VIBRATIONS OF 1D NON-CONSERVATIVE ELASTIC CONTINUUM DUE TO PASSAGE OF A MDOF DAMPED OSCILLATOR

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INTRODUCTION

Dynamic interaction between moving vehicles and supporting structures over which they travel is an important problem in bridge engineering. Pesterev and Bergman (1997) considered vibrations of a conservative 1D elastic continuum carrying moving mass sprung on a linear spring. They presented a formulation, which enabled the solution of the interaction problem to be expanded in a series in terms of the eigenfunction of the isolated continuum. The present study extends their work by introduction of proportional damping and a MDOF vehicle model, interacting with the continuum at several contact points. The solution of the interaction problem is obtained in terms of modal expansion using the eigenfunctions and eigenvectors of the isolated continuum and oscillator. The time dependent terms of the modal expansion are found through integration of a system of linear differential equations. The obtained method is tested on several numerical examples.

THEORY

Equation of motion

Consider a 1D continuous system and a MDOF oscillator moving over it (Fig. 1). The deflections of the continuum are described by the function $u_c(x,t)$. External forces acting on the continuum are denoted by $f_c(x,t)$. The equation of motion of the isolated continuum is

$$\hat{A}_c u_c(x,t) = f_c(x,t) \tag{1}$$

$\hat{A}_c$ is a partial differential operator of the form

$$\hat{A}_c u_c(x,t) = m(x)\partial^2 u_c(x,t)/\partial t^2 + \hat{C}\,\partial u_c(x,t)/\partial t + \hat{L}u_c(x,t) \tag{2}$$

where $m(x)$ is the mass distribution function, and $\hat{C}$ and $\hat{L}$ are, respectively, linear homogenous differential damping and stiffness operators. Damping operator, $\hat{C}$, is assumed to share the well-known orthogonality properties with $m(x)$ and $\hat{L}$. The displacements of the vehicle under the action of external forces, $f_v(t)$, are denoted by vector $u_v(t)$. The equation of motion of the isolated vehicle is

$$\hat{A}_v u_v(t) = f_v(t) \tag{3}$$

Operator $\hat{A}_v$ is an ordinary differential operator with respect to time variable defined by

$$\hat{A}_v u_v(t) = M\,d^2 u_v(t)/dt^2 + C\,du_v(t)/dt + Ku_v(t) \tag{4}$$

M, C and K are $N_v \times N_v$ mass, damping and stiffness matrices, respectively. Damping matrix C is assumed to share the well-known orthogonality properties with M and K.

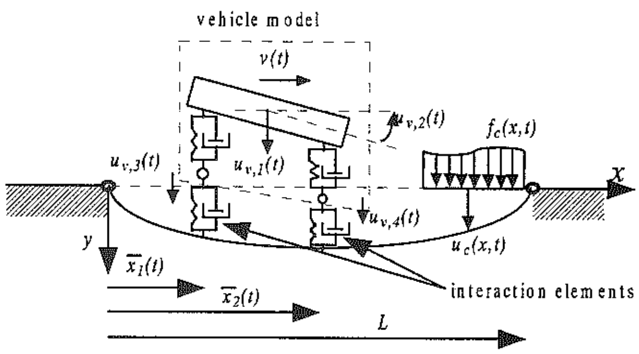

Fig. 1 Interaction of 1D continuum and MDOF oscillator.

The interaction forces are assumed to be linear elastic and viscous ones, where stiffness and damping coefficients are denoted respectively by $k_{cv,i}$ and $c_{cv,i}$ (i=1, …, N_{cv}, and N_{cv} is the number of contact points). Define interaction stiffness and damping matrices, respectively, as

$$K_{cv} = diag\left(k_{cv,1} \ \ \dots \ \ k_{cv,N_{cv}}\right) \tag{5}$$

$$C_{cv} = diag\left(c_{cv,1} \ \ \dots \ \ c_{cv,N_{cv}}\right) \tag{6}$$

as well as operator $\hat{K}$ as

$$\hat{K} = K_{cv} + C_{cv}\, d/dt \tag{7}$$

Introduce sensor operator

$$\hat{\Pi}_x[\bar{x}(t)]u_c(x,t) = \left[u_c(\bar{x}_1(t),t) \ \ \dots \ \ u_c(\bar{x}_{N_{cv}}(t),t)\right]^T \tag{8}$$

where $\bar{x}(t) = \left[\bar{x}_1(t) \ \ \dots \ \ \bar{x}_{N_{cv}}(t)\right]^T$ is the vector of contact point locations.

