

# On Unipotent Radicals of Pseudo-Reductive Groups

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ABSTRACT. We establish some results on the structure of the geometric unipotent radicals of pseudo-reductive  $k$ -groups. In particular, our main theorem gives bounds on the nilpotency class of geometric unipotent radicals of standard pseudo-reductive groups, which are sharp in many cases. A major part of the proof rests upon consideration of the following situation: let  $k'$  be a purely inseparable field extension of  $k$  of degree  $p^e$ , and let  $G$  denote the Weil restriction of scalars  $\mathbf{R}_{k'/k}(G')$  of a reductive  $k'$ -group  $G'$ . When  $G = \mathbf{R}_{k'/k}(G')$ , we also provide some results on the orders of elements of the unipotent radical  $\mathcal{R}_u(G_{\bar{k}})$  of the extension of scalars of  $G$  to the algebraic closure  $\bar{k}$  of  $k$ .

## 1. Introduction

Let  $G$  be a smooth affine algebraic  $k$ -group over an arbitrary field  $k$ . Then  $G$  is said to be *pseudo-reductive* if  $G$  is connected and the largest  $k$ -defined connected smooth normal unipotent subgroup  $\mathcal{R}_{u,k}(G)$  of  $G$  is trivial. Tits [Tit91/92; Tit92/93] introduced pseudo-reductive groups to the literature some time ago in a series of courses at the Collège de France, but they have resurfaced rather dramatically in recent years thanks to the monograph [CGP15], many of whose results were used in B. Conrad’s proof of the finiteness of the Tate–Shafarevich sets and Tamagawa numbers of arbitrary linear algebraic groups over global function fields [Con12, Thm. 1.3.3]. The main result of that monograph is [CGP15, Thm. 5.1.1], which says that unless one is in some special situation over a field of characteristic 2 or 3, then any pseudo-reductive group is *standard*. This means that it arises after a process of modification of a Cartan subgroup of a certain Weil restriction of scalars of a given reductive group (we assume that reductive groups are connected). More specifically, a standard pseudo-reductive group  $G$  can be expressed as a quotient group of the form

$$G = (\mathbf{R}_{k'/k}(G') \rtimes C) / \mathbf{R}_{k'/k}(T')$$

corresponding to a 4-tuple  $(G', k'/k, T', C)$ , where  $k'$  is a nonzero finite reduced  $k$ -algebra,  $G'$  is a  $k'$ -group with reductive fibers over  $\text{Spec } k'$ ,  $T'$  is a maximal  $k'$ -torus of  $G'$ , and  $C$  is a commutative pseudo-reductive  $k$ -group occurring in a

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factorization

$$\mathbf{R}_{k'/k}(T') \xrightarrow{\phi} C \xrightarrow{\psi} \mathbf{R}_{k'/k}(T'/Z_{G'}) \tag{*}$$

of the natural map  $\varpi : \mathbf{R}_{k'/k}(T') \rightarrow \mathbf{R}_{k'/k}(T'/Z_{G'})$ . Here  $Z_{G'}$  is the (scheme-theoretic) center of  $G'$  (note that  $\varpi$  is *not* surjective when  $Z_{G'}$  has nonétale fiber at a factor field  $k'_i$  of  $k'$  that is not separable over  $k$ ), and  $C$  acts on  $\mathbf{R}_{k'/k}(G')$  via  $\psi$  followed by the functor  $\mathbf{R}_{k'/k}$  applied to the conjugation action of  $T'/Z_{G'}$  on  $G'$ ; we regard  $\mathbf{R}_{k'/k}(T')$  as a central subgroup of  $\mathbf{R}_{k'/k}(G') \rtimes C$  via the map  $h \mapsto (i(h)^{-1}, \phi(h))$ , where  $i$  is the natural inclusion of  $\mathbf{R}_{k'/k}(T')$  in  $\mathbf{R}_{k'/k}(G')$ .

The structure of general connected linear algebraic groups over perfect fields  $k$  is well understood: the geometric unipotent radical  $\mathcal{R}_u(G_{\bar{k}}) = \mathcal{R}_{u,\bar{k}}(G_{\bar{k}})$  descends to a subgroup  $\mathcal{R}_u(G)$  of  $G$ , and the quotient  $G^{\text{red}} := G/\mathcal{R}_u(G)$  is reductive. If we further insist  $k$  be separably closed one even has that  $G^{\text{red}}$  is split, so  $G^{\text{red}}$  is the central product of its semisimple derived group  $\mathcal{D}(G)$  and a central torus with  $G^{\text{red}}/Z_G$  semisimple—in fact, the direct product of its simple factors.

Most of this theory goes wrong over imperfect fields  $k$ , hence in particular the need to consider pseudo-reductive groups, whose geometric unipotent radicals may not be defined over  $k$ . To understand the structure of a given smooth affine algebraic group  $G$  over  $k$ , it is therefore instructive to extend scalars to the (perfect) algebraically closed field  $\bar{k}$  and analyze the structure of  $G_{\bar{k}}$ , where, for example, we see the full unipotent radical. We pursue this approach in this paper and discuss the structure of  $G_{\bar{k}}$  where  $G$  is a standard pseudo-reductive group arising from a 4-tuple  $(G', k'/k, T', C)$ . The reductive part  $G_{\bar{k}}^{\text{red}} = G_{\bar{k}}/\mathcal{R}_u(G_{\bar{k}})$  is not especially interesting in that the universal property of Weil restriction implies that  $G_{\bar{k}}^{\text{red}}$  has the same root system as  $G'$ . Further results [CGP15, Thm. 3.4.6, Cor. A.5.16] even furnish us, under some restrictive conditions, with a Levi subgroup for  $G$ , a smooth subgroup  $H$  such that  $H_{\bar{k}}$  is a complement to the geometric unipotent radical  $\mathcal{R}_u(G_{\bar{k}})$ . However, the precise structure of  $\mathcal{R}_u(G_{\bar{k}})$  is rather more mysterious. Although we know that there is a composition series of  $\mathcal{R}_u(G_{\bar{k}})$  whose composition factors are related to the adjoint  $G'_{\bar{k}}$ -module  $\mathfrak{g}' = \text{Lie}(G'_{\bar{k}})$  (see Lemma 3.1), it is unclear what the structure of  $\mathcal{R}_u(G_{\bar{k}})$  is *qua* group. Since  $\mathcal{R}_u(G_{\bar{k}})$  is a  $p$ -group, we may consider some standard invariants, which measure the order of its elements and the extent to which it is non-abelian. One major purpose of this paper is to show that as soon as the root system associated with  $G$  is nontrivial and  $k'/k$  is finite and inseparable, then  $\mathcal{R}_u(G_{\bar{k}})$  is almost always non-abelian (in a way that depends on the characteristic of the field amongst other things).

To state our results, let  $(G', k'/k, T', C)$  be a 4-tuple giving rise to a pseudo-reductive group  $G$  and denote the separable closure of  $k$  by  $k_s$ . Then  $k'_s = k' \otimes_k k_s$  is a finite nonzero reduced  $k_s$ -algebra, isomorphic to a product of factor fields  $\prod_i k_i$ , where each  $k_i/k_s$  is a purely inseparable extension. Let  $G'_i$  (resp.,  $T_i$ ) denote the fiber of  $G'_{k'_s}$  (resp.,  $T'_{k'_s}$ ) over  $k_i$ . By standard results,  $\mathbf{R}_{k'/k}(G')_{k_s} \cong \mathbf{R}_{k'_s/k_s}(G')$  (see Lemma 2.1). If

$$p = 2 \quad \text{and} \quad G'_i \cong \text{SL}'_2 \times S'$$

for  $S'$  a torus and  $r > 0$ , then we say  $G'_i$  is *unusual*. We define a number  $\ell_i$  depending on  $G'_i$  and  $k_i$ . First, if  $G'_i$  is commutative, then set  $\ell_i = 1$ . Now, for each field  $k_i$ , form the  $k_s$ -algebra  $B_i = k_i \otimes_{k_s} k_i$ . Each  $B_i$  is a local  $k_i$ -algebra, which by virtue of the pure inseparability of  $k_i/k_s$  means that its maximal ideal,  $\mathfrak{m}_i$  say, consists of nilpotent elements. Hence there is a minimal  $n$  such that  $\mathfrak{m}_i^n = 0$ ; in case  $G'_i$  is not unusual, let  $\ell_i := n - 1$ . In case  $G'_i$  is unusual, define  $\ell_i$  to be the minimal  $n$  such that  $({}^2\mathfrak{m}_i)^{n-1} \cdot \mathfrak{m}_i^2 = 0$ , where  $({}^2\mathfrak{m}_i)$  is the ideal of squares in  $\mathfrak{m}_i$ . Finally, define the integer

$$N := \max_i \{\ell_i\}.$$

**THEOREM 1.1.** *Let  $G$  be a noncommutative standard pseudo-reductive group over a field  $k$  of characteristic  $p$ , arising from a 4-tuple  $(G', k'/k, T', C)$  with  $k'$  a nonzero finite reduced  $k$ -algebra. Let the  $G'_i$  (resp., the  $T'_i$ ) be the fibers of  $G'$  (resp.,  $T'$ ) over the factor fields of  $k_s$ , as before. Suppose that for every unusual fiber  $G'_i$ , the map  $\phi$  in the factorization  $(*)$  is an injection on restriction to  $T'_i \cap G'_i$ . Then the nilpotency class of  $\mathcal{R}_u(G_{\bar{k}})$  is  $N$ .*

What the theorem indicates is that the precise structure of  $\mathcal{R}_u(G_{\bar{k}})$ , including its nilpotency class, appears to depend in a very particular way on the nature of the extension  $k'/k$  used in Weil restriction, rather than on, for instance, the reductive group  $G'$  (provided that the reductive group contains at least one fiber that is not unusual or commutative).

**REMARKS 1.2.** (i) There is one set of cases we have excluded. After passing to an appropriate direct factor of a fiber of  $G'$  over  $k'$ , this happens when  $G$  arises from a 4-tuple  $(G', k'/k, T', C)$ , where  $G' = \text{SL}_2$ . The kernel of the map  $\text{R}_{k'/k}(T') \rightarrow \text{R}_{k'/k}(T'/Z_{G'})$  is  $\text{R}_{k'/k}(\mu_2)$  (see [CGP15, Prop. 1.3.4]). For example, let us suppose in the factorization  $\text{R}_{k'/k}(T') \xrightarrow{\phi} C \xrightarrow{\psi} \text{R}_{k'/k}(T'/Z_{G'})$  that  $\ker(\phi) = \text{R}_{k'/k}(\mu_2)$  and  $\psi$  is injective. Then we may regard  $C$  as a subgroup of  $\text{R}_{k'/k}(T'/Z_{G'})$  sitting in between  $\text{R}_{k'/k}(T')/\text{R}_{k'/k}(\mu_2)$  and  $\text{R}_{k'/k}(T'/Z_{G'})$ . If  $C = \text{R}_{k'/k}(T')/\text{R}_{k'/k}(\mu_2)$ , then  $G \cong \text{R}_{k'/k}(\text{SL}_2)/\text{R}_{k'/k}(\mu_2)$ , and the nilpotency class of the unipotent radical of  $G_{\bar{k}}$  is at most the minimal  $n$  such that  $({}^2\mathfrak{m}_i)^{n-1} \cdot \mathfrak{m}_i^2 = 0$  (see Remark 3.8)—in other words, it is bounded above by the nilpotency class of the unipotent radical of  $\text{R}_{k'/k}(\text{SL}_2)_{\bar{k}}$ . If  $C = \text{R}_{k'/k}(T'/Z_{G'})$ , then  $G \cong \text{R}_{k'/k}(\text{PGL}_2)$  (cf. Remark 3.17), so the unipotent radical of  $G_{\bar{k}}$  has nilpotency class  $n - 1$  for the minimal  $n$  such that  $\mathfrak{m}^n = 0$ . These numbers are usually different. In between these two extremes, there is a range of possibilities: there may be many subgroups  $C$  sitting in between  $\text{R}_{k'/k}(T')/\text{R}_{k'/k}(\mu_2)$  and  $\text{R}_{k'/k}(T'/Z_{G'})$ —note that the quotient  $\text{R}_{k'/k}(T'/Z_{G'})/(\text{R}_{k'/k}(T')/\text{R}_{k'/k}(\mu_2))$  is a smooth commutative unipotent group—and we have, a priori, no control over these.

(ii) Given any standard pseudo-reductive group  $G$ , by [CGP15, Thm. 4.1.1], we may find a *standard presentation* for it; that is, a 4-tuple  $(G', k'/k, T', C)$  giving rise to  $G$  through the standard construction such that  $G'$  has absolutely simple, simply-connected fibers over  $\text{Spec } k'$ . In this case, we may simplify the

statement of the theorem somewhat, since then a fiber of  $G'_i$  of  $G'$  is unusual precisely when  $(G'_i)_{k_s} \cong \mathrm{SL}_2$  and  $p = 2$ .

