G-COMPLETE REDUCIBILITY IN NON-CONNECTED GROUPS

MICHAEL BATE, SEBASTIAN HERPEL, BENJAMIN MARTIN, AND GERHARD RÖHRLE

Abstract. In this paper we present an algorithm for determining whether a subgroup H of a non-connected reductive group G is G-completely reducible. The algorithm consists of a series of reductions; at each step, we perform operations involving connected groups, such as checking whether a certain subgroup of G^0 is G^0-cr. This essentially reduces the problem of determining G-complete reducibility to the connected case.

1. Introduction

Let G be a connected reductive linear algebraic group over an algebraically closed field of characteristic 0 or characteristic $p > 0$. Following Serre [14], we say a subgroup H of G is *G-completely reducible* (G-cr) if whenever H is contained in a parabolic subgroup P of G, then H is contained in some Levi subgroup of P. The definition extends to non-connected reductive G as well: one replaces parabolic and Levi subgroups with so-called Richardson parabolic and Richardson Levi subgroups respectively (see [2], [13] and Section 2).

Even if one is interested mainly in connected reductive groups, one must sometimes consider non-connected groups. For instance, natural subgroups of a connected group, such as normalizers and centralizers, are often non-connected. The notion of G-complete reducibility is much better understood in the connected case, e.g., see [1], [10], and [11]. In this paper we present an algorithm for determining whether a subgroup H of a non-connected reductive group G is G-cr. The algorithm consists of a series of reductions; at each step, we perform operations involving connected groups, such as checking whether a certain subgroup of G^0 is G^0-cr. This essentially reduces the problem of determining G-complete reducibility to the connected case.

An important special case of the general problem described above is the following. Let H be a subgroup of G. We say *H acts on G^0 by outer automorphisms* if for each $1 \neq h \in H$, conjugation by h gives a non-inner automorphism of G^0. In this case, we may identify H with a subgroup of $\mathrm{Out}(G^0)$. Now suppose also that G^0 is simple; then H is cyclic except for possibly when G^0 is of type D_4. It is convenient when studying conjugacy classes in G^0 to determine the fixed point set of a non-inner automorphism. See e.g., [12, Lem. 2.9] when H is cyclic and semisimple (that is, H is of order coprime to p); note that if H is generated by a semisimple element then H is G-cr by Theorem 2.6, as semisimple conjugacy classes are closed. On the other hand, if H is cyclic and unipotent (that is, H is a p-group) then H can be G-cr or non-G-cr.

We prove the following result, which gives a criterion for G-complete reducibility of H. It is an ingredient in our algorithm. In case H is cyclic, this follows from a recent result

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due to Guralnick and Malle, cf. Theorem 3.1 and Corollary 3.2, which is valid without the simplicity assumption on G^0.

Theorem (Corollary 4.5). *Suppose G^0 is simple and H acts on G^0 by outer automorphisms. Then H is G-completely reducible if and only if $C_{G^0}(H)$ is reductive.*

Our work fits into a study begun in our earlier papers [2], [3]. It was shown in [2, Thm. 3.10] that if H is a G-cr subgroup of G and N is a normal subgroup of H then N is also G-cr. In [3] we considered a complementary question: if H is a subgroup of G, N is a normal subgroup of H and N is G-cr then under what hypotheses is H also G-cr? We gave an example (due to Liebeck) with H of the form $M \times N$, where M and N are both G-cr but H is not [3, Ex. 5.3]. We also showed this kind of pathological behaviour does not happen when G is connected and p is good for G [3, Thm. 1.3]. Here we study the above question in the case when N is the normal subgroup $H \cap G^0$ of H (see the algorithm in Theorem 5.3).

2. Preliminaries

2.1. **Notation.** Throughout, we work over an algebraically closed field k of characteristic $p \geq 0$; we let k^* denote the multiplicative group of k. Let H be a linear algebraic group. By a subgroup of H we mean a closed subgroup. In particular, the (topologically) cyclic subgroup generated by h in H is the closure of the subgroup generated by h. We let $Z(H)$ denote the centre of H and H^0 the connected component of H that contains 1. For $h \in H$, we let Int_h denote the automorphism of H given by conjugation with h. Frequently, we abbreviate $\mathrm{Int}_h(g)$ by $h \cdot g$. If S is a subset of H and K is a subgroup of H, then $C_K(S)$ denotes the centralizer of S in K and $N_K(S)$ the normalizer of S in K. Likewise, if S is a group of algebraic automorphisms of H, then we denote the fixed point subgroup of S in H by $C_H(S)$. For a subgroup K of H, we denote the commutator subgroup of K by $[K, K]$. If H acts on a set X, then we also write $C_H(x)$ for the stabilizer of a point $x \in X$ in H.

For the set of cocharacters (one-parameter subgroups) of H we write $Y(H)$; the elements of $Y(H)$ are the homomorphisms from k^* to H.

The *unipotent radical* of H is denoted $R_u(H)$; it is the maximal connected normal unipotent subgroup of H. The algebraic group H is called *reductive* if $R_u(H) = \{1\}$; note that we do not insist that a reductive group is connected. In particular, H is reductive if it is simple as an algebraic group. Here, H is said to be *simple* if H is connected and all proper normal subgroups of H are finite. The algebraic group H is called *linearly reductive* if all rational representations of H are semisimple.

Throughout the paper G denotes a reductive algebraic group, possibly non-connected.

Definition 2.1. Let $H \subseteq G$ be a subgroup. We say that H *acts on G^0 by outer automorphisms* if for every $1 \neq h \in H$, the automorphism $\mathrm{Int}_h|_{G^0}$ of G^0 is non-inner, i.e., is not given by conjugation with an element of G^0. This is equivalent to the condition that H maps bijectively onto its image under the natural map $G \to \mathrm{Aut}(G^0) \to \mathrm{Out}(G^0)$.

2.2. ***G*-Complete Reducibility.** In [2, §6], Serre's original notion of G-complete reducibility is extended to include the case when G is reductive but not necessarily connected (so that G^0 is a connected reductive group). The crucial ingredient of this extension is the use of so-called *Richardson-parabolic subgroups* (*R-parabolic subgroups*) of G. We briefly recall the main definitions here; for more details on this formalism, see [2, §6].

For a cocharacter $\lambda \in Y(G)$, the *R-parabolic subgroup* corresponding to λ is defined by

$$P_\lambda := \{g \in G \mid \lim_{a\to 0} \lambda(a) g \lambda(a)^{-1} \text{ exists}\}.$$

Here, for a morphism of algebraic varieties $\phi : k^* \to X$, we say that $\lim_{a\to 0} \phi(a)$ exists provided that ϕ extends to a morphism $\widehat{\phi} : k \to X$; in this case we set $\lim_{a\to 0} \phi(a) = \widehat{\phi}(0)$. Then P_λ admits a Levi decomposition $P_\lambda = R_u(P_\lambda) \rtimes L_\lambda$, where

$$L_\lambda = \{g \in G \mid \lim_{a\to 0} \lambda(a) g \lambda(a)^{-1} = g\} = C_G(\lambda(k^*)).$$

We call L_λ an *R-Levi subgroup* of P_λ. For an R-parabolic subgroup P of G, the different R-Levi subgroups of P correspond in this way to different choices of $\lambda \in Y(G)$ such that $P = P_\lambda$; moreover, the R-Levi subgroups of P are all conjugate under the action of $R_u(P)$. An R-parabolic subgroup P is a parabolic subgroup in the sense that G/P is a complete variety; the converse is true when G is connected, but not in general ([13, Rem. 5.3]).

