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SUMMAND ABSORBING SUBMODULES OF A MODULE OVER A SEMIRING

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Abstract. An R-module V over a semiring R lacks zero sums (LZS) if $x + y = 0 \Rightarrow x = y = 0$. More generally, a submodule W of V is summand absorbing in V if $\forall x, y \in V: \ x + y \in W \ \Rightarrow \ x \in W, \ y \in W$. These arise in tropical algebra and modules over idempotent semirings. We explore the lattice of summand absorbing submodules of a given LZS module, especially those that are finitely generated, in terms of the lattice-theoretic Krull dimension, and describe their explicit generation.

Contents

1. Introduction

Throughout this paper, R is a semiring [2], and V is a (left) module (sometimes called "semimodule") over R; i.e., $(V, +)$ is a semigroup satisfying the familiar module axioms as well as $r0_V = 0_R x = 0_V$ for all $r \in R$, $x \in V$. The zero submodule $\{0_V\}$ is usually written as 0. We denote the set of all submodules of V by $\mathrm{Mod}(V)$. Since we cannot take quotient modules V/W over semirings, we also write $\mathrm{Mod}(V; W)$ for the submodules of V containing W.

An R-module V over a semiring R **lacks zero sums** (where the term "zerosumfree" is used in [2]), abbreviated LZS, if

$$\forall\, x, y \in V: \ x + y = 0 \ \Rightarrow \ x = y = 0. \qquad \text{(LZS)}$$

In this paper we continue the theory of LZS modules from [4], which applies immediately to tropical algebra and modules over idempotent semirings. Our results apply in particular

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to additive semigroups (with neutral element), which can be viewed as modules over the semiring $\mathbb{N}_0$ of natural numbers including zero.

The condition (LZS) follows at once from the condition, called **upper bound** (ub), that $a+b+c=a$ implies $a+b=a$. (We take $a=0$.) In [4, §6] the obstruction in an R-module V to the ub condition was studied in terms of Green's partial pre-order, a binary relation $\preceq$ (more precisely denoted $\preceq_V$) given by

$$x \preceq y \quad \Leftrightarrow \quad \exists\, z \in V : x+z=y.$$

This is a pre-order (also called a "quasi-ordering") on V; in other words, $\preceq$ is reflexive ($x \preceq x$) and transitive ($x \preceq y$, $y \preceq z \Rightarrow x \preceq z$), but not necessarily antisymmetric. (Green's partial preorder is generated by the relations $0_V \preceq z$, since $x+z=y$ implies $x = x+0_V \preceq x+z=y$.)

To get a better understanding of this property, we introduce an equivalence relation $\equiv$ on V as follows:

$$x \equiv y \quad \Leftrightarrow \quad x \preceq y, \ y \preceq x.$$

As observed after [4, Definition 6.1], $\equiv$ is a congruence, implying $\overline{V} := V/\equiv$ is again an R-module in the obvious way, with the induced operations $\bar{x}+\bar{y} = \overline{x+y}$ and $r\bar{x} = \overline{rx}$, where $\bar{x}$ denotes the equivalence class of x. We have the well-defined partial ordering $\leq$ on the R-module $\overline{V}$ induced by $\preceq$:

$$\bar{x} \leq \bar{y} \quad \Leftrightarrow \quad x \preceq y$$

for any $x, y \in V$, which is compatible with addition and scalar multiplication.

Proposition 1.1 ([4, Proposition 6.2])**.**

(i) *The monoid $\overline{V}$ is upper bound.*

(ii) *V is upper bound if and only if the congruence $\equiv$ is trivial.*

For $x \preceq y$ we want to examine those $z \in V$ such that $x \preceq z \preceq y$. For example, if $\alpha+\beta=1$, then $x \preceq \alpha x+\beta y \preceq y$, since $x = \alpha x+\beta x \preceq \alpha x+\beta y$ and $\alpha x+\beta y \preceq \alpha y+\beta y = y$.

Definition 1.2. *A subset S of V is **convex** (in V) if for any $x, y, z \in V$:*

$$x \in S, \ y \in S, \ x \preceq z \preceq y \quad \Rightarrow \quad z \in S.$$

It is easily seen that a set $S \subset V$ is convex in V iff S partitions into full equivalence classes (of $\equiv$), and the image $\overline{S}$ in the partially ordered set $\overline{V}$ is convex.

1.1. Summand absorbing submodules.

We turn to the main notion of this paper, examining the LZS condition in terms of a related property which we call SA.

Definition 1.3. *A submodule W of V is **summand absorbing** (abbreviated **SA**) in V if*

$$\forall\, x, y \in V: \ x+y \in W \quad \Rightarrow \quad x \in W, \ y \in W; \tag{SA}$$

*we then say that W is an **SA-submodule** V. A submodule U of V is a ΣSA-**submodule** of V (or: an SA-**sum** in V) if $U = \sum_{i \in I} W_i$ for some family $(W_i \mid i \in I)$ of SA-submodules.*

Remarks 1.4.

a) *The R-module V is LZS iff 0 is an SA-submodule of V.*

b) *V is ub iff the semigroup $V_y := \{x \in V : x+y=y\}$ is SA (as an additive semigroup) for each $y \in V$.*

We also recall the notion of a weak complement of a submodule W of V.

Definition 1.5 ([4, Definition 1.2]). *A submodule T of V is a **weak complement** of W (in V), denoted $V = W \oplus_w T$, if $V = W + T$ and for every $w \in W \setminus \{0\}$ the intersection $(w + T) \cap T$ is empty.*

Thus

$$V = W \oplus T \quad \Rightarrow \quad V = W \oplus_w T,$$

although $V = W \oplus_w T$ does not necessarily imply that $V = T \oplus_w W$.

Lemma 1.6 ([4, Lemma 2.2]). *Suppose that W is a submodule of an LZS module V. Then T is a weak complement of W, if and only if T is SA with $T \cap W = 0$.*

Thus the SA property is a natural continuation of the research in [4]. The following result, proved in [4, Lemma 2.3], leads to a theory of decompositions in tropical algebra and related structures, much stronger than the classical theory, since one gets unique decompositions.

Lemma 1.7. *Suppose V has an SA-submodule T. Then any decomposition of V descends to a decomposition of T, in the sense that if $V = Y + Z$, then $T = (T \cap Y) + (T \cap Z)$.*

Accordingly, we are led to study SA-submodules in their own right, particularly when they are finitely generated.

$\mathrm{SA}(V)$ denotes the poset consisting of all SA-submodules of V, partially ordered by inclusion. We also write $\mathrm{SA}(V; W)$ for the SA-submodules of V containing W. SA_f denotes the finitely generated SA-submodules. The set of all SA-sum submodules is denoted as $\Sigma\,\mathrm{SA}(V)$, regarded again as a poset by the inclusion relation (containing $\mathrm{SA}(V)$ as a sub-poset). $\Sigma\,\mathrm{SA}_f$ is the set of all sums of finitely generated SA-submodules.

Proposition 1.8 ([4, Proposition 5.7]). *A submodule W of V is in $\mathrm{SA}(V)$ iff W is a union of equivalence classes and $\overline{W}$ is SA in $\overline{V}$.*

In other words, the elements of $\mathrm{SA}(V)$ are just the convex submodules of V under the relation $\preceq$. We denote the convex hull $\mathrm{conv}_V(W)$ of an R-submodule W of V more concisely as $\widehat{W}$.

Any family $(W_i \mid i \in I)$ in the poset $\mathrm{SA}(V)$ has the infimum $\bigcap_{i \in I} W_i$ and the supremum $\big(\sum_{i \in I} W_i\big)^\wedge$ in $\mathrm{SA}(V)$, and so $\mathrm{SA}(V)$ is a complete lattice (in contrast to $\Sigma\,\mathrm{SA}(V)$, cf. Remark 2.10) below. Furthermore, we shall show in Proposition 3.1 that $\mathrm{SA}(V)$ is a modular lattice. Accordingly many tools of classical module theory become available.

The first part of the paper (§2–§3) covers the general theory of SA-submodules. In §2 we continue the theory of [4], and introduce decompositions of SA-modules, called "SA-decompositions," proving the following results:

Theorem 2.2. Assume that W and T are submodules of V with $W + T = V$, $W \cap T = 0$, and furthermore, that T is an SA-submodule of V. Then $V = W \oplus_w T$.

Theorem 2.13. Any R-module V has at most one SA-decomposition $(T_i \mid i \in I)$, where all T_i are SA-indecomposable. This is the finest SA-decomposition of V.

Theorem 2.16. *Assume that W and T are submodules of V with $T \in \mathrm{SA}(V)$, $W + T = V$, $W \cap T = 0$, (whence $V = W \oplus_w T$ by Theorem 2.2). Let $(v_\lambda \,|\, 1 \leq \lambda \leq d)$ be a system of generators of V. Write $v_\lambda = w_\lambda + t_\lambda$ with $w_\lambda \in W$, $t_\lambda \in T$. Then $(t_\lambda \,|\, 1 \leq \lambda \leq d)$ is a system of generators of T.*

Theorem 2.18. *The SA-decompositions of R correspond uniquely to the complete orthogonal systems of idempotents of R.*

In §3 we develop a structure theory of SA-submodules along the classical lines of the socle and various analogs of dimension theory, including SA-Kdim, which is Krull dimension in the sense of [3] (or, more precisely, [5]), but based on SA-submodules. The theory is made more explicit in §3.2 by use of **SA-uniform** submodules and the **SA-uniformity dimension** (Definition 3.28), described in Theorem 3.30.

In the second of the paper (§4–§6) we exhibit a reasonably broad class of R-modules V, over an arbitrary semiring R, called **finitely** SA**-accessible**, for which we can obtain some stronger results. We say that V is SA_f**-hereditary** if submodules of SA_f-submodules are SA_f-submodules. Those SA_f-hereditary modules with SA_f-Kdim, called SAF-accessible, are examined in §4, and put to use in §5 and §6, where the **height** of a module is defined in terms of the Krull dimension, and decisive results are obtained for modules in terms of their height.

In the third part of the paper (§7–§9) we study generation of SA-submodules. §7 brings in a somewhat technical condition, called **spines** on R, built from **halos**, which permit rather efficient generation of modules, cf. Theorem 7.10, and these are presented over V in §8. One main result:

Theorem 8.3. *Assume that S is an additive spine of an R-module V. Then every SA-submodule W of V is generated by $W \cap S$, and moreover $W \cap S$ is an additive spine of W.*

For rings, this specializes to:

Theorem 7.8. *Assume that S is a set of generators of a (left) R-module V, and M is an additive spine of R. Then any SA-submodule W of V is generated by the set $W \cap (MS)$.*

An application to matrices is given in Theorem 7.14, and more generally to monoid semirings in Theorem 7.17.

In §9 we obtain rather satisfactory results about the SA-submodules and the ΣSA-submodules of a finitely generated module V over a semiring R which has a finite additive spine. The main reason is that in this case all SA-submodules of V are finitely generated.

2. Preliminary results

We make a fresh start, reworking easy facts from [4, §4] in a slightly different way which fits better into the present chain of arguments than a mere citation of parts of [4] would do. Our goal is to compare different "weak" decompositions[1] of V into SA-submodules.

2.1. Three basic principles.

For basic facts on SA-submodules we refer to [4], now being content to recall three general principles from that paper.

A) *If $f : V \to V'$ is an R-linear map between R-modules and W' is an SA-submodule of V' then (clearly) $f^{-1}(W')$ is SA in V.*

Using this principle we immediately see that if V is LZS and W is a direct summand of V, i.e., $V = W \oplus T$, then W is in $\mathrm{SA}(V)$, since W is the kernel of a projection $p : V \to V$ with $p(V) = T$. (But usually an R-module V lacking zero sums has many more SA-submodules than direct summands.) We note in passing that for this conclusion it suffices to assume that T (instead of V) is LZS.

[1]In contrast to direct decompositions, a formal definition of such decompositions, named "SA-decompositions", will be given only later (Definition 2.8).

B) *If* $(W_i \mid i \in I)$ *is a family of SA-submodules of* V *then (evidently)* $\bigcap_{i\in I} W_i$ *is SA in* V. *If the family* (W_i) *is upward directed, i.e., for* $i, j \in I$ *there exists* $k \in I$ *with* $W_i \subset W_k$, $W_j \subset W_k$, *then also* $\bigcup_{i\in I} W_i$ *is in* $\mathrm{SA}(V)$.

C) *A subset* S *containing* 0_V *is convex in* V *if and only if* S *is SA* [4, Lemma 6.6].

Our paper [4] contains various facts about weak complements, but more can be done. Our results below are rooted in analyzing the situation where both $V = W \oplus_w T$ and $V = T \oplus_w W$ hold, cf. Definition 1.5.

Theorem 2.1 ([4, Lemmas 2.1 and 2.2])**.** *Assume that* W *is a submodule of* V *which is LZS, and that* T *is a weak complement of* W *in* V. *Then the module* T *is in* $\mathrm{SA}(V)$. *If moreover* T *also is LZS, then* V *is LZS.*

Proof. The first assertion is by Lemma 1.6. Assume that $v_1, v_2 \in V$ and $v_1 + v_2 = t \in T$. Then $v_i = w_i + t_i$ with $w_i \in W$, $t_i \in T \quad (i = 1, 2)$. By adding we obtain

$$(w_1 + w_2) + (t_1 + t_2) = t.$$

By Definition 1.5 this implies $w_1 + w_2 = 0$ and $t_1 + t_2 = t$. Since W is LZS, it follows that $w_1 = w_2 = 0$, and so $v_i = t_i \in T$. Thus T is in $\mathrm{SA}(V)$.

