} V_{S\cap O^p(G)}$ is injective. Then H controls fusion in G and the restriction map $\mathrm{res}_{G,H} : H^*(G) \longrightarrow H^*(H)$ is an isomorphism.*

Note that Theorem C says in particular that the map $\mathrm{res}^*_{G,H}\colon V_H \to V_G$ is an isomorphism of varieties if its restriction to $\mathrm{res}^*_{H,S\cap O^p(G)} V_{S\cap O^p(G)}$ is injective. One sees easily that the converse of Theorem C holds as well: If $\mathrm{res}_{G,H} : H^*(G) \longrightarrow H^*(H)$ is an isomorphism then $\mathrm{res}^*_{G,H} : V_H \longrightarrow V_G$ is an isomorphism. In particular, the restriction of $\mathrm{res}^*_{G,H}$ to $\mathrm{res}_{H,S\cap O^p(G)} V_{S\cap O^p(G)}$ is injective.

We remark also that a theorem analogous to Theorem C can be proved for saturated fusion systems rather than for groups. For more details, we refer the reader to Remark 3.5.

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2. Proof of Theorem A and Theorem B

The proof of Theorem A is very similar to the proof of Theorem B in [5]. We need the following variation of [5, Theorem 2.1].

Theorem 2.1. *Let P be a finite p-group and let G be a subgroup of $\mathrm{Aut}(P)$ containing the group $\mathrm{Inn}(P)$ of inner automorphism. Then there exists a G-invariant subgroup D of $[P, O^p(G)]$, of exponent p if p is odd and exponent at most 4 if $p = 2$, such that $[D, P] \leq Z(D)$, and such that every non-trivial p'-automorphism in G restricts to a non-trivial p'-automorphism of D. Furthermore, for any such D and any maximal (with respect to inclusion) abelian subgroup A of D it follows that $A \trianglelefteq P$ and $C_G(A)$ is a p-group.*

Proof. By [5, Theorem 2.1], there exists a characteristic subgroup D_1 of P, of exponent p if p is odd and exponent at most 4 if $p = 2$, such that $[D_1, P] \leq Z(D_1)$, and such that every non-trivial p'-automorphism of P restricts to a non-trivial p'-automorphism of D_1. Set $D := [D_1, O^p(G)]$. As D_1 is G-invariant and as $O^p(G)$ is normal in G, the subgroup D is G-invariant. In particular, as $\mathrm{Inn}(P) \leq G$ by assumption, we have $[D, P] \leq D$. Using $[D_1, P] \leq Z(D_1)$ we obtain thus $[D, P] \leq [D_1, P] \cap D \leq Z(D_1) \cap D \leq Z(D)$. If φ is a p'-automorphism of P in G with $\varphi|_D = \mathrm{Id}_D$ then $[D, \varphi] = 1$ and $[D_1, \varphi] \leq [D_1, O^p(G)] = D$. Thus, by [11, Theorem 5.3.6], we have $[D_1, \varphi] = [D_1, \varphi, \varphi] \leq [D, \varphi] = 1$ and $\varphi|_{D_1} = \mathrm{Id}_{D_1}$. Because of the way D_1 was chosen, this implies that $\varphi = \mathrm{Id}_P$. So we have shown that every non-trivial p'-automorphism in G restricts to a non-trivial automorphism of D.

For the last part let A be a maximal abelian subgroup of D with respect to inclusion. Then $[A, P] \leq Z(D) \leq A$ and thus $A \trianglelefteq P$. Furthermore, if $B \leq C_G(A)$ is a p'-subgroup, then $A \times B$ acts on D. Since A is maximal abelian, we have $C_D(A) = A \leq C_D(B)$. Note also that A and D are p-groups and B is a p'-group. So Thompson's $A \times B$-lemma [11, Theorem 5.3.4] says now that $[D, B] = 1$. The choice of D yields now $B = 1$. Since B was arbitrary, it follows that $C_G(A)$ is a p-group. □

We need the following crucial lemma, which is [5, Main Lemma 2.4].

