Supplementary Material for

Uncovering hidden flows in physical networks

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11th September 2017

1 Example of Flow Tracing in a DC Network

We build up a MATLAB model to simulate a direct current (DC) network shown in Fig. 1 to illustrate the flow tracing process. The flow quantity f is given by the electric current I in this model. Nodes 1 and 2 are two nodes with current sources where $I_1^s = 3\text{A}$ and $I_2^s = 5\text{A}$, respectively. The resistances of resistors are randomly chosen within the set of integer numbers [1,10], shown in Tab. 1. The sink flow leaving from the sink nodes 9 and 10 are measured by the current scopes as $I_9^t = 4.51\text{A}$ and $I_{10}^t = 3.49\text{A}$. The current directions are shown in Fig. 2. Next, we show how to calculate the source-to-sink hidden currents from the current source I_1^s and I_2^s to the sink I_9^t and I_{10}^t by different methods.

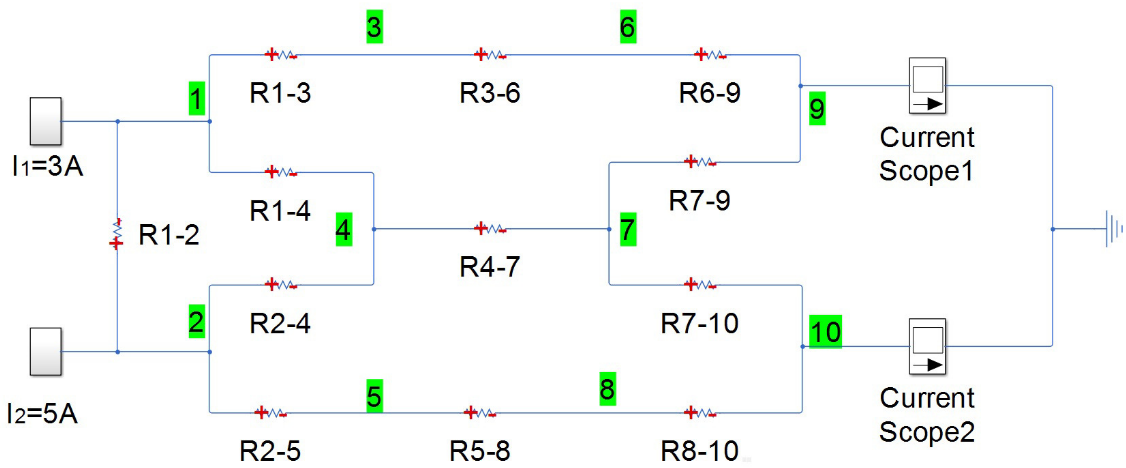

Figure 1: The MATLAB/Simulink model for a DC network with 10 nodes.

Table 1: Resistances of the resistors in Fig. 1.

Resistor	R_{1-2}	R_{1-3}	R_{1-4}	R_{2-4}	R_{2-5}	R_{3-6}
Resistance/Ω	7	9	7	4	6	5
Resistor	R_{4-7}	R_{5-8}	R_{6-9}	R_{7-9}	R_{7-10}	R_{8-10}
Resistance/Ω	1	3	2	2	3	8

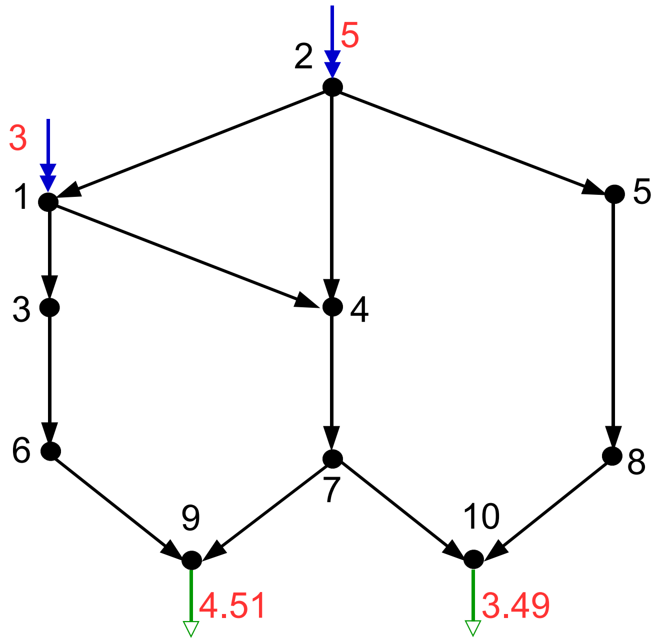

Figure 2: The current directions in the DC network shown in Fig. 1.