Remark 2.2. For a subgroup H of G, there is a natural inclusion $Y(H) \subseteq Y(G)$. If $\lambda \in Y(H)$, and H is reductive, we can therefore associate to λ an R-parabolic subgroup of H as well as an R-parabolic subgroup of G. To avoid confusion, we reserve the notation P_λ for R-parabolic subgroups of G, and distinguish the R-parabolic subgroups of H by writing $P_\lambda(H)$ for $\lambda \in Y(H)$. The notation $L_\lambda(H)$ has the obvious meaning. Note that $P_\lambda(H) = P_\lambda \cap H$ and $L_\lambda(H) = L_\lambda \cap H$ for $\lambda \in Y(H)$. In particular, $P_\lambda^0 = P_\lambda(G^0)$ and $L_\lambda^0 = L_\lambda(G^0)$. If $\lambda \in Y(H)$ then the R-Levi subgroups of $P_\lambda(H)$ are the $R_u(P_\lambda(H))$-conjugates of $L_\lambda(H)$; in particular, any R-Levi subgroup of $P_\lambda(H)$ is of the form $L \cap H$ for some R-Levi subgroup L of P_λ.

For later use, we record the following way to construct R-Levi subgroups.

Lemma 2.3. *Let P be an R-parabolic subgroup of G, and let M be a Levi subgroup of P^0. Then $N_P(M)$ is an R-Levi subgroup of P.*

Proof. We may choose $\lambda \in Y(G)$ such that $P = P_\lambda$, $P^0 = P_\lambda(G^0)$ and $M = L_\lambda(G^0) = L_\lambda^0$. We have the Levi decomposition $P = R_u(P_\lambda) \rtimes L_\lambda = R_u(P_\lambda^0) \rtimes L_\lambda$. Since $L_\lambda \subseteq N_P(L_\lambda^0)$ and $R_u(P_\lambda^0) \cap N_P(L_\lambda^0) = 1$ (as $R_u(P_\lambda^0)$ acts simply transitively on the set of Levi subgroups of P_λ^0), we conclude that $N_P(M) = N_P(L_\lambda^0) = L_\lambda$. ☐

Definition 2.4. Suppose H is a subgroup of G. We say H is *G-completely reducible* (G-cr for short) if whenever H is contained in an R-parabolic subgroup P of G, then there exists an R-Levi subgroup L of P with $H \subseteq L$.

Since all parabolic subgroups (respectively all Levi subgroups of parabolic subgroups) of a connected reductive group are R-parabolic subgroups (respectively R-Levi subgroups of R-parabolic subgroups), Definition 2.4 coincides with Serre's original definition for connected groups [15].

The following consequence of Lemma 2.3 gives a converse of [4, Lem. 5.1] in the case of the normal subgroup G^0 of G.

Corollary 2.5. *Let H be a subgroup of G. Suppose that whenever H normalizes P^0 for an R-parabolic subgroup P of G, then H also normalizes a Levi subgroup of P^0. Then H is G-completely reducible.*

Proof. Let P be an R-parabolic subgroup of G containing H. Then $H \subseteq N_G(P^0)$. Thus, by our assumption, there is a Levi subgroup M of P^0 which is normalized by H. Consequently, $H \subseteq P \cap N_G(M) = N_P(M)$ which is an R-Levi subgroup of P, by Lemma 2.3. □

Let H be a subgroup of G and let $G \hookrightarrow \mathrm{GL}_m$ be an embedding of algebraic groups. Let $\mathbf{h} \in H^n$ be a tuple of generators of the associative subalgebra of Mat_m spanned by H (such a tuple exists for n sufficiently large). Then $\mathbf{h}$ is called a *generic tuple* of H, see [5, Def. 5.4]. We recall the following geometric criterion for G-complete reducibility [5, Thm. 5.8]; it provides a link between the theory of G-complete reducibility and the geometric invariant theory of reductive groups.

Theorem 2.6. *Let H be a subgroup of G and let $\mathbf{h} \in H^n$ be a generic tuple for H. Then H is G-completely reducible if and only if the orbit $G \cdot \mathbf{h}$ under simultaneous conjugation is closed in G^n. In particular, if $H = \overline{\langle h \rangle}$ is a cyclic subgroup of G, then H is G-completely reducible if and only if the conjugacy class $G \cdot h \subseteq G$ is closed.*

The following result has been proved with methods from geometric invariant theory (see [5, Def. 5.17]):

Theorem 2.7. *Assume that the subgroup H of G is not G-completely reducible. Then there exists an R-parabolic subgroup P of G with the following two properties:*
- (i) *H is not contained in any R-Levi subgroup of P,*
- (ii) *$N_G(H) \subseteq P$.*

The geometric construction of P in [5, §4] is roughly as follows: there is a class of so-called *optimal destabilizing cocharacters* $\Omega \subseteq Y(G)$ such that if $\lambda \in \Omega$ then $P := P_\lambda$ has properties (i) and (ii) as in Theorem 2.7. We call such an R-parabolic subgroup P of G an *optimal destabilizing R-parabolic subgroup for H*.

2.3. **Criteria for G-complete reducibility.** In this subsection we study criteria for G-complete reducibility in terms of some smaller group.

We first recall the following results about normal subgroups (cf. [3, Thm. 3.4, Cor. 3.7(ii)]).

Theorem 2.8. *Suppose that $N \subseteq H$ are subgroups of G and N is normal in H.*
- (i) *If N is also normal in G, then H is G-completely reducible if and only if H/N is G/N-completely reducible.*
- (ii) *If H/N is linearly reductive, then H is G-completely reducible if and only if N is G-completely reducible.*

A homomorphism $\pi : G_1 \to G_2$ is called *non-degenerate* provided that $\ker(\pi)^0$ is a torus. The next result is contained in [2, Lem. 2.12 and §6]:

Lemma 2.9. *Let $\pi : G_1 \to G_2$ be a non-degenerate epimorphism of reductive groups. Let $H \subseteq G_1$ be a subgroup. Then H is G_1-completely reducible if and only if $\pi(H)$ is G_2-completely reducible.*

As an immediate consequence, we obtain the following result which allows us to focus on the part of G that is effectively acting on G^0. Note that $C_G(G^0)^0 \subseteq Z(G^0)^0$ is a torus.

Corollary 2.10. *Let $H \subseteq G$ be a subgroup. Let $\pi : G \to G/C_G(G^0)$ be the natural projection. Then H is G-completely reducible if and only if $\pi(H)$ is $\pi(G)$-completely reducible.*

The following lemma gives two necessary conditions for a subgroup of G to be G-completely reducible, both of which can be checked in the connected group G^0.