Lemma 2.2. *Let $\mathcal{G} \subseteq \mathcal{F}$ be two saturated fusion systems on the same finite p-group S, and $P \leq S$ an $\mathcal{F}$-centric and fully $\mathcal{F}$-normalized subgroup, with $\mathrm{Aut}_{\mathcal{F}}(R) = \mathrm{Aut}_{\mathcal{G}}(R)$ for every $P < R \leq N_S(P)$. Suppose that there exists a subgroup $Q \trianglelefteq P$ with $\mathrm{Hom}_{\mathcal{F}}(Q, S) = \mathrm{Hom}_{\mathcal{G}}(Q, S)$. Then $\mathrm{Aut}_{\mathcal{F}}(P) = \langle \mathrm{Aut}_{\mathcal{G}}(P), C_{\mathrm{Aut}_{\mathcal{F}}(P)}(Q) \rangle$.*

Proof of Theorem A. By Alperin's fusion theorem [1, Theorem I.3.6], $\mathcal{F}$ is generated by $\mathcal{F}$-automorphisms of fully $\mathcal{F}$-normalized $\mathcal{F}$-centric subgroups. We want to show that $\mathrm{Aut}_{\mathcal{F}}(P) = \mathrm{Aut}_{\mathcal{G}}(P)$ for all $P \leq S$. By induction on $|S : P|$, we can assume that $\mathrm{Aut}_{\mathcal{F}}(R) = \mathrm{Aut}_{\mathcal{G}}(R)$ for all $R \leq S$ with $|R| > |P|$. Furthermore, by Alperin's fusion theorem, we can choose P to be fully $\mathcal{F}$-normalized and $\mathcal{F}$-centric. By Theorem 2.1, we can pick an $\mathrm{Aut}_{\mathcal{F}}(P)$-invariant subgroup D of $[P, O^p(\mathrm{Aut}_{\mathcal{F}}(P))]$, of exponent p if p is odd and of exponent at most 4 if $p = 2$, such that every non-trivial p'-automorphism $\varphi \in \mathrm{Aut}_{\mathcal{F}}(P)$ restricts to a non-trivial automorphism of D and, for any maximal (with respect to inclusion) abelian subgroup A of D, $A \trianglelefteq P$ and $C_{\mathrm{Aut}_{\mathcal{F}}(P)}(A)$ is a p-group. As P is fully $\mathcal{F}$-normalized, $\mathrm{Aut}_S(P)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$, and so if we replace A by a conjugate of A under $\mathrm{Aut}_{\mathcal{F}}(P)$, we can arrange that $C_{\mathrm{Aut}_{\mathcal{F}}(P)}(A) \leq \mathrm{Aut}_S(P) \leq \mathrm{Aut}_{\mathcal{G}}(P)$. As D has exponent p if p is odd and exponent at most 4 if $p = 2$, we have by assumption

$\mathrm{Hom}_{\mathcal{F}}(A, S) = \mathrm{Hom}_{\mathcal{G}}(A, S)$. So by Lemma 2.2 applied with A in place of Q, we obtain that $\mathrm{Aut}_{\mathcal{F}}(P) = \langle \mathrm{Aut}_{\mathcal{G}}(P), C_{\mathrm{Aut}_{\mathcal{F}}(P)}(A) \rangle = \mathrm{Aut}_{\mathcal{G}}(P)$ as wanted. □

Let $\mathcal{P}$ be a set of representatives of the $\mathcal{F}$-conjugacy classes of $\mathcal{F}$-essential subgroups. A version of the Alperin–Goldschmidt Theorem for fusion systems states that $\mathcal{F}$ is generated by the $\mathcal{F}$-automorphism groups of the elements of $\mathcal{P} \cup \{S\}$. Analyzing what is used in the proof above, one sees that we only need the following condition in Theorem A: For every $P \in \mathcal{P} \cup \{S\}$ and every abelian subgroup A of the commutator subgroup $[P, O^p(\mathrm{Aut}_{\mathcal{F}}(P))]$ which is of exponent p or 4, we have $\mathrm{Hom}_{\mathcal{F}}(A, S) = \mathrm{Hom}_{\mathcal{G}}(A, S)$.

The proof of Theorem B is essentially the same as the one of [2, Theorem A] except that we use Theorem 2.1 instead of [5, Theorem 2.1]. Essentially, Theorem B is a consequence of the following lemma:

Lemma 2.3. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. Suppose that $\mathrm{Hom}_{\mathcal{F}}(A, B) \subseteq \mathrm{Hom}_{N_{\mathcal{F}}(S)}(A, B)$ for all subgroups $A, B \leq \mathfrak{hyp}(\mathcal{F})$ which are cyclic of order p or 4. Then $\mathcal{F} = N_{\mathcal{F}}(S)$.*