1.1 Using the Downstream Flow Tracing Method

As shown in Fig. 2, there are two paths from node 1 to node 9, which are $P_1(1,9) = 1\ \{1,3\}\ 3\ \{3,6\}\ 6\ \{6,9\}\ 9$, and $P_2(1,9) = 1\ \{1,4\}\ 4\ \{4,7\}\ 7\ \{7,9\}\ 9$.

Using the downstream flow tracing method, we calculate the current from node 1 to node 9 through the path $P_1(1,9)$ by

$$I^{(1)}_{1\to 9} = I^{in}_1 \frac{I^{out}_{13}}{I^{out}_1} \frac{I^{out}_{36}}{I^{out}_3} \frac{I^{out}_{69}}{I^{out}_6} = I^{in}_1 \kappa^d_{13} \kappa^d_{36} \kappa^d_{69}, \tag{1}$$

and through the path $P_2(1,9)$ by

$$I^{(2)}_{1\to 9} = I^{in}_1 \frac{I^{out}_{14}}{I^{out}_1} \frac{I^{out}_{47}}{I^{out}_4} \frac{I^{out}_{79}}{I^{out}_7} = I^{in}_1 \kappa^d_{14} \kappa^d_{47} \kappa^d_{79}. \tag{2}$$

Thus, the total node-to-node hidden current from node 1 to node 9 is

$$I_{1\to 9} = I^{(1)}_{1\to 9} + I^{(2)}_{1\to 9}. \tag{3}$$

The source-to-sink hidden current is calculated by

$$I_{s1\to t9} = \iota^s_1 \cdot I_{1\to 9} \cdot \iota^t_9. \tag{4}$$

By doing this type of calculation, we obtain $I_{s1\to t9} = 2.35$, $I_{s1\to t10} = 0.65$, $I_{s2\to t9} = 2.16$ and $I_{s2\to t10} = 2.84$.

1.2 Using the Upstream Flow Tracing Method

Using the upstream flow tracing method, we have

$$I^{(1)}_{1\to 9} = I^{out}_9 \frac{I^{in}_{69}}{I^{in}_9} \frac{I^{in}_{36}}{I^{in}_6} \frac{I^{in}_{13}}{I^{in}_3} = I^{out}_9 \kappa^u_{96} \kappa^u_{63} \kappa^u_{31}, \tag{5}$$

and

$$I^{(2)}_{1\to 9} = I^{out}_9 \frac{I^{in}_{79}}{I^{in}_9} \frac{I^{in}_{47}}{I^{in}_7} \frac{I^{in}_{14}}{I^{in}_4} = I^{out}_9 \kappa^u_{97} \kappa^u_{74} \kappa^u_{41}. \tag{6}$$

The node-to-node hidden current from node 1 to 9 is calculated by Eq. (3), and source-to-sink hidden current is calculated by Eq. (4).

Table 2 illustrates the results of flow tracing using the downstream flow tracing method and the upstream flow tracing method. The numbers in the following table indicate source-to-sink hidden currents. As we can see, the two methods imply the same results.

Table 2: Flow tracing in the DC network shown in Fig. 1, where nodes 1 and 2 are source nodes, and nodes 9 and 10 are sink nodes. Numbers in the table shows source-to-sink hidden flows.

Downstream			Upstream		
Node	9	10	Node	9	10
1	2.35	0.65	1	2.35	0.65
2	2.16	2.84	2	2.16	2.84

1.3 Using the Downstream Extended Incidence Matrix

From the MATLAB simulation results of the DC network, the downstream extended incidence matrix, $\mathbf{K}$, is

$$\mathbf{K} = \begin{bmatrix} 1 & -0.0378 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.4571 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.5429 & -0.6722 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.2900 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -0.6000 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.4000 & -1 & 0 & 1 \end{bmatrix},$$

and the downstream contribution matrix, $\mathbf{C}$, is

$$\mathbf{C} = \begin{bmatrix} 1 & 0.0378 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.4571 & 0.0173 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.5429 & 0.6927 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2900 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0.4571 & 0.0173 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0.5429 & 0.6927 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0.2900 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0.7828 & 0.4329 & 1 & 0.6000 & 0 & 1 & 0.6000 & 0 & 1 & 0 \\ 0.2172 & 0.5671 & 0 & 0.4000 & 1 & 0 & 0.4000 & 1 & 0 & 1 \end{bmatrix}.$$