To prove the theorem, we reduce to the case that  $G$  is a Weil restriction, but in order to deal with cases where the center  $Z_{G'}$  of  $G'$  is not smooth, we show how to present  $G$  through the standard construction, starting from a central extension  $\widehat{G}'$  of  $G'$  such that the center of  $\widehat{G}'$  is smooth (see Section 3.5). This may be of independent interest.

In Section 4, for  $G$  of the form  $R_{k'/k}(G')$ , we consider another invariant of  $\mathcal{R}_u(G_{\bar{k}})$  usually attached to  $p$ -groups—the *exponent*, that is, the minimal integer  $s$  such that the  $p^s$ -power map on  $\mathcal{R}_u(G_{\bar{k}})$  factors through the trivial group. For this invariant, we do not have as precise a result as we do for the nilpotency class, but we establish some upper and lower bounds and identify a possible form of a precise result; cf. Question 4.6.

Finally, let us remark that part of the proof of Theorem 1.1 reduces to some bare-hands calculations with matrices arising from the Weil restriction  $R_{k'/k}(\mathrm{GL}_2)$  and may be of interest to anyone who would like to see some examples of pseudo-reductive groups in an explicit description by matrices.

## 2. Notation and Preliminaries

We follow the notation of [CGP15]. In particular,  $k$  is always a field of characteristic  $p > 0$ . All algebraic groups are assumed to be affine and of finite type over the ground ring, and all subgroups are closed. Reductive and pseudo-reductive groups are assumed to be smooth and connected.

Let us recall the definition of Weil restriction and some relevant features. We consider algebraic groups scheme-theoretically, so that an algebraic  $k$ -group  $G$  is a functor  $\{k\text{-algebras} \rightarrow \text{groups}\}$ , which is representable via a finitely presented  $k$ -algebra  $k[G]$ . Let  $k'$  be a nonzero finite reduced  $k$ -algebra. Then for any smooth  $k'$ -group  $G'$  with connected fibers over  $\mathrm{Spec} k'$ , the Weil restriction  $G = R_{k'/k}(G')$  is a smooth connected  $k$ -group of dimension  $[k' : k] \dim G'$ , characterized by the property  $G(A) = G'(k' \otimes_k A)$  functorially in  $k$ -algebras  $A$ . If  $H'$  is a subgroup of  $G'$ , then  $R_{k'/k}(H')$  is a subgroup of  $R_{k'/k}(G')$ . For a thorough treatment of this, we refer to [CGP15, A.5]. Important for us is the fact that Weil restriction is right adjoint to base change; that is, we have a bijection

$$\mathrm{Hom}_k(M, R_{k'/k}(G')) \cong \mathrm{Hom}_{k'}(M_{k'}, G')$$

natural in the  $k$ -group scheme  $M$ . We need two particular cases. First, if  $M = G = R_{k'/k}(G')$ , then the identity morphism  $G \rightarrow G$  corresponds to a map  $q_{G'} : G_{k'} \rightarrow G'$ . When  $k'$  is a finite purely inseparable field extension of  $k$  and  $G'$  is reductive, then by [CGP15, Thm. 1.6.2],  $q_{G'}$  is smooth and surjective,  $\ker q_{G'}$  coincides with  $\mathcal{R}_{u,k'}(G_{k'})$ , and  $\mathcal{R}_{u,k'}(G_{k'})$  is a descent of  $\mathcal{R}_u(G_{\bar{k}})$  (so  $\mathcal{R}_u(G_{\bar{k}})$  is defined over  $k'$ ); it follows from the naturality of the maps concerned that if  $H'$  is a connected reductive subgroup of  $G'$ , then  $\ker q_{H'} \subseteq \ker q_{G'}$ , so  $\mathcal{R}_{u,k'}(H_{k'}) \subseteq \mathcal{R}_{u,k'}(G_{k'})$ . In particular,  $\dim \mathcal{R}_{u,k'}(G_{k'}) = ([k' : k] - 1) \dim G'$ . Second, if  $H$  is a reductive  $k$ -group and  $M = G' = H_{k'}$ , then the identity morphism  $H_{k'} \rightarrow H_{k'}$  corresponds to

a map  $i_H : H \rightarrow \mathbf{R}_{k'/k}(H_{k'})$ . When  $k'$  is a finite purely inseparable field extension of  $k$ , then by [CGP15, Cor. A.5.16] the composition  $H_{k'} \xrightarrow{(i_H)_{k'}} \mathbf{R}_{k'/k}(H_{k'}) \xrightarrow{q_{H_{k'}}} H_{k'}$  is the identity, so we may regard  $H$  as a subgroup of  $\mathbf{R}_{k'/k}(H_{k'})$  via  $i_H$ ; in fact,  $H$  is a Levi subgroup of  $\mathbf{R}_{k'/k}(H_{k'})$ .

Let  $k' = \prod_{i=1}^n k_i$  be a nonzero finite reduced  $k$ -algebra with factor fields  $k_i$ . Then any algebraic  $k'$ -group  $G'$  decomposes as a product  $G' = \prod_{i=1}^n G'_i$  where each  $G'_i$  is an algebraic  $k_i$ -group (it is the fiber of  $G'$  over  $\text{Spec } k_i$ ). In such a situation, we let  $\mathcal{R}_{u,k'}(G')$  denote the unipotent subgroup  $\prod_{i=1}^n \mathcal{R}_{u,k_i}(G'_i)$ . We have  $\mathbf{R}_{k'/k}(G') = \prod_{i=1}^n \mathbf{R}_{k_i/k}(G'_i)$ ; in particular,

$$\mathcal{R}_{u,k'}(\mathbf{R}_{k'/k}(G')_{k'}) = \prod_{i=1}^n \mathcal{R}_{u,k'}(\mathbf{R}_{k_i/k}(G'_i)_{k'}).$$

This allows us to reduce immediately to the case that  $k'$  is a field in the proof of Theorem 1.1.

Suppose now that  $H$  is a smooth connected algebraic group over  $k$ , and let  $[g, h] = ghg^{-1}h^{-1}$  denote the commutator of the elements  $g, h \in H(\bar{k})$ . Let  $M, K$  be smooth connected subgroups of  $H$ . Then the commutator subgroup  $[M, K]$  is a smooth connected subgroup of  $H$ , and we may identify  $[M, K](\bar{k})$  with the commutator subgroup  $[M(\bar{k}), K(\bar{k})]$ . Moreover,  $[M_{k'}, K_{k'}] = ([M, K])_{k'}$  for any field extension  $k'/k$  (cf. [Bor91, I.2.4]). In particular, we may form the lower central series  $\{\mathcal{D}_m(H)\}_{m \geq 0}$  in the usual way, and  $\mathcal{D}_m(H)(\bar{k})$  is the  $m$ th term in the lower central series for the abstract group  $H(\bar{k})$ . We say that  $H$  is *nilpotent* if there exists an integer  $m$  such that  $\mathcal{D}_m(H) = 1$ . The *nilpotency class*  $\text{cl}(H)$  of  $H$  is the smallest integer  $m$  such that  $\mathcal{D}_m(H) = 1$ . These arguments show that extending the base field does not change the nilpotency class of  $H$ .

In proving Theorem 1.1 and intermediate results, we usually want to reduce to the case that  $k$  is separably closed, guaranteeing that  $k_i/k$  is purely inseparable for  $k_i$  a factor field of  $k'$ . This also allows us to assume that the group  $G'$  has *split* reductive fibers. We denote by  $k_s$  the separable closure of  $k$  in its algebraic closure  $\bar{k}$ , and we set  $k'_s = k' \otimes_k k_s$ , a nonzero finite reduced  $k_s$ -algebra. Note that even when  $k'$  is a field,  $k'_s$  need not be a field.

LEMMA 2.1. *Let  $G$  be a standard pseudo-reductive group arising from  $(G', k'/k, T', C)$ . Then*

- (i)  $G_{k_s}$  is isomorphic to the standard pseudo-reductive group arising from  $(G'_{k'_s}, k'_s/k_s, T'_{k'_s}, C_{k_s})$ ;
- (ii)  $\mathcal{R}_{u,k'}(G_{k'})_{k'_s} = \mathcal{R}_{u,k'_s}(G_{k'_s})$ .

*Proof.* (i) We have  $G = (\mathbf{R}_{k'/k}(G') \rtimes C)/\mathbf{R}_{k'/k}(T')$ . To see that

$$G_{k_s} \cong (\mathbf{R}_{k'_s/k_s}(G'_{k'_s}) \rtimes C_{k_s})/\mathbf{R}_{k'_s/k_s}(T'_{k'_s}),$$

it suffices to see that: (a) the sequences

$$1 \rightarrow \mathbf{R}_{k'/k}(T') \rightarrow \mathbf{R}_{k'/k}(G') \rtimes C \rightarrow G \rightarrow 1$$

and

$$1 \rightarrow R_{k'/k}(G') \rightarrow R_{k'/k}(G') \rtimes C \rightarrow C \rightarrow 1$$

remain exact after taking the base change to  $k_s$ , with the second remaining split; and (b) the formation of the Weil restriction  $R_{k'/k}(G')$  commutes with base change to  $k_s$ .

These facts are standard: for (a), this follows since algebras over a field  $k$  are flat, and (b) is [CGP15, A.5.2(1)].

(ii) Without loss, we can assume that  $k'$  is a field. Write  $k'_s = \prod_{i \in I} k_i$ . Since

$$\mathcal{R}_{u,k'}(G_{k'})_{\bar{k}} = \mathcal{R}_u(G_{\bar{k}}) = \mathcal{R}_u((G_{k_s})_{\bar{k}}) = \mathcal{R}_u((G_{k_i})_{\bar{k}}) = \mathcal{R}_{u,k_i}(G_{k_i})_{\bar{k}},$$

we have  $\mathcal{R}_{u,k'}(G_{k'})_{k_i} = \mathcal{R}_{u,k_i}(G_{k_i})$  for each  $i \in I$ . The result now follows.  $\square$

### 3. Proof of Theorem 1.1

#### 3.1. Upper Bound

The following result provides an upper bound for the nilpotency class of  $\mathcal{R}_u(R_{k'/k}(G')_{\bar{k}})$  in case  $k'$  is a purely inseparable field extension. We will later reduce to this case.

PROPOSITION 3.1. *Let  $G = R_{k'/k}(G')$  for  $k'$  a purely inseparable field extension of  $k$ . Then  $B := k' \otimes_k k'$  is local with maximal nilpotent ideal  $\mathfrak{m}$ . Suppose  $n$  is minimal such that  $\mathfrak{m}^n = 0$ . Then  $\mathcal{R}_u(G_{\bar{k}})$  has a filtration  $\mathcal{R}_u(G_{\bar{k}}) = U_1 \supseteq U_2 \supseteq \dots \supseteq U_n = 1$  satisfying the following:*

- (a) *the successive quotients are  $G'_k$ -equivariantly isomorphic to a direct sum of copies of the adjoint  $G'_k$ -module;*
- (b) *the commutator subgroup  $[U_1, U_i] \subseteq U_{i+1}$  for any  $i \geq 1$ . In particular, we have  $\text{cl}(\mathcal{R}_u(G_{\bar{k}})) \leq n - 1$ .*

*Proof.* Following [Oes84, A.3.6], we can identify  $G_{k'} = R_{k'/k}(G')_{k'}$  with  $R_{B/k'}(G'_B)$  where  $B := k' \otimes_k k'$ . For the convenience of the reader, we reproduce some of the constructions from [Oes84, Sec. A.3.5] in this setting.

We begin with some basic observations and notation. The maximal ideal  $\mathfrak{m}$  of  $B$  is the kernel of the map  $B \rightarrow k'$  that sends  $b_1 \otimes b_2 \rightarrow b_1 b_2$ , so  $\mathfrak{m}$  is generated by elements of the form  $b \otimes 1 - 1 \otimes b$ . This is an ideal of nilpotent elements (everything is killed by a suitably high power of  $p$ ), so there is  $n \in \mathbb{N}$  such that  $\mathfrak{m}^n = \{0\}$ ; choose  $n$  minimal subject to this. Note that  $n \leq [k' : k]$  since  $B$  has dimension  $[k' : k]$  as a (left) vector space over  $k'$ , so this is the largest possible size for a strictly decreasing chain of proper subspaces of  $B$ . Denote the quotient  $B/\mathfrak{m}^i$  for  $1 \leq i \leq n$  by  $B_i$  and let  $G_i := R_{B_i/k'}(G'_{B_i})$ . Note that since  $B_1 = B/\mathfrak{m} \cong k'$ , we have that  $G_1 \cong G'$ , and since  $\mathfrak{m}^n = \{0\}$ , we have that  $G_n \cong G_{k'}$ . It is proved in [Oes84, Prop. A.3.5] that the canonical surjection  $B_i \rightarrow B_{i-1}$  for each  $1 \leq i \leq n$  gives rise to a surjective homomorphism of algebraic groups  $G_i \rightarrow G_{i-1}$ ; let  $V_i$  denote the kernel of this map for each  $i$ . Let  $A := k'[G'] \otimes_{k'} B$  denote the coordinate algebra of  $G'_B$ , and let  $A_i := k'[G'] \otimes_{k'} B_i$  denote the coordinate algebra of  $G'_{B_i}$  for each  $i$  (recall that  $A_n = A$ ).