Proof. Suppose that Q is an $\mathcal{F}$-essential subgroup. Then by definition, Q is in particular fully normalized and thus $\mathrm{Aut}_S(Q)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$. By Theorem 2.1, there is an $\mathrm{Aut}_{\mathcal{F}}(Q)$-invariant subgroup $D \leq [Q, O^p(\mathrm{Aut}_{\mathcal{F}}(Q))] \leq Q \cap \mathfrak{hyp}(\mathcal{F})$ such that every non-trivial p'-element of $\mathrm{Aut}_{\mathcal{F}}(Q)$ restricts to a non-trivial automorphism of D, and D is of exponent p or 4. Let $Z_i(S)$ be the i-th center of S and $D_i = D \cap Z_i(S)$. We argue now that D_i is $\mathrm{Aut}_{\mathcal{F}}(Q)$-invariant: For every $x \in D_i$ and any $\varphi \in \mathrm{Aut}_{\mathcal{F}}(Q)$, $\varphi|_{\langle x \rangle}$ extends by hypothesis to an element of $\mathrm{Aut}_{\mathcal{F}}(S)$ which clearly normalizes $Z_i(S)$. As φ normalizes D, it follows $\varphi(x) \in Z_i(S) \cap D = D_i$. So D_i is indeed $\mathrm{Aut}_{\mathcal{F}}(Q)$-invariant. Thus, for some $n \in \mathbb{N}$, the series $1 = D_0 \leq D_1 \leq \cdots \leq D_n = D$ is $\mathrm{Aut}_{\mathcal{F}}(Q)$-invariant. So the stabilizer H of this series (i.e. the set of elements in $\mathrm{Aut}_{\mathcal{F}}(Q)$ acting trivially on D_i/D_{i-1} for each $i \leq n$) forms a normal subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$. For any p'-element φ of H, we have $\varphi|_D = \mathrm{Id}_D$ by [11, Theorem 5.3.2], and thus $\varphi = \mathrm{Id}_Q$ by the choice of D. Therefore, the stabilizer H is a p-group and so $H \leq O_p(\mathrm{Aut}_{\mathcal{F}}(Q))$. Since $\mathrm{Aut}_S(Q)$ stabilizes the series $D_0 \leq D_1 \leq \cdots \leq D_n = D$, it follows that $\mathrm{Aut}_S(Q) = O_p(\mathrm{Aut}_{\mathcal{F}}(Q))$, which is a contradiction as every $\mathcal{F}$-essential subgroup is centric and radical. Hence there is no $\mathcal{F}$-essential subgroup. Thus, $\mathcal{F} = N_{\mathcal{F}}(S)$ by Alperin's fusion theorem [1, Theorem I.3.6]. □

Proof of Theorem B. By Lemma 2.3, $\mathcal{F} = N_{\mathcal{F}}(S)$. So for $\mathcal{G} = N_{\mathcal{F}}(S)$ the assertion follows immediately. Assume now $\mathcal{G} = \mathcal{F}_S(S)$. As $\mathcal{F} = N_{\mathcal{F}}(S)$, it is sufficient to show that $\mathrm{Aut}_{\mathcal{F}}(S) = \mathrm{Inn}(S)$. By Theorem 2.1, there is an $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant subgroup $D \leq [S, O^p(\mathrm{Aut}_{\mathcal{F}}(S))] \leq \mathfrak{hyp}(\mathcal{F})$ such that every non-trivial p'-element of $\mathrm{Aut}_{\mathcal{F}}(S)$ restricts to a non-trivial automorphism of D, and D is of exponent p or 4. Let $D_i = D \cap Z_i(S)$ and $n \in \mathbb{N}$ such that $D_n = D$. By hypothesis, every element of $\mathrm{Aut}_{\mathcal{F}}(S)$ acts on every element of D as conjugation by an element of S. Hence, $\mathrm{Aut}_{\mathcal{F}}(S)$ stabilizes the series $1 = D_0 \leq D_1 \leq \cdots \leq D_n = D$ and is thus a p-group by [11, Theorem 5.3.2]. Since $\mathrm{Inn}(S) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(S))$, it follows $\mathrm{Aut}_{\mathcal{F}}(S) = \mathrm{Inn}(S)$ as required. □

3. Proof of Theorem C

Throughout, assume that G is a finite group and that Ω is an algebraically closed field of prime characteristic p. Let $H^*(G)$ and V_G be as in the introduction. Recall that, for any subgroup H of G, we write $\operatorname{res}_{G,H}\colon H^*(G) \to H^*(H)$ for the map induced by the inclusion map from H to G, and $\operatorname{res}^*_{G,H}\colon V_H \to V_G$ for the corresponding map of varieties.