Lemma 2.11. *Let H be a G-completely reducible subgroup of G. Then the following hold:*
(i) *$H \cap G^0$ is G^0-completely reducible;*
(ii) *$C_{G^0}(H)$ is G^0-completely reducible.*

Proof. (i). This is the content of [2, Lem. 6.10 (ii)]. (ii). Since H is G-cr, so is its centralizer $C_G(H)$, by [2, Cor. 3.17 and §6]. Now the result follows from part (i). □

Under the assumption that assertion (i) or (ii) of Lemma 2.11 holds, the next two lemmas allow us to replace the ambient group G with a potentially smaller subgroup M.

Lemma 2.12. *Let H be a subgroup of G and suppose that $H \cap G^0$ is G^0-completely reducible. Then $M = HC_{G^0}(H \cap G^0)$ is reductive. Moreover, H is G-completely reducible if and only if it is M-completely reducible.*

Proof. First note that M is a subgroup of G, since H normalizes $C_{G^0}(H \cap G^0)$. As $C_{G^0}(H \cap G^0)$ is a G^0-cr subgroup (Lemma 2.11(ii)), it is reductive, by [14, Property 4]. The same is true for $H \cap G^0$ by assumption. Hence $(H \cap G^0)C_{G^0}(H \cap G^0)$ is the product of two reductive groups and thus is reductive. As this group contains M^0 as a normal subgroup, the group M is reductive as well.

Now first suppose that H is not G-cr. Let $\lambda \in Y(G) = Y(G^0)$ be an optimal destabilizing cocharacter for H in G. Then $P_\lambda(G^0)$ contains the subgroup $H \cap G^0$, which is G^0-cr by assumption. Hence after replacing λ with an $R_u(P_\lambda)$-conjugate, we may assume that λ centralizes $H \cap G^0$. This implies that $\lambda \in Y(M)$, and $H \subseteq P_\lambda(M) \subseteq P_\lambda$. Since H is not contained in an R-Levi subgroup of P_λ (cf. Theorem 2.7), it is not contained in an R-Levi subgroup of $P_\lambda(M)$. We conclude that H is not M-cr.

Conversely, suppose that H is G-cr. Let $\lambda \in Y(M)$ and suppose that $H \subseteq P_\lambda(M)$. To show that H is M-cr we need to show that H is contained in an R-Levi subgroup of $P_\lambda(M)$. Since H is G-cr, we can find a cocharacter $\mu \in Y(G)$, such that $H \subseteq L_\mu \subseteq P_\mu = P_\lambda$. In particular, μ centralizes $H \cap G^0$, so that $\mu \in Y(M)$. Hence $H \subseteq L_\mu(M) \subseteq P_\mu(M) = P_\lambda(M)$, as required. □

Remark 2.13. In the situation of Lemma 2.12, the subgroup $N = H \cap G^0$ of H is normal in M. Thus, by Theorem 2.8(i), we deduce that H is G-completely reducible if and only if $H/(H \cap G^0)$ is $M/(H \cap G^0)$-completely reducible.

Lemma 2.14. *Let H be a subgroup of G and suppose that $C_{G^0}(H)$ is G^0-completely reducible. Then $M = HC_{G^0}(C_{G^0}(H))$ is reductive. Moreover, H is G-completely reducible if and only if it is M-completely reducible.*

Proof. We proceed as in the proof of Lemma 2.12: Again, M is a subgroup since H normalizes $C_{G^0}(C_{G^0}(H))$. Moreover, $M^0 \subseteq (H \cap G^0)C_{G^0}(C_{G^0}(H)) = C_{G^0}(C_{G^0}(H)) \subseteq M$, yielding $M^0 = (C_{G^0}(C_{G^0}(H)))^0$. Since $C_{G^0}(H)$ is G^0-cr by assumption, as before we may conclude that its centralizer is reductive, so that M is reductive.

Suppose that H is not G-cr. Let $\lambda \in Y(G) = Y(G^0)$ be an optimal destabilizing cocharacter for H in G. By Theorem 2.7(ii), P_λ contains $C_{G^0}(H)$, which is G^0-cr. Thus we may again assume that λ centralizes $C_{G^0}(H)$, so that $\lambda \in Y(M)$. As before, we conclude that H is not M-cr.

Conversely, suppose that H is G-cr. Let $\lambda \in Y(M)$ such that $H \subseteq P_\lambda(M)$. As λ evaluates in M^0, it centralizes $C_{G^0}(H)$, so that $C_{G^0}(H)$ is contained in P_λ. On the other hand, since H is G-cr, we may find $\mu \in Y(G)$ such that $P_\mu = P_\lambda$ and such that $\mu \in Y(C_{G^0}(H))$. But then $P_\mu(C_{G^0}(H)) = C_{G^0}(H)$, which forces μ to centralize $C_{G^0}(H)$ [2, Lem. 2.4]. So $\mu \in Y(M)$, and $H \subseteq L_\mu(M) \subseteq P_\mu(M) = P_\lambda(M)$. As before, this shows that H is M-cr. □

Remark 2.15. Let H be a subgroup of G and let $\pi\colon G \to G'$ be an isogeny. Then $\pi(C_{G^0}(H)^0) = C_{G'^0}(\pi(H))^0$ (see the proof of [4, Lem. 3.1]), so $C_{G'^0}(\pi(H))^0$ is reductive if and only if $C_{G^0}(H)^0$ is.

We may write the connected reductive group G^0 in the form

$$G^0 = SG_1 \cdots G_n, \tag{2.16}$$

where S is the radical of G^0 and $G_1, \ldots, G_n$ are the simple components of the derived group of G^0. Any subgroup H of G acts via conjugation on the derived subgroup of G^0 and hence permutes the simple components. We obtain an induced action of H on the set of indices $\{1, \ldots, n\}$. For $1 \leq i \leq n$, we use the shorthand

$$\widehat{G}_i = S \prod_{j \neq i} G_j$$

for the product of all factors in G^0 above with the exception of G_i.

Our next lemma allows us to replace G with a collection of reductive groups whose identity components are simple.

Lemma 2.17. *Let H be a subgroup of G. For $1 \leq i \leq n$, let $H_i := N_H(G_i)$ and let $\pi_i : H_iG^0 \to H_iG^0/\widehat{G}_i$ be the natural projection. Let $I \subseteq \{1, \ldots, n\}$ be a subset meeting each H-orbit. Then H is G-completely reducible if and only if $\pi_i(H_i)$ is $\pi_i(H_iG^0)$-completely reducible for each $i \in I$.*

Proof. First note that, by construction, H_i and G^0 both normalize the group $\widehat{G}_i$. Hence the map π_i is well-defined. Since H_iG^0 is reductive, so is its image under π_i.

To prove the forward implication, suppose the assertion fails for some $i \in I$. Up to reordering the indices, we may assume that $\pi_1(H_1)$ is not $\pi_1(H_1G^0)$-cr and that H acts transitively on the set $\{1, \ldots, r\}$ for some $r \geq 1$. Let Q be an optimal destabilizing R-parabolic subgroup of $\pi_1(H_1G^0)$ for $\pi_1(H_1)$. To obtain a contradiction, we show that $\pi_1(H_1)$ is contained in an R-Levi subgroup of Q. Consider the group Q^0. This is a parabolic subgroup of $\pi_1(H_1G^0)^0 = \pi_1(G^0) = \pi_1(G_1)$, hence it is of the form $Q^0 = \pi_1(P_1)$, where P_1 is a parabolic subgroup of G_1.