The adjoint of the selector operator, named an effector operator, is

$$\Delta_x[\bar{x}(t)] = \hat{\Pi}_x^*[\bar{x}(t)] = \left[\delta[x-\bar{x}_1(t)] \ \ \dots \ \ \delta[x-\bar{x}_{N_{cv}}(t)]\right] \tag{9}$$

The interaction forces, P, can be expressed as follows

$$P = -\hat{\Theta}^*\hat{K}\hat{\Theta}u \tag{10}$$

and the continuum-vehicle interaction governing equation of motion is

$$\left(\hat{A} + \hat{\Theta}^*\hat{K}\hat{\Theta}\right)u = F \tag{11}$$

The symbols introduced in (10) and (11) are as follow

$$\hat{A} = block\ diag\left(\hat{A}_c, \hat{A}_v\right),\ \hat{\Theta} = \left[\Pi_x[\bar{x}(t)] \ \ -T\right],\ \hat{\Theta}^* = \left[\Delta_x^T[\bar{x}(t)] \ \ -T\right]^T \qquad (12, 13, 14)$$

$$u = \left[u_c(x,t) \ \ u_v(t)\right]^T,\ F = \left[f_c(x,t) \ \ f_v(t)\right]^T \qquad (15, 16)$$

where T is the matrix that transforms displacements of the vehicle into displacements inducing interaction forces.

Solution by reduction to ordinary differential equations

The solution of the interaction problem (11) can be given by the formula

$$u = \left(\hat{A} + \hat{\Theta}^*\hat{K}\hat{\Theta}\right)^{-1}F \tag{17}$$

The inverse operator appearing in (17) can be found as

$$\left(\hat{A} + \hat{\Theta}^*\hat{K}\hat{\Theta}\right)^{-1} = \hat{A}^{-1} - \hat{A}^{-1}\hat{\Theta}^*\left(\hat{I} + \hat{K}\hat{\Theta}\hat{A}^{-1}\hat{\Theta}^*\right)^{-1}\hat{K}\hat{\Theta}\hat{A}^{-1} \tag{18}$$

where $\hat{I}$ is the identity operator. The inverse operators $\hat{A}_c^{-1}$ and $\hat{A}_v^{-1}$ can be expressed as

$$\hat{A}_c^{-1}f_c(x,t) = \int_0^t\int_0^L\sum_{i=1}^{\infty}\varphi_{c,i}(x)\varphi_{c,i}(\xi)h_i\left(t,\tau,\omega_{0c,i},\omega_{c,i},\zeta_{c,i}\right)f_c(\xi,\tau)d\xi d\tau \tag{19}$$

$$\hat{A}_v^{-1}f_v(t) = \int_0^t\sum_{i=1}^{N_v}\varphi_{v,i}\varphi_{v,i}^T h_i\left(t,\tau,\omega_{0v,i},\omega_{v,i},\zeta_{v,i}\right)f_v(\tau)d\tau \tag{20}$$

where

$$h_i\left(t,\tau,\omega_{0,i},\omega_i,\zeta_i\right) = \begin{cases}\left[e^{-\zeta_i\omega_{0,i}(t-\tau)}\sin\omega_i(t-\tau)\right]/\omega_i & \omega_{0,i} \neq 0 \\ t-\tau & \omega_{0,i} = 0\end{cases} \tag{21}$$

The symbols introduced in (19) – (21) stand for: $\varphi_{c,i}(x)$, eigenfunctions of continuum; $\varphi_{v,i}$, eigenvectors of oscillator; $\omega_{0c,i}$ and $\omega_{0v,i}$, undamped natural frequencies; $\zeta_{c,i}$ and $\zeta_{v,i}$, damping ratios; $\omega_{c,i}$ and $\omega_{v,i}$, damped natural frequencies, respectively, where superscripts c and v correspond to the continuum and vehicle, respectively.