For the proof of Theorem C we will need some more notation: For every elementary abelian p-group A, we set

$$V_A^+ := V_A \setminus \bigcup_{A' < A} \operatorname{res}^*_{A,A'} V_{A'}.$$

If A is an elementary abelian subgroup of G, set

$$V_{G,A}^+ = \operatorname{res}^*_{G,A} V_A^+.$$

We start with the following elementary observation:

Remark 3.1. Let $A \le K \le G$ such that A is elementary abelian. Then $\operatorname{res}^*_{G,K} V^+_{K,A} = V^+_{G,A}$.

Proof. As $\operatorname{res}^*_{G,K} \circ \operatorname{res}^*_{K,A} = \operatorname{res}^*_{G,A}$, we have $V^+_{G,A} = \operatorname{res}^*_{G,A} V^+_A = \operatorname{res}^*_{G,K}(\operatorname{res}^*_{K,A} V^+_A) = \operatorname{res}^*_{G,K} V^+_{K,A}$. □

Remark 3.2. Write $H^{ev}(G)$ for the subring of $H^*(G)$ of elements of even degree. If $k = \mathbb{F}_p$ notice that the k-algebra homomorphisms from $H^*(G)$ to Ω are the same as the ring homomorphisms from $H^*(G)$ to Ω. So if $k = \mathbb{F}_p$ then, upon replacing $H^*(G)$ by $H^{ev}(G)$ if p is odd, the variety V_G corresponds to the variety $H_G(X)(\Omega)$ studied by Quillen [18] in the special case that X is a point. If $k = \Omega$, it follows from Hilbert's Nullstellensatz that V_G is homeomorphic to the maximal ideal spectrum of $H^*(G)$ via the map sending every homomorphism in V_G to its kernel; see Theorem 5.4.2 and the surrounding discussion in [3]. So again upon replacing $H^*(G)$ by $H^{ev}(G)$, the variety V_G as defined in this paper corresponds to the variety V_G as defined by Benson [3].

It is common to study the variety of $H^{ev}(G)$ rather than the variety of $H^*(G)$, because $H^{ev}(G)$ is commutative, whereas $H^*(G)$ is only graded commutative, and texts on commutative algebra are written for strictly commutative rings. As pointed out by Benson [4, p.9], the results from commutative algebra which are needed in the theory hold accordingly for graded commutative rings. Moreover, it is pointed out that any graded commutative ring A is commutative modulo its nilradical, and every element of odd degree lies in the nilradical if p is odd. So writing $\mathfrak{Nil}$ for the nilradical of $H^*(G)$, it follows that $H^*(G)/\mathfrak{Nil}$ is isomorphic to $H^{ev}(G)/(H^{ev}(G) \cap \mathfrak{Nil})$. As the nilradical $\mathfrak{Nil}$ is contained in the kernel of every k-algebra homomorphism from $H^*(G)$ to Ω, the variety $\operatorname{Hom}_k(H^*(G), \Omega)$ is canonically homeomorphic to the variety $\operatorname{Hom}_k(H^{ev}(G), \Omega)$.

In particular, the Quillen Stratification Theorem as stated in [18, Theorem 10.2] and [3, Theorem 5.6.3] can be proved accordingly with our definitions:

Theorem 3.3 (Quillen's Stratification Theorem)**.** *Let $\mathcal{A}$ be a set of representatives of the G-conjugacy classes of elementary abelian subgroups of G. Then V_G is the disjoint union*

$$V_G = \coprod_{A \in \mathcal{A}} V^+_{G,A}.$$

of locally closed subvarieties $V^+_{G,A}$. Moreover, for every $A \in \mathcal{A}$, the automorphism group $\operatorname{Aut}_G(A)$ acts freely on V^+_A and the map $\operatorname{res}^*_{G,A}$ induces a homeomorphism $V^+_A/\operatorname{Aut}_G(A) \to V^+_{G,A}$.

The fact that $V_G = \coprod_{A\in\mathcal{A}} V_{A,G}^+$ for any set $\mathcal{A}$ of representatives of the G-conjugacy classes of the elementary abelian subgroups of G, will be used in our proof in the following form:

Remark 3.4. Let A and A' be elementary abelian subgroups of G. If A and A' are G-conjugate then we have $V_{G,A}^+ = V_{G,A'}^+$, and if A and A' are not G-conjugate then $V_{G,A}^+$ and $V_{G,A'}^+$ are disjoint. □

Proof of Theorem C. Assume that the restriction of the map $\mathrm{res}_{G,H}^*\colon V_H \to V_G$ to the subvariety $\mathrm{res}_{H,S\cap O^p(G)}^* V_{S\cap O^p(G)}$ of V_H is injective.