In [Oes84, Prop. A.3.5], Oesterlé observes that each  $V_i$  is a vector group naturally isomorphic to the additive group  $\text{Lie}(G') \otimes_{k'} \mathfrak{m}^{i-1}/\mathfrak{m}^i$ . We recall how to see this. Given any  $k'$ -algebra  $R$ , the identity element of  $G_i(k')$  gives a homomorphism of  $B_i$ -algebras  $A_i \rightarrow B_i$ , which we can compose with the canonical injection  $B_i \rightarrow R \otimes_{k'} B_i$  to get a homomorphism  $e_i : A_i \rightarrow R \otimes_{k'} B_i$ . Note that an  $R$ -point  $v \in V_i(R)$  corresponds to a  $B_i$ -algebra homomorphism  $v : A_i \rightarrow R \otimes_{k'} B_i$  whose difference with  $e_i$  has image in  $R \otimes_{k'} \mathfrak{m}^{i-1}/\mathfrak{m}^i$ .

The ring  $R \otimes_{k'} \mathfrak{m}^{i-1}/\mathfrak{m}^i$  is also an  $A_i$ -module via the identity element of  $G_i(k')$ : multiplication by an element  $a \in A_i$  under this action coincides with multiplication by  $e_i(a)$  in  $R \otimes_{k'} \mathfrak{m}^{i-1}/\mathfrak{m}^i$ . Now, given  $a, b \in A_i$ , we write

$$\begin{aligned} (e_i - v)(ab) &= e_i(a)e_i(b) - v(a)v(b) \\ &= (e_i - v)(a)e_i(b) + e_i(a)(e_i - v)(b) - (e_i - v)(a)(e_i - v)(b). \end{aligned}$$

The last term is a product of terms in  $R \otimes_{k'} \mathfrak{m}^{i-1}/\mathfrak{m}^i$ , so is zero, and hence we can conclude that  $e_i - v$  acts like a  $B_i$ -linear derivation  $\delta : A_i \rightarrow R \otimes_{k'} \mathfrak{m}^{i-1}/\mathfrak{m}^i$  (with the  $A_i$ -module structure we have given). Conversely, any such derivation will give rise to an element of  $V_i(R)$  by reversing the above process.

Any  $B_i$ -linear derivation of  $A_i$  in  $R \otimes_{k'} \mathfrak{m}^{i-1}/\mathfrak{m}^i$  will kill the ideal  $(\mathfrak{m}/\mathfrak{m}^i)A_i$  in  $A_i$  (since it is  $B_i$ -linear and multiplication by  $\mathfrak{m}/\mathfrak{m}^i$  in  $R \otimes_{k'} \mathfrak{m}^{i-1}/\mathfrak{m}^i$  kills everything); this means that we can identify this set of derivations with the  $k'$ -linear derivations of  $A_i/(\mathfrak{m}/\mathfrak{m}^i)A_i \cong k'[G']$  in  $R \otimes_{k'} \mathfrak{m}^{i-1}/\mathfrak{m}^i$ —write  $\text{Der}_{k'}(k'[G'], R \otimes_{k'} \mathfrak{m}^{i-1}/\mathfrak{m}^i)$  for this set. Further, since the  $A_i$ -module structure on  $R \otimes_{k'} \mathfrak{m}^{i-1}/\mathfrak{m}^i$  comes from the identity element in  $G_i(R) = G'(R \otimes B_i)$ , we see that the induced  $k'[G']$ -module structure permits an identification

$$\text{Der}_{k'}(k'[G'], R \otimes_{k'} \mathfrak{m}^{i-1}/\mathfrak{m}^i) \cong \text{Der}_{k'}(k'[G'], k') \otimes_{k'} R \otimes_{k'} \mathfrak{m}^{i-1}/\mathfrak{m}^i.$$

Now the set of derivations on the right-hand side is just  $\text{Lie}(G')$ , so we have the required identification of  $V_i(R)$  with  $\text{Lie}(G') \otimes_{k'} R \otimes_{k'} \mathfrak{m}^{i-1}/\mathfrak{m}^i$ .

We now show that the identifications in the previous paragraphs turn group multiplication in  $V_i(R)$  into addition. Given  $v_1, v_2 \in V_i$ , the idea is to rewrite

$$e_i - v_1 v_2 = (e_i - v_1) + (e_i - v_2) - (e_i - v_1)(e_i - v_2)$$

and then to note that when we apply this equation to  $a \in A_i$ , the final term is zero in  $R \otimes_{k'} \mathfrak{m}^{i-1}/\mathfrak{m}^i$ . To justify this, given  $a \in A_i$ , let  $\sum_{r,s} a_r \otimes a_s$  denote the image of  $a$  under the comultiplication map  $A_i \rightarrow A_i \otimes A_i$ . Then recalling that  $e_i$  comes from the identity of  $G_i(k')$ , we can write:

$$\begin{aligned} (e_i - v_1 v_2)(a) &= \sum_{r,s} (e_i(a_r)e_i(a_s) - v_1(a_r)v_2(a_s)) \\ &= \sum_{r,s} (e_i(a_r)e_i(a_s) - v_1(a_r)e_i(a_s)) \\ &\quad + \sum_{r,s} (e_i(a_r)e_i(a_s) - e_i(a_r)v_2(a_s)) \end{aligned}$$

$$\begin{aligned}
 & - \sum_{r,s} (e_i(a_r) - v_1(a_r))(e_i(a_s) - v_2(a_s)) \\
 & = (e_i - v_1)(a) + (e_i - v_2)(a),
 \end{aligned}$$

where the third sum on the penultimate line is zero in  $R \otimes \mathfrak{m}^{i-1}/\mathfrak{m}^i$ , being a sum of products. So we see that each successive quotient  $V_i$  is a vector group as claimed.

We now recall that there is a natural copy of  $G'$  inside  $G_i$ ; on the level of  $R$ -points for a  $k'$ -algebra  $R$ , this comes from the observation that any  $k'$ -algebra homomorphism  $k'[G'] \rightarrow R$  has a canonical extension to a  $B_i$ -algebra homomorphism  $A_i = k'[G'] \otimes B_i \rightarrow R \otimes B_i$ . This copy of  $G'$  acts on  $V_i$  by conjugation, and the replacement of  $v \in V_i(R)$  with  $e_i - v$  is clearly equivariant with respect to this action. On the other hand, under the identification of  $\text{Lie}(G')$  with  $\text{Der}_{k'}(k'[G'], k')$ , the adjoint action comes from the action of  $G'$  on  $k'[G']$  induced by the conjugation action of  $G'$  on itself. This in turn gives rise to the action of  $G_i$  on  $A_i = k'[G'] \otimes B_i$  induced by conjugation after Weil restriction, and the canonical copy of  $G'$  acts via its action on the first factor in this tensor product. Hence we see that the isomorphism of vector groups we have established above is  $G'$ -equivariant.

Once this is established, we show how to piece together the  $V_i$  to give the claims in the proposition. Since the natural map  $B = B_n \rightarrow B_i$  for each  $1 \leq i \leq n$  is the composition of the natural maps  $B_n \rightarrow B_{n-1} \rightarrow \dots \rightarrow B_i$ , we also have surjective maps  $G_{k'} \cong G_n \rightarrow G_i$ ; let  $U_i$  denote the kernel of this map for each  $i$ . Then  $U_1$  is the geometric unipotent radical of  $G$  and  $U_1 \supset U_2 \supset \dots \supset U_n$ . We also see that  $U_i/U_{i+1} \cong V_{i+1}$  for each  $1 \leq i \leq n - 1$ . So we have a filtration of the unipotent radical with successive quotients equal to the groups  $V_i$ .

The final step is to check that the commutator  $[U_1, U_i] \subseteq U_{i+1}$  for each  $i$ , so that this filtration is compatible with the lower central series for  $U_1$ . For this, given a  $k'$ -algebra  $R$ , let  $u \in U_1(R)$  and  $v \in U_i(R)$  and as usual identify  $u$  and  $v$  with  $B$ -algebra homomorphisms  $A \rightarrow R \otimes B$ . Analogously with above, we want to rewrite the commutator in a nice way. Recall that in this setup, we have denoted by  $e_i$  the homomorphisms  $A_i \rightarrow R \otimes B_i$  coming from the identity of  $G_i(k')$ . To ease notation, denote  $e_n = e$ . Note that for each  $i$ ,  $e_i$  coincides with  $e$  modulo  $\mathfrak{m}^i$ , and the homomorphisms  $e, u, v$  have the same domain and codomain. Then one equation to use (of the many possible) is

$$e - uvu^{-1}v^{-1} = (e - v^{-1})(e - vu^{-1}v^{-1}) - (e - u)(e - v)u^{-1}v^{-1}.$$

Apply this to  $a \in A$ . Note that since  $u \in U_1$ ,  $(e - u)(a) \in R \otimes_{k'} \mathfrak{m}$ . Similarly, since  $v \in U_i$ ,  $(e - v) \in R \otimes_{k'} \mathfrak{m}^i$  modulo  $\mathfrak{m}^{i+1}$ . Therefore the second term on the right-hand side is zero modulo  $\mathfrak{m}^{i+1}$ . Similarly, the first term is zero, noting that  $vu^{-1}v^{-1} \in U_1$ . Therefore, the whole right-hand side, and hence the left-hand side, is zero in  $U_i/U_{i+1}$ , which is what we wanted. Again, this calculation can be justified using the comultiplication. □

When  $G'$  is not unusual—that is, it is noncommutative, and either  $p \neq 2$  or  $G'$  is not a direct product of some copies of  $\text{SL}_2$  with a torus—then we show in the next



sections that the upper bound for  $\text{cl}(\mathcal{R}_u(G_{\bar{k}}))$  in Proposition 3.1 is in fact also a lower bound.

EXAMPLE 3.2. Note that the filtration from Proposition 3.1 need not exist for an arbitrary standard pseudo-reductive group. For example, if  $p = 2$ ,  $G' = \text{SL}_2$ ,  $k'/k$  is a purely inseparable field extension of degree 2, and  $G$  is the standard pseudo-reductive group  $\mathbf{R}_{k'/k}(G')/\mathbf{R}_{k'/k}(\mu_2)$ , then  $G$  has dimension  $\dim G' \cdot [k' : k] - (p - 1) = 5$ ; see [CGP15, Ex. 1.3.2]. Since  $G' = \text{SL}_2$  is defined over  $k$ , there is a canonical copy of  $G'$  inside  $\mathbf{R}_{k'/k}(G')_{k'}$ . Factoring out by  $\mathbf{R}_{k'/k}(\mu_2)_{k'}$  kills the copy of  $\mu_2$  inside  $G'$ , so we obtain a copy of  $\text{PGL}_2$  inside  $G_{k'}$ . This gives rise to a copy of  $\text{Lie}(\text{PGL}_2)$  in  $\text{Lie}(G_{k'})$ , so we see that  $\text{Lie}(\mathcal{R}_u(G_{\bar{k}}))$  cannot be a sum of copies of  $\text{Lie}(G')_{\bar{k}}$ , just for dimensional reasons. (Similar examples can be given for any  $G'$  whose center is not smooth.)

### 3.2. Lower Bound

Let  $k'/k$  be a purely inseparable field extension, and set  $B = k' \otimes_k k'$  with maximal ideal  $\mathfrak{m}$ . In this section, we do a calculation in  $\text{GL}_2(B)$ , which shows that  $\mathcal{R}_u(\mathbf{R}_{k'/k}(\text{GL}_2)_{\bar{k}})$  has nilpotency class exactly  $n - 1$ , where  $n$  is minimal such that  $\mathfrak{m}^n = 0$ . (Of course, the upper bound has already been established in the previous section.)