If T is LZS and $t = 0$, i.e. $v_1 + v_2 = 0$, we obtain from $t_1 + t_2 = 0$ that $t_1 = t_2 = 0$, whence $v_1 = v_2 = 0$. So V is LZS. □

Theorem 2.2. *Assume that* W *and* T *are submodules of* V *with* $W + T = V$, $W \cap T = 0$, *and furthermore, that* T *is an SA-submodule of* V. *Then* $V = W \oplus_w T$.

Proof. Let $w \in W \setminus \{0\}$, $t \in T$, and suppose that $w + t \in T$. Then $w \in T$ since T is SA in V. We conclude that $w \in T \cap W = \{0\}$, a contradiction. Thus $(w + T) \cap T = \emptyset$. □

Lemma 2.3. *Assume that* W, T, U *are submodules of* V, *such that* $V = W \oplus_w T$ *and* $V = W + U$. *Then* $T \subset U$.

Proof. The module T is in $\mathrm{SA}(V)$ (Theorem 2.1). Thus $V = W + U$ implies that

$$T = (T \cap W) + (T \cap U).$$

But $T \cap W = \{0\}$, whence $T = T \cap U$, i.e., $T \subset U$. □

Theorem 2.4 (Uniqueness of weak complements, cf. [4, Corollary 2.5])**.** *Assume that* $V = W \oplus_w T$ *and* $V = W \oplus_w U$. *Then* $T = U$.

Proof. By Lemma 2.3 we have $T \subset U$ and $U \subset T$. □

Theorem 2.5. *Assume that* W *and* T *are submodules of* V *with* $W + T = V$ *and* $W \cap T = 0$, *and furthermore that both* W *and* T *are LZS. Then the following conditions are equivalent.*

(1) W, T *are SA-submodules of* V.

(2) $V = W \oplus_w T = T \oplus_w W$.

If (1), (2) hold, then V *is LZS.*

Proof. (1) ⇒ (2) is clear by Theorem 2.2, and (2) ⇒ (1) is clear by Theorem 2.1. If (1), (2) hold, then it follows by the last sentence in Theorem 2.1 that V is LZS. □

The following lemma will be useful.

Lemma 2.6. *Assume that* W, T, Y, Z *are submodules of* V *with*

$$V = W \oplus_w T = Y + Z.$$

Assume also that W *is LZS.*

a) *Then* $T = (T \cap Y) + (T \cap Z)$.

b) *If moreover* $V = Y \oplus_w Z$, *then* $T = (T \cap Y) \oplus_w (T \cap Z)$.

Proof. a): This is clear since T is in $\mathrm{SA}(V)$ (Theorem 2.1).

b): If $y \in T \cap Y$ and $y \neq 0$ then $(y + Z) \cap Z = \emptyset$, and so

$$[y + (T \cap Z)] \cap T \cap Z = \emptyset.$$

□

2.2. SA**-decompositions.**

We assume throughout that the R-**module** V **is LZS** (and so all submodules of V also are LZS), a natural hypothesis in view of the preceding Theorems 2.1 and 2.5. Given two decompositions

$$V = W_1 \oplus_w W_1' = W_2 \oplus_w W_2', \tag{2.1}$$

we know by Theorems 2.1 and 2.5 that all four submodules W_1, W_1', W_2, W_2' are in $\mathrm{SA}(V)$, and that

$$V = W_1' \oplus_w W_1 = W_2' \oplus_w W_2.$$

By Lemma 2.6 we have decompositions

$$\begin{aligned} W_1 &= W_1 \cap W_2 \ \oplus_w \ W_1 \cap W_2', \\ W_1' &= W_1' \cap W_2 \ \oplus_w \ W_1' \cap W_2', \end{aligned} \tag{2.2}$$

and analogous decompositions of W_2, W_2'. By adding we obtain

$$V = [(W_1 \cap W_2) \oplus_w (W_1 \cap W_2')] \oplus_w [(W_1' \cap W_2) \oplus_w (W_1' \cap W_2')]. \tag{2.3}$$

Since $W_1' \cap W_2'$ is in $\mathrm{SA}(V)$ we conclude by Theorem 2.2 that

$$V = [(W_1 \cap W_2) + (W_1 \cap W_2') + (W_1' \cap W_2)] \oplus_w W_1' \cap W_2'. \tag{2.4}$$

Furthermore, by adding the equalities

$$\begin{aligned} W_1 &= W_1 \cap W_2 \ + \ W_1 \cap W_2', \\ W_2 &= W_1 \cap W_2 \ + \ W_1' \cap W_2, \end{aligned}$$

we obtain

$$W_1 + W_2 = W_1 \cap W_2 \ + \ W_1 \cap W_2' \ + \ W_1' \cap W_2. \tag{2.5}$$

Comparing (2.4) and (2.5) we learn the following.

Proposition 2.7. *The present situation* (2.1) *implies that*

$$V = (W_1 + W_2) \oplus_w (W_1' \cap W_2'). \tag{2.6}$$

But we do not know whether $W_1 + W_2$ is in $\mathrm{SA}(V)$ or not. If W_1, W_2 are **direct** summands of V then we know from [4, Theorem 2.9] that $W_1 + W_2$ is a direct summand of V, and so $W_1 + W_2$ is an SA-submodule of V. This difficulty prompts the following somewhat unusual definition of "SA-decompositions" and "SA-summands."

Definition 2.8.

a) *We call a family* $(T_i \mid i \in I)$ *in* $\mathrm{SA}(V)$ ***orthogonal***, *if for any two indices* $i \neq j$, *the intersection* $T_i \cap T_j$ *equals* 0.

b) *An* SA-***decomposition*** *of* V *is an orthogonal family* $(T_i \mid i \in I)$ *in* $\mathrm{SA}(V)$ *with all* $T_i \neq 0$, *which spans* V, *i.e.,* $V = \sum_{i \in I} T_i$.

c) *We call a submodule* T *of* V *an* SA-***summand*** *of* V *if* T *is a member of an SA-decomposition* $(T_i \mid i \in I)$ *of* V, *i.e.,* $T = T_i$ *for some* $i \in I$.

d) *We say that* V *is* SA-***indecomposable****, if* V *does not have any SA-summand* $T \neq V$*, i.e.,* $\{V\}$ *is the unique SA-decomposition of* V*.*

Remark 2.9. *If* $(T_i \mid i \in I)$ *is an orthogonal family in* $\mathrm{SA}(V)$ *with all* $T_i \neq 0$*, then* $(T_i \mid i \in I)$ *is an SA-decomposition of* $W = \sum_i T_i$*, since every* T_i *is also an SA-submodule of* W*.*

Any family $(U_\lambda \mid \lambda \in \Lambda)$ in $\Sigma\,\mathrm{SA}(V)$ has the supremum $\sum_{\lambda\in\Lambda} U_\lambda$ in $\Sigma\,\mathrm{SA}(V)$, but even for two SA-sums U_1, U_2 no infimum in $\Sigma\,\mathrm{SA}(V)$ is in sight. This changes if one of the modules U_1, U_2 is in $\mathrm{SA}(V)$.

Remark 2.10. *If* $W \in \mathrm{SA}(V)$*,* $U \in \Sigma\,\mathrm{SA}(V)$ *and* $U = \sum_{i\in I} W_i$ *with* $W_i \in \mathrm{SA}(V)$*, then*

$$W \cap U = \sum_{i\in I}(W \cap W_i), \tag{2.7}$$

and so $W \cap U$ *is an SA-sum in* V*. The module* $W \cap U$ *is the infimum of* W *and* U *in* $\Sigma\,\mathrm{SA}(V)$*. Furthermore* $W + U$ *is the supremum of* W *and* U *in* $\Sigma\,\mathrm{SA}(V)$*.*

We cannot build up finite SA-decompositions from binary SA-decompositions, as is common for finite *direct* decompositions, but nevertheless a finite SA-decomposition may be viewed as an iterated formation of weak complements of $\Sigma\,\mathrm{SA}$-modules, due to the following fact.

Proposition 2.11. *Assume that* $(T_i \mid 1 \leq i \leq n)$ *is a finite orthogonal family in* $\mathrm{SA}(V)$*. Let* $U := \sum_{i=1}^{n} T_i \in \Sigma\,\mathrm{SA}(V)$*. Then the chain in* $\Sigma\,\mathrm{SA}(V)$

$$U_0 = \{0_V\} \subset U_1 \subset U_2 \subset \cdots \subset U_n = U \tag{$*$}$$

with $U_r := \sum_{i=1}^{r} T_i \quad (1 \leq r \leq n)$ *has the property*

$$U_{r+1} = U_r \oplus_w T_{r+1} \quad (0 \leq r \leq n). \tag{2.8}$$

Proof. This follows from Theorem 2.2 since $U_{r+1} = U_r + T_{r+1}$ and

$$U_r \cap T_{r+1} = \sum_{i=1}^{r}(T_i \cap T_{r+1}) = 0.$$

□

Note that conversely the chain $(*)$ in $\Sigma\,\mathrm{SA}(V)$ determines the family $(T_i \mid 1 \leq i \leq n)$, due to the uniqueness of weak complements (Theorem 2.4). If an infinite orthogonal family $(T_i \mid i \in I)$ in $\mathrm{SA}(V)$ is given, then we see in the same way that for any module $U_J := \sum_{i\in J} T_i$ and $k \in J$ we have

$$U_J = U_{J\setminus\{k\}} \oplus_w T_k. \tag{2.9}$$

Definition 2.12. *Given two SA-decompositions* $(T_i \mid i \in I)$*,* $(S_j \mid j \in J)$ *of* V *we say that the second SA-decomposition* ***refines*** *the first one, if every module* S_j *is contained in some module* T_i*.*

If this happens then clearly every S_j is contained in a unique module T_i, since different members of $(T_i \mid i \in I)$ have intersection zero. We thus have a unique map $\lambda : J \to I$ with

$S_j \subset T_{\lambda(j)}$ for each $j \in J$. This map λ is surjective, since otherwise $(S_j \mid j \in J)$ would not span V. It follows that for every $i \in T$

$$T_i = \sum_{\lambda(j)=i} S_j \tag{2.10}$$

and so $(S_j \mid \lambda(j) = i)$ is an SA-decomposition of T_i. Now the following is obvious.

Theorem 2.13. *Any R-module V has at most one SA-decomposition $(T_i \mid i \in I)$, where all T_i are SA-indecomposable. This is the finest SA-decomposition of V.*

Proposition 2.14. *Any two SA-decompositions have a common refinement.*

Proof. Assume that $(T_i \mid i \in I)$ and $(S_j \mid j \in J)$ are two decompositions of V. We have

$$V = \sum_{i \in I} T_i = \sum_{j \in J} S_j.$$

Then, since $T_i \in \mathrm{SA}(V)$, we have

$$T_i = \sum_{j \in J} T_i \cap S_j$$

(cf. Lemma 2.6.a), and so

$$V = \sum_{(i,j) \in I \times J} T_i \cap S_j. \tag{$*$}$$

Furthermore, $(T_i \cap S_j) \cap (T_k \cap S_\ell) = 0$ if $(i,j) \neq (k,\ell)$. Let K denote the subset of $I \times J$ consisting of all (i,j) with $T_i \cap S_j \neq 0$. Then $(T_i \cap S_j \mid (i,j) \in K)$ is a common refinement of the SA-decompositions $(T_i \mid i \in I)$ and $(S_j \mid j \in J)$. □

It is evident that the SA-decomposition just constructed is the **coarsest common refinement** of the SA-decompositions $(T_i \mid i \in I)$ and $(S_j \mid j \in J)$ of V.

Proposition 2.15. *If V is finitely generated, then every SA-decomposition $(T_i \mid i \in I)$ of V is finite (i.e., I is finite).*

Proof. We pick a set of generators $\{s_1, \dots, s_r\}$ of V. For every $k \in \{1, \dots, r\}$ there is a finite subset I_k of I such that $s_k \in \sum_{i \in I_k} T_i$, whence $Rs_k \subset \sum_{i \in I_k} T_i$. Thus

$$V = \sum_{k=1}^{r} Rs_k \subset \sum_{i \in J} T_i$$

with $J := \bigcup_{k=1}^{r} I_k$ finite. Suppose that $J \neq I$. Then choosing some $\ell \in I \setminus J$ we have $T_\ell \subset \sum_{i \in J} T_i$. But this is impossible since all intersections $T_\ell \cap T_i$ with $i \in J$ are zero, and so

$$T_\ell = \bigcup_{i \in J} (T_\ell \cap T_i) = 0.$$

Thus $J = I$. □

Theorem 2.16. *Assume that W and T are submodules of V with $T \in \mathrm{SA}(V)$, $W + T = V$, $W \cap T = 0$, (whence $V = W \oplus_w T$ by Theorem 2.2). Let $(v_\lambda \,|\, 1 \leq \lambda \leq d)$ be a system of generators of V. Write $v_\lambda = w_\lambda + t_\lambda$ with $w_\lambda \in W$, $t_\lambda \in T$. Then $(t_\lambda \,|\, 1 \leq \lambda \leq d)$ is a system of generators of T.*

Proof. $V = \sum_{\lambda\in\Lambda} R(w_\lambda + t_\lambda)$. It follows that

$$V = \sum_{\lambda\in\Lambda} Rw_\lambda + \sum_{\lambda\in\Lambda} Rt_\lambda.$$

Intersecting with T we obtain

$$T = \underbrace{(\Sigma Rw_\lambda) \cap T}_{0} + \left(\sum_{\lambda\in\Lambda} Rt_\lambda \right) \cap T = \sum_{\lambda\in\Lambda} Rt_\lambda.$$

□

Corollary 2.17. *Let $(T_i \mid i \in I)$ be an SA-decomposition of V. Then V is finitely generated iff I is finite and each R-module T_i is finitely generated.*

Proof. This is an immediate consequence of Proposition 2.15 and Theorem 2.16. □

In the case that R is commutative and $V = R$, considered as an R-module, we obtain the following explicit description of all SA-decompositions of R. First note that by Proposition 2.15 all SA-decompositions of R are finite.