Step 1: Let A be an elementary abelian subgroup of $S\cap O^p(G)$. We show that the map $\mathrm{res}_{G,H}^*$ induces a bijection from $V_{H,A}^+$ to $V_{G,A}^+$. Moreover, if A' is another elementary abelian subgroup of $S\cap O^p(G)$ such that $V_{G,A}^+ = V_{G,A'}^+$, then we show $V_{H,A}^+ = V_{H,A'}^+$.

To see this note that, by Remark 3.1, we have that $\mathrm{res}_{G,H}^* V_{H,A}^+ = V_{G,A}^+$ and that $V_{H,A}^+ = \mathrm{res}_{H,S\cap O^p(G)}^* V_{S\cap O^p(G),A}^+$ is contained in $\mathrm{res}_{H,S\cap O^p(G)}^* V_{S\cap O^p(G)}$. By a symmetric argument, it follows that $V_{H,A'}^+$ is contained in $\mathrm{res}_{H,S\cap O^p(G)}^* V_{S\cap O^p(G)}$ and $\mathrm{res}_{G,H}^* V_{H,A}^+ = V_{G,A}^+ = V_{G,A'}^+ = \mathrm{res}_{G,H}^* V_{H,A'}^+$. As we assume that the restriction of $\mathrm{res}_{G,H}^*$ to $\mathrm{res}_{H,S\cap O^p(G)}^* V_{S\cap O^p(G)}$ is injective, the above assertion follows.

Step 2: Let A and A' be two G-conjugate elementary abelian subgroups of $S\cap O^p(G)$. We show that A and A' are H-conjugate. For the proof note that $V_{G,A}^+ = V_{G,A'}^+$ by Remark 3.4. So by Step 1, we have $V_{H,A}^+ = V_{H,A'}^+$. Thus, again by Remark 3.4 now used with H in place of G, the subgroups A and A' need to be H-conjugate. This completes the proof of Step 2.

Step 3: Let A be an elementary abelian subgroup of $S\cap O^p(G)$. We show that $\mathrm{Aut}_G(A) = \mathrm{Aut}_H(A)$. By the Quillen stratification theorem Theorem 3.3, the group $\mathrm{Aut}_G(A)$ acts freely on V_A^+, and the map $\mathrm{res}_{G,A}^*$ induces a homeomorphism $V_A^+/\mathrm{Aut}_G(A) \to V_{G,A}^+$. In particular, the fibres of the map $\mathrm{res}_{G,A}^*\colon V_A^+ \to V_{G,A}^+$ are precisely the orbits of $\mathrm{Aut}_G(A)$ on V_A^+. Similarly, applying the Quillen stratification theorem with H in place of G, we get that $\mathrm{Aut}_H(A)$ acts freely on V_A^+, and the fibres of the map $\mathrm{res}_{H,A}^*\colon V_A^+ \to V_{H,A}^+$ are precisely the orbits of $\mathrm{Aut}_H(A)$ on V_A^+. Note that $\mathrm{res}_{G,A}^* = \mathrm{res}_{G,H}^* \circ \mathrm{res}_{H,A}^*$. As the map $\mathrm{res}_{G,H}^*\colon V_{H,A}^+ \to V_{G,A}^+$ is by Step 1 a bijection, it follows that the maps $\mathrm{res}_{G,A}^*\colon V_A^+ \to V_{G,A}^+$ and $\mathrm{res}_{H,A}^*\colon V_A^+ \to V_{H,A}^+$ have the same fibres. So the $\mathrm{Aut}_G(A)$-orbits on V_A^+ are the same as the $\mathrm{Aut}_H(A)$-orbits. As the actions of $\mathrm{Aut}_G(A)$ and $\mathrm{Aut}_H(A)$ on V_A^+ are free, this implies that $|\mathrm{Aut}_G(A)| = |\mathrm{Aut}_H(A)|$. Thus, since $\mathrm{Aut}_H(A) \subseteq \mathrm{Aut}_G(A)$, it follows $\mathrm{Aut}_G(A) = \mathrm{Aut}_H(A)$.