Let  $G = \mathbf{R}_{k'/k}(\text{GL}_2)$ . It suffices to compute the nilpotency class of  $\mathcal{R}_{u,k'}(G_{k'})$  since this is a  $k'$ -form of the geometric unipotent radical. As in the proof of Proposition 3.1, we have  $\mathbf{R}_{k'/k}(\text{GL}_2)_{k'} = \mathbf{R}_{B/k}(\text{GL}_2)$ , and the  $k'$ -radical of this is the kernel of the map induced from  $B \rightarrow k'$  with kernel  $\mathfrak{m}$ . Thus the  $k'$ -points of  $\mathcal{R}_{u,k'}(G_{k'})$  are simply the matrices

$$\begin{pmatrix} 1 + m_1 & m_2 \\ m_3 & 1 + m_4 \end{pmatrix} \quad \text{for } m_i \in \mathfrak{m}.$$

By minimality of  $n$  we can find  $m \in \mathfrak{m}$  and  $m_i \in \mathfrak{m}$  for  $1 \leq i \leq n - 2$  such that  $m \cdot m_1 \cdots m_{n-2} \neq 0$ . Certainly, it suffices to find the desired lower bound for  $\text{cl}(\mathcal{R}_u(G_{\bar{k}}))$  on the level of the  $k'$ -points of  $\mathcal{R}_{u,k'}(G_{k'})$ , so it suffices to see that there exist  $z, y_1, \dots, y_{n-2} \in \mathcal{R}_{u,k'}(G_{k'})(k')$  with  $[y_1, [y_2, [\dots, [y_{n-2}, z]]]] \neq 1$ . In the light of that observation, take

$$z := \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad y_i := \begin{pmatrix} 1 + m_i & 0 \\ 0 & 1 \end{pmatrix} \quad \text{for } 1 \leq i \leq n - 2;$$

then an elementary calculation yields

$$[y_1, [y_2, [\dots, [y_{n-2}, z]]]] = \begin{pmatrix} 1 & m \cdot m_1 \cdots m_{n-2} \\ 0 & 1 \end{pmatrix} \neq 1.$$

Combining with the lower bound from Proposition 3.1, we have proved the following:

LEMMA 3.3. *Let  $k'/k$  be a purely inseparable field extension. Then*

$$\text{cl}(\mathcal{R}_u((\mathbf{R}_{k'/k}(\text{GL}_2))_{\bar{k}})) = n - 1,$$

where  $n$  is minimal such that  $\mathfrak{m}^n = 0$ .

We wish to deduce the analogous result for  $\text{PGL}_2$  and  $\text{SL}_2$  (the latter when  $p \neq 2$ ). To do this, we need the following two lemmas, which we also use in the proof of Theorem 1.1.

LEMMA 3.4. *Let  $k'/k$  be a finite field extension, and let  $G'$  be a semisimple  $k'$ -group such that  $Z' := Z_{G'}$  is smooth. Let  $G'_{\text{ad}} = G'/Z'$ , and let  $G := \mathbf{R}_{k'/k}(G')$ . Then there is a natural isomorphism*

$$\mathcal{R}_{u,k'}(G_{k'}) \cong \mathcal{R}_{u,k'}(\mathbf{R}_{k'/k}(G'_{\text{ad}})_{k'}).$$

*Proof.* Since  $Z'$  is smooth, there is a smooth isogeny giving rise to the exact sequence

$$1 \rightarrow Z' \rightarrow G' \xrightarrow{\pi} G'_{\text{ad}} \rightarrow 1.$$

By [CGP15, A.5.4(3)] Weil restriction preserves the exactness of this sequence, and so there is an exact sequence

$$1 \rightarrow \mathbf{R}_{k'/k}(Z') \rightarrow G \xrightarrow{\mathbf{R}_{k'/k}(\pi)} \mathbf{R}_{k'/k}(G'_{\text{ad}}) \rightarrow 1.$$

This gives, after base change to  $k'$ , an exact sequence

$$1 \rightarrow \mathbf{R}_{k'/k}(Z')_{k'} \rightarrow G_{k'} \xrightarrow{\mathbf{R}_{k'/k}(\pi)_{k'}} \mathbf{R}_{k'/k}(G'_{\text{ad}})_{k'} \rightarrow 1.$$

Now  $Z'$  is a smooth finite group scheme, so  $Z' \cong \mathbf{R}_{k'/k}(Z')_{k'}$  as algebraic groups. This implies that  $\mathbf{R}_{k'/k}(\pi)_{k'}$  is a smooth isogeny. It follows that the map  $\mathbf{R}_{k'/k}(\pi)_{\bar{k}}: G_{\bar{k}} \rightarrow \mathbf{R}_{k'/k}(G'_{\text{ad}})_{\bar{k}}$  obtained by base change to  $\bar{k}$  gives rise to an isomorphism from  $\mathcal{R}_u(G_{\bar{k}})$  to  $\mathcal{R}_u(\mathbf{R}_{k'/k}(G'_{\text{ad}})_{\bar{k}})$ . But  $\mathcal{R}_u(G_{\bar{k}})$  and  $\mathcal{R}_u(\mathbf{R}_{k'/k}(G'_{\text{ad}})_{\bar{k}})$  are defined over  $k'$ , so  $\mathbf{R}_{k'/k}(\pi)_{k'}$  gives an isomorphism from  $\mathcal{R}_{u,k'}(G_{k'})$  to  $\mathcal{R}_{u,k'}(\mathbf{R}_{k'/k}(G'_{\text{ad}})_{k'})$ , as required.  $\square$

LEMMA 3.5. *Let  $k', k, G', Z'$ , and  $G'_{\text{ad}}$  be as in Lemma 3.4, but without the assumption that  $Z'$  is smooth. Further, let  $C$  be a commutative pseudo-reductive  $k$ -group occurring in a factorization of the map  $\mathbf{R}_{k'/k}(T') \rightarrow \mathbf{R}_{k'/k}(T'/Z')$  for  $T'$  a maximal  $k'$ -torus of  $G'$ . Then:*

(i) *We have*

$$\begin{aligned} \text{cl}(\mathcal{R}_u(((\mathbf{R}_{k'/k}(G') \rtimes C)/\mathbf{R}_{k'/k}(T'))_{\bar{k}})) &\leq \text{cl}(\mathcal{R}_u((\mathbf{R}_{k'/k}(G') \rtimes C)_{\bar{k}})) \\ &= \text{cl}(\mathcal{R}_u(\mathbf{R}_{k'/k}(G')_{\bar{k}})). \end{aligned}$$

(ii) *If moreover  $Z'$  is smooth, then we have*

$$\begin{aligned} \text{cl}(\mathcal{R}_{u,k'}((\mathbf{R}_{k'/k}(G') \rtimes C)_{k'})) &= \text{cl}(\mathcal{R}_{u,k'}(\mathbf{R}_{k'/k}(G'_{\text{ad}})_{k'})) \\ &= \text{cl}(\mathcal{R}_{u,k'}(((\mathbf{R}_{k'/k}(G') \rtimes C)/\mathbf{R}_{k'/k}(T'))_{k'})), \end{aligned}$$

where in the final term we quotient by the usual central copy of  $\mathbf{R}_{k'/k}(T')$  occurring in the standard construction.

*Proof.* (i) Let  $H := (\mathbf{R}_{k'/k}(G') \rtimes C)_{\bar{k}}$  and  $G := ((\mathbf{R}_{k'/k}(G') \rtimes C)/\mathbf{R}_{k'/k}(T'))_{\bar{k}}$ . We have an epimorphism  $f : H \rightarrow G$ . Let  $U = \mathcal{R}_u(H)$ . Then  $U' := f(U)$  is a smooth, connected normal unipotent subgroup of  $G$ , and so we get an epimorphism from the reductive group  $H/U$  onto  $G/U'$ . But  $H/U$  is reductive, so  $G/U'$  must be reductive. This means that  $U'$  contains (and therefore equals)  $\mathcal{R}_u(G)$ . Thus  $U'$  is a quotient of  $U$ , and so  $\text{cl}(U') \leq \text{cl}(U)$ , proving the inequality.

For the equality, we have trivially  $\text{cl}(U) \geq \text{cl}(\mathcal{R}_{u,k'}(\mathbf{R}_{k'/k}(G')_{k'}))$ , so we will be done if we can show that  $\mathcal{D}(U) \subseteq \mathcal{D}(\mathcal{R}_u(\mathbf{R}_{k'/k}(G')_{\bar{k}}))$ . Now  $C$  factorizes the map  $\mathbf{R}_{k'/k}(T') \rightarrow \mathbf{R}_{k'/k}(T'/Z')$ , whose kernel is  $\mathbf{R}_{k'/k}(Z')$  (due to left exactness of Weil restriction). If  $A$  is any  $k$ -algebra and  $g \in C(A)$ , then for any  $q \in \mathbf{R}_{k'/k}(G')(A)$ , we can find  $h \in \mathbf{R}_{k'/k}(T')(A)$  such that  $hgh^{-1} = gqg^{-1}$ . In particular,  $[\mathcal{R}_{u,k'}(\mathbf{R}_{k'/k}(G'))_{k'}, C_{k'}] \subseteq [\mathcal{R}_{u,k'}(\mathbf{R}_{k'/k}(G'))_{k'}, \mathcal{R}_{u,k'}(\mathbf{R}_{k'/k}(G'))_{k'}]$ , as required.

(ii) Let  $\phi : \mathbf{R}_{k'/k}(T') \rightarrow C$  be the map in the factorization. Set  $Z = \mathbf{R}_{k'/k}(Z')$ . Thanks to the isomorphism of groups  $Z' \cong Z_{k'}$  arising from the hypothesis on  $Z'$ , we have by the arguments of Lemma 3.4 a smooth isogeny

$$1 \rightarrow Z' \times \phi_{k'}(Z') \rightarrow \mathbf{R}_{k'/k}(G')_{k'} \rtimes C_{k'} \rightarrow \mathbf{R}_{k'/k}(G'_{\text{ad}})_{k'} \rtimes C_{k'}/\phi_{k'}(Z') \rightarrow 1$$

or, equivalently,

$$1 \rightarrow Z' \times \phi_{k'}(Z') \rightarrow (\mathbf{R}_{k'/k}(G') \rtimes C)_{k'} \rightarrow (\mathbf{R}_{k'/k}(G'_{\text{ad}}) \rtimes D)_{k'} \rightarrow 1, \quad (*)$$

where we write  $D$  in place of  $C/\phi(Z)$ . The argument of Lemma 3.4 yields an isomorphism

$$\mathcal{R}_{u,k'}((\mathbf{R}_{k'/k}(G') \rtimes C)_{k'}) \cong \mathcal{R}_{u,k'}((\mathbf{R}_{k'/k}(G'_{\text{ad}}) \rtimes D)_{k'}).$$

The map  $C \rightarrow \mathbf{R}_{k'/k}(T'/Z')$  gives rise to a map  $\kappa : D \rightarrow \mathbf{R}_{k'/k}(G'_{\text{ad}})$ . We claim that the map  $\tau : \mathbf{R}_{k'/k}(G'_{\text{ad}}) \rtimes D \rightarrow \mathbf{R}_{k'/k}(G'_{\text{ad}}) \times D$  given on the level of  $A$ -points by  $(g, d) \mapsto (g\kappa(d), d)$  is an isomorphism. To see this, first recall that  $(g, d)(h, e) = (g\kappa(d)h\kappa(d)^{-1}, de)$ . Then  $(g, d)(h, e) \mapsto (g\kappa(d)h\kappa(d)^{-1}\kappa(de), de) = (g\kappa(d)h\kappa(e), de)$ , which is the product of  $(g\kappa(d), d)$  and  $(h\kappa(e), e)$  in the direct product, as required.

With this in mind, we get

$$\begin{aligned} \mathcal{R}_{u,k'}((\mathbf{R}_{k'/k}(G'_{\text{ad}}) \rtimes D)_{k'}) &\cong \mathcal{R}_{u,k'}((\mathbf{R}_{k'/k}(G'_{\text{ad}}) \times D)_{k'}) \\ &\cong \mathcal{R}_{u,k'}(\mathbf{R}_{k'/k}(G'_{\text{ad}})_{k'}) \times \mathcal{R}_{u,k'}(D_{k'}); \end{aligned}$$

but the commutativity of  $D$  implies that  $\text{cl}(\mathcal{R}_{u,k'}((\mathbf{R}_{k'/k}(G'_{\text{ad}}) \times D)_{k'})) = \text{cl}(\mathcal{R}_{u,k'}(\mathbf{R}_{k'/k}(G'_{\text{ad}})_{k'}))$ . This proves the first equality of the lemma.