Theorem 2.18. *Assume that the semiring R is commutative. Then the SA-decompositions $(T_i \mid 1 \le i \le n)$ of R are the families $(e_iR \mid 1 \le i \le n)$ given by the complete finite orthogonal systems $(e_i \mid 1 \le i \le n)$ of idempotents of R $\left(e_ie_j = \delta_{ij}e_i,\ \sum_{i=1}^n e_i = 1\right)$. In this way the SA-decompositions of R correspond uniquely to the complete orthogonal systems of idempotents of R. Every SA-decomposition of R is a direct decomposition of R.*

Proof. If $(e_i \mid 1 \le i \le n)$ is a complete family of orthogonal idempotents of R, then it is plain that

$$R = \bigoplus_{i=1}^n e_iR,$$

and so $(e_iR \mid 1 \le i \le n)$ is also an SA-decomposition of R.

Conversely assume that $(T_i \mid 1 \le i \le n)$ is an SA-decomposition of R. Then all T_i are ideals of R, and so $T_iT_j \subset T_i \cap T_j = 0$ for $i \ne j$. We pick elements $e_1, \dots, e_n$ of R with $e_i \in T_i$ and

$$1 = e_1 + \dots + e_n.$$

Multiplying by e_k for some $k \in \{1, \dots, n\}$ we obtain

$$e_k = \sum_{i=1}^n e_ke_i.$$

But for $k \ne i$ we have $e_ke_i \in T_k \cap T_i = 0$, and conclude that $e_k = e_k^2$. Thus $\{e_1, \dots, e_n\}$ is a complete system of orthogonal idempotents of R. If $x \in T_i$ then

$$x = \left(\sum_{j=1}^n e_j \right) x = \sum_{j=1}^n e_jx = e_ix,$$

since $e_jx \in T_jT_i = 0$ for $j \ne i$. Conversely if $x \in R$ and $x = e_ix$ then $x \in T_i$, since $e_iR \subset T_i$. This proves that $T_i = e_iR$. The e_i are uniquely determined by the family of submodules $(T_i \mid 1 \le i \le n)$ of R, since $R = \sum_{i=1}^n T_i$ and $e_ix = x$ for $x \in T_i$, while $e_jx = 0$ for $x \in T_j$. □

Remark 2.19. *If $(T_i \mid i \leq i \leq n)$ is an SA-decomposition of a commutative semiring R, viewed as an R-module, then the T_i are ideals of the semiring R with $T_i \cap T_j = T_iT_j = 0$ for $i \neq j$, and they can be viewed as semirings having as unit elements the idempotents e_i from above. Thus an SA-decomposition of R as an R-module is the same as a finite direct product decomposition*

$$R = T_1 \times \cdots \times T_n = \prod_{i=1}^{n} T_i$$

of R as a semiring.

We illustrate Theorem 2.18 by some examples. Let X be a topological space, and, as common, let $C(X)$ denote the ring of continuous $\mathbb{R}$-valued functions on X. This ring is equipped with the "function ordering", where $f \leq g$ iff $f(x) \leq g(x)$ for all $x \in X$. Furthermore let $C^+(X)$ denote the positive cone of this partial ordering on $C(X)$, i.e.,

$$C^+(X) := \{f \in C(X) \mid f \geq 0\},$$

a semiring lacking zero sums. Our interest is in the SA-submodules of the semiring $C^+(X)$, viewed as a $C^+(X)$-module. Note that $C^+(X)$ is the set of all continuous functions on X with values in $\mathbb{R}_{\geq 0} = [0, \infty[$. The restriction of the function ordering to $C^+(X)$ coincides with the minimal ordering, since for $f \leq g$ in $C^+(X)$ we have

$$g = f + (g - f),$$

and $g - f \in C^+(X)$.

It is plain that the function $f \in C(X)$ is an idempotent of $C(X)$ iff f has only values in $\{0, 1\}$, and so f is the characteristic function χ_U of a clopen (= closed and open) subset of X_j; $\chi_U(x) = 1$ if $x \in U$, $\chi_U(x) = 0$ if $x \in X \setminus U$. All these idempotents lie in $C^+(X)$. Thus a complete orthogonal system $(e_i \mid i \leq i \leq n)$ of $C^+(X)$ corresponds uniquely to a finite disjoint decomposition $X = \dot{\bigcup}_i U_i$ of X into clopen subsets via $e_i = \chi_{U_i}$. In particular $C^+(X)$ itself is SA-indecomposable iff the topological space X is connected. In consequence of Theorem 2.18 we can describe all SA-decompositions of $C^+(X)$ when the clopen subsets of X are known. We give three examples. Some more notation: $]\alpha, \beta[$ (resp. $[\alpha, \beta]$) denotes the open (resp. closed) interval from α to β. Likewise for the half-closed intervals $]\alpha, \beta]$ and $[\alpha, \beta[$.

Examples 2.20. *We fix a topological subspace X of the real line $\mathbb{R}$. Let $R := C^+(X)$.*

a) *If X is an interval of $\mathbb{R}$ (open, half-open, closed), then R is SA-indecomposable.*
b) *Assume that $(x_n \mid n \in \mathbb{N})$ is a strictly increasing sequence in $\mathbb{R}$ converging to $x_\infty := \sup_{n \in \mathbb{N}} x_n \in \mathbb{R}$. Let $X = \{x_n \mid n \in \mathbb{N}\} \cup \{x_\infty\}$. The primitive idempotents of R are precisely all elements*

$$e_n := \chi_{\{x_n\}} \qquad (n \in \mathbb{N}),$$

and so the SA-indecomposable summands of R are the ideals

$$T_n := e_n R \qquad (n \in \mathbb{N})$$

of R, consisting of the $\mathbb{R}_+$-valued functions f on X with $f(x) = 0$ for $x \in X \setminus \{x_n\}$. For every $n \in \mathbb{N}$ we also have an idempotent g_n of R with

$$e_1 + \cdots + e_n + g_n = 1$$

and $e_i g_n = 0$ for $1 \leq i \leq n$, namely the characteristic function χ_{Y_n} of

$$Y_n := \{x_i \mid i > n\} \cup \{x_\infty\}.$$

These clopen sets Y_n constitute a fundamental system of neighborhoods of x_∞ in X. The SA-decompositions $(T_1, \dots, T_n, S_n)$ of R which correspond to the orthogonal systems $(e_1, \dots, e_n, g_n)$, i.e., $T_i = Re_i$, $S_n = Rg_n$, are co-final in the set of all SA-decompositions of R under refinement. Note that the "decomposition socle"

$$\mathrm{dsoc}(R) = \bigoplus_{i \in \mathbb{N}} Re_i$$

of R (cf. [4, Definition 2.15]*) is not an SA-summand of R, but is an SA-submodule of R. It is the set $\{f \in R \mid f(x_\infty) = 0\}$.*

c) *Let $X = \mathbb{Q} \subset \mathbb{R}$. The clopen subsets of X are the disjoint unions of intervals*

$$]\alpha, \beta[\,:= \{x \in \mathbb{Q} \mid \alpha < x < \beta\}$$

with $\alpha, \beta \in (\mathbb{R} \cup \{-\infty, \infty\}) \setminus \mathbb{Q}$ and $\alpha < \beta$. Every such interval $]\alpha, \beta[$ provides an idempotent $e_{\alpha,\beta} := \chi_{]\alpha,\beta[}$ of $R = C^+(\mathbb{Q})$, and so an SA-submodule

$$T_{\alpha,\beta} := e_{\alpha,\beta} R$$

consisting of all $f \in R$ with $f(x) = 0$ for $x < \alpha$ or $x > \beta$. These submodules $T_{\alpha,\beta}$ are a co-final system of SA-summands of R (with respect to reverse inclusion). If $\gamma \in]\alpha, \beta[\setminus \mathbb{Q}$, then

$$e_{\alpha,\beta} = e_{\alpha,\gamma} + e_{\gamma,\beta}, \qquad e_{\alpha,\gamma} \cdot e_{\gamma,\beta} = 0.$$

Thus every module $T_{\alpha,\beta}$ is SA-decomposable. It follows that R contains no SA-indecomposable SA-summands altogether.

Every finite sequence $\alpha_1 < \alpha_2 < \cdots < \alpha_n$ in $\mathbb{R} \setminus \mathbb{Q}$ gives a partition of $\mathbb{Q}$ into clopen intervals $]-\infty, \alpha_1[\ ,\ ]\alpha_1, \alpha_2[\ , \dots,\ ]\alpha_n, \infty[$, and thus a complete orthogonal system of idempotents $\{e_{-\infty,\alpha_1}, e_{\alpha_1,\alpha_2}, \dots, e_{\alpha_n,\infty}\}$ which corresponds to a direct sum decomposition

$$R = T_{-\infty,\alpha_1} \oplus T_{\alpha_1,\alpha_2} \oplus \cdots \oplus T_{\alpha_n,\infty}.$$

These decompositions are a co-final system in the set of all finite SA-decompositions of R with respect to refinement.

3. The lattice SA(V)

Proposition 3.1. SA(V) *is a modular lattice. More precisely, if W_i are submodules with $W_1 \le W_2$ and W_2 is SA, then*

$$W_1 + (W_2 \cap W_3) = W_2 \cap (W_1 + W_3).$$

Proof. $(\subseteq)$ is immediate. To prove $(\supseteq)$, suppose $x_2 = x_1 + x_3 \in W_2$ for $x_i \in W_i$. By hypothesis $x_1, x_3 \in W_2$, so $x_3 \in W_2 \cap W_3$ and $x_2 \in W_1 + (W_2 \cap W_3)$. □

Corollary 3.2. *Suppose $W_1 \le W_2, W$ are SA submodules of V, with $W_1 \cap W = W_2 \cap W$ and $W_1 + W = W_2 + W$. If $W_1 + W$ is SA, then $W_1 = W_2$.*

Corollary 3.3. *Given a set $\{W_i : i \in I\}$ of distinct submodules of an SA-module V, for I infinite, and any submodule W, either $\{W_i \cap W : i \in I\}$ or $\{W_i + W : i \in I\}$ contains infinitely many distinct submodules.*

Definition 3.4. *Let $(T_i \mid i \in I)$ denote the set of all* ***minimal*** *non-zero SA-submodules of V. (It can happen that this set is empty.) We define the SA-**socle** of V by*

$$P := \mathrm{soc}_{\mathrm{SA}}(V) := \sum_{i \in I} T_i$$

(read $P = 0$ if $I = \emptyset$).

Note that if $i, j \in I$ are different indices then $T_i \cap T_j = 0$, and that $T_i \in \mathrm{SA}(P)$ for every $i \in I$. Thus $(T_i \mid i \in I)$ is an SA-decomposition of the SA-socle P.

3.1. **Krull dimension (in the sense of Lemonnier-Gordon-Robson).**

SA(V), being a modular lattice, admits a satisfying dimension theory, denoted SA-Kdim, along the lines of Krull dimension, as defined and exposed elegantly in [3], which we use as our model. Since quotient modules do not play an effective role over semirings, we need to consider instead pairs (V, W) where $V \supsetneqq W$. Fortunately, this theory already was developed at the level of lattices by Lemonnier [5], and is developed in this generality in [6], so all we need to do is put it in the present context.

Definition 3.5.

a) *The pair (V, W) is SA-**artinian** if every descending chain*

$$S_0 \supsetneqq S_1 \supsetneqq S_2 \supsetneqq \dots$$

*of finitely generated non-zero SA-submodules of V containing W stops after finitely many steps. The R-module V is SA-**artinian** if $(V, 0)$ is SA-artinian.*

b) *The pair (V, W) is SA-**noetherian** if every ascending chain*

$$S_0 \subsetneqq S_1 \subsetneqq S_2 \subsetneqq \dots$$

*of finitely generated non-zero SA-submodules of V containing W stops after finitely many steps. The R-module V is SA-**noetherian** if $(V, 0)$ is SA-noetherian.*

c) *The pair (V, V) has* SA-Kdim *equal to -1. For $W \neq V$,* SA-Kdim$(V, W) = 0$ *if for every descending chain*

$$W_0 \supsetneqq W_1 \supsetneqq W_2 \supsetneqq \dots \tag{3.1}$$

in SA$(V; W)$ *stops after finitely many steps.*

In general, SA-Kdim(V, W) *(if it exists) is the smallest ordinal θ for which for every chain* (3.1) *one must have* SA-Kdim$(W_i, W_{i+1}) < \theta$ *for almost all i. Such a chain is called θ-**stable**.* SA-Kdim(V) *(if it exists) is* SA-Kdim$(V, 0)$.

d) *A pair (W, W') of SA-sumodules is called **SA-critical** if* SA-Kdim$(W, W') = \theta$ *but* SA-Kdim$(W, W'') < \theta$ *whenever $W \supsetneqq W'' \supsetneqq W'$.*

e) *The submodule W is θ-**SA-critical** if $(W, 0)$ is SA-critical.*

This leads to a natural generalization of the socle (cf. Definition 3.4). One can define the **SA-critical socle** [6, p. 146] to be the sum of all SA-critical submodules of V, of minimal SA-Kdim, but we do not go in that direction.

Unfortunately, SA-artinian R-modules seem to be not very frequent, but here is an instance.

Definition 3.6. *A set of generators T of V is **SA-adapted** if every SA-submodule W of V is generated by the set $W \cap T$.*

Example 3.7. *If V has an SA-adapted finite set of generators, then V is certainly SA-artinian.*

Hence, some of the results emerge more neatly for finitely generated SA-modules, and we need some more terminology.