Step 4: We are now in a position to complete the proof. Let A and A' be elementary abelian subgroups of $S\cap O^p(G)$. We want to show that $\mathrm{Hom}_G(A,A') = \mathrm{Hom}_H(A,A')$ and can assume without loss of generality that A and A' are G-conjugate. Then A and A' are H-conjugate by Step 2 and thus there exists $\psi \in \mathrm{Hom}_H(A,A')$. Let $\varphi \in \mathrm{Hom}_G(A,A')$. Note that $\varphi = \psi\circ(\psi^{-1}\circ\varphi)$ and $\psi^{-1}\circ\varphi \in \mathrm{Aut}_G(A) = \mathrm{Aut}_H(A)$ by Step 3. So it follows that $\varphi \in \mathrm{Hom}_H(A,A')$ which proves $\mathrm{Hom}_G(A,A') = \mathrm{Hom}_H(A,A')$. By Puig's hyperfocal subgroup theorem [16, §1.1], we have $S\cap O^p(G) = \mathfrak{hyp}(\mathcal{F}_S(G))$. So using Theorem A, we can conclude that $\mathcal{F}_S(G) = \mathcal{F}_S(H)$. Thus, by the Cartan–Eilenberg stable elements formula [9, XII.10.1], the map $\mathrm{res}_{G,H}\colon H^*(G) \to H^*(H)$ is an isomorphism. □

Remark 3.5. A version of Theorem C can also be formulated and proved for abstract saturated fusion systems rather than for groups. Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. Assume that k is an algebraically closed field of characteristic p. The cohomology ring $H^*(\mathcal{F}) = H^*(\mathcal{F}, k)$ of the saturated fusion system $\mathcal{F}$ is the subring of $\mathcal{F}$-stable elements in $H^*(S) = H^*(S, k)$, which is the subring of $H^*(S)$ consisting of elements $\xi \in H^*(S)$ such that $\mathrm{res}^S_P(\xi) = \mathrm{res}_\varphi(\xi)$ for any $\varphi \in \mathrm{Hom}_\mathcal{F}(P, S)$ and any subgroup $P \leq S$. The ring $H^*(\mathcal{F})$ is a graded commutative ring. We write $V_\mathcal{F}$ for the maximal ideal spectrum of $H^*(\mathcal{F})$, or alternatively for the variety of k-algebra homomorphisms from $H^*(\mathcal{F})$ to k.

Let $\mathcal{G}$ be a saturated fusion subsystem of $\mathcal{F}$. Note that any $\mathcal{F}$-stable element of $H^*(S)$ is in particular $\mathcal{G}$-stable, so we can consider the inclusion map $\mathrm{res}_{\mathcal{F},\mathcal{G}} \colon H^*(\mathcal{F}) \to H^*(\mathcal{G})$ which then gives us a map $\mathrm{res}^*_{\mathcal{F},\mathcal{G}} \colon V_\mathcal{G} \to V_\mathcal{F}$ of varieties. Similarly, if $Q \leq S$, we are given a k-algebra homomorphism $\mathrm{res}_{\mathcal{F},Q} \colon H^*(\mathcal{F}) \to H^*(Q)$ by composing the inclusion map $H^*(\mathcal{F}) \hookrightarrow H^*(S)$ with the restriction map $\mathrm{res}_{S,Q} \colon H^*(S) \to H^*(Q)$. Again, this induces a map of varieties $\mathrm{res}^*_{\mathcal{F},Q} \colon V_Q \to V_\mathcal{F}$. In particular, if $A \leq S$ is elementary abelian, one can define $V^+_{\mathcal{F},A} = \mathrm{res}^*_{\mathcal{F},A} V^+_A$. In an unpublished preprint, Markus Linckelmann [15, Theorem 1] proves a version of the Quillen stratification theorem; see also Theorem 1.3 and Remark 1.1 in [19]. Using this, one can similarly prove the following version of Theorem C for fusion systems:

Let $\mathcal{G} \subseteq \mathcal{F}$ be an inclusion of saturated fusion systems over the same finite p-group S, and p an odd prime. If the restriction of the map $\mathrm{res}^*_{\mathcal{F},\mathcal{G}} \colon V_\mathcal{G} \to V_\mathcal{F}$ to $\mathrm{res}^*_{\mathcal{G},\mathfrak{hyp}(\mathcal{F})} V_{\mathfrak{hyp}(\mathcal{F})}$ is injective, then $\mathcal{F} = \mathcal{G}$ and in particular $H^*(\mathcal{F}) = H^*(\mathcal{G})$.

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