To see the second equality, observe that  $\mathbf{R}_{k'/k}(T'/Z') \cong \mathbf{R}_{k'/k}(T')/Z$  since  $Z'$  is smooth. We see that the map  $(\mathbf{R}_{k'/k}(G') \rtimes C)_{k'} \rightarrow (\mathbf{R}_{k'/k}(G'_{\text{ad}}) \rtimes D)_{k'}$  from  $(*)$  takes the copy of  $\mathbf{R}_{k'/k}(T')_{k'}$  onto the copy of  $\mathbf{R}_{k'/k}(T'/Z')_{k'}$ , so the induced

map of quotient groups yields an isogeny

$$\begin{aligned} 1 \rightarrow N \rightarrow (\mathbf{R}_{k'/k}(G') \rtimes C)_{k'}/\mathbf{R}_{k'/k}(T')_{k'} \\ \rightarrow (\mathbf{R}_{k'/k}(G'_{\text{ad}}) \rtimes D)_{k'}/\mathbf{R}_{k'/k}(T'/Z')_{k'} \rightarrow 1 \end{aligned}$$

for some  $N$ . This isogeny is smooth because the canonical projections to the quotient groups are smooth. By the argument of Lemma 3.4 again, we get an isomorphism

$$\begin{aligned} \mathcal{R}_{u,k'}((\mathbf{R}_{k'/k}(G') \rtimes C)_{k'}/\mathbf{R}_{k'/k}(T')_{k'}) \\ \cong \mathcal{R}_{u,k'}((\mathbf{R}_{k'/k}(G'_{\text{ad}}) \rtimes D)_{k'}/\mathbf{R}_{k'/k}(T'/Z')_{k'}). \end{aligned}$$

Now  $\phi_{k'}$  induces a map  $\Phi$  from  $\mathbf{R}_{k'/k}(T'/Z')_{k'}$  onto a smooth subgroup of  $D$ , and it is easily checked that  $\tau$  gives rise to an isomorphism

$$(\mathbf{R}_{k'/k}(G'_{\text{ad}}) \rtimes D)_{k'}/\mathbf{R}_{k'/k}(T'/Z')_{k'} \cong \mathbf{R}_{k'/k}(G'_{\text{ad}}) \times D/\Phi(\mathbf{R}_{k'/k}(T'/Z')_{k'}).$$

The commutativity of  $D$  again gives the result. □

We can now tackle the following:

**PROPOSITION 3.6.** *Let  $k'/k$  be a purely inseparable field extension, and let  $G = \mathbf{R}_{k'/k}(G')$ , where  $G'$  is one of  $\text{GL}_2$  or  $\text{PGL}_2$  or  $\text{SL}_2$  (the final case only if  $\text{char } k \neq 2$ ). Let  $n$  be minimal such that  $\mathfrak{m}^n = 0$ . Then  $\text{cl}(\mathcal{R}_u(G_{\bar{k}})) = n - 1$ .*

*Proof.* In the cases considered,  $Z_{G'}$  is smooth, so by Lemmas 3.4 and 3.5 the nilpotency classes of  $\mathcal{R}_{u,k'}(\mathbf{R}_{k'/k}(\text{GL}_2)_{k'})$ ,  $\mathcal{R}_{u,k'}(\mathbf{R}_{k'/k}(\text{PGL}_2)_{k'})$ , and  $\mathcal{R}_{u,k'}(\mathbf{R}_{k'/k}(\text{SL}_2)_{k'})$  are all equal. Lemma 3.3 now supplies the result. □

### 3.3. $\text{SL}_2$ in Characteristic 2

In this section, we wish to study the case  $G' = \text{SL}_2$  and  $G = \mathbf{R}_{k'/k}(G')$ , where  $\text{char } k = 2$ , and  $k'/k$  is a purely inseparable field extension.

We will show

**LEMMA 3.7.** *Let  $k'/k$  be a purely inseparable finite field extension, and let  $B = k' \otimes_k k'$  with maximal ideal  $\mathfrak{m}$ . Furthermore, denote by  ${}^2\mathfrak{m}$  the ideal generated by the squares of the elements in  $\mathfrak{m}$ . Suppose  $n$  is the minimum such that  $({}^2\mathfrak{m})^{n-1} \cdot \mathfrak{m}^2 = 0$ . Then  $\text{cl}(\mathcal{R}_u(G_{\bar{k}})) = n$ .*

*Proof.* By Lemma 2.1 we may assume that  $k = k_s$ . We use arguments analogous to those in the previous section. First, we establish the upper bound. Recall that we have  $R := \mathbf{R}_{u,k'}(G_{k'}) = \ker \varphi$ , where  $\varphi : G_{k'} \rightarrow G'$  is the map induced by reduction modulo  $\mathfrak{m}$ . Since we will be working with commutator subgroups, it is convenient to work with the  $\bar{k}$ -points of  $R$ . Set  $\bar{B} = B \otimes_{k'} \bar{k} = k' \otimes_k \bar{k}$  and  $\bar{\mathfrak{m}} = \mathfrak{m} \otimes_{k'} \bar{k}$ . Then  $\bar{B}$  is local with unique maximal ideal  $\bar{\mathfrak{m}}$  and quotient field  $\bar{k}$ . The ideal  $\bar{\mathfrak{m}}$  is nilpotent, and the minimal  $n$  with  $\bar{\mathfrak{m}}^n = 0$  is the same as the minimal  $n$  with  $\mathfrak{m}^n = 0$ .

Define the ideals  $I_r$  and  $J_r$  of  $B$  for  $r \geq 0$  by  $I_0 = J_0 = \mathfrak{m}$ ,  $I_r = ({}^2\mathfrak{m})^r \cdot \mathfrak{m}$  for  $r \geq 1$ , and  $J_r = ({}^2\mathfrak{m})^{r-1} \cdot \mathfrak{m}^2$  for  $r \geq 1$ . Set  $\bar{I}_r = I_r \otimes_{k'} \bar{k} = ({}^2\bar{\mathfrak{m}})^r \cdot \bar{\mathfrak{m}}$  and

also  $\bar{J}_r = J_r \otimes_{k'} \bar{k} = ({}^2\bar{m})^{r-1} \cdot \bar{m}^2$ . It is clear that  $\bar{J}_r = 0$  (resp.,  $\bar{I}_r = 0$ ) if and only if  $J_r = 0$  (resp.,  $I_r = 0$ ) for each  $r$ . Observe that  $\bar{I}_r \subseteq \bar{J}_r$ ,  ${}^2\bar{m} \cdot \bar{I}_r \subseteq \bar{I}_{r+1}$ ,  ${}^2\bar{m} \cdot \bar{J}_r \subseteq \bar{J}_{r+1}$ , and  $\bar{m} \cdot \bar{I}_r \subseteq \bar{J}_{r+1}$  for all  $r$ .

Define  $R_r \subseteq G(\bar{k})$  by

$$R_r = \left\{ \begin{pmatrix} 1 + m_1 & m_2 \\ m_3 & 1 + m_4 \end{pmatrix} \mid m_1, m_4 \in \bar{J}_r, m_2, m_3 \in \bar{I}_r, m_1 + m_4 + m_1 m_4 + m_2 m_3 = 0 \right\}.$$

It is easily checked that  $R_0 = R(\bar{k})$  and each  $R_r$  is a subgroup of  $R(\bar{k})$ . Let  $M = \begin{pmatrix} 1+m_1 & m_2 \\ m_3 & 1+m_4 \end{pmatrix} \in R_r$  for some given  $r$ . Consider the following matrices of  $R_0$ :

$$M_1 := \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \quad \text{with } m \in \bar{m};$$

$$M_2 := \begin{pmatrix} 1+m & 0 \\ 0 & (1+m)^{-1} \end{pmatrix} \quad \text{with } m \in \bar{m}.$$

We have  $M^{-1} = \begin{pmatrix} 1+m_4 & m_2 \\ m_3 & 1+m_1 \end{pmatrix}$ , and so we find the commutator

$$[M_1, M] = \begin{pmatrix} 1 + mm_3 + mm_1 m_3 + m^2 m_3^2 & mm_1^2 + m^2 m_3 + m^2 m_1 m_3 \\ mm_3^2 & 1 + mm_3 + mm_1 m_3 \end{pmatrix}.$$

We see that  $[M_1, M]$  belongs to  $R_{r+1}$ . By symmetry the commutator  $[M_1^\top, M]$  also belongs to  $R_{r+1}$  (here  $M_1^\top$  denotes the transpose of  $M_1$ ). Similarly, noting that  $(1+m)^{-1} = 1+m+m'$  for some  $m' \in {}^2\bar{m}$ , we have

$$[M_2, M] = \begin{pmatrix} 1 + m^2 m_2 m_3 & m^2 (1 + m_1) m_2 \\ (m^2 + m'^2) m_3 (1 + m_4) & 1 + (m^2 + (m')^2) m_2 m_3 \end{pmatrix},$$

which belongs to  $R_{r+1}$ . These calculations together show that if  $n$  is such that  $\bar{J}_n = 0$ , then  $[X_1, [X_2, \dots, [X_n, g]]] = 1$  for arbitrary  $g \in R(\bar{k})$  and arbitrary  $X_i$  chosen from elements of the form  $M_1, M_1^\top$ , and  $M_2$ .

Let  $(U')^-$  denote the lower root group of  $G'$ ,  $T'$  denote the diagonal torus, and  $U'$  denote the upper root group, and set  $U := R_{k'/k}(U')$ ,  $C := R_{k'/k}(T')$ ,  $U^- := R_{k'/k}((U')^-)$ . Then the multiplication map  $\mu : U^- \times C \times U \rightarrow G$  is an open immersion by [CGP15, Cor. 3.3.16]. Moreover, the intersections  $R_{k'/k}(U)_{\bar{k}} \cap R_{\bar{k}}$ ,  $R_{k'/k}(T)_{\bar{k}} \cap R_{\bar{k}}$ , and  $R_{k'/k}(U^-)_{\bar{k}} \cap R_{\bar{k}}$  have codimension 1 in  $R_{k'/k}(U)_{\bar{k}}$ ,  $R_{k'/k}(T)_{\bar{k}}$ , and  $R_{k'/k}(U^-)_{\bar{k}}$ , respectively; this can be seen by looking at the kernel of the restriction of  $\varphi_{\bar{k}}$  to these subgroups, where  $\varphi$  is the map above. It follows that  $\mu_{\bar{k}}(D)$  has codimension 3 in  $R_{k'/k}(G')_{\bar{k}}$ , where  $D := (R_{k'/k}(U)_{\bar{k}} \cap R_{\bar{k}}) \times (R_{k'/k}(T)_{\bar{k}} \cap R_{\bar{k}}) \times (R_{k'/k}(U^-)_{\bar{k}} \cap R_{\bar{k}})$ . In particular,  $\mu_{\bar{k}}(D)$  is an open subset of  $R_{\bar{k}}$  and hence generates  $R_{\bar{k}}$ . But, in light of the identity  $[x, zy] = [x, y] \cdot [x, z]^y$  and the fact from the previous paragraph that  $[X_1, [X_2, \dots, [X_n, R(\bar{k})]]] = 1$  for generators  $X_i$  of  $R(\bar{k})$ , we are done for the upper bound.

For the lower bound, we exhibit an explicit nonzero commutator in an analogous way to Lemma 3.3. Let  $n$  be as in the statement. If  $n = 1$ , then there is

nothing to do, so suppose  $n > 1$ . Since we assume that  $n$  is minimal such that  $J_n = 0$ , we may pick  $m_1, \dots, m_{n-2} \in \mathfrak{m}$  and  $m, m' \in \mathfrak{m}$  such that  $m_1^2 \cdots m_{n-2}^2 mm'$  is nonzero. Take

$$z = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}, \quad z' = \begin{pmatrix} 1 & 0 \\ m' & 1 \end{pmatrix}, \quad y_i = \begin{pmatrix} 1 + m_i & 0 \\ 0 & (1 + m_i)^{-1} \end{pmatrix}$$

for  $1 \leq i \leq n - 2$ . Set  $w = [y_1, [y_2, [\dots, [y_{n-2}, z]]]]$ . Then a calculation yields

$$w = \begin{pmatrix} 1 & m_1^2 \cdots m_{n-2}^2 m \\ 0 & 1 \end{pmatrix}$$

and

$$[z', w] = \begin{pmatrix} 1 + m_1^2 \cdots m_{n-2}^2 mm' & 0 \\ 0 & 1 + m_1^2 \cdots m_{n-2}^2 mm' \end{pmatrix} \neq 1.$$

This gives the lower bound. □

REMARK 3.8. Consider the pseudo-reductive group  $R_{k'/k}(\mathrm{SL}_2)/R_{k'/k}(\mu_2)$  with  $k'/k$  inseparable. The canonical projection  $R_{k'/k}(\mathrm{SL}_2) \rightarrow R_{k'/k}(\mathrm{SL}_2)/R_{k'/k}(\mu_2)$  is surjective on  $\bar{k}$ -points, so  $\mathcal{R}_u((R_{k'/k}(\mathrm{SL}_2)/R_{k'/k}(\mu_2))_{\bar{k}})(\bar{k})$  is a quotient of  $\mathcal{R}_u(R_{k'/k}(\mathrm{SL}_2)_{\bar{k}})(\bar{k})$ , so the nilpotency class  $c$  of the latter is at most  $n$  (where  $n$  is as in Lemma 3.7). The image of the commutator  $w$  from above in  $\mathcal{R}_u((R_{k'/k}(\mathrm{SL}_2)/R_{k'/k}(\mu_2))_{\bar{k}})(\bar{k})$  does not vanish, so  $c$  is at least  $n - 1$  (note that we may identify  $\mathcal{R}_u(R_{k'/k}(\mu_2)_{\bar{k}})(\bar{k})$  with the group of matrices of the form  $\begin{pmatrix} 1+m & 0 \\ 0 & 1+m \end{pmatrix}$  with  $m \in \bar{\mathfrak{m}}$  and  $m^2 = 0$ ). The image of  $[z', w]$ , however, does vanish. It follows that  $c$  is either  $n$  or  $n - 1$ ; we do not know which in general.