Notation 3.8.

a) *Given a module V over a semiring R, we denote the set of all finitely generated SA-modules of V by* $\mathrm{SA}_f(V)$. *We furthermore denote the set of all sums* $\sum_{i\in I} W_i$ *of families* $(W_i \mid i \in I)$ *in* $\mathrm{SA}_f(V)$ *by* $\Sigma\,\mathrm{SA}_f(V)$ *and the subset of all such sums with finite* I *by* $\Sigma_f\,\mathrm{SA}_f(V)$.

b) *Observe that the modules* $U \in \Sigma_f\,\mathrm{SA}_f(V)$ *are again finitely generated. Moreover it is easily seen that* $\Sigma_f\,\mathrm{SA}_f(V)$ *is the set of all finitely generated modules* $U \in \Sigma\,\mathrm{SA}_f(V)$.

c) *We often call a module* $W \in \mathrm{SA}_f(V)$ *an* SA_f***-submodule*** *of* V, *and call a module* $U \in \Sigma\,\mathrm{SA}_f(V)$ *a* $\Sigma\,\mathrm{SA}_f$***-submodule*** *of* V, *or a* SA_f***-sum*** *in* V, *furthermore call a module* $U \in \Sigma_f\,\mathrm{SA}_f(V)$ *a* ***finite*** SA_f***-sum*** *in* V.

d) *The pair* (V,V) *has* SA_f-Kdim *equal to* -1. *For* $W \neq V$, $\mathrm{SA}_f\text{-Kdim}(V,W) = 0$ *if for every descending chain*

$$W_0 \supsetneqq W_1 \supsetneqq W_2 \supsetneqq \dots \tag{3.2}$$

in $\mathrm{SA}_f(V;W)$ *stops after finitely many steps. In general,* $\mathrm{SA}_f\text{-Kdim}(V,W)$ *(if it exists) is the smallest ordinal* θ *for which for every chain* (3.2) *one must have* $\mathrm{SA}_f\text{-Kdim}(W_i,W_{i+1}) < \theta$ *for almost all* i. $\mathrm{SA}_f\text{-Kdim}(V)$ *(if it exists) is* $\mathrm{SA}_f\text{-Kdim}(V,0)$.

The following results are really special cases of results in [5, 6] as indicated above.

Proposition 3.9 ([6, 3.1.8]). *Every SA-noetherian module has* SA-Kdim.

Proposition 3.10 ([6, 1.3.7]). *Define the* ***composition length*** $\ell(V,W)$ *from* V *to* $W \subset V$ *to be the length* m *of a chain (if it exists)*

$$V = W_0 \supsetneqq W_1 \supsetneqq \cdots \supsetneqq W_m = W \tag{3.3}$$

for which, for each i, *the chain* $W_i \supsetneqq W_{i+1}$ *cannot be refined to* $W_i \supsetneqq W_i' \supsetneqq W_{i+1}$. *Then* $\ell(V,W)$ *is well-defined (independent of the choice of chain* (3.3)*), and additive in the sense that*

$$\ell(V,W) = \ell(V,W') + \ell(W',W), \qquad \forall V \supsetneqq W' \supsetneqq W.$$

(This holds in either context, $\mathrm{SA}(V)$ or $\mathrm{SA}_f(V)$.) Analogously, we have

Proposition 3.11. $\mathrm{SA\text{-}Kdim}(V) = \sup\{\mathrm{SA\text{-}Kdim}(V,W), \mathrm{SA\text{-}Kdim}(W)\}$.

3.2. **SA-equivalence and SA-uniform modules.**

A neuralgic point, for the sake of brevity often not adequately reflected in our terminology, is the fact that for $W \in \mathrm{Mod}(V)$ the set $\mathrm{SA}(W)$ is definitely bigger than $\mathrm{Mod}(W) \cap \mathrm{SA}(V)$ except in the case that $W \in \mathrm{SA}(V)$.

We introduce on $\mathrm{Mod}(V)$ an equivalence relation which plays a central role throughout the subsection. For the remainder of this section, the module V is LZS.

Definition 3.12. *Given* $W_1, W_2 \in \mathrm{Mod}(V)$, *we say that* W_1 *and* W_2 *are SA-**equivalent** (**in** V) if for any* $S \in SA(V)$ *either* $W_1 \cap S = W_2 \cap S = 0$ *or both* $W_1 \cap S$ *and* $W_2 \cap S$ *are nonzero (where "0" means the zero module* $\{0_V\}$ *). We then write* $W_1 \sim_e W_2$. *In the rare case where a second module* V' *is under consideration and* W_1, W_2 *are also submodules of* V', *we speak more precisely about the above equivalence relation as an* $SA(V)$***-equivalence***, *or specify "in* V*".*

SA-equivalence is closely related to a notion of "SA-essential extension" of R-modules, to be defined now, which vaguely resembles the all-important notion of "essential extension" in the theory of modules over rings.

Definition 3.13. *If $W, W' \in \mathrm{Mod}(V)$, we say that W' is an SA-**essential extension** of W (in V), and write $W \subset_e W'$, if $W \subset W'$ and for $S \cap W \neq 0$ every $S \in \mathrm{SA}(V)$ with $S \cap W' \neq 0$.*[2]

Remark 3.14. *If W, W' are submodules of V with $W \subset W'$, then $W \subset_e W'$ means the same as $W \sim_e W'$.*

We list easy facts about SA-equivalence and SA-extensions.

Lemma 3.15. *Assume that W_1, W_2, X are submodules of V with $W_1 \subset X \subset W_2$. Then the following are equivalent.*

(1) $W_1 \subset_e W_2$.
(2) $W_1 \subset_e X$ *and* $X \subset_e W_2$.

Proof. (1) $\Rightarrow$ (2): If $T \in \mathrm{SA}(V)$ with $T \cap X \neq 0$, then $T \cap W_2 \neq 0$, whence $T \cap W_1 \neq 0$. This proves that $W_1 \subset_e X$. We have $X \sim_e W_1 \sim_e W_2$, and so $X \sim_e W_2$, whence $X \subset_e W_2$. (2) $\Rightarrow$ (1): We have $W_1 \sim_e X \sim_e W_2$, whence $W_1 \sim_e W_2$, and so $W_1 \subset_e W_2$. □

Remark 3.16. *Assume that $(W_i \mid i \in I)$ and $(W_i' \mid i \in I)$ are families in $\mathrm{Mod}(V)$ with $W_i \sim_e W_i'$ for every $i \in I$. Then*

$$\sum_{i\in I} W_i \sim_e \sum_{i \in I} W_i'. \tag{3.4}$$

Proof. Let $S \in \mathrm{SA}(V)$. Then

$$S \cap \left(\sum_i W_i\right) = \sum_i S \cap W_i, \; S \cap \left(\sum_i W_i'\right) = \sum_i S \cap W_i',$$

and so

$$S \cap \sum_i W_i \neq 0 \quad \Leftrightarrow \quad \underset{i\in I}{\exists}\, S \cap W_i \neq 0 \quad \Leftrightarrow \quad \underset{i\in I}{\exists}\, S \cap W_i' \neq 0 \quad \Leftrightarrow \quad S \cap \sum_i W_i' \neq 0.$$

□

Note in particular that

$$W \sim_e W' \quad \Rightarrow \quad W + X \sim_e W' + X \tag{3.5}$$

for any $X \in \mathrm{Mod}(V)$.

Remark 3.17. *Let $W_1, W_2 \in \mathrm{Mod}(V)$ and $S \in \mathrm{SA}(V)$ be given. Then*

$$W_1 \sim_e W_2 \quad \Rightarrow \quad S \cap W_1 \sim_e S \cap W_2.$$

Proof. If $T \in \mathrm{SA}(V)$ and $W_1 \sim_e W_2$, then $S \cap T \in \mathrm{SA}(V)$, and so

$$(W_1 \cap S) \cap T \neq 0 \quad \Leftrightarrow \quad (W_2 \cap S) \cap T \neq 0.$$

□

Proposition 3.18. *Assume that W_1 and W_2 are SA-submodules of V. Then*

$$W_1 \sim_e W_2 \quad \Leftrightarrow \quad W_1 \cap W_2 \subset_e W_1 \quad \textit{and} \quad W_1 \cap W_2 \subset_e W_2.$$

[2]It may seem appropriate to reserve the letter "e" for a straight generalization of "essential extensions" to modules over semirings, as defined in [2, p.95] (there called "essential module-monomorphisms"), and to label SA-equivalences and SA-extensions by "*sae*" instead of "e". But in the present paper the true essential extensions do not show up, and so we feel free to use the simpler label "e".

Proof. ($\Leftarrow$): By use of Remark 3.14 we see that $W_1 \cap W_2 \sim_e W_1$ and $W_1 \cap W_2 \sim_e W_2$, whence $W_1 \sim_e W_2$.

($\Rightarrow$): Using Remark 3.17 we see that $W_1 \cap W_2 \sim_e W_2 \cap W_2 = W_2$ and in the same way that $W_1 \cap W_2 \sim_e W_1$. Thus, again by Remark 3.14,

$$W_1 \cap W_2 \subset_e W_1, \; W_1 \cap W_2 \subset_e W_2.$$

□

Proposition 3.19. *Assume that $(V_i \mid i \in I)$ is an orthogonal family in $SA(V)$, and furthermore that $(W_i \mid i \in I)$ is a family in* $\mathrm{Mod}(V)$ *with $W_i \subset V_i$ for each $i \in I$. Then*

$$\sum_i W_i \subset_e \sum_i V_i \quad \Leftrightarrow \quad \forall\, i \in I : W_i \subset_e V_i.$$

Proof. ($\Rightarrow$): Pick some $k \in I$. Then

$$V_k \cap \left(\sum_i W_i \right) = \sum_i V_k \cap W_i = V_k \cap W_k = W_k,$$

and $V_k \cap \big(\sum_i V_i \big) = V_k$. We conclude by means of Remarks 3.14 and 3.17 that $W_k \subset_e V_k$.

($\Leftarrow$): We have $W_i \sim_e V_i$ for each $i \in I$ and conclude by Remark 3.16 that $\sum_i W_i \sim_e \sum_i V_i$, whence by Remark 3.14 that $\sum_i W_i \subset_e \sum_i V_i$. (Here we did not need the orthogonality assumption on $(V_i \mid i \in I)$.) □

Lemma 3.20. *If V is SA-artinian, Then* $\mathrm{soc}_{\mathrm{SA}}(V) \subset_e V$.

Proof. It is immediate that every non-zero $S \in \mathrm{SA}(V)$ contains a minimal non-zero SA-module T_i. Thus $S \cap \mathrm{soc}_{\mathrm{SA}}(V) \neq 0$. □

Definition 3.21. *We call a submodule W of V SA-**uniform** (in V), if for every $S \in \mathrm{SA}(V)$ with $S \cap W \neq 0$ the extension $S \cap W \subset W$ is SA-essential (in V). We denote the set of all these submodules W of V by* $\mathrm{Mod}_{\mathrm{u}}(V)$ *and its subset* $\mathrm{SA}(V) \cap \mathrm{Mod}_{\mathrm{u}}(V)$ *by* $\mathrm{SA}_{\mathrm{u}}(V)$.[3]

Note that the zero submodule 0 is SA-uniform in V, and furthermore, that $V \in \mathrm{Mod}_{\mathrm{u}}(V)$ iff $S \cap T \neq 0$ for any two non-zero SA-submodules S, T of V.

Theorem 3.22. $\mathrm{Mod}_{\mathrm{u}}(V)$ *is a union of full SA-equivalence classes in* $\mathrm{Mod}(V)$. *In other words, if $W, W' \in$* $\mathrm{Mod}(V)$ *are SA-equivalent (in V) and W is SA-uniform, then W' is SA-uniform.*

Proof. Let $S \in \mathrm{SA}(V)$ be given with $S \cap W' \neq 0$. From $W \sim_e W'$ we conclude by Remark 3.17 that $W \cap S \sim_e W' \cap S$, whence $W \cap S \neq 0$, and so $W \cap S \sim_e W$. Since also $W \sim_e W'$ we conclude that $W' \cap S \sim_e W'$. □

Theorem 3.23. *Let ξ be an SA-equivalence class in* $\mathrm{Mod}_{\mathrm{u}}(V)$. *Then there exists a unique member $M(\xi)$ of ξ such that*

$$\xi = \{ W \in \mathrm{Mod}(V) \mid W \subset_e M(\xi) \}. \tag{3.6}$$

[3]These notions can be viewed in terms of the general theory from [6]. By [6, 3.2.4] any pair (M, N) with SA-Kdim has SA-critical submodules (cf. Definition 3.5). By [6, 3.2.6] every SA-critical module is SA-uniform.

Proof. We choose a labeling of all elements of ξ, $\xi = (W_i \mid i \in I)$, and fix an index $0 \in I$. We then define $M(\xi) = \sum_{i\in I} W_i$. Since $W_i \sim_e W_0$ for every $i \in I$, we conclude by Remark 3.16 that

$$M(\xi) \sim_e \sum_{i\in I} W_0 = W_0.$$

Thus $M(\xi) \in \xi$, and more precisely $M(\xi)$ is the unique maximal element of the poset ξ. If W is a submodule of $M(\xi)$ then $W \in \xi$ iff $W \sim_e M(\xi)$ iff $W \subset_e M(\xi)$. □

By this theorem the $M(\xi)$ are precisely all **maximal** SA-**uniform submodules** of V.

We know nearly nothing about the equivalence classes ξ in $\mathrm{Mod_u}(V)$ with $\xi \cap \mathrm{SA}(V) = \emptyset$, but when ξ contains SA-submodules of V we get more insight about ξ (than provided by Theorems 3.22 and 3.23) by SA-restricting ξ to $\Sigma\,\mathrm{SA}(V) \cap \mathrm{Mod_u}(V)$, as we explain now. We first give a description of the SA-uniform modules in $\Sigma\,\mathrm{SA}(V)$.

Proposition 3.24. *Let $(W_i \mid i \in I)$ be a family of non-zero SA-submodules of V. The following are equivalent.*

(1) $U := \sum_{i\in I} W_i$ *is SA-uniform in* V.