Since P_1 contains the centre of G_1 and $\pi_1(P_1) = Q^0$ is normalized by $\pi_1(H_1) \subseteq Q$, it follows that P_1 is normalized by H_1. Indeed, let $h \in H_1$. Then $h \cdot P_1 \subseteq G_1$. On the other hand, $\pi_1(h \cdot P_1) = \pi_1(h) \cdot \pi_1(P_1) = \pi_1(P_1)$. Since $\ker(\pi_1) = \widehat{G}_1$, this implies that $h \cdot P_1 \subseteq P_1\widehat{G}_1$. We conclude that $h \cdot P_1 \subseteq P_1(G_1 \cap \widehat{G}_1) = P_1$, where we have used that the last intersection is central in G_1.

For $2 \leq j \leq r$, let $h_j \in H$ be an element satisfying $h_j \cdot G_1 = G_j$. Let $P_j = h_j \cdot P_1$, which is a parabolic subgroup of G_j. Since we have just verified that H_1 normalizes P_1, the definition of P_j does not depend on the choice of h_j that transports G_1 to G_j.

We now consider the parabolic subgroup $P = SP_1 \cdots P_n$ of G^0, where we take $P_j = G_j$ for $j > r$. By construction, P is normalized by H. Indeed, any $h \in H$ fixes S under conjugation, and permutes the groups $G_1, \ldots, G_n$. If h maps G_i to G_j with $i, j \in \{1, \ldots, r\}$, then $(hh_i) \cdot G_1 = G_j$, and hence $h \cdot P_i = (hh_i) \cdot P_1 = P_j$. So h also permutes the groups $P_1, \ldots, P_r$, and thus normalizes P.

The group $N_G(P)$ is thus an R-parabolic subgroup of G containing H with $N_G(P)^0 = P$ (see [2, Prop. 6.1]). Since H is G-cr, it is contained in an R-Levi subgroup L of $N_G(P)$, hence it normalizes the Levi subgroup L^0 of P. We may write $L^0 = SL_1 \cdots L_n$ for certain Levi subgroups L_j of P_j. Then H_1 normalizes L_1, since $L_1 = L^0 \cap G_1$. This forces $\pi_1(H_1)$ to normalize a Levi subgroup of $Q^0 = \pi_1(P_1)$. By Lemma 2.3, $\pi_1(H_1)$ is contained in an R-Levi subgroup of Q, yielding a contradiction.

To prove the reverse implication, we again assume after reordering the indices that $1 \in I$ and that H permutes the set $\{1, \ldots, r\}$ transitively for some $r \geq 1$. Assume that H is not G-cr, and that $Q \subseteq G$ is an optimal destabilizing R-parabolic subgroup of G containing H. Again we want to deduce that H is contained in an R-Levi subgroup of Q, contradicting our assumption.

Write $Q^0 = SP_1 \cdots P_n$, where the P_i are parabolic subgroups of G_i. Since H normalizes Q^0, H_1 normalizes $P_1 = Q^0 \cap G_1$. This means that $\pi_1(H_1)$ is contained in $N_{\pi_1(H_1 G^0)}(\pi_1(P_1))$, and the latter is an R-parabolic subgroup of $\pi_1(H_1 G^0)$. Since $1 \in I$, the subgroup $\pi_1(H_1)$ is contained in an R-Levi subgroup, say M, of this normalizer. Thus $\pi_1(H_1)$ normalizes the Levi subgroup M^0 of $\pi_1(P_1)$, which is hence of the form $\pi_1(L_1)$ for some Levi subgroup L_1 of P_1.

Since L_1 contains the centre of G_1, as in the proof of the forward implication (where we have proved that H_1 normalizes P_1), we may conclude that H_1 normalizes L_1. Choosing again elements $h_j \in H$ with $h_j \cdot G_1 = G_j$ for $2 \leq j \leq r$, we obtain well-defined Levi subgroups $L_j := h_j \cdot L_1$ of $h_j \cdot P_1 = P_j$, where the latter equality follows from $P_j = Q^0 \cap G_j$. Proceeding similarly for the other H-orbits on $\{1, \ldots, n\}$ (each of which contains an element of I by assumption), we construct an H-stable Levi subgroup $L = SL_1 \cdots L_n$ of Q^0. As before, by Lemma 2.3, H is contained in an R-Levi subgroup of Q, which gives the desired contradiction. This finishes the proof. □

3. Cyclic subgroups

Following Steinberg [17, §9], we say that g in G is *quasi-semisimple* provided g normalizes a pair (B, T) consisting of a Borel subgroup B of G and a maximal torus $T \subseteq B$. The next result is contained in recent work of Guralnick and Malle (see [8, Thm. 2.3]).

Theorem 3.1. *For $g \in G$, the following properties are equivalent:*

(i) *the conjugacy class $G \cdot g$ is closed;*
(ii) *the centralizer $C_G(g)$ is reductive;*
(iii) *g is quasi-semisimple.*

According to Theorem 2.6, a cyclic subgroup of G is G-completely reducible if and only if the conjugacy class of a generator is closed. With this characterization we may reformulate the equivalence of (i) and (ii) from Theorem 3.1 as follows.

Corollary 3.2. *Let H be a cyclic subgroup of G. Then H is G-completely reducible if and only if $C_{G^0}(H)$ is reductive.*

By Lemma 2.11(ii), we deduce a criterion for the complete reducibility of the fixed-point set.

Corollary 3.3. *Let H be a cyclic subgroup of G. Then $C_{G^0}(H)$ is G^0-completely reducible if and only if it is reductive.*

Combining some of our previous reductions, we give a different proof of Corollary 3.2. We believe this is of independent interest, as our arguments allow us to avoid the case-by-case considerations that are needed for the proof of Theorem 3.1. Here is the first ingredient of the proof.

Proposition 3.4. *Let U be a (not necessarily connected) unipotent algebraic group with $\dim U > 0$. Then the centre $Z(U)$ satisfies $\dim Z(U) > 0$.*

Proof. Let $V = U^0$. As V is nilpotent, connected and of positive dimension, we have $\dim Z(V) > 0$ (see, e.g., [9, Prop. 17.4]). Hence it is enough to prove the result when V is abelian, so we assume this. The finite p-group $H = U/V$ acts on V and it suffices to show that H has infinitely many fixed points on V. Up to passing to a characteristic subgroup (the subgroup of elements of order dividing p), we may assume that V has exponent p. Thus V has the structure of an $\mathbb{F}_p$-vector space of infinite dimension with an $\mathbb{F}_p$-linear H-action.