EXAMPLE 3.9. (i) Consider the case that  $k' = k(t)$  with  $t^2 \in k \setminus k^2$ . Then  $\mathfrak{m}^2 = 0$ , so  $J_1 = 0$ . We can calculate:

$$\begin{pmatrix} 1 + m_1 & m_2 \\ m_3 & 1 + m_4 \end{pmatrix} \begin{pmatrix} 1 + m'_1 & m'_2 \\ m'_3 & 1 + m'_4 \end{pmatrix} = \begin{pmatrix} 1 + m_1 + m'_1 & m_2 + m'_2 \\ m_3 + m'_3 & 1 + m_4 + m'_4 \end{pmatrix}$$

for all  $m_i \in \bar{\mathfrak{m}}$ . Thus the unipotent radical is abelian in this case, so the nilpotency class is indeed 1.

(ii) Now let  $k' = k(s, t)$  of degree 4, where  $s^2, t^2 \in k \setminus k^2$ . We see that  $\mathfrak{m}^2 = 0$  but  $\mathfrak{m}^2 \neq 0$ , so  $J_1 \neq 0$  and  $J_2 = 0$ . Choose  $m_1 = s \otimes 1 + 1 \otimes s \in \mathfrak{m}$  and  $m_2 = t \otimes 1 + 1 \otimes t \in \mathfrak{m}$ . Note that  $m_1 m_2 \neq 0$ , and hence if we set

$$u_1 := \begin{pmatrix} 1 & m_1 \\ 0 & 1 \end{pmatrix}, \quad u_2 := \begin{pmatrix} 1 & 0 \\ m_2 & 1 \end{pmatrix},$$

then we have  $u_1 u_2 \neq u_2 u_1$ , so the unipotent radical in this case is not abelian. In fact, we see from Lemma 3.7 that the nilpotency class is 2.

### 3.4. The Case of Weil Restrictions

We will treat here the case that  $G$  is a Weil restriction  $R_{k'/k}(G')$  for  $G'$  a reductive  $k'$ -group, where  $k'$  is a finite reduced nonzero  $k$ -algebra. We deal with some special situations relating to the case where some fiber of  $G'$  is unusual.

LEMMA 3.10. *Suppose  $p = 2$ . Let  $H' = \mathrm{SL}_2$  and suppose  $G' = S'H'$  is a split reductive group, where  $S'$  is a central torus of  $G'$ . Then either the product map  $S' \times H' \rightarrow G'$  is an isomorphism, or  $G'$  contains a copy of  $\mathrm{GL}_2$ .*

*Proof.* The group  $G'$  has semisimple rank 1, so by [Mil12, Thm. 21.55],  $G'$  is isomorphic to one of  $\mathrm{SL}_2 \times T'$ ,  $\mathrm{GL}_2 \times T'$ , or  $\mathrm{PGL}_2 \times T'$  for some torus  $T'$ . The result follows.  $\square$

LEMMA 3.11. *Let  $k'$  be a field, and let  $G'$  a noncommutative split reductive  $k'$ -group. Then exactly one of the following holds:*

- (i)  $G'$  has a subgroup isomorphic to  $\mathrm{GL}_2$  or  $\mathrm{PGL}_2$ ;
- (ii)  $G' = (\mathrm{SL}_2)^r \times S'$  for some  $r \geq 1$ .

*Proof.* Suppose  $G'$  has no subgroup isomorphic to  $\mathrm{GL}_2$  or  $\mathrm{PGL}_2$ . If  $G'$  contains a subgroup of type  $A_2$ , then  $G'$  contains a copy of  $\mathrm{GL}_2$ , which is impossible; note that this also implies that  $G'$  does not contain a subgroup of type  $G_2$ . If  $G'$  has a simple factor  $H'$  of type  $B_2$ , then the long-root Levi subgroup of  $H'$  is isomorphic to  $\mathrm{GL}_2$  if  $H'$  is simply connected and to  $\mathbb{G}_m \times \mathrm{PGL}_2$  if  $H'$  is adjoint, which gives a contradiction in both cases. So every simple factor of  $G'$  is isomorphic to  $\mathrm{SL}_2$ .

Let  $H'_1, \dots, H'_r$  be the simple factors of  $G'$ , and let  $S' = Z(G')^0$ . Multiplication gives a central isogeny  $\varphi: H'_1 \times \dots \times H'_r \times S' \rightarrow G'$ . Suppose  $\varphi$  is not an isomorphism. Then the projection of  $\ker(\varphi)$  to  $H'_i$ , say, must be nontrivial. Let  $T'$  be a split maximal torus of  $H'_2 \times \dots \times H'_r \times S'$ . Since  $\ker(\varphi)$  is contained in a maximal torus of  $H'_1 \times T'$ , we must have that  $\varphi$  is not injective on  $H'_1 \times T'$ . It follows from Lemma 3.10 that  $H'_1 T' \subseteq G'$  contains a copy of  $\mathrm{GL}_2$ .  $\square$

We can now prove Theorem 1.1 for the case of Weil restrictions of reductive groups. Note that such groups are of course standard pseudo-reductive, taking  $C = R_{k'/k}(T')$ , and  $\phi$  an isomorphism.

PROPOSITION 3.12. *Theorem 1.1 holds when  $G = R_{k'/k}(G')$ .*

*Proof.* By Lemma 2.1 we may assume that  $k = k_s$ . Since  $G$  is a direct product of  $R_{k_i/k}(G'_i)$  for some inseparable field extensions  $k_i/k$  and fibers  $G'_i$  of  $G'$  over  $k'$ , we clearly have  $\mathrm{cl}(\mathcal{R}_u(G_{\bar{k}})) = \max_i(\mathcal{R}_u(R_{k_i/k}(G'_i)_{\bar{k}}))$ . Hence it suffices to show that the nilpotency class of the unipotent radical of  $G_i := R_{k_i/k}(G'_i)$  is the number  $\ell_i$  from the introduction. The case that  $G'_i$  is commutative is obvious, so assume that  $G'_i$  is noncommutative. It is of course split since  $k = k_s$ . If  $G'_i$  is not unusual, then Lemma 3.11 says that  $G'_i$  contains a copy of  $\mathrm{PGL}_2$  or  $\mathrm{GL}_2$ . In this case, Proposition 3.6 gives  $\mathrm{cl}(\mathcal{R}_u((G_i)_{\bar{k}})) \geq \ell_i$  and Proposition 3.1 gives

$\text{cl}(\mathcal{R}_u((G_i)_{\bar{k}})) \leq \ell_i$ . Lastly, if  $G'_i$  is unusual, then by Lemma 3.11  $G_i$  is isomorphic to  $\mathbf{R}_{k_i/k}(\text{SL}_2)^r \times \mathbf{R}_{k_i/k}(S')$  for some torus  $S'$  and some  $r \geq 1$ . Then the result follows from Lemma 3.7.  $\square$

### 3.5. Centrally Smooth Enlargements

The following is essentially a definition from [Mar99].

DEFINITION 3.13. Let  $G'$  be a reductive  $k'$ -group, where  $k'$  is a field. A *centrally smooth enlargement* of  $G'$  is a reductive  $k'$ -group  $\widehat{G}'$  such that  $Z(\widehat{G}')$  is smooth,  $G' \subseteq \widehat{G}'$ , and  $\widehat{G}'$  is generated by  $G'$  and a central torus.

LEMMA 3.14. *Let  $G'$  be a split reductive  $k'$ -group. There exists a centrally smooth enlargement  $\widehat{G}'$  of  $G'$ .*

*Proof.* This follows from [Mar99, Thm. 4.5]. The result in loc. cit. is stated only for algebraically closed fields, but the proof works for arbitrary split reductive groups by the existence and isogeny theorems for split reductive groups [Spr98, 16.3.2, 16.3.3].  $\square$

REMARK 3.15. There is also another, simpler, proof, which works for the nonsplit case as well: embed  $Z(G')$  in a torus  $S'$ : for example, we could choose  $S'$  to be a maximal torus of  $G'$ . Then set  $\widehat{G}' = (G' \times S')/N$ , where  $N$  is the image of  $Z(G')$  under the obvious diagonal embedding. (Since  $N \cap (1 \times S') = 1$ ,  $S'$  appears as a subgroup of  $\widehat{G}'$ .)

Let  $(G', k'/k, T', C)$  be a 4-tuple corresponding to a standard pseudo-reductive group

$$G = (\mathbf{R}_{k'/k}(G') \rtimes C) / \mathbf{R}_{k'/k}(T'),$$

with factorization  $\mathbf{R}_{k'/k}(T') \xrightarrow{\phi} C \xrightarrow{\psi} \mathbf{R}_{k'/k}(T'/Z_{G'})$ . Assume that  $k'$  is a field. Suppose  $\widehat{G}' = G'S'$  is a central enlargement of  $G'$ , where  $S'$  is a central torus of  $\widehat{G}'$ . Set  $Z' = Z(G')$ ,  $\widehat{Z}' = Z(\widehat{G}')$ , and  $\widehat{T}' = T'S'$ . It is easy to see that  $\widehat{Z}' = Z'S' \subseteq G'S'$ . Therefore the composition  $T' \rightarrow \widehat{T}' \rightarrow \widehat{T}'/\widehat{Z}'$  induces an isomorphism  $\beta: T'/Z' \rightarrow \widehat{T}'/\widehat{Z}'$ .

LEMMA 3.16.  $\text{cl}(\mathcal{R}_{u,k'}(G_{k'})) \leq \text{cl}(\mathcal{R}_{u,k'}(\mathbf{R}_{k'/k}(\widehat{G}')_{k'}))$ .

*Proof.* The idea is to embed  $\mathbf{R}_{k'/k}(G') \rtimes C$  in a group of the form  $\mathbf{R}_{k'/k}(\widehat{G}') \rtimes \widehat{C}$  arising from a 4-tuple  $(\widehat{G}', k'/k, \widehat{T}', \widehat{C})$  and a factorization  $\mathbf{R}_{k'/k}(\widehat{T}') \xrightarrow{\widehat{\phi}} \widehat{C} \xrightarrow{\widehat{\psi}} \mathbf{R}_{k'/k}(\widehat{T}'/\widehat{Z}')$ ; the latter group is easier to understand than the former because  $\widehat{Z}'$  is smooth. Define the homomorphism  $\kappa: \mathbf{R}_{k'/k}(T') \rightarrow C \times \mathbf{R}_{k'/k}(\widehat{T}')$  by  $\kappa = \phi \times \iota^{-1}$ , where  $\iota: \mathbf{R}_{k'/k}(\widehat{T}') \rightarrow \mathbf{R}_{k'/k}(\widehat{T}')$  is inversion, and we identify  $\mathbf{R}_{k'/k}(T')$  naturally as a subgroup of  $\mathbf{R}_{k'/k}(\widehat{T}')$ , say via a map  $i_{T'}$ . Note that  $\kappa$  maps  $\mathbf{R}_{k'/k}(T')$  isomorphically onto its image, which we denote by  $N$ . Set



$\widehat{C} = (C \times R_{k'/k}(\widehat{T}'))/N$  and let  $\pi_{\widehat{C}} : C \times R_{k'/k}(\widehat{T}')) \rightarrow \widehat{C}$  be the canonical projection. We claim that  $\widehat{C}$  is pseudo-reductive. To see this, let  $V$  denote the preimage of  $\mathcal{R}_{u,k}(\widehat{C})$  in  $C \times R_{k'/k}(\widehat{T}'))$ . Projection onto the second factor here maps  $N$  isomorphically onto the natural copy of  $R_{k'/k}(T')$  in the second factor. Hence if we project to the second factor and then quotient by this copy of  $R_{k'/k}(T')$ , then the subgroup  $V$  gets mapped to a smooth unipotent  $k$ -subgroup of the commutative group  $R_{k'/k}(\widehat{T}')/R_{k'/k}(T')$ . Since this last group is a subgroup of the commutative pseudo-reductive group  $R_{k'/k}(\widehat{T}'/T')$ , it is pseudo-reductive, and hence we conclude that the projection of  $V$  to the second factor of  $C \times R_{k'/k}(\widehat{T}'))$  is contained in the projection of  $N$ . Hence  $V \subseteq C \times R_{k'/k}(T')$ .