(2) *All* $W_i \in \mathrm{SA_u}(V)$, *and* $W_i \sim_e W_j$ *for* $i \neq j$.

(3) *All* $W_i \in \mathrm{SA_u}(V)$, *and* $W_i \cap W_j \neq 0$ *for* $i \neq j$.

Proof. (1) ⇒ (2): Since W_i is in $\mathrm{SA}(V)$ and $W_i \neq 0$, we have $W_i = W_i \cap U \subset_e U$, whence $W_i \sim_e U$ and W_i is SA-uniform (cf. Theorem 3.22). It follows that $W_i \sim_e W_j$ for $i \neq j$.

(2) ⇔ (3): Evident.

(2) ⇒ (1): Let $S \in \mathrm{SA}(V)$ and $S \cap U \neq 0$. We have $S \cap U = \sum_{i\in I} S \cap W_i$, and so there exists $i \in I$ with $S \cap W_i \neq 0$, implying $S \cap W_i \subset_e W_i$. Fixing an index $0 \in I$ we have $U = \sum_{i\in I} W_i \sim_e \sum_{i\in I} W_0 = W_0$ and conclude by Theorem 3.22 that U is SA-uniform. □

Assume now that ξ is an SA-equivalence class in $\mathrm{Mod_u}(V) \setminus \{0\}$ with $\xi \cap \mathrm{SA}(V) \neq \emptyset$. We write $\xi \cap \mathrm{SA}(V) = \xi \cap SA_u(V) = \{W_i \mid i \in J\}$ and define

$$U(\xi) := \sum_{i\in J} W_i \in \Sigma\,\mathrm{SA}(V). \tag{3.7}$$

Choosing an index $0 \in J$ we have

$$U(\xi) \sim_e \sum_{i\in J} W_0 = W_0. \tag{3.8}$$

Thus $U(\xi)$ is the unique biggest module in the set $\xi \cap \Sigma\,\mathrm{SA}(V)$.

Proposition 3.25. *Assume again that ξ contains a non-zero SA-submodule of V. Then*

$$\xi \cap \mathrm{SA}(V) = \{W \in \mathrm{SA_u}(V) \mid W \neq 0,\ W \subset U(\xi)\}, \tag{3.9}$$

$$\xi \cap \Sigma\,\mathrm{SA}(V) = \{U \in \Sigma\,\mathrm{SA}(V) \mid U \subset_e U(\xi)\}. \tag{3.10}$$

If in addition V is SA-artinian, then $\xi \cap \Sigma\,\mathrm{SA}(V)$ contains a smallest module $P(\xi)$, and

$$\xi \cap \Sigma\,\mathrm{SA}(V) = \{U \in \Sigma\,\mathrm{SA}(V) \mid P(\xi) \subset U \subset U(\xi)\}. \tag{3.11}$$

Proof. In (3.9) the inclusion "⊃" is obvious, while "⊂" follows from (3.8). If $U \in \Sigma\,\mathrm{SA}(V)$ and $U \in \xi$ then $U \subset U(\xi)$ and $U \sim_e U(\xi)$, whence $U \subset_e U(\xi)$. Thus (3.10) is evident. If V is SA-artinian, then the set $\xi \cap \mathrm{SA}(V)$ contains a smallest module $P(\xi)$ and so $\xi \cap \mathrm{SA}(V)$

is the set of all $U \in \Sigma\,\mathrm{SA}(V)$ with $P(\xi) \subset_e U \subset_e U(\xi)$. Since we know that $P(\xi) \subset_e U(\xi)$ the "e" in these inclusions can be omitted. □

Lemma 3.26. *Assume that S and T are non-zero SA-uniform SA-submodules of V. Then $S \sim_e T$ iff $S \cap T \neq 0$.*

Proof. If $S \sim_e T$, then $S \cap T \subset_e S$ and thus certainly $S \cap T \neq 0$. Conversely, if $S \cap T \neq 0$ then, due to the SA-uniformity of S and T, we have $S \cap T \subset_e S$ and $S \cap T \subset_e T$, and so $S \sim_e T$. □

Theorem 3.27.

a) *Assume that $(T_i \mid i \in I)$ is a maximal orthogonal family of non-zero SA-uniform submodules of V. Then $(T_i \mid i \in I)$ is a system of representatives of all SA-equivalence classes in $\mathrm{SA_u}(V) \setminus \{0\}$.*
b) *If $(S_j \mid j \in J)$ is a second such family, then there is a bijection $\lambda\colon I \to J$ with $T_i \sim_e S_{\lambda(i)}$ for all $i \in I$, and $\sum\limits_{i\in I} T_i \sim_e \sum\limits_{j \in J} S_j$.*

Proof. This is an immediate consequence of the preceding Lemma 3.26. □

We are ready to define an invariant for R-modules lacking zero sums.

Definition 3.28. *The **SA-uniformity dimension** $\dim_{\mathrm{sau}}(V)$ of V is the cardinality of the set of all SA-equivalence classes of nonzero SA-uniform submodules of V. In other terms,*

$$1 + \dim_{\mathrm{sau}}(V) = \mathrm{card}(\mathrm{SA_u}(V)/\sim_e).$$

(In particular, $\dim_{\mathrm{sau}}(V) = 0$ iff V does not contain any non-zero SA-uniform submodule.)

Theorem 3.27 provides the following more elementary description of this invariant.

Corollary 3.29. $\dim_{\mathrm{sau}}(V)$ *is the cardinality $|I|$ of any maximal orthogonal family $(T_i \mid i \in I)$ of non-zero SA-uniform submodules of V.*

Theorem 3.30. *Assume that $(V_\lambda \mid \lambda \in \Lambda)$ is a family of SA-submodules of V with $\sum\limits_{\lambda\in\Lambda} V_\lambda = V$.*

a) *Then*
$$\dim_{\mathrm{sau}}(V) \leq \sum_{\lambda\in\Lambda} \dim_{\mathrm{sau}}(V_\lambda). \tag{3.12}$$
b) *If in addition the family (V_λ) is orthogonal (i.e., $V_\lambda \cap V_\mu = 0$ for $\lambda \neq \mu$), then we have equality,*
$$\dim_{\mathrm{sau}}(V) = \sum_{\lambda\in\Lambda} \dim_{\mathrm{sau}}(V_\lambda). \tag{3.13}$$

Proof. a): Let a non-zero SA-uniform module S be given. Then

$$S = \sum_{\lambda\in\Lambda} S \cap V_\lambda,$$

and thus $S \cap V_\lambda \neq 0$ for at least one index λ. This implies that $S \cap V_\lambda \in \mathrm{SA_u}(V_\lambda)$ and $S \sim_e S \cap V_\lambda$. Thus the natural map

$$\dot{\bigcup_{\lambda\in\Lambda}} (\mathrm{SA_u}(V_\lambda) \setminus \{0\})/\sim_e \;\longrightarrow\; (\mathrm{SA_u}(V) \setminus \{0\})/\sim_e$$

is surjective. Comparing cardinalities gives the first claim (3.12).

b): If $V_\lambda \cap V_\mu = 0$ for $\lambda \neq \mu$, then any two non-zero modules $S \in \mathrm{SA}(V_\lambda)$, $T \in \mathrm{SA}(V_\mu)$ have intersection zero and thus certainly are **not** SA-equivalent. Thus now the map $(*)$ is also injective, and in (3.12) holds equality. □

Remark 3.31. *It is clear that for every SA-submodule V' of V we have*

$$\dim_{\mathrm{sau}}(V') \leq \dim_{\mathrm{sau}}(V).$$

Thus we can complement (3.12) *by the inequality*

$$\sup_{\lambda\in\Lambda} \dim_{\mathrm{sau}}(V_\lambda) \leq \dim_{\mathrm{sau}}(V). \tag{3.14}$$

If $\dim_{\mathrm{sau}}(V)$ *is infinite it follows from* (3.12), (3.14) *that*

$$\dim_{\mathrm{sau}}(V) = \max_{\lambda\in\Lambda} \dim_{\mathrm{sau}}(V_\lambda) = \sum_{\lambda\in\Lambda} \dim_{\mathrm{sau}}(V_\lambda).$$

In a similar vein we see, in the case that the set Λ is finite, that $\dim_{\mathrm{sau}}(V)$ *is finite iff* $\dim_{\mathrm{sau}}(V_\lambda)$ *is finite for each λ. Then* (3.13) *holds iff the family* $(V_\lambda \mid \lambda \in \Lambda)$ *is orthogonal.*

4. SA_f-hereditary modules with SA-Kdim

By working only with finitely generated SA-submodules of V, we obtain results on a wide class of submodules U of V, to be put to use in §5 and §6. Throughout §4–§6, we assume for simplicity that **the R-module V is LZS.**

Definition 4.1.

a) *We say that the R-module V is* SA_f***-hereditary*** *if for any submodule $W \in \mathrm{SA}_f(V)$, $W' \in \mathrm{SA}_f(V)$ for all $W' \in \mathrm{SA}(V)$ with $W' \subset W$.*

b) *We say that V is* ***finitely*** SA***-accessible*** *(= SAF-**accessible** for short) if V is both* SA_f*-hereditary and* SA-Kdim(V) *exists.*

Note that if V is finitely generated and SA_f-hereditary, then $\mathrm{SA}(V) = \mathrm{SA}_f(V)$. We present some ways to obtain new SAF-accessible modules from old ones.

Proposition 4.2. *Assume that V is an R-module and $(V_i \mid i \in I)$ is a family of submodules of V with $V = \bigcup_{i\in I} V_i$. Assume that this family is upwardly directed, i.e., for every $i, j \in I$ there is some $k \in I$ with $V_i \subset V_k$, $V_j \subset V_k$. Then, if every V_i is* SA_f*-hereditary, V also is* SA_f*-hereditary; and if every* SA_f-Kdim$(V_i) \leq \theta$, *then V also has* SA_f-Kdim $\leq \theta$.

Proof. a): Assume that the V_i are SA_f-hereditary. Let $W \in \mathrm{SA}_f(V)$, $W' \in SA(V)$ and $W' \subset W$. Since W is finitely generated, there exists some $i \in I$ with $W \subset V_i$. Both W and W' are in $\mathrm{SA}(V)$, and so are SA in V_i. Because V_i is SA_f-hereditary and W is finitely generated, W' also is finitely generated.

b): Assume now that V has SA_f-Kdim $\leq \theta$. Given a descending chain $(W_i \mid i \in I)$ in $\mathrm{SA}_f(V)$ then for any W_0 there exist some $i \in I$ with $W_0 \in \mathrm{SA}_f(V_i)$. By the same argument as above all $W_i \in \mathrm{SA}_f(V_i)$ for all $i > i_0$. Since SA_f-Kdim$(V_i) \leq \theta$, the chain is θ-stable. □

For later reference we also quote an obvious fact.

Lemma 4.3. *Assume again that V is the union of an upward directed family $(V_i \mid i \in I)$ of submodules. If U is a finitely generated submodule of V then $U \subset V_i$ for some $i \in I$.*

Proposition 4.4. *Assume that a direct decomposition $V = \bigoplus_{i\in I} V_i$ of an R-module V is given. If each V_i is* SA_f*-hereditary, then V is* SA_f*-hereditary. If each* SA_f-Kdim$(V_i) \leq \theta$, *then V has* SA_f-Kdim $\leq \theta$.

Proof. a) It is immediate from the definition of the summand absorbing property (cf. (SA)) that the SA-submodules of V are the direct sums $\bigoplus_{i\in I} W_i$ with each W_i an SA-submodule of V_i. It follows by use of Lemma 4.3, that the finitely generated SA-submodules of V are the direct sums $\bigoplus_{i\in J} W_i$ with $W_i \in \mathrm{SA}_f(V_i)$ and $J \subset I$ finite.

b) If the modules V_i are SA_f-hereditary and $W = \bigoplus_{i\in J} W_i \in \mathrm{SA}_f(V)$, then every SA-submodule W' of W has the form $W' = \bigoplus_{i\in J} W'_i$ with $W'_i \subset W_i$, and so $W'_i \in \mathrm{SA}_f(V_i)$, whence $W' \in \mathrm{SA}_f(V)$. Thus V is SA_f-hereditary.

c) Assume now that each SA_f-$\mathrm{Kdim}(V_i) \leq \theta$ and I is finite. Let $(W'_k \mid k \in \mathbb{N}_0)$ be a decreasing chain in $\mathrm{SA}_f(V)$. We want to verify that this chain is θ-stable. We have $W'_k = \bigoplus_{i\in I} W'_{i,k}$ with $W'_{i,k} \in \mathrm{SA}_f(V_i)$. Since SA_f-$\mathrm{Kdim}(V_i) \leq \theta$, each chain $(W'_{i,k} \mid k \in \mathbb{N}_0)$ is θ-stable. It follows that the chain (W'_k) is θ-stable. This proves that V has SA_f-$\mathrm{Kdim} \leq \theta$ for I finite. If I is infinite, and all SA_f-$\mathrm{Kdim}(V_i) \leq \theta$, then for every finite $J \subset I$ the submodule $V_J := \bigoplus_{i\in J} V_i$ has SA_f-$\mathrm{Kdim} \leq \theta$, as proved. Invoking Proposition 4.2 we see that V has SA_f-$\mathrm{Kdim} \leq \theta$. □

Theorem 4.5. *Assume that U is an $\Sigma\,\mathrm{SA}_f$-submodule of an R-module V.*

a) *If V is SA_f-hereditary, then U is SA_f-hereditary, and*

$$\Sigma\,\mathrm{SA}_f(U) = \{X \in \Sigma\,\mathrm{SA}_f(V) \mid X \subset U\}. \tag{4.1}$$

b) *If V is SAF-accessible then U is SAF-accessible.*

Proof. We choose a family $(W_i \mid i \in I)$ in $\mathrm{SA}_f(V)$ with $U = \sum_{i\in I} W_i$.
a): Let $W \in \mathrm{SA}_f(U)$, $W' \in \mathrm{SA}(U)$ and $W' \subset W$. Then

$$W = \sum_{i\in I} W \cap W_i\ , \quad W' = \sum_{i\in I} W' \cap W_i.$$

Every W_i is in $\mathrm{SA}(U)$ and so $W \cap W_i$ and $W' \cap W_i$ are in $\mathrm{SA}(U)$. These modules are contained in the SA-submodule W_i of U and so are SA in W_i. Since W_i is SA in V, they are SA in V. Moreover, since V is SA_f-hereditary, all the modules $W \cap W_i$, $W' \cap W_i$ are finitely generated. Since $W = \sum_{i\in I} W \cap W_i$ this proves that W is an SA_f-sum in V, whence $\mathrm{SA}_f(U) \subset \Sigma\,\mathrm{SA}_f(V)$, and thus

$$\Sigma\,\mathrm{SA}_f(U) \subset \Sigma\,\mathrm{SA}_f(V).$$

On the other hand, every SA_f-submodule X of V which is contained in U is a SA_f-sum in U. This proves assertion (4.1).