We also obtain, from the experiments, that $f_1^{in} = 3.1891$, $f_2^{in} = 5$, $\iota_1^s = 0.9407$, $\iota_2^s = 1$, $\iota_9^t = 1$ and $\iota_{10}^t = 1$. Thus, we calculate $f_{sj \to ti}$ for $j = 1,\ 2$ and $i = 9,\ 10$ by $f_{s1\to t9} = \iota_9^t \cdot C_{91} f_1^{in} \cdot \iota_1^s = 2.35$, $f_{s2\to t9} = \iota_9^t \cdot C_{92} f_2^{in} \cdot \iota_2^s = 2.16$, $f_{s1\to t10} = \iota_{10}^t \cdot C_{10\ 1} f_1^{in} \cdot \iota_1^s = 0.65$, and $f_{s2\to t10} = \iota_{10}^t \cdot C_{10\ 2} f_2^{in} \cdot \iota_2^s = 2.84$. We note that all these numbers coincide with that in Tab. 2.

1.4 Using the Upstream Extended Incidence Matrix

Define the *upstream extended incidence matrix*, $\mathbf{K}'$, by

$$K'_{ij} = \begin{cases} -f_{ij}^{out}/f_j^{in} & \text{if } i \neq j, \text{ and } f_{ij} > 0, \\ 1 & \text{if } i = j, \\ 0 & \text{else.} \end{cases} \quad (7)$$

We know $f_i^{out} = \sum_{j=1}^{N} f_{ij}^{out} + f_i^t$, implying, $f_i^{out} - \sum_{j=1}^{N} f_{ij}^{out}/f_j^{in} \cdot f_j^{in} = f_i^t$. Since $f_i^{out} = f_i^{in}$, we have

$$f_i^{in} - \sum_{j=1}^{N} f_{ij}^{out}/f_j^{in} \cdot f_j^{in} = f_i^t. \tag{8}$$

Equations (7) and (8) imply

$$\mathbf{K'F^{in}} = \mathbf{F^t}, \tag{9}$$

where $\mathbf{F^{in}} = [f_1^{in},\ f_2^{in}, \cdots, f_N^{in}]^T$ and $\mathbf{F^t} = [f_1^t,\ f_2^t, \cdots, f_N^t]^T$. From $\mathbf{F^{in}} = \mathbf{K'^{-1}F^t}$, we have

$$\begin{aligned} f_i^{in} &= \sum_{j=1}^{N} \left[\mathbf{K'^{-1}}\right]_{ij} f_j^t \\ &= \sum_{j=1}^{N} \left[\mathbf{K'^{-1}}\right]_{ij} f_j^{out} \cdot \iota_j^t. \end{aligned} \tag{10}$$

Let $\mathbf{C'} = \mathbf{K'^{-1}}$ be the *upstream contribution matrix* whose element, $C_{ij} = \left[\mathbf{K'^{-1}}\right]_{ij}$, is a *upstream contribution factor* indicating how much proportion of the total outflow at node j is coming from node i, i.e., $f_{i\to j} = C'_{ij} f_j^{out}$. Then, $f_{si\to tj} = \iota_i^s \cdot C'_{ij} f_j^{out} \cdot \iota_j^t$.

The upstream extended incidence matrix, $\mathbf{K'}$, of the DC network is

$$\mathbf{K} = \begin{bmatrix} 1 & 0 & -1 & -0.3400 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.0593 & 1 & 0 & -0.6600 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -0.3230 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -0.6770 & -0.5842 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -0.4158 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and the upstream contribution matrix, $\mathbf{C'}$, is

$$\mathbf{C'} = \begin{bmatrix} 1 & 0 & 1 & 0.3400 & 0 & 1 & 0.3400 & 0 & 0.5532 & 0.1986 \\ 0.0593 & 1 & 0.0593 & 0.6802 & 1 & 0.0593 & 0.6802 & 1 & 0.4796 & 0.8132 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0.3230 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0.6770 & 0.5842 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0.4158 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0.3230 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0.6770 & 0.5842 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0.4158 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

We also obtain $f_9^{out} = 4.5132$, $f_{10}^{out} = 3.4868$, $\iota_1^s = 0.9407$, $\iota_2^s = 1$, $\iota_9^t = 1$ and $\iota_{10}^t = 1$. Then, $f_{s1\to t9} = \iota_1^s \cdot C'_{19} f_9^{out} \cdot \iota_9^t = 2.35$, $f_{s2\to t9} = \iota_2^s \cdot C'_{29} f_9^{out} \cdot \iota_9^t = 2.16$, $f_{s1\to t10} = \iota_1^s \cdot C'_{1\ 10} f_{10}^{out} \cdot \iota_{10}^t = 0.65$, and $f_{s2\to t10} = \iota_2^s \cdot C'_{2\ 10} f_{10}^{out} \cdot \iota_{10}^t = 2.84$. The results are the same as that in Tab. 2.