If $H \neq 1$, let $h \in H$ be a central element of order p. Then h fixes infinitely many points on V. Indeed, on any h-stable finite dimensional subspace W of V the automorphism induced by h may be brought into Jordan normal form with block sizes bounded by p (the Jordan normal form exists as the only eigenvalue of h is $1 \in \mathbb{F}_p$). As each block contributes at least $p-1$ fixed points, h has at least $(p-1)(\dim W)/p$ fixed points on W, and we can make $\dim_{\mathbb{F}_p} W$ arbitrarily large. Hence the fixed-point set V^h is positive-dimensional.

Finally consider the exact sequence $1 \to \langle h\rangle \to H \to H' \to 1$, with $H' = H/\langle h\rangle$. The group H' acts on $C_h(V)$. Using induction on the order of H, we may conclude that $V^H = (V^h)^{H'}$ is infinite. □

Proof of Corollary 3.2. The forward implication is clear, by Lemma 2.11(ii).

Conversely, assume that $C_{G^0}(H)$ is reductive. Let $H = \overline{\langle g\rangle}$ and let $g = g_s g_u$ be the Jordan decomposition of g and let H_s and H_u be the closed subgroups of G generated by g_s and g_u, respectively. Then H_s is linearly reductive, H is isomorphic to $H_s \times H_u$ and $C_{G^0}(H) = C_{C_{G^0}(H_s)}(H_u)$. By Lemma 2.11(ii), $C_G(H_s)$ is reductive (note that H_s is G-cr by [2, Lem. 2.6]). By [3, Prop. 3.9], H is G-cr if and only if H_u is $C_G(H_s)$-cr. So we can replace the pair (G, H) by $(C_G(H_s), H_u)$ and hence we can assume that $H = H_u$ is finite and unipotent.

We first show that $C_{G^0}(H)$ is G^0-cr. Suppose this fails, and let $P \subseteq G^0$ be an optimal destabilizing parabolic subgroup for $C_{G^0}(H)$ in G^0. Then H normalizes P, by Theorem 2.7(ii). Let U be the unipotent radical of P. We may apply Proposition 3.4 to the unipotent group HU to obtain a positive-dimensional centre $Z = Z(HU)$. Since H is finite, the identity component Z^0 of Z lies in U. Thus $C_{G^0}(H)$ normalizes U and so Z^0 yields a non-trivial, connected, normal, unipotent subgroup of $C_{G^0}(H)$, contradicting the reductivity assumption. We thus conclude that $C_{G^0}(H)$ is G^0-cr.

By Lemma 2.14, it therefore suffices to show that H is M-cr, where $M = HC_{G^0}(C_{G^0}(H))$. Suppose M^0 is not a torus. Let $M_1, \dots, M_r$ be the simple components of M^0 and set $H_1 := N_H(M_1)$. Then by a result of Steinberg (cf. [17, Thm. 10.13]), the fixed-point set $C_1 := C_{M_1}(H_1)$ of the cyclic group H_1 is positive-dimensional. Without loss of generality, we can assume H acts transitively on the set $M_1, \dots, M_s$ for some $1 \leq s \leq r$. For each $1 \leq i \leq s$, choose $h_i \in H$ such that $h_i M_1 h_i^{-1} = M_i$. Set $C := \{\prod_{i=1}^{s} h_i c h_i^{-1} \mid c \in C_1\}$.

Then C is a positive-dimensional subgroup of $C_{[M^0,M^0]}(H)$. But $C_{M^0}(H)$ is contained in $Z(M^0)$, so $C_{[M^0,M^0]}(H)$ is finite, a contradiction. We deduce that M^0 is a torus, and the result follows. □

Finally, using Corollary 3.2, we can give a short alternative proof of the implication (ii) ⇒ (iii) in Theorem 3.1 which is free of any case-by-case considerations.

Remark 3.5. Fix $g \in G$ and let $H = \overline{\langle g \rangle}$. Thanks to [17, Thm. 7.2], g normalizes a Borel subgroup B of G^0. Suppose that $C_{G^0}(g)$ is reductive. Then H is G-cr, by Corollary 3.2. Thus, since H normalizes B, it normalizes a Levi subgroup T of B, thanks to [4, Lem. 5.1]. Thus g is quasi-semisimple.

4. Outer automorphisms for D_4

In this section, let D_4 denote an adjoint simple group of type D_4. Amongst the simple groups D_4 has the largest outer automorphism group, in that $\mathrm{Out}(D_4) \cong S_3$, the symmetric group on 3 letters. We may identify $\mathrm{Out}(D_4)$ with the set of graph automorphisms in $\mathrm{Aut}(D_4)$ induced by the symmetries of the Dynkin diagram. However, there are other subgroups isomorphic to S_3 in $\mathrm{Aut}(D_4)$ that act via outer automorphisms. As this is the only situation where outer automorphisms of a simple group arise that is not covered by Corollary 3.2, we treat this case separately in this section.

Let T be a maximal torus of D_4 with associated root system Φ. Let $\Delta = \{\alpha, \beta, \gamma, \delta\}$ be a set of simple roots for Φ, where δ is the unique simple root that is non-orthogonal to every other simple root. Let $\lambda = \omega_\delta^\vee \in Y(T)$ be the fundamental dominant coweight determined by $\langle \alpha, \lambda \rangle = \langle \beta, \lambda \rangle = \langle \gamma, \lambda \rangle = 0$, $\langle \delta, \lambda \rangle = 1$. For $\epsilon \in \Phi$ we denote by $u_\epsilon : \mathbb{G}_a \to U_\epsilon$ a fixed root homomorphism onto the corresponding root subgroup of G.

Let $\sigma \in \mathrm{Aut}(D_4)$ be the triality graph automorphism determined by requiring that for all $c \in k$,

$$\sigma(u_\alpha(c)) = u_\beta(c), \qquad \sigma(u_\beta(c)) = u_\gamma(c), \qquad \sigma(u_\gamma(c)) = u_\alpha(c), \qquad \sigma(u_\delta(c)) = u_\delta(c).$$

Then $C_{D_4}(\sigma)$ is a simple group of type G_2. In fact, $\tilde{T} = C_T(\sigma)$ is a maximal torus of $C_{D_4}(\sigma)$, and $\tilde{\alpha} = \alpha|_{\tilde{T}} = \beta|_{\tilde{T}} = \gamma|_{\tilde{T}}$ and $\tilde{\beta} = \delta|_{\tilde{T}}$ form a pair of simple roots with respect to $\tilde{T}$, with corresponding root groups given by $u_{\tilde{\alpha}}(c) = u_\alpha(c)u_\beta(c)u_\gamma(c)$, $u_{\tilde{\beta}}(c) = u_\delta(c)$. Since λ evaluates in $\tilde{T}$, we may regard it as an element of $Y(\tilde{T})$; we denote this element by $\tilde{\lambda}$. We have $\langle \tilde{\alpha}, \tilde{\lambda} \rangle = 0$, $\langle \tilde{\beta}, \tilde{\lambda} \rangle = 1$.

We begin with a detailed description of triality in the particular case where the ground field has characteristic three, using the results of [6] and [7].