Moreover, if I is finite, we conclude from $W' = \sum_{i\in I} W' \cap W_i$ that $W' \in \mathrm{SA}_f(U)$, since all $W' \cap W_i \in \mathrm{SA}_f(U)$. This proves for I finite that U is SA_f-hereditary. If I is infinite, then the R-module $U_J := \sum_{i\in J} W_i$ is SA_f-hereditary for every finite $J \subset I$. Invoking Proposition 4.2 we see that U is SA_f-hereditary.

b): Assume now that V is SAF-accessible. We first consider the case that I is finite. We proceed as in the last part of the proof of Proposition 4.4. Assume that $(W'_k \mid k \in \mathbb{N}_0)$ is a decreasing chain in $\mathrm{SA}_f(U)$ with $W'_0 = W$. For every $i \in I$ the modules $W_i \cap W'_k$ are SA_f-submodules of V, as just proved, and so $(W_i \cap W'_k \mid k \in \mathbb{N}_0)$ is a decreasing chain in $\mathrm{SA}_f(V)$. Since V has SA_f-$\mathrm{Kdim} \leq \theta$, all these chains $(W'_i \cap W'_k \mid k \in \mathbb{N})$ with i running through the finite set I, are θ-stable. Since $W'_k = \sum_{i\in I} W \cap W'_k$ it follows that the chain $(W'_k \mid k \in \mathbb{N})$ is θ-stable. This proves that U has SA_f-$\mathrm{Kdim} \leq \theta$ and so is SAF-accessible.

If I is infinite, then for any finite $J \subset I$ the module $U_J = \sum_{i\in J} W_i$ is SAF-accessible, and so U is SAF-accessible by Proposition 4.2. □

5. The height filtration

Again let V be any LZS module over a semiring R.

Definition 5.1. *Given a submodule U and an SA-submodule W of V, we say that U* ***dominates*** *W, if W is contained in the convex hull $\widehat{U}$ of U in V, i.e., in the smallest SA-submodule of V containing U, cf. Proposition 1.8.*

Let On denote the set of ordinal numbers of cardinality $\leq 2^{2^{|V|}}$. In §6 we will gain some insight in the dominance relation for submodules $W, U \in \Sigma\,\mathrm{SA}_f(V)$ (cf. Notations 3.8) by use of a "**height function**"

$$h : \Sigma\,\mathrm{SA}_f(V) \to \mathrm{On},$$

to be established now. The modules V, in which this works well, are the SAF-accessible modules defined in §4.

We first construct a family $(V_t^0 \mid t \leq \omega)$ and a strictly increasing chain $V_0 \subsetneqq V_1 \subsetneqq \cdots \subsetneqq V_\omega$ in $\Sigma\,\mathrm{SA}_f(V)$, indexed by ordinal numbers. We proceed by transfinite induction. We do not assume anything about the R-module V, except that V is LZS, as always, but it seems that the construction is really useful only if V has some SA-Kdim.

Construction 5.2. *Let*

$$\begin{aligned} V_0 &:= V_0^0 = \{0\}, \\ V_1^0 &:= \text{the sum of all minimal } W \neq 0 \text{ in } \mathrm{SA}_f(V), \\ V_1 &:= V_0 + V_1^0 = V_1^0. \end{aligned}$$

Assume that V_s^0 and V_s are already defined for all $s < t \in \mathrm{On}$.

- A) *Assume that t is not a limit ordinal, so $t = \tau + 1$ for a unique $\tau \in \mathrm{On}$.*
  - Case I: *There exists* ***no*** *SA-critical $W \in \mathrm{SA}_f(V)$ with $W \not\subset V_\tau$. The construction stops with $\omega := \tau$.*
  - Case II: *Otherwise. We define $V_t^0 = V_{\tau+1}^0$ as the sum of all $W \in \mathrm{SA}_f(V)$ which are SA-critical and not contained in V_τ, and define $V_t := V_\tau + V_{\tau+1}^0$.*
- B) *Assume that t is a limit ordinal. We put $V_t^0 = \{0\}$, $V_t = \bigcup_{s<t} V_s$.*

Note that $V_s \subsetneqq V_t$ for all $s < t \leq \omega$, and that all modules V_t^0 and V_t are elements of $\Sigma\,\mathrm{SA}_f(V)$. The strictly ascending chain (V_t) stops with a module V_ω, $\omega \in \mathrm{On}$, which may or may not be a limit ordinal.

Definition 5.3.

- a) *The* ***height*** *$h_V(U)$ of a submodule U of V with $U \subset V_\omega$ is the minimum of all ordinals $t \leq \omega$ with $U \subset V_t$. This minimum exists, since the set $\{t \in \mathrm{On} \mid t \leq \omega\}$ is well ordered.*
- b) *Clearly, if $U \subset V_\omega$ and $t \in O_n$, $t \leq \omega$, then*

$$h_V(U) \leq t \quad \Leftrightarrow \quad U \subset V_t. \tag{5.1}$$

We call the family $(V_t \mid t \leq \omega)$ the ***height filtration in*** *V (or:* ***of*** *V_ω).*

Given any module V over a semiring R we denote by $\widetilde{V}$ the sum of all $W \in \mathrm{SA}_f(V)$. This is the **top element** of the poset $\Sigma\,\mathrm{SA}_f(V)$. Clearly $\widetilde{V}$ is also the union of all $U \in \Sigma_f\,\mathrm{SA}_f(V)$ (cf. Notations 3.8). If V is finitely generated then, of course, $\widetilde{V} = V$.

Theorem 5.4. *Assume that* $\mathrm{SA\text{-}Kdim}(V) \le \theta$*. Then* V_ω *coincides with the maximal* SA_f*-sum* $\widetilde{V}$ *in* V*.*

Proof. Clearly $V_\omega \subset \widetilde{V}$. Suppose that $V_\omega \neq \widetilde{V}$. Then there exists some $W \in \mathrm{SA}_f(V)$ with $W \not\subset V_\omega$ Since $\mathrm{SA\text{-}Kdim}(V) \le \theta$, there exists some SA-critical $W' \in \mathrm{SA}_f(V)$ with $W' \subset W$, with $W' \not\subset V_\omega$. But this contradicts the definition of V_ω. Thus $V_\omega = \widetilde{V}$. □

In the following we usually write $h(U)$ instead of $h_V(U)$, whenever it is clear from the context, which R-module V is under consideration. We concentrate on a study of the heights of the SA_f-sums in V. We will assume almost everywhere that $\mathrm{SA\text{-}Kdim}(V) \le \theta$, so that we catch all SA_f-sums in the height filtration due to Theorem 5.4.

Proposition 5.5. *Assume that* $\mathrm{SA\text{-}Kdim}(V) \le \theta$*. Assume furthermore that* $(U_\lambda \mid \lambda \in \Lambda)$ *is a family in* $\Sigma\,\mathrm{SA}_f(V)$ *and* $U := \sum_{\lambda\in\Lambda} U_\lambda$*. Then*

$$h(U) = \sup_{\lambda\in\Lambda} h(U_\lambda). \tag{5.2}$$

Proof. Let $t := h(U)$. Of course $h(U_\lambda) \le t$ for all $\lambda \in \Lambda$. Suppose there exists an ordinal number $\tau < t$ with $h(U_\lambda) \le \tau$ for all $\lambda \in \Lambda$. Then $U_\lambda \subset V_\tau$ for all λ, and so $U \subset V_\tau$. But this means that $h(U) \le \tau$, a contradiction. Thus t is the least upper bound of $(h(U_\lambda) \mid \lambda \in \Lambda)$. □

Corollary 5.6. *Assume again that* $\mathrm{SA\text{-}Kdim}(V) \le \theta$ *and* $(U_\lambda \mid \lambda \in \Lambda)$ *is a family in* $\Sigma\,\mathrm{SA}_f(V)$*. Assume furthermore that the height of* $U := \sum_{\lambda\in\Lambda} U_\lambda$ *is* ***not*** *a limit ordinal. Then*

$$h(U) = \max_{\lambda\in\Lambda} h(U_\lambda). \tag{5.3}$$

Proof. Let $t := h(U)$, $t_\lambda := h(U_\lambda)$ for $\lambda \in \Lambda$. Then $\sup_{\lambda\in\Lambda} t_\lambda = r$, as we have seen. If t is not a limit ordinal, this implies that there exists $\lambda \in \Lambda$ with $t_\lambda = t$. □

In the following proposition we do not need the assumption that $\mathrm{SA\text{-}Kdim}(V) \le \theta$.

Proposition 5.7. *Assume that* U *is a sum of finitely many finitely generated SA-submodules of* V *(i.e.,* $U \in \Sigma_f\,\mathrm{SA}_f(V)$*, cf. Definition 3.8). Then* $h(U)$ *is not a limit ordinal.*

Proof. U has a finite system S of generators, $S = \{s_1, \dots, s_m\}$. For each $i \in \{1, \dots, m\}$ there is a smallest ordinal t_i with $s_i \in V_{t_i}$, and clearly t_i is not a limit ordinal. Let t_k denote the largest of the t_i. This is the smallest ordinal τ of $\le \omega$ with $S \subset V_\tau$, whence $U \subset V_\tau$. Thus $h(U) = t_k$. □

Definition 5.8.

a) *Assume that* t *is an ordinal number with* $t \le \omega$*, and that* t *is not a limit ordinal. As common we denote the ordinal number* τ *with* $\tau + 1 = t$ *by* $t - 1$*. We call a module* $W \in \mathrm{SA}_f(V)$ t*-**critical**, if* W *is SA-critical with* $W \not\subset V_{t-1}$*. We denote the set of all* t*-critical* SA_f*-modules in* V *by* $\mathrm{SA}_t(V)$*.*

b) *If* $\tau \le \omega$ *is a limit ordinal we put* $\mathrm{SA}_\tau(V) := \emptyset$*.*
We furthermore define

$$\mathrm{SA}_{\min}(V) := \bigcup_{t\le\omega} \mathrm{SA}_t(V),$$

and we call the elements of this set the ***height-critical*** SA_f***-submodules*** *of* V*.*

Theorem 5.9. *Assume that V is SAF-accessible, and that U is an SA-submodule of V of height $h(U) = t$. Then for any $\tau \leq t$ the following holds:*

$$\mathrm{SA}_\tau(U) = \{W \in \mathrm{SA}_\tau(V) \mid W \subset U\}, \tag{5.4}$$

$$U_\tau = U \cap V_\tau. \tag{5.5}$$

Proof. We verify this by induction on τ. For $\tau = 0$ both assertions are obvious. Let $\tau > 0$, and assume first that τ is not a limit ordinal and (5.4), (5.5) are true for $\tau - 1$. If W is an SA-submodule of U, then

$$W \in \mathrm{SA}_f(V) \quad \Leftrightarrow \quad W \in \mathrm{SA}_f(U),$$

since U is in $\mathrm{SA}(V)$. We conclude from $U_{\tau-1} = U \cap V_{\tau-1}$ that $W \in \mathrm{SA}_\tau(U)$ iff $W \not\subset U_{\tau-1}$ iff $W \not\subset V_{t-1}$ iff $W \in \mathrm{SA}_\tau(V)$. This proves (5.4) for the ordinal τ. Let $\{W_i \mid i \in I\}$ denote the set of all τ-critical submodules of U and $\{W'_k \mid k \in K\}$ denote the set of τ-critical submodules of V not contained in U. Thus

$$\begin{aligned} U_\tau &= U_{\tau-1} + \sum_{i\in I} W_i, \\ V_\tau &= V_{\tau-1} + \sum_{i\in I} W_i + \sum_{k\in K} W'_k, \end{aligned}$$

whence

$$\begin{aligned} U \cap V_\tau &= U \cap V_{\tau-1} + \sum_{i\in I} W_i + \sum_{k\in K} U \cap W'_k \\ &= U_{\tau-1} + \sum_{i\in I} W_i + \sum_{k\in K} U \cap W'_k \\ &= U_\tau + \sum_{k\in K} U \cap W'_k. \end{aligned}$$

Now $U \cap W'_k \in \mathrm{SA}_f(V)$ since V is SA_f-hereditary and $U \cap W'_k \subsetneqq W'_k$. Due to the τ-criticality of W'_k it follows that $U \cap W'_k \subset V_{\tau-1}$, and thus $U \cap W'_k \subset U \cap V_{\tau-1} = U_{\tau-1}$, so that altogether we obtain that $U \cap V_\tau = U_\tau$.