We have the map  $\psi : C \times R_{k'/k}(T') \rightarrow C$  defined by  $\psi(c, x) = c\phi(x)^{-1}$  for  $c \in C(A)$  and  $x \in R_{k'/k}(T')(A)$ , where  $A$  is any  $k$ -algebra. Clearly, the kernel of  $\psi$  is  $N$ , which is smooth, so  $\psi$  is smooth. Hence  $\psi(V)$  is a smooth unipotent  $k$ -subgroup of  $C$  and is therefore trivial. It follows that  $V \subseteq N$ , so  $\mathcal{R}_{u,k}(\widehat{C}) = 1$ , as required.

We have obvious inclusions  $i_C : C \rightarrow C \times R_{k'/k}(\widehat{T}'))$  and  $i' : R_{k'/k}(\widehat{T}') \rightarrow C \times R_{k'/k}(\widehat{T}'))$ . Define  $j : C \rightarrow \widehat{C}$  by  $j = \pi_{\widehat{C}} \circ i_C$ ; then  $j$  is an embedding of  $C$  in  $\widehat{C}$  since  $N \cap (C \times 1) = 1$ . Now define  $\widehat{\phi} : R_{k'/k}(\widehat{T}') \rightarrow \widehat{C}$  by  $\widehat{\phi} = \pi_{\widehat{C}} \circ i'$ . It follows from the construction that  $j \circ \phi = \widehat{\phi} \circ R_{k'/k}(i_{T'})$ .

Consider the homomorphism  $C \times R_{k'/k}(\widehat{T}')) \rightarrow R_{k'/k}(\widehat{T}'/\widehat{Z}')$  given by the composition

$$\begin{aligned}
 C \times R_{k'/k}(\widehat{T}')) &\xrightarrow{\psi \times \text{id}} R_{k'/k}(T'/Z') \times R_{k'/k}(\widehat{T}') \\
 &\xrightarrow{R_{k'/k}(\beta) \times R_{k'/k}(\pi_{\widehat{T}'})} R_{k'/k}(\widehat{T}'/\widehat{Z}') \times R_{k'/k}(\widehat{T}'/\widehat{Z}') \rightarrow R_{k'/k}(\widehat{T}'/\widehat{Z}'),
 \end{aligned}$$

where the third map is multiplication, and  $\pi_{\widehat{T}'} : \widehat{T}' \rightarrow \widehat{T}'/\widehat{Z}'$  is the canonical projection. The kernel of this homomorphism contains  $N$ , so we get an induced homomorphism  $\widehat{\psi} : \widehat{C} \rightarrow R_{k'/k}(\widehat{T}'/\widehat{Z}')$ . It follows from the construction that  $\widehat{\psi} \circ j = R_{k'/k}(\beta) \circ \psi$ . We can now check that the map  $R_{k'/k}(i_{G'}) \times j : R_{k'/k}(G') \rtimes C \rightarrow R_{k'/k}(\widehat{G}') \rtimes \widehat{C}$  is an embedding of groups, where  $i_{G'}$  is the inclusion of  $G'$  in  $\widehat{G}'$ .

Now  $G$  is a quotient of  $R_{k'/k}(G') \rtimes C$ , so

$$\begin{aligned}
 \text{cl}(\mathcal{R}_{u,k'}(G_{k'})) &\leq \text{cl}(\mathcal{R}_{u,k'}((R_{k'/k}(G') \rtimes C)_{k'})) \leq \text{cl}(\mathcal{R}_{u,k'}((R_{k'/k}(\widehat{G}') \rtimes \widehat{C})_{k'})) \\
 &= \text{cl}(\mathcal{R}_{u,k'}(R_{k'/k}(\widehat{G}'/\widehat{Z}')_{k'})) \leq \text{cl}(\mathcal{R}_{u,k'}(R_{k'/k}(\widehat{G}')_{k'}));
 \end{aligned}$$

here the equality follows from Lemma 3.5(ii) applied to  $(\widehat{G}', k'/k, \widehat{T}', \widehat{C})$ , and the final inequality holds because the canonical projection  $\widehat{G}' \rightarrow \widehat{G}'/\widehat{Z}'$  is smooth. This proves the result. □

*Proof of Theorem 1.1.* As in the proof of Proposition 3.12, we may reduce to the case that  $k = k_s$ , and by passing to a fiber of  $G'$  above  $k'$ , we may assume that  $k'/k$  is a purely inseparable field extension. Certainly, we are done if  $G'$  is commutative, so assume otherwise.

If  $G'$  is unusual, then by assumption the map  $R_{k'/k}(T') \rightarrow C$  is a monomorphism. As in the proof of Lemma 3.5, we have that the semidirect product  $R_{k'/k}(G') \rtimes C$  is actually isomorphic to a direct product via  $(g, d) \mapsto (gd, d)$ , and so the quotient by the central subgroup  $N := R_{k'/k}(T') \subseteq R_{k'/k}(G') \rtimes C$  from the standard construction gives the group  $R_{k'/k}(G') \times C'$ , where  $C' \cong C/R_{k'/k}(T')$ . In particular, by Proposition 3.12 the nilpotency class of the unipotent radical is as claimed. Hence we can assume that  $G'$  is not unusual.

We get an upper bound for  $\text{cl}(\mathcal{R}_u(R_{k'/k}(G')_{\bar{k}}))$  from Lemma 3.16. Proposition 3.12 applied to  $\widehat{G}'$ , where  $\widehat{G}'$  is a central enlargement of  $G'$ , then gives the desired upper bound.

To complete the proof, it suffices to show that  $\text{cl}(\mathcal{R}_u(R_{k'/k}(G')_{k'})) \leq \text{cl}(\mathcal{R}_u(G_{k'}))$ .

By Lemma 3.11 and Proposition 3.12,  $G'$  has a subgroup  $H'$  isomorphic to one of  $\text{SL}_2$ ,  $\text{GL}_2$ , or  $\text{PGL}_2$  such that

$$\text{cl}(\mathcal{R}_u(R_{k'/k}(G')_{k'})) = \text{cl}(\mathcal{R}_u(R_{k'/k}(H')_{k'})).$$

Furthermore, if  $p = 2$ , then since  $G'$  is not unusual, it follows that  $H$  is  $\text{GL}_2$  or  $\text{PGL}_2$ . We may regard  $R_{k'/k}(H') \subseteq R_{k'/k}(G')$  as subgroups of  $R_{k'/k}(G') \rtimes C$  in the obvious way. Let  $\pi : R_{k'/k}(G') \rtimes C \rightarrow G$  be the projection of the standard construction. Now  $\ker(\pi)$  is a smooth central subgroup of  $R_{k'/k}(G') \rtimes C$ , and it follows from the definition of  $\pi$  that  $\ker(\pi) \cap R_{k'/k}(G')$  is contained in the kernel of the map  $R_{k'/k}(T') \rightarrow R_{k'/k}(T'/Z')$ , which is  $R_{k'/k}(Z')$  by [CGP15, Prop. 1.3.4]. So  $M := \ker(\pi) \cap R_{k'/k}(H')$  is a central subgroup of  $R_{k'/k}(H')$ . Now  $R_{k'/k}(H')/M$  is a subgroup of  $G$ , so  $\mathcal{R}_u((R_{k'/k}(H')/M)_{\bar{k}})$  is a subgroup of  $\mathcal{R}_u(G_{\bar{k}})$ . But if  $Z_{H'}$  denotes the center of  $H'$ , then

$$\begin{aligned} \text{cl}(\mathcal{R}_u((R_{k'/k}(H')/M)_{\bar{k}})) &\geq \text{cl}(\mathcal{R}_u((R_{k'/k}(H')/R_{k'/k}(Z_{H'}))_{\bar{k}})) \\ &= \text{cl}(\mathcal{R}_u(R_{k'/k}(H'/Z_{H'})_{\bar{k}})) \\ &= \text{cl}(\mathcal{R}_u(R_{k'/k}(\text{PGL}_2)_{\bar{k}})) \end{aligned}$$

since  $Z_{H'}$  is smooth, and we are done by Proposition 3.12. □

### 3.6. When $k'/k$ is a Primitive Field Extension

Here we work out the consequences of Theorem 1.1 in case  $k' = k(t)$  for  $q = p^e$  the order of  $t$  in  $k'/k$ . As ever, we may assume that  $k$  is separably closed so that  $k'/k$  is purely inseparable. In this situation the ideal  $\mathfrak{m} = \langle 1 \otimes t - t \otimes 1 \rangle$  of  $B$  is a principal ideal, and it is easy to see that the minimal  $n$  such that  $\mathfrak{m}^n = 0$  is just  $q$ , since  $(1 \otimes t - t \otimes 1)^q = 1 \otimes t^q - t^q \otimes 1 = t^q \otimes 1 - t^q \otimes 1 = 0$ . Hence the value of  $N$  in Theorem 1.1 is  $q - 1$ . In particular, if  $G'$  is not unusual or commutative, then  $\text{cl}(R_{k'/k}(G')) = q - 1$ .

On the other hand, if  $G'$  is unusual, then  $p = 2$  and  ${}^2\mathfrak{m} = \mathfrak{m}^2$ . From this we readily deduce that the value of  $N$  in Theorem 1.1 is just  $q/2$ , and in particular  $\text{cl}(\mathcal{R}_u(R_{k'/k}(G')_{\bar{k}})) = q/2$ .

REMARK 3.17. Applying the standard construction to the Weil restriction of a reductive group can change the nilpotency class of the unipotent radical; in other words, it can happen that  $\text{cl}(\mathcal{R}_u(\mathbf{R}_{k'/k}(G')_{\bar{k}}))$  is strictly less than  $\text{cl}(\mathcal{R}_u(G_{\bar{k}}))$ . For instance, suppose  $p = 2$  and  $k'/k$  is a primitive purely inseparable field extension of degree  $q \geq 4$ . Let  $G' = \text{SL}_2$ , let  $T'$  be a maximal torus of  $G'$ , and let  $Z' = Z(G')$ . Let  $C = \mathbf{R}_{k'/k}(T'/Z')$  and take the factorization  $\mathbf{R}_{k'/k}(T') \rightarrow C \rightarrow \mathbf{R}_{k'/k}(T'/Z')$  to be the obvious one. Then  $G = (\mathbf{R}_{k'/k}(G') \rtimes C)/\mathbf{R}_{k'/k}(T') \cong \mathbf{R}_{k'/k}(\text{PGL}_2)$ , so  $\text{cl}(\mathcal{R}_u(G_{k'})) = q - 1$ ; but  $\text{cl}(\mathcal{R}_u(\mathbf{R}_{k'/k}(G')_{k'})) = \text{cl}(\mathcal{R}_u(\mathbf{R}_{k'/k}(\text{SL}_2)_{k'})) = \frac{q}{2}$  by the previous remarks.

Note, however, that by Lemma 3.16—which holds for an arbitrary standard pseudo-reductive group—the nilpotency class of  $\mathcal{R}_u(G_{\bar{k}})$  is bounded above by  $n$ , where  $n$  is the smallest positive integer such that  $\mathfrak{m}^n$  vanishes. This is the case even where  $k'/k$  is not primitive.

### 4. Orders of Elements in Unipotent Radicals of Weil Restrictions

In this section, we collect some observations about the orders of elements in unipotent radicals  $\mathcal{R}_u(G_{\bar{k}})(\bar{k})$ , where  $G = \mathbf{R}_{k'/k}(G')$  is the Weil restriction of a reductive group  $G'$ . More precisely, we wish to say something about the exponent  $e$  of  $G$ ; that is, the minimal integer such that the  $p^e$ -power map on  $G$  factors through the trivial group. Clearly,  $e$  is stable under base change, so by Lemma 2.1 it suffices to consider the case where  $k$  is separably closed. Of course, we could let  $k'$  be a nonzero reduced  $k$ -algebra, but by passing to a fiber it does no harm to assume that  $k'/k$  is a purely inseparable field extension.