Introduce the notation

$$\left(\hat{I} + \hat{K}\hat{\Theta}\hat{A}^{-1}\hat{\Theta}^{*}\right)^{-1}\hat{K}\hat{\Theta}\hat{A}^{-1}P = y(t) \tag{22}$$

and define new variables

$$q_{c,i}(t) = \int_0^t \left[Q_{c,i}(\tau) - \overline{\varphi}_{c,i}^T(\tau)y(\tau)\right]h_i\left(t,\tau,\omega_{0c,i},\omega_{c,i},\zeta_{c,i}\right)d\tau \tag{23}$$

$$q_{v,i}(t) = \int_0^t \left[Q_{v,i}(\tau) + \varphi_{v,i}^T T^T y(\tau)\right]h_i\left(t,\tau,\omega_{0v,i},\omega_{v,i},\zeta_{v,i}\right)d\tau \tag{24}$$

where

$$\overline{\varphi}_{c,i}(t) = \Pi_x\left[\overline{x}(t)\right]\varphi_{c,i}(x),\ Q_{c,i}(t) = \int_0^L \varphi_{c,i}(x)f_c(x,t)dx\,,\ Q_{v,i}(t) = \varphi_{v,i}^T f_v(t) \quad (25, 26, 27)$$

Expanding (22) and using the new variables (23) and (24), function $y(t)$ can be evaluated as

$$y(t) = \hat{K}\left\{\sum_{i=1}^{\infty}\overline{\varphi}_{c,i}(t)q_{c,i}(t) - \sum_{i=1}^{N_v}T\varphi_{v,i}q_{v,i}(t)\right\} \tag{28}$$

Differentiating (23) and (24) with respect to t twice, one obtains

$$\ddot{q}_{c,i}(t) + 2\zeta_{c,i}\omega_{0c,i}\dot{q}_{c,i}(t) + \omega_{0c,i}^2 q_{c,i}(t) = Q_{c,i}(t) - \overline{\varphi}_{c,i}^T(t)y(t) \tag{29}$$

$$\ddot{q}_{v,i}(t) + 2\zeta_{v,i}\omega_{0v,i}\dot{q}_{v,i}(t) + \omega_{0v,i}^2 q_{v,i}(t) = Q_{v,i}(t) + \varphi_{v,i}^T T^T y(t) \tag{30}$$

Expanding (17) and substituting (23) and (24), the solution for the interaction problem can be found as

$$u_c(x,t) = \sum_{i=1}^{\infty}\varphi_{c,i}(x)q_{c,i}(t),\ u_v(t) = \sum_{i=1}^{N_v}\varphi_{v,i}q_{v,i}(t) \quad (31, 32)$$

Equations (28) - (32) present exact solution of the problem of interaction of a MDOF proportionally damped oscillator moving over a 1D proportionally damped elastic continuum, and interacting with it through linear elastic and viscous forces.

NUMERICAL EXAMPLE

The numerical example considered by Green and Cebon (1994) is studied. The continuum is a simply supported Euler-Bernoulli beam of length L=40 m, bending stiffness EI=1.275×10^{11} Nm^2 and mass per unit length m=1.2×10^4 kg/m. Modal damping ratios are computed according to the formula $\zeta_{c,i}$=0.322/$\omega_{0c,i}$+0.0002$\omega_{0c,i}$. The MDOF vehicle and interaction model is depicted in Fig. 2, where numerical values of parameters are also shown.