Assume finally that τ is a limit ordinal. Then $\mathrm{SA}_\tau(U) = \mathrm{SA}_\tau(V) = \emptyset$, and so (5.4) holds trivially. By induction hypothesis $U \cap V_\sigma = U_\sigma$ for $\sigma < \tau$. Thus

$$U \cap V_\tau = U \cap \left(\sum_{\sigma<\tau} V_\sigma\right) = \sum_{\sigma<\tau} U \cap V_\sigma = \sum_{\sigma<\tau} U_\sigma = U_\tau,$$

which proves (5.5). □

Corollary 5.10. *Assume again that V is SAF-accessible. Let $U \in \Sigma\,\mathrm{SA}_f(V)$. Recall from Theorem 4.5.(b) that U is SAF-accessible. Let $t := h_V(U)$.*

- a) *If $U' \in \Sigma\,\mathrm{SA}_f(U)$ then $U' \in \Sigma\,\mathrm{SA}_f(V)$ and $h_U(U') = h_V(U')$. In particular $t = h_U(U)$.*
- b) *U is the sum of all modules $W \in \mathrm{SA}_\tau(V)$ with $W \subset U$, $\tau \leq t$.*

Proof. a): The height $h_U(U')$ is the minimal ordinal τ such that $U' \subset U_\tau$. Since $U_\tau = U \cap V_\tau$ (cf. (5.5)), this is also the minimal ordinal τ with $U' \subset V_\tau$, and so $h_U(U') = h_V(U')$.

b): We have $t = h_U(U)$. Now U is the sum of all $W \in \mathrm{SA}_\tau(U)$ with $\tau \leq t$, as is clear by Construction 5.2 and Theorem 5.4. By (5.4) these are the $W \in \mathrm{SA}_\tau(V)$ with $W \subset U$ and $\tau \leq t$. □

6. Primitive SA_f-modules

Definition 6.1. *Assume that* $\mathrm{SA}\text{-Kdim}(V) \le \theta$.

a) *We call a module* $W \in \mathrm{SA}_f(V)$ ***primitive*** *in* V *if* W *is* τ*-critical for some* $\tau \le \omega$ *(and so* $W \subset V_\tau^0$, $h(W) = \tau$*), but* $W \not\subset \widehat{V}_{\tau-1}$ *(i.e.,* W *is* ***not*** *dominated by* $V_{\tau-1}$*, cf. Definition 5.1). We define*

$$\mathrm{SA}_{\tau,\mathrm{prim}}(V) := \text{ set of all primitive } W \in \mathrm{SA}_f(V) \text{ of height } \tau. \tag{6.1}$$

$$\mathrm{SA}_{\mathrm{prim}}(V) := \bigcup_{\tau \le \omega} \mathrm{SA}_{\tau,\mathrm{prim}}(V). \tag{6.2}$$

b) *If* $T \in \Sigma\,\mathrm{SA}_f(V)$, $h(T) = r$, *we set for* $\tau \le t$

$$\mathrm{SA}_{\mathrm{prim}}(T, V) = \{W \in \mathrm{SA}_{\mathrm{prim}}(V) \mid W \subset T\} \tag{6.3}$$

and for $\tau \le t$

$$\mathrm{SA}_{\tau,\mathrm{prim}}(T, V) = \{W \in \mathrm{SA}_{\tau,\mathrm{prim}}(V) \mid W \subset T\}. \tag{6.4}$$

Theorem 6.2. *Assume that* V *is SAF-accessible. Let* $T, U \in \Sigma\,\mathrm{SA}_f(V)$. *Assume furthermore that all primitive* SA_f*-submodules of* V*, which are contained in* T*, are also contained in* U*. Then* $T \subset \widehat{U}$.

Proof. We know by Corollary 5.10.(b) that T is the sum of all $W \in \mathrm{SA}_\tau(T)$ with $\tau \le t := h(T)$. Furthermore, it is clear by Definition 7.1, that every $W \in \mathrm{SA}_\tau(T)$ is dominated by the sum X_τ of all $W' \in \mathrm{SA}_{\mathrm{prim}}(T, V)$ with $h(W') \le \tau$. Since we assume that every $W' \in \mathrm{SA}_{\mathrm{prim}}(T, V)$ is contained in U, it follows that $T \subset \left(\sum_{\tau \le t} X_\tau\right)^\wedge \subset \widehat{U}$. □

Theorem 6.3. *Assume conversely that* $T \subset \widehat{U}$. *Then all primitive* SA_f*-submodules of* V *which are contained in* T *are contained in* U.

Proof. Let $W \in \mathrm{SA}_{\tau,\mathrm{prim}}(T)$ be given, i.e. $W \in \mathrm{SA}_f(V)$, $h(W) = \tau$, $W \subset T$, W primitive in V. We have $W \subset V_\tau^0$, $W \not\subset \widehat{V}_{\tau-1}$, but $W \subset \widehat{U}$. This is only possible if $W \subset U$ (and so $W \in \mathrm{SA}_{\tau,\mathrm{prim}}(U)$). □

Definition 6.4. *The* ***primitivity socle*** $\mathrm{prsoc}(T)$ *of a module* $T \in \Sigma\,\mathrm{SA}_f(V)$ *is the sum of all primitive* SA_f*-submodules* W *of* V *contained in* T.

We state an immediate consequence of Theorems 6.2 and 6.3.

Corollary 6.5. *Assume that* V *is SAF-accessible. For modules* $T, U \in \Sigma\,\mathrm{SA}_f(V)$ *the following are equivalent:*

(1) $\widehat{T} \subset \widehat{U}$,
(2) $T \subset \widehat{U}$,
(3) $\mathrm{prsoc}(T) \subset \mathrm{prsoc}(U)$.

Proof. (1) $\Leftrightarrow$ (2): Obvious, cf. Proposition 1.8.
(2) $\Leftrightarrow$ (3): Clear by Theorems 6.2 and 6.3. □

Proposition 6.6. *Assume again that* V *is SAF-accessible. Let* $T \in \Sigma\,\mathrm{SA}_f(V)$. *The primitivity socle* $\mathrm{prsoc}(T)$ *is the smallest module* $U \in \Sigma\,\mathrm{SA}_f(V)$ *contained in* T *which dominates* T.

Proof. Let $T_0 := \mathrm{prsoc}(T)$. By definition of the primitivity socle it is evident that $\mathrm{prsoc}(T_0) = T_0$, and thus $\mathrm{prsoc}(T) = \mathrm{prsoc}(T_0)$. It follows by Corollary 6.5, that $\widehat{T} = \widehat{T}_0$, and so T_0 dominates T. If $U \in \Sigma\,\mathrm{SA}_f(V)$ and $U \subset T \subset \widehat{U}$, then $\widehat{U} \subset \widehat{T} \subset \widehat{U}$, and so $\widehat{U} = \widehat{T}$. Again by Corollary 6.5 we conclude that $\mathrm{prsoc}(U) = \mathrm{prsoc}(T) = T_0$. Thus certainly $T_0 \subset U$. □

7. Generating SA-submodules by use of additive spines

Given an R-module V and a set S of generators of V we want to establish a new set T of generators of V, which is "small" in some sense if S is "small", and gives us sets of generators of all SA-submodules W of V in a coherent way. Recall **SA-adapted** from Definition 3.6.

We will obtain a reasonable SA-adapted set of generators T from a given set of generators S by employing the so-called **additive spine** M of a module (Definition 8.1) the semiring R (Definition 7.2). In the special case that both M and S are finite it will turn out that also T is finite, and so all SA-submodules W of V are generated by $|T|$ elements.

We first define additive spines of R, state basic facts about them, and give first examples.

Notation 7.1. *Given (nonempty) subsets A, B of R, we denote the set of products ab with $a \in A$, $b \in B$ by AB (or $A \cdot B$). Similarly, if $A \subset R$, $X \subset V$ then AX denotes the set of products ax with $a \in A$, $x \in X$. Furthermore $\sum^{\infty} A$ and $\sum^{\infty} X$ denote the set of all finite sums of elements of A in R and of X in V respectively. Admitting also the empty sum of elements of A or X, we always have $0_R \in \sum^{\infty} A$, $0_V \in \sum^{\infty} X$. If necessary we write more precisely $\sum_R^{\infty} A$ and $\sum_V^{\infty} X$ instead of $\sum^{\infty} A$ and $\sum^{\infty} X$.*

In this notation a set $S \subset V$ generates the R-module V if $V = \sum^{\infty} RS$.

Definition 7.2. *Given a subset M of R,*

a) *We define the set*

$$\widetilde{M} := \{x \in R \mid \exists\, y, z \in R : \ yx \in M, \ zyx = x\},$$

*which we call the **halo** of M in R.*

b) *If the halo $\widetilde{M}$ generates R additively, i.e., $R = \sum^{\infty} \widetilde{M}$, we call M an **additive spine of** R.*

We state some facts about halos which are immediate consequences of Definition 7.2.a.

Remarks 7.3.

i) *$M \subset \widetilde{M}$ for any set $M \subset R$.*

ii) *If $M \subset N \subset R$ then $\widetilde{M} \subset \widetilde{N}$.*

iii) *If $(M_i \mid i \in I)$ is a family of subsets of R, then*

$$\Big(\bigcup_{i \in I} M_i\Big)^{\sim} = \bigcup_{i \in I} \widetilde{M_i}.$$

iv) *$\{0\}^{\sim} = \{0\}$ and $(M \setminus \{0\})^{\sim} = \widetilde{M}$.*

Due to the last remark we may assume in any study of halos that $0 \in M$ or $0 \notin M$, whatever is more convenient.

Here are the perhaps most basic examples of halos deserving interest.

Example 7.4. *Let $M = \{1_R\}$. Then $\widetilde{M}$ is the set of left invertible elements of R. Indeed, if $x \in \widetilde{M}$, then there exists $y \in R$ with $yx = 1$. Conversely, if x is left-invertible there exists $y \in R$ with $yx = 1$, and so $xyx = x$, which proves that $x \in \widetilde{M}$.*

Example 7.5. *Let $M = \{e\}$ with e an idempotent of R. If $x \in \widetilde{M}$, then there exist $y, z \in R$ with $yx = e$, $ze = x$. It follows that $xe = x$, yielding the von Neumann condition $xyx = x$. Conversely, if $yx = e$ and $xyx = x$, then clearly $x \in \widetilde{M}$. This proves that*

$$\{e\}^{\sim} = \{x \in R \mid \exists\, y \in R : \; yx = e, \; xyx = x\}.$$

Let $\mathrm{Id}(R)$ denote the set of all idempotents of R. Starting from Example 7.5, we obtain the following fact.

Proposition 7.6. *If R is any semiring then*

$$\mathrm{Id}(R)^{\sim} = \{x \in R \mid \exists\, y \in R : xyx = x\}.$$

Proof. $\mathrm{Id}(R)^{\sim}$ is the union of the sets $\{e\}^{\sim}$ with e an idempotent of R (cf. Remark 7.3.iii). Thus it is clear from Example 2.6 that for every $x \in \mathrm{Id}(R)^{\sim}$ there exists some $y \in R$ with $xyx = x$.

Conversely, if $xyx = x$, then $yx \cdot yx = yx$, and so $e := yx$ is an idempotent of R. Moreover $xe = x$, and so $x \in \{e\}^{\sim}$. □

We state an immediate consequence of this proposition.

Corollary 7.7. *For any subset M of R we have*

$$[M \cap \mathrm{Id}(R)]^{\sim} = \{x \in \widetilde{M} \mid \exists\, y \in R : xyx = x\},$$

*and $\widetilde{M}$ is the **disjoint** union of this set and $[M \setminus \mathrm{Id}(R)]^{\sim}$.*

The set $[M \cap \mathrm{Id}(R)]^{\sim}$ may be regarded as the "easy part" of the halo $\widetilde{M}$.

We are ready for a central result.

Theorem 7.8. *Assume that S is a set of generators of a (left) R-module V, and M is an additive spine of R. Then any SA-submodule W of V is generated by the set $W \cap (MS)$.*

Proof. Since $V = \sum^{\infty} RS$ and $R = \sum^{\infty} \widetilde{M}$, we have $V = \sum^{\infty} \widetilde{M}S$.
Let $w \in W$, $w \neq 0$, be given. Then

$$w = \sum_{i=1}^{n} x_i s_i \tag{A}$$

with $n \in \mathbb{N}$, $s_i \in S$, $x_i \in \widetilde{M}$. Since W is in $\mathrm{SA}(V)$, it follows that

$$x_i s_i \in W \qquad \text{for } 1 \leq i \leq n.$$

Now choose $y_i, z_i \in R$ such that $m_i := y_i x_i \in M$ and $x_i = z_i m_i$. Then

$$y_i(x_i s_i) = m_i s_i \in W \cap (MS) \tag{B}$$

and

$$z_i m_i s_i = z_i y_i x_i s_i = x_i s_i.$$

From (A) we obtain that

$$w = \sum_{i=1}^{n} z_i (m_i s_i). \tag{C}$$

We conclude from (B) and (C) that $W \cap (MS)$ generates W. □

Corollary 7.9. *Assume that R has a finite additive spine M and V has a finite set of generators S. Then every SA-submodule W of V is finitely generated, more precisely, generated by at most $|M| \cdot |S|$ elements (independent of the choice of W!).*

Theorem 7.10. *Assume that V is a module over a semiring R which is additively generated by the set of its left invertible elements. Then every set of generators S of V is SA-adapted.*

Proof. We read off from Example 7.4 that $\{1_R\}$ is an additive spine of R. So by Theorem 7.8 every SA-submodule W of V is generated by $W \cap S = W \cap (1_R S)$. □

We take a look at additive spines of matrix semirings.