Since  $k'/k$  is purely inseparable, Proposition 3.1 provides a crude upper bound: as usual, letting  $B = k' \otimes_k k'$  with maximal ideal  $\mathfrak{m}$ , we have a filtration of the unipotent radical  $\mathcal{R}_u(G_{\bar{k}})$  with  $n$  abelian quotients, where  $n$  is the minimal positive integer such that  $\mathfrak{m}^n = 0$ . Since each quotient is a vector group isomorphic to a direct sum of copies of the adjoint module for  $G'$ , all elements in the quotients have order  $p$ , and we see that  $p^n$  is an upper bound for the order of elements in  $\mathcal{R}_u(G)$ . Our next result shows that we can immediately do better than this with quite an elementary argument.

LEMMA 4.1. *With notation as before, let  $s$  be minimal such that  $p^s \geq n$ . Then the exponent of  $\mathcal{R}_u(G)$  is at most  $s$ .*

*Proof.* Let  $\bar{B}$  and  $\bar{\mathfrak{m}}$  be as in Lemma 3.7. Since  $G$  is smooth, it suffices to consider elements of  $G(\bar{k})$ , which we can identify with  $G'(\bar{B})$ . By embedding  $G'$  into  $\text{GL}_r$  for some  $r$ , we may assume that the elements of interest are matrices with entries in  $\bar{B}$ , and we may identify points of the unipotent radical as the kernel of the map  $G'(\bar{B}) \rightarrow G'(\bar{k})$  given by reduction modulo  $\bar{\mathfrak{m}}$ . Then a typical element of the unipotent radical has the form  $I + M$ , where  $I$  is the  $r \times r$  identity matrix, and  $M$  is an  $r \times r$  matrix with entries in  $\bar{\mathfrak{m}}$ . We are therefore after a  $p^s$ th power  $p^s$ , which kills all such matrices  $M$ —for then  $(I + M)^{p^s} = I + M^{p^s} = I$  in all cases.

Now note that the entries of a power  $M^{p^s}$  are homogeneous polynomials in the entries of  $M$  of degree  $p^s$ . Since  $\overline{m}^n = 0$ , all such polynomials vanish as soon as  $p^s$  is at least  $n$ . Hence, choosing  $s$  as in the statement, we are done.  $\square$

The following example shows that the bound provided by Lemma 4.1 is sharp when the rank of  $G'$  is large enough.

EXAMPLE 4.2. Suppose  $G' = \text{GL}_n$  where  $n$  is as before. Then we can find elements  $m_1, \dots, m_{n-1} \in \overline{m}$  such that  $m_1 \cdots m_{n-1} \neq 0$ , and the element

$$x = \begin{pmatrix} 0 & m_1 & 0 & \cdots & 0 \\ 0 & 0 & m_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & m_{n-1} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

has  $x^{n-1} \neq 0$ , so the bound in Lemma 4.1 cannot be improved in this case.

In contrast to this example, we can see in some easy examples that the bound provided by Lemma 4.1 can be much too large.

EXAMPLE 4.3. (i) Suppose  $G = \text{R}_{k'/k}(\text{GL}_2)$  when  $p = 2$  and  $k'/k$  is a purely inseparable field extension of exponent 1 (recall that the *exponent* of such a purely inseparable extension is the minimal  $e$  such that  $x^{p^e} \in k$  for all  $x \in k'$ ). The fact that  $k'/k$  has exponent 1 means in this case that  $m^2 = 0$  for all  $m \in \overline{m}$ , irrespective of the value of  $n$ . Therefore, if we consider any four elements  $m_1, m_2, m_3, m_4 \in \overline{m}$  arranged in a  $2 \times 2$  matrix

$$x = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix},$$

then we have  $x^4 = 0$ , so the maximal order of elements in  $\mathcal{R}_u(G)$  is 4 in this situation. However, we can make the minimal  $n$  such that  $\overline{m}^n = 0$  as large as we like (for example, by letting  $k$  be a field of rational functions in several variables  $T_1, T_2, \dots$  and then adjoining elements  $t_i$  with  $t_i^2 = T_i$  for each  $i$ ).

(ii) Suppose  $G = \text{R}_{k'/k}(\mathbb{G}_m)$ . If the exponent of the extension  $k'/k$  is  $e$ , then because the  $p^e$  power map sends  $(k')^\times$  to  $k^\times$ , we see that elements of  $\mathcal{R}_u(G)$  have order at most  $p^e$ , and some elements do indeed have this order (by the argument for  $\text{GL}_r$  above, taking  $r = 1$ ). Hence, in this case, the exponent of  $\mathcal{R}_u(G)$  coincides with the exponent of the extension  $k'/k$ .

Motivated by the previous examples, which show dependence of the extension  $k'/k$  on the exponent, we finish with a bound for matrices, which depends on that exponent and the rank of the matrices, rather than the number  $n$ .

LEMMA 4.4. *Suppose  $k'/k$  has exponent  $e$ , and let  $G = \text{R}_{k'/k}(\text{GL}_r)$ . Then the exponent of  $\mathcal{R}_u(G)$  is at most  $s$ , where  $s$  is chosen so that  $p^s \geq r^2(p^e - 1)$ . In particular, if  $p \geq r^2$ , then this exponent is at most  $e + 1$ .*

*Proof.* As before, we are after a power  $p^s$ , which kills all  $r \times r$  matrices  $M$  with entries in  $\overline{\mathfrak{m}}$ . Since the exponent is  $e$ , it is true that  $m^{p^e} = 0$  for all  $m \in \overline{\mathfrak{m}}$ . Again observe that a power  $M^{p^s}$  has entries that are homogeneous polynomials of degree  $p^s$  in the  $r^2$  entries of  $M$ . If we take  $p^s$  large enough so that any monomial of degree  $p^s$  in  $r^2$  variables must contain at least one of the variables at least  $p^e$  times, then we know that  $M^{p^s} = 0$ . The claimed bound now follows.  $\square$

REMARKS 4.5. A few comments on this last result:

- (i) When  $p = 2, e = 1$ , and  $r = 2$  in Lemma 4.4, we obtain the bound  $4 = p^{e+1}$ , which was observed in Example 4.3(i). On the other hand, for larger  $e$ , we have to go up to  $p^{e+2}$ .
- (ii) When  $r = 1$  in Lemma 4.4, we obtain the bound  $p^e$ , which was already observed in Example 4.3(ii).
- (iii) The bound from Lemma 4.1 will obviously be much better in some cases than that provided by Lemma 4.4. The tension between these two observations seems to be something that merits further study, and we will return to it in future work.
- (iv) Computer calculations suggest that if  $G := R_{k'/k}(G')$ , then the exponent of  $\mathcal{R}_u(G_{\bar{k}})$  is always the same as the exponent of  $\mathcal{R}_u(G_{\bar{k}}) \cap B_{\bar{k}}$ , where  $B := \mathcal{R}_u(R_{k'/k}(B'))$  for  $B' = T' \times U'$  a Borel subgroup of  $G'$  with split maximal torus  $T'$  (recall that we assumed  $k$  to be separably closed). We do not know a proof that these exponents are the same—assuming, of course, that it is even true. In any case, we can make a little more conceptual progress in obtaining an upper bound for the exponent of  $\mathcal{R}_u(G_{\bar{k}}) \cap B_{\bar{k}}$ , as follows.

The group  $B$  is connected and solvable (since  $B'$  is), so any unipotent subgroup of  $B_{\bar{k}}$  is contained in  $\mathcal{R}_u(B_{\bar{k}})$ . In particular,  $\mathcal{R}_u(G_{\bar{k}}) \cap B_{\bar{k}}$  is contained in  $\mathcal{R}_u(B_{\bar{k}})$ . We now give an upper bound for the exponent of the latter. Observe that  $B \cong C \times U$ , where  $C := R_{k'/k}(T')$  and  $U := R_{k'/k}(U')$ . Then  $\mathcal{R}_u(B_{\bar{k}}) = \mathcal{R}_u(C_{\bar{k}}) \times U_{\bar{k}}$ . Now from Example 4.3(ii) we get that the exponent of  $\mathcal{R}_u(C_{\bar{k}})$  is precisely  $e$ , where  $e$  is the exponent of  $k'/k$ . Thus the  $p^e$ -power map sends  $\mathcal{R}_u(B_{\bar{k}})$  into  $U_{\bar{k}}$ ; denote the image of this map by  $J$ . We know that the exponent of  $U'$  is the smallest integer  $s$  such that  $p^s > h - 1$ , where  $h$  is the Coxeter number of  $G'$ ; see [Tes95, Order Formula 0.4]. Thus the  $p^s$ -power map on  $U'$  factors through 1, and so the  $p^s$ -power map on  $U = R_{k'/k}(U')$  also factors through 1. Since  $J \subseteq U_{\bar{k}}$ , the exponent of  $J$  is at most  $s$ . So we get an upper bound of  $e + s$  for the exponent of  $\mathcal{R}_u(B_{\bar{k}})$  (and hence for the exponent of  $\mathcal{R}_u(G_{\bar{k}}) \cap B_{\bar{k}}$ ) though this bound can be further improved with knowledge of the structure of  $k'$ .

Elementary calculations show that when  $G' = \text{GL}_2$ , we have  $J = 1$  when  $k'/k$  is primitive, giving  $e$  as the exponent, and  $J \neq 1$  when  $k'/k$  is imprimitive; since all elements of  $J$  have order  $p$ , we then get  $e + 1$  as the exponent. To explain what happens in the imprimitive case, find  $t \in k'$  with exponent  $e$

in  $k'/k$ ; since  $k'$  is imprimitive, we may find also  $u \in k' \setminus k(t)$ . We check

$$\begin{aligned} & \begin{pmatrix} 1 + 1 \otimes t - t \otimes 1 & 1 \otimes u - u \otimes 1 \\ 0 & 1 \end{pmatrix}^{p^e} \\ &= \begin{pmatrix} 1 & (1 \otimes u - u \otimes 1)(1 \otimes t - t \otimes 1)^{p^e - 1} \\ 0 & 1 \end{pmatrix} \neq 1. \end{aligned}$$

More generally, we ask the following:

QUESTION 4.6. Suppose  $G'$  is not unusual and let  $r = \log_p([k' : k(k')^p]) - 1$ . Let  $s = \max\{\lceil \log_p(h_{L'} - 1) \rceil\}$ , where the maximum is over all Coxeter numbers  $h_{L'}$  of Levi subgroups  $L'$  of  $G'$  with rank at most  $r$ . Is the exponent of  $\mathcal{R}_u(B_{\bar{k}})$  equal to  $\max\{e + s, \lceil \log_p(h_{G'} - 1) \rceil\}$ ?

REMARK 4.7. Note that the bounds above fail in the context of a general standard pseudo-reductive group. For instance, given a finite purely inseparable field extension  $k'/k$  with exponent  $e$ , we can take  $G$  to be the standard group arising from the 4-tuple  $(G', k'/k, T', C)$ , where  $G' = T' = 1$  and  $C = \mathbf{R}_{k'/k}(\mathbb{G}_m)$  with  $k'/k$  a finite purely inseparable field extension of arbitrarily large exponent  $f$ : for then  $G \cong C$  has exponent  $f$  by Example 4.3(ii).

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## References

- [Bor91] A. Borel, *Linear algebraic groups*, Grad. Texts in Math., 126, second edition, Springer-Verlag, New York, 1991.
- [Con12] B. Conrad, *Finiteness theorems for algebraic groups over function fields*, Compos. Math. 148 (2012), no. 2, 555–639.
- [CGP15] B. Conrad, O. Gabber, and G. Prasad, *Pseudo-reductive groups*, New Math. Monogr., 26, second edition, Cambridge University Press, Cambridge, 2015.
- [Mar99] B. M. S. Martin, *Étale slices for representation varieties in characteristic  $p$* , Indag. Math. (N.S.) 10 (1999), no. 4, 555–564.
- [Mil12] J. S. Milne, *Basic theory of affine group schemes*, 2012, ([www.jmilne.org/math/](http://www.jmilne.org/math/)).
- [Oes84] J. Oesterlé, *Nombres de Tamagawa et groupes unipotents en caractéristique  $p$* , Invent. Math. 78 (1984), no. 1, 13–88.
- [Spr98] T. A. Springer, *Linear algebraic groups*, Progr. Math., 9, second edition, Birkhäuser Boston Inc., Boston, MA, 1998.
- [Tes95] D. M. Testerman,  *$A_1$ -type overgroups of elements of order  $p$  in semisimple algebraic groups and the associated finite groups*, J. Algebra 177 (1995), no. 1, 34–76.

[Tit91/92] J. Tits, *Théorie des groupes*, Annu. Coll. Fr. 92 (1991/92), 115–133 (1993).

[Tit92/93] ———, *Théorie des groupes*, Annu. Coll. Fr. 93 (1992/93), 113–131 (1994).

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