Proposition 4.1. *Assume that $p = 3$. In $\mathrm{Aut}(D_4)$ there are exactly two conjugacy classes of cyclic groups of order three generated by outer automorphisms. Let $\langle \sigma_1 \rangle$, $\langle \sigma_2 \rangle$ be representatives of the respective classes, and let $M_i = C_{D_4}(\sigma_i)$ ($i = 1, 2$). Then we may choose the labelling such that the following holds:*

(i) *M_1 is a simple group of type G_2; moreover $\mathrm{Aut}(D_4) \cdot \sigma_1$, the orbit of σ_1 under conjugation, is closed in $\mathrm{Aut}(D_4)$.*
(ii) *M_2 is an 8-dimensional group with 5-dimensional unipotent radical and corresponding reductive quotient isomorphic to SL_2; the orbit $\mathrm{Aut}(D_4) \cdot \sigma_2$ is not closed and contains σ_1 in its closure.*

(iii) *We may take* $\sigma_1 = \sigma$.
(iv) *Let* $u = u_{\alpha+\beta+\gamma+2\delta}(1)$. *Then we may take* $\sigma_2 = \sigma u$.
(v) *With the choices in (iii) and (iv), we have* $M_2 = C_{M_1}(u) = \langle U_{\tilde\alpha}, U_{-\tilde\alpha}\rangle \ltimes R_u(P_{\tilde\lambda}) \subseteq P_{\tilde\lambda}$, *where* $P_{\tilde\lambda}$ *denotes* $P_\lambda(M_1)$.

Proof. By [6, Cor. 6.5, Thm. 9.1], there are precisely two conjugacy classes of cyclic groups of order three generated by outer automorphisms, which are denoted by type I and type II, respectively. They are distinguished by the structure of the corresponding fixed point groups, where type I yields a group of type G_2, whereas type II in characteristic 3 gives a group with the structure described in (ii) (see [6, §9] together with [7, Thm. 7]). This implies the first statements of (i) and (ii), as well as (iii).

Working in the algebraic group $\mathrm{Aut}(D_4)$, using $\sigma(\lambda) = \lambda$ and $\langle \alpha+\beta+\gamma+2\delta, \lambda\rangle = 2 > 0$ we compute that

$$\lim_{a\to 0} \lambda(a)\sigma u\lambda(a)^{-1} = \sigma \lim_{a\to 0} \lambda(a)u\lambda(a)^{-1} = \sigma,$$

which proves that σ is contained in the closure of the orbit through σu. By [5, Thm. 3.3], to show that σ and σu are not conjugate it is enough to show that they are not D_4-conjugate. This non-conjugacy follows from [16, §I, Prop. 3.2], as σ and σu are given in *loc. cit.* as class representatives for distinct unipotent classes in σD_4. Moreover, by [16, §II, Lem. 1.15], the orbit through σ is closed. This proves (iv) and the remaining assertions of (i) and (ii).

To prove (v), we first note that $R_u(P_{\tilde\lambda})$ consists of the root groups for the roots $\tilde\beta, \tilde\alpha+\tilde\beta$, $2\tilde\alpha+\tilde\beta$, $3\tilde\alpha+\tilde\beta$ and $3\tilde\alpha+2\tilde\beta$. In particular, the semi-direct product $\langle U_{\tilde\alpha}, U_{-\tilde\alpha}\rangle \ltimes R_u(P_{\tilde\lambda})$ has dimension 8 and is contained in $C_{M_1}(u) = C_{M_1}(u_{3\tilde\alpha+2\tilde\beta}(1))$. As $\tilde\lambda$ centralizes $\pm\tilde\alpha$, the semi-direct product is also contained in $P_{\tilde\lambda}$. Since clearly $C_{M_1}(u) \subseteq M_2$, the assertion (v) follows by comparing dimensions. This finishes the proof. □

We can now characterize G-complete reducibility in the case where $G^0 = D_4$ and H maps isomorphically onto the full group of outer automorphisms of D_4. The following result is the analogue of Corollary 3.2.

Theorem 4.2. *Let* H *be a subgroup of* G. *Assume that* $G^0 = D_4$ *and* $H \cong S_3$ *acts by outer automorphisms on* G^0. *Then* H *is* G*-completely reducible if and only if* $C_{G^0}(H)$ *is reductive.*

Proof. The forward implication is clear by Lemma 2.11. Conversely, assume that H is not G-cr. Let $h \in H$ be an element of order 3, so that $K = \langle h\rangle$ is a normal subgroup of index 2 in H. By the assumption on H, the map $\pi : G \to \mathrm{Aut}(D_4), g \mapsto \mathrm{Int}(g)|_{G^0}$ is surjective. Since $\ker(\pi) = C_G(G^0)$, π is an isogeny. Hence $\pi(H)$ is not $\mathrm{Aut}(D_4)$-cr, by Lemma 2.9. It now follows from Remark 2.15 that we can take G to be $\mathrm{Aut}(D_4)$.

First assume that $p = 3$. Let $M_1 = C_{D_4}(\sigma)$ and $M_2 = C_{D_4}(\sigma u)$ with notation as in Proposition 4.1. Then K is a normal subgroup of order 3 and index 2 in H. Since $p = 3$ is coprime to 2, we have by Theorem 2.8(ii) that K is not $\mathrm{Aut}(D_4)$-cr. This implies (by Theorem 2.6) that the orbit $\mathrm{Aut}(D_4)\cdot h$ is not closed, whence by Proposition 4.1 there exists $g \in G$ with $ghg^{-1} = \sigma u$. Replacing H with gHg^{-1}, we may assume that $h = \sigma u$. Let $s \in H$ be an element of order 2 such that h and s generate H. Let $\tau \in \mathrm{Aut}(D_4)$ be the graph automorphism of order 2 determined by s, i.e., the graph automorphism that induces the same element as s in $\mathrm{Out}(D_4)$. Let $t = \tilde\beta^\vee(-1) \in \tilde T \subseteq M_1$. Since τ and σ fix M_1, both elements commute with t and u. Moreover, by construction $tut = u^{-1}$. This implies that τt has order 2 and $(\tau t)(\sigma u)(\tau t) = \tau\sigma\tau u^{-1} = \sigma^{-1}u^{-1} = (\sigma u)^{-1}$. As τt induces the same element as s in $\mathrm{Out}(D_4)$, we can find $x \in D_4$ with $s = \tau t x$. We conclude that both pairs of elements $h = \sigma u, s = \tau t x$ as well as $\sigma u, \tau t$ generate a group isomorphic to S_3. In particular, $x(\sigma u)x^{-1} = (\tau t)(\tau t x)(\sigma u)(\tau t x)^{-1}(\tau t) = (\tau t)(\sigma u)^{-1}(\tau t) = \sigma u$. Thus $x \in M_2 \subseteq M_1$ (cf. Proposition 4.1(v)), so that $s = \tau t x$ normalizes M_1. As M_1 is simple of type G_2, it has no outer automorphisms. Therefore we may find $s' \in M_1$ with $\mathrm{Int}(s)|_{M_1} = \mathrm{Int}(s')|_{M_1}$. Since M_1 is adjoint, s' is of order 2. Now

$$C_{G^0}(H) = C_{M_2}(s'). \tag{4.3}$$

Since $s = \tau t x$ normalizes M_2, s normalizes $N_{M_1}(M_2) = P_{\tilde{\lambda}}$ (see Proposition 4.1(v)). Hence $s' \in P_{\tilde{\lambda}}$ and $C_{M_2}(s') \subseteq P_{\tilde{\lambda}}$. Up to conjugation in $P_{\tilde{\lambda}}$ we may thus assume $s' \in \tilde{T}$. As $\tilde{T}$ is generated by the images of $\tilde{\alpha}^\vee$ and $\tilde{\beta}^\vee$, this reduces the possibilities to $s' \in \{\tilde{\alpha}^\vee(-1), \tilde{\beta}^\vee(-1), \tilde{\alpha}^\vee(-1)\tilde{\beta}^\vee(-1)\}$. But then s' centralizes $U_{3\tilde{\alpha}+2\tilde{\beta}}$, or $U_{\tilde{\beta}}$, or $U_{\tilde{\alpha}+\tilde{\beta}}$ respectively. We deduce that $C_{M_2}(s') \cap R_u(P_{\tilde{\lambda}})$ is positive-dimensional in each case. Thus $C_{M_2}(s')$ is not reductive, as required.