Figs. 3-5 show results of the simulations for the vehicle traveling with speed of v=50 m/s. In Fig. 3, mid-span deflections obtained through solution of (29) and (30) with different number of modes of the continuum, N_c, considered are shown. In Fig. 4, the coefficients $q_{c,i}(t)$ are shown for the case of N_c=3. It can be seen that a very good approximation is obtained for a small number of continuum modes considered, and that values of coefficients $q_{c,i}(t)$ rapidly converge to zero for increasing i. Thus, the proposed method is computationally efficient. Fig. 5 depicts the mid-span deflections due to passage of the considered MDOF vehicle model as well as two constant forces of values S=1.962×10^5 N, computed for N_c=3. The results of Fig. 5 match perfectly those reported by Green and Cebon (1994).

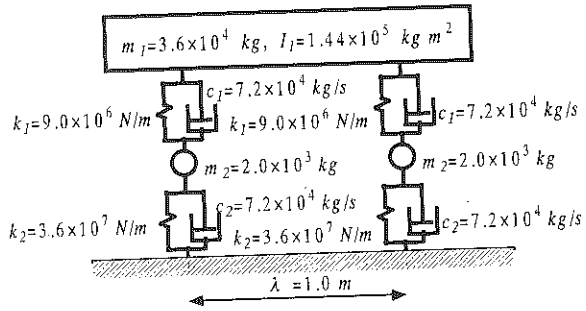

Fig. 2 MDOF vehicle model.

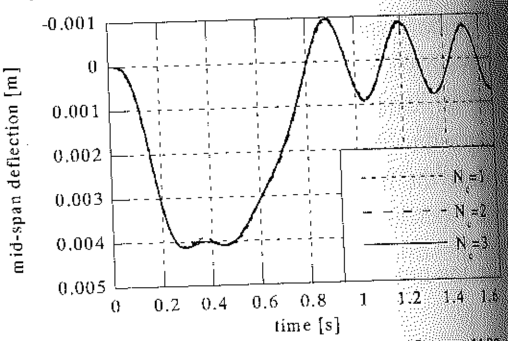

Fig. 3 Mid-span deflection for different number of continuum modes.

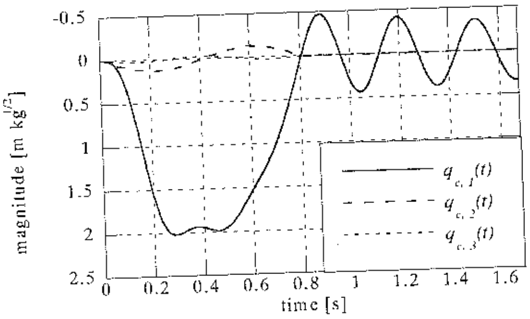

Fig. 4 Coefficients $q_{c,i}(t)$ for N_c=3.

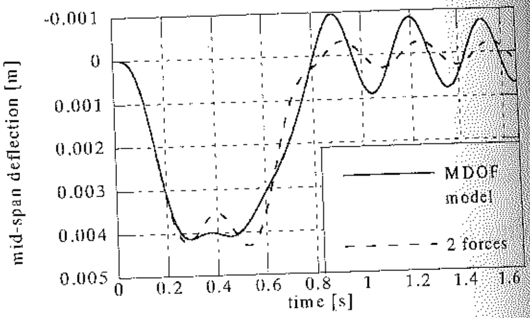

Fig. 5 Mid-span deflection for MDOF vehicle model and 2 constant forces.

CONCLUSIONS

A method for computing the response of a 1D elastic continuum induced by a MDOF oscillator traveling over it has been proposed. The continuum and the oscillator are non-conservative systems with the proportional damping. The interaction is realized through linear elastic and viscous forces. The exact solution has been obtained in a form of a series using eigenfunctions and eigenvectors of the isolated continuum and oscillator, respectively. The time dependent terms of the series are solutions of a system of linear differential equations. The method is tested on numerical examples and results are successfully compared to those available in the literature. Numerical examples show that the number of terms in the modal expansion required for high accuracy is small, and thus the method is efficient.

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