Now let $p \neq 3$. Then the subgroup K of H of order 3 is linearly reductive, in particular it is G-cr and $C_{G^0}(K)$ is reductive. Moreover, the group $C_{G^0}(K)$ is connected being the fixed point group under a triality automorphism (cf. [6, §9]). Let $M = HC_{G^0}(K)$. By [3, Thm. 3.1(b)(ii)] applied to $K \subseteq H \subseteq M$, we deduce that H is not M-cr. Since K is normal in M, by Theorem 2.8(i), H/K is not M/K-cr. But H/K is cyclic of order 2, so we may apply Corollary 3.2 to conclude that $C_{(M/K)^0}(H/K)$ is not reductive. By construction, $(M/K)^0 \cong C_{G^0}(K)$ and

$$C_{(M/K)^0}(H/K) \cong C_{G^0}(H). \tag{4.4}$$

This finishes the proof. ∎

Having settled the case of D_4, we can combine Corollary 3.2 and Theorem 4.2 to characterize G-complete reducibility in case G^0 is simple and the subgroup H acts by outer automorphisms.

Corollary 4.5. *Let $H \subseteq G$ be a subgroup acting on G^0 by outer automorphisms. Assume that G^0 is simple. Then H is G-completely reducible if and only if $C_{G^0}(H)$ is reductive.*

Proof. We may assume that G^0 is adjoint (cf. the first paragraph of the proof of Theorem 4.2). Since H acts via outer automorphisms, we may identify it as an abstract group with a subgroup of $\mathrm{Out}(G^0)$, the finite group of outer automorphisms of G^0. As G^0 is simple, $\mathrm{Out}(G^0)$ is either simple of prime order or G^0 is of type D_4 and $\mathrm{Out}(G^0) \cong S_3$. The result now follows from Corollary 3.2 and Theorem 4.2. ∎

Remark 4.6. In the situation of Corollary 4.5, we always have $\dim C_{G^0}(H) > 0$. This follows again from the theorem of Steinberg ([17, Thm. 10.13]) in case H is cyclic. The general case follows from the identities for $C_{G^0}(H)$ in (4.3) and (4.4).

5. The Algorithm

We return to the general situation where $H \subseteq G$ is a subgroup of a possibly non-connected reductive group. In this section, we are going to establish an algorithm that reduces the question of whether H is G-cr to the question of whether certain subgroups of certain *connected* reductive groups are G-completely reducible.

We start with a proposition that recasts some of our earlier results as operations for a potential algorithm. We say that the pair (H, G) is *completely reducible* provided that H is G-completely reducible.

Proposition 5.1. *Let (H, G) be a pair consisting of a reductive group G and a subgroup $H \subseteq G$. Then each of the following operations replaces (H, G) with pairs of the same form (i.e., consisting of a group and a reductive group containing it as a subgroup):*

(O1) *Let $\pi : G \to G/C_G(G^0)$ be the canonical projection. Replace (H, G) with $(\pi(H), \pi(G))$.*

(O2) *Let $G^0 = SG_1 \cdots G_n$ be the decomposition as in (2.16), and let H_i, π_i for $1 \leq i \leq n$ be defined as in Lemma 2.17. Replace (H, G) with the pairs*
$$(\pi_1(H_1), \pi_1(H_1G^0)), \ldots, (\pi_n(H_n), \pi_n(H_nG^0)).$$

(O3) *If $H \cap G^0$ is G^0-completely reducible, replace (H, G) with*
$$(H/(H \cap G^0), HC_{G^0}(H \cap G^0)/(H \cap G^0)).$$

Moreover, H is G-completely reducible if and only if each of the pairs obtained through one of these operations is completely reducible.

Proof. The results follow from Corollary 2.10, Lemma 2.17 and Remark 2.13. □

Remark 5.2. In the situation of (O2), suppose we are given a set I as in Lemma 2.17. Then it is enough to replace (H, G) with the pairs $(\pi_i(H_i), \pi_i(H_iG^0))_{i \in I}$ in (O2).

We are now in a position to give an algorithm that determines whether H is G-completely reducible.

Theorem 5.3. *Let H be a subgroup of G. The following algorithm, starting with the pair (H, G), reduces the question of whether H is G-completely reducible in a finite number of steps to questions of complete reducibility in connected groups:*

Algorithm. *Input: a pair (H', G'), where H' is a subgroup of a reductive group G'.*

Step 1 *If G'^0 is not simple, or $C_{G'}(G'^0) \neq 1$, apply (O2) and then (O1) to each of the newly obtained pairs. Restart instances of the algorithm for each of the new pairs. Then H' is G'-cr if and only if each of these pairs turns out to be completely reducible.*

Step 2 *Identify $H'/(H' \cap G^0)$ with a subgroup of $\mathrm{Out}(G'^0)$. Let $n \in \{1, 2, 3, 6\}$ be the order of $H'/(H' \cap G'^0)$. If p does not divide n, H' is G'-cr if and only if $H' \cap G'^0$ is G'^0-cr, and the algorithm stops.*

Step 3 *If $H' \cap G'^0$ is not G'^0-cr, the algorithm stops with the conclusion that H' is not G'-cr.*

Step 4 *If $H' \cap G'^0 = 1$, H' is G'-cr if and only if $C_{G'^0}(H')$ is reductive. The algorithm stops.*

Step 5 *If $1 \neq H'/(H' \cap G'^0) \ncong S_3$, let $M = H'C_{G'^0}(H' \cap G'^0)/(H' \cap G'^0)$. Then H' is G'-cr if and only if $C_{M^0}(H'/(H' \cap G'^0))$ is reductive. The algorithm stops.*

Step 6 *If $H'/(H' \cap G'^0) \cong S_3$, apply (O3) and restart the algorithm with the new pair.*

Proof. Step 1 is covered by Proposition 5.1. Moreover, each of the new pairs (H'', G'') produced in this step satisfies G''^0 simple and $C_{G''}(G''^0) = 1$, so that the algorithm moves on to Step 2 after a possible application of Step 1.

From Step 2 on, we may assume that G'^0 is simple and $C_{G'}(G'^0) = 1$. This allows us to identify $H'/(H' \cap G'^0)$ with a subgroup of $\mathrm{Out}(G'^0)$, and yields the constraints on its order. If p does not divide n, the quotient $H'/(H' \cap G'^0)$ is linearly reductive. By Theorem 2.8(ii), H' is indeed G'-cr if and only if $H' \cap G'^0$ is G'^0-cr.

From Step 3 on, we may assume in addition that $p \in \{2,3\}$ and that H is not contained in G^0. The conclusion of Step 3 is correct by Lemma 2.11(i).

Step 4 is an application of Corollary 4.5.

Since we have passed Step 3, we may assume that $H' \cap G'^0$ is G'^0-cr. Under the condition of Step 5, $H'/(H' \cap G'^0)$ is cyclic. The conclusion of Step 5 thus follows from Remark 2.13 and Corollary 3.2.

Finally, Step 6 is again covered by Proposition 5.1. Moreover, this step is only applicable for G'^0 simple of type D_4. As we may assume $H' \cap G'^0 \neq 1$ and $Z(G'^0) = 1$, the group $M = H'C_{G'^0}(H' \cap G'^0)/(H' \cap G'^0)$ featuring in (O3) satisfies $\dim M < \dim G'$.

It remains to show that the algorithm terminates. Step 1 may restart finitely many instances of the algorithm. In each instance the algorithm terminates in Step 2–Step 5 if Step 6 is not reached. If Step 6 is applicable, it replaces G'—which is simple of type D_4—with a group of smaller dimension. This implies that after Step 1 is applied again, Step 6 cannot be reached a second time, and the algorithm terminates. □

Remark 5.4. (i). It follows from the proof of Theorem 5.3 that Step 1, the only step that replaces a pair with several new pairs, need only be done at most twice along a path through the algorithm. Also, Step 6 only occurs at most once.

(ii). There are some situations where shortcuts may be applied to reduce to a connected group. First of all, if H^0 is not reductive, then H cannot be G-cr. On the other hand, if H is cyclic, then we may apply Corollary 3.2 to deduce that H is G-cr if and only if $C_{G^0}(H)$ is G^0-cr. Finally, if $H/(H \cap G^0)$ is linearly reductive, we can apply Theorem 2.8(ii) to deduce that H is G-cr if and only if $H \cap G^0$ is G^0-cr. However, the proposed algorithm gives a systematic approach that deals with all possible cases.

(iii). If $p = 0$, then a subgroup H is G-cr if and only if it is reductive ([15, Prop. 4.2]). Of course, H is reductive if and only if H^0 is reductive, which in turn is equivalent to H^0 being G^0-completely reducible.

6. Examples

We conclude with some examples of the algorithm outlined in Theorem 5.3.

Example 6.1. Let $p = 3$, $G = \mathrm{Aut}(D_4)$. Let σ be the triality graph automorphism as in Section 4. Let $H = \langle\sigma\rangle K$, where $K = C_{D_4}(\sigma)$ is the fixed point subgroup of type G_2. We follow through the algorithm to deduce that H is G-cr:

Step 1 is not applicable, as $G^0 = D_4$ is simple and $C_G(G^0) = 1$. In Step 2 we obtain $n = 3 = p$ as the order of $\langle\sigma\rangle \cong H/(H \cap G^0)$. Now $H \cap G^0 = K$ is G-cr (see Corollary 3.3), hence Steps 3 and 4 are not applicable. Step 5 applies and leads us to consider the group $M = HC_{D_4}(K)/K \cong \langle\sigma\rangle C_{D_4}(K)$. As K is adjoint, we obtain $C_{M^0}(\sigma) = 1$ and thus this group is clearly reductive. The algorithm stops with the conclusion that H is G-cr.

Here we have two commuting G-cr subgroups $\langle\sigma\rangle$ and K of G and their product is also G-cr. This is not always the case: see [3, Ex. 5.1].

Example 6.2. Let Γ be a finite group acting transitively on a finite set I. Let $i_0 \in I$. Let $\rho : \Gamma \to M$ be a homomorphism to a simple group M such that $\rho(C_\Gamma(i_0))$ is not M-cr. We set

$$G = \Gamma \ltimes \prod_{i \in I} M,$$

where Γ acts on the product by permuting the indices. Clearly, $G^0 = \prod_i M$. Let $d : M \to G^0$ be the diagonal embedding. As Γ commutes with the image of d, we may form the subgroup

$$H = \{\gamma d(\rho(\gamma)) \mid \gamma \in \Gamma\},$$

a finite subgroup of G. We claim that H is not G-cr, and use our algorithm to prove it.

To apply Step 1, we compute that for $i \in I$, $N_G(G_i) = C_\Gamma(i) \ltimes G^0$. In particular, $H_i = \{\gamma d(\rho(\gamma)) \mid \gamma \in C_\Gamma(i)\}$. We obtain

$$\pi_i(H_i G^0) = H_i G^0 / \prod_{j \neq i} G_j \cong C_\Gamma(i) \times M =: G'_i,$$

and correspondingly

$$\pi_i(H_i) \cong \{\gamma\rho(\gamma) \mid \gamma \in C_\Gamma(i)\} =: H'_i.$$

Since the action of Γ on I is transitive, we may by Remark 5.2 replace (H, G) with (H', G'), where $G' = G'_{i_0}$, $H' = H'_{i_0}$.

Now $C_{G'}(G'^0) = C_\Gamma(i_0)$, so Step 1 is again applicable and replaces (H', G') with the pair (H'', M), where

$$H'' = \rho(C_\Gamma(i_0)).$$

By assumption, H'' is not M-cr and hence H is not G-cr, by Step 2.

As a concrete realisation of this example, take $\Gamma = \mathrm{PGL}_2(q)$ for q a sufficiently large power of p, $I = \mathrm{PGL}_2(q)/B(q)$ where B is a Borel subgroup of PGL_2, and consider I as a transitive Γ-set by left translation. Let $M = \mathrm{PGL}_2$, and let $\rho : \Gamma \hookrightarrow M$ be the canonical embedding. If we take $i_0 = B(q)$, then $C_\Gamma(i_0) = B(q)$ is not M-cr, for q large enough, as B is not M-cr. In this example, we have $H \cap G^0 = 1$ and $C_{G^0}(H) = 1$ (as $C_M(B(q)) = 1$ for q sufficiently large). In particular, this example shows that H may fail to be G-cr even if $C_{G^0}(H)$ and $H \cap G^0$ both are G^0-cr.

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Department of Mathematics, University of York, York YO10 5DD, United Kingdom
E-mail address: `michael.bate@york.ac.uk`

Fachbereich Mathematik, Technische Universität Kaiserslautern, Postfach 3049, 67653 Kaiserslautern, Germany
E-mail address: `herpel@mathematik.uni-kl.de`

Department of Mathematics, University of Auckland, Private Bag 92019, Auckland 1142, New Zealand
E-mail address: `Ben.Martin@auckland.ac.nz`

Fakultät für Mathematik, Ruhr-Universität Bochum, D-44780 Bochum, Germany
E-mail address: `gerhard.roehrle@rub.de`