Example 7.11. *Assume that C is a semiring which is additively generated by $\{1_C\}$,*

$$C = \sum^{\infty} \{1_C\}.$$

In other terms, the unique homomorphism $\varphi : \mathbb{N}_0 \to C$ with $\varphi(1) = 1_C$ is surjective. Then the semiring

$$R = M_n(C) = \sum_{i,j=1}^{n} C e_{ij}$$

of $(n \times n)$-matrices with entries in C, and e_{ij} the usual matrix units, has the additive spine

$$D := \{e_{11}, e_{22}, \dots, e_{nn}\}.$$

Indeed, for every $j \in \{1, \dots, n\}$

$$\{e_{jj}\}^{\sim} \supset \{e_{ij} \mid 1 \leq i \leq n\},$$

since $e_{ji}e_{ij} = e_{jj}$, $e_{ij}e_{jj} = e_{ij}$, and so $\widetilde{D} = \bigcup_j \{e_{jj}\}^{\sim}$ contains the set $E := \{e_{ij} \mid 1 \leq i, j \leq n\}$ of all matrix units, which by the nature of C generates $M_n(C)$ additively.

This example can be amplified to a theorem about additive spines in arbitrary matrix rings $M_n(A)$ by use of a general principle to "multiply" additive spines, which runs as follows:

Proposition 7.12. *Assume that R_1 and R_2 are subsemirings of a semiring R, such that R is additively generated by R_1R_2, i.e., $R = \sum^{\infty} R_1R_2$. Assume furthermore that the elements of R_1 commute with those of R_2. Assume finally that M_i is an additive spine of R_i. Let $\widetilde{M_i}$ denote the halo of M_i in $R_i (i = 1, 2)$. Then $\widetilde{M_1}\widetilde{M_2}$ is contained in the halo $(M_1M_2)^{\sim}$ of M_1M_2 in R, and M_1M_2 is an additive spine of R.*

Proof. Let $x_i \in \widetilde{M_i}$ $(i = 1, 2)$ be given. We have elements y_i, z_i of R_i with $m_i := y_i x_i \in M_i$ and $z_i m_i = x_i$. Now

$$(y_1y_2)(x_1x_2) = (y_1x_1)(y_2x_2) = m_1m_2$$

and

$$(z_1z_2)(m_1m_2) = (z_1m_1)(z_2m_2) = x_1x_2.$$

This proves that $x_1x_2 \in (M_1M_2)^{\sim}$. It follows that

$$\left(\sum^{\infty} \widetilde{M_1}\right) \cdot \left(\sum^{\infty} \widetilde{M_2}\right) = R_1R_2$$

and then that

$$R = \sum^{\infty} R_1R_2 = \sum^{\infty} \widetilde{M_1}\widetilde{M_2}.$$

□

Theorem 7.13. *Assume that R is the semiring of $(n\times n)$-matrices over any semiring A, so*

$$R := M_n(A) = \sum_{i,j=1}^{n} Ae_{ij}$$

with the usual matrix units e_{ij}. Let N be an additive spine of A. Then the set $M := \bigcup_{i=1}^{n} Ne_{ii}$, consisting of the diagonal matrices with entries in N, is an additive spine of R.

Proof. Let C denote the smallest subsemiring of A, $C = \{n \cdot 1_A \mid n \in \mathbb{N}\}$. We have seen that $R_1 := M_n(C)$ has the additive spine $D := \{e_{ii} \mid 1 \leq i \leq n\}$ (Example 7.11). Let $R_2 := A \cdot 1_R$. This is the subsemiring of R consisting of all matrices aI with $a \in A$, where I is the identity matrix. It has the additive spine $N \cdot 1_{R_2}$. Now $R = \sum R_1R_2$, and the elements of R_1 commute with those of R_2. Thus, by Proposition 7.12, R has the additive spine $D \cdot (N1_{R_2}) = \bigcup_{i=1}^{n} Ne_{ii}$. □

Recalling Theorem 7.8 we obtain

Theorem 7.14. *Assume that V is an $M_n(A)$-module, A any semiring, and S a system of generators of V. Assume furthermore that N is an additive spine of A. Then any SA-submodule W of $M_n(A)$ is generated by the set*

$$W \cap \left(\bigcup_{i=1}^{n} Ne_{ii}\right) = \bigcup_{i=1}^{n} W \cap (Ne_{ii}).$$

If N is finite then W can be generated by at most $n \cdot |N|$ elements.

The proof of Theorem 2.14 can be seen in a much wider context, as we explain now.

Definition 7.15. *Let $S = (S, \cdot)$ be a monoid, in multiplicative notation. We call a subset T of S a* ***spine of*** *S (= monoid spine), if for any $s \in S$ there exist $s_1, s_2 \in S$ such that $t := s_1 s \in T$ and $s_2 t = s$.*

Given any semiring A and monoid $S = (S, \cdot)$ we denote, as common, the **monoid-semiring** of S over A by $A[S]$.

In the case that the monoid S is **without zero**, i.e., S does **not** contain an absorbing element 0, ($0 \cdot S = S \cdot 0 = 0$ for all $s \in S$), the elements x of $R := A[S]$ are the formal sums

$$x = \sum_{s\in S} a_s s,$$

with coefficients $a_s \in A$ uniquely determined by x, only finitely many non-zero. The multiplication is determined by the rule $(as) \cdot (bt) = (ab)(st)$ for $a, b \in A$, $s, t \in S$. Identifying $a = a \cdot 1_S$, $s = 1_A \cdot s$, we regard A as a subsemiring of R and S as a submonoid of $(R, \cdot)$.

If the monoid S has a zero $0 = 0_S$, we take for $R = A[S]$ the free A-module with base $S \setminus \{0\}$ and multiplication rule $(as) \cdot (bt) = (ab)(st)$ if $st \neq 0_S$, $(as)(bt) = 0$ otherwise. Now the nonzero elements of $R = A[S]$ are formal sums $\sum_{s\neq 0} a_s s$. We identify again $a = a \cdot 1_S$, $s = 1_A \cdot s$ for $s \in S \setminus \{0\}$, and now also $0_S = 0_A$. Then again A becomes a subsemiring of R and S a submonoid of $(R, \cdot)$. We have $R = \sum^{\infty} AS$ in both cases.

Example 7.16. *The matrix semiring $M_n(A)$ coincides with $A[S]$, where S is the monoid $\{e_{ij} \mid 1 \leq i, j \leq n\} \cup \{0\}$ with multiplication rule $e_{ij}e_{kl} = \delta_{jk}e_{il}$. Note that S has the monoid spine $\{e_{11}, \dots, e_{nn}\} \cup \{0\}$.*

Theorem 7.17. *Assume that S is a multiplicative monoid (with zero or without zero) and T is a spine of S. Assume furthermore that A is a semiring and N is an additive spine of A. Then $N \cdot T$ is an additive spine of $A[S]$.*

Proof. Let $R := A[S]$ and $R_1 := C[S] \subset R$, with C the image of the (unique) homomorphism $\mathbb{N}_0 \to A$. It is obvious that $R_1 = \sum^{\infty} S$ and that S is contained in the halo $\widetilde{T}$ of T in R_1. Thus T is a spine of R_1. {In fact it can be verified that $\widetilde{T} = \widetilde{S} = S$.} Let $R_2 := A \subset R$. Then $R = \sum^{\infty} R_1 \cdot R_2$ and the elements of R_1 commute with those of R_2. The assertion follows from Proposition 7.12. □

8. Halos and additive spines in R-modules

Halos and additive spines can be defined and studied on any R-module instead of the semiring R itself. Although at present perhaps of limited practical value, this will make the theory of generators of SA-submodules more transparent.

Definition 8.1. *Assume that S is a subset of V.*

a) *The **halo** $\widetilde{S}$ of S in V is the set of all $v \in V$ such that there exist $\lambda, \mu \in R$ with $\lambda v \in S$ and $\mu\lambda v = v$.*
b) *S is called an **additive spine** of the R-module V if V is additively generated by $\widetilde{S}$, $V = \sum^{\infty} \widetilde{S}$.*

Thus the additive spines on ${}_R R$, i.e., of R considered as left R-module, are the same objects as the additive spines on R as defined in §2.

Example 8.2. *If S is a set of generators of the R-module V and M is an additive spine of R, then we know by Theorem 7.8 that MS is an additive spine of V.*

Theorem 7.8 generalizes as follows:

Theorem 8.3. *Assume that S is an additive spine of an R-module V. Then every SA-submodule W of V is generated by $W \cap S$, and moreover $W \cap S$ is an additive spine of W.*

Proof. a) We first verify that V itself is generated by S. Since V is additively generated by $\widetilde{S}$, for given nonzero $v \in V$ we have

$$v = \sum_{i=1}^{n} v_i, \tag{A}$$

with $n \in \mathbb{N}$, $v_i \in \widetilde{S}$. There exist $\lambda_i, \mu_i \in R$ such that

$$s_i := \lambda_i v_i \in S, \tag{B}$$

$$v_i = \mu_i s_i, \tag{C}$$

and so by (A)

$$v = \sum_{i=1}^{n} \mu_i s_i,$$

and we are done.

b) If now W is an SA-submodule of V, and the above element v lies in W, then in (A) all summands v_i are in W, and so the s_i from (7) are in $W \cap S$. We conclude from (B) and (C) that all v_i are in the halo $(W \cap S)^{\sim}$ of $W \cap S$ in W, and we infer from (A) that W is additively generated by $(W \cap S)^{\sim}$, i.e., $W \cap S$ is an additive spine of W. As proved in a) the set $W \cap S$ generates the R-module W. □

We write down a chain of propositions which turn out to be useful in working with halos and additive spines. For clarity we sometimes denote the halo of a set S in an V more elaborately by $\mathrm{hal}_V(S)$ instead of $\widetilde{S}$.

Proposition 8.4. *If S is a subset of an R-module V and W a submodule of V, then*

$$W \cap \mathrm{hal}_V(S) = \mathrm{hal}_W(W \cap S) = \mathrm{hal}_V(W \cap S).$$

Proof. Let $v \in \mathrm{hal}_V(S)$ be given. We choose $\lambda, \mu \in R$ with $\lambda v = s \in S$ and $\mu s = v$. If now $v \in W$ then $\lambda v = s \in W \cap S$, and so $v \in \mathrm{hal}_W(W \cap S)$. This proves that

$$W \cap \mathrm{hal}_V(S) \subset \mathrm{hal}_W(W \cap S). \tag{A}$$

Trivially

$$\mathrm{hal}_W(W \cap S) \subset \mathrm{hal}_V(W \cap S). \tag{B}$$

If $v \in \mathrm{hal}_V(W \cap S)$, then there exist $\lambda, \mu \in R$ with $\lambda v = s \in W \cap S$ and $\mu s = v$. It follows that $v \in W \cap \mathrm{hal}_V(S)$. This proves

$$\mathrm{hal}_V(W \cap S) \subset W \cap \mathrm{hal}_V(S). \tag{C}$$

(A)–(C) together imply the assertion of the proposition. □

In case $S \subset W$ the proposition reads as follows:

Corollary 8.5. *Let $S \subset V$. Then the halo of S in any submodule $W \supset S$ of V coincides with the halo of S in V.*

Thus in practice the notation $\mathrm{hal}_V(S)$ instead of $\widetilde{S}$ is rarely needed.

Proposition 8.6. *Let $(V_i \mid i \in I)$ be a family of submodules of the R-module V and assume that for every $i \in I$ there is given a set $S_i \subset V_i$.*

a) *Then*

$$\bigcup_{i \in I} \mathrm{hal}_{V_i}(S_i) = \mathrm{hal}_V\left(\bigcup_{i \in I} S_i\right).$$

b) *If $\sum\limits_{i \in I} V_i = V$ and each S_i is an additive spine of V_i, then $\bigcup\limits_{i \in I} S_i$ is an additive spine of V.*

Proof. Let $S := \bigcup\limits_{i \in I} S_i$.

a): We have $\mathrm{hal}_V(S) = \bigcup_{i \in I} \mathrm{hal}_V(S_i)$ in complete analogy to Remark 7.3.iii. Furthermore $\mathrm{hal}_V(S_i) = \mathrm{hal}_{V_i}(S_i)$ by Corollary 8.5.

b): Let $\widetilde{S}_i := \mathrm{hal}_{V_i}(S_i)$. Then $\bigcup \widetilde{S}_i = \widetilde{S}$, $\overset{\infty}{\sum} \widetilde{S}_i = V_i$, and so

$$\overset{\infty}{\sum} \widetilde{S} = \sum_{i \in I} \left(\overset{\infty}{\sum} \widetilde{S}_i \right) = \sum_{i \in I} V_i = V.$$

□

We now have a good hold on all additive spines of a free R-module as follows:

Proposition 8.7. *Assume that V is a free R-module with base $(v_i \mid i \in I)$. Then every additive spine S of V has the shape*

$$S = \bigcup_{i \in I} M_i v_i$$

with every M_i an additive spine of R, as defined in §7.

Proof. We have $V = \bigoplus_{i \in I} V_i$ with $V_i = Rv_i \cong {}_RR$. The claim follows from Proposition 8.6. □

Proposition 8.8 (Functoriality of halos and additive spines). *Let $\varphi : V \to V'$ be an R-linear map between R-modules.*

a) *If S is a subset of V then*
$$\varphi(\widetilde{S}) \subset \varphi(S)^{\sim}.$$
b) *If S is an additive spine of V, then the R-module $\varphi(V)$ is additively generated by $\varphi(\widetilde{S})$, and so $\varphi(S)$ is an additive spine of $\varphi(V)$.*

Proof. a): Let $x \in \widetilde{S}$. We have $\lambda, \mu \in R$ with $\lambda x = s \in S$, $\mu s = x$. It follows that $\lambda\varphi(x) = \varphi(s)$, $\mu\varphi(s) = \varphi(x)$, whence $\varphi(x) \in \varphi(S)^{\sim}$.

b): By Corollary 8.5 we may replace V by $\varphi(V)$, and so assume that φ is surjective. We have $\sum^{\infty} \widetilde{S} = V$. Applying φ we obtain
$$\sum^{\infty} \varphi(\widetilde{S}) = \varphi(V).$$
It follows by a) that $\sum^{\infty} \varphi(S)^{\sim} = \varphi(V)$. □

Corollary 8.9. *Assume that R, T are semirings and V is an (R,T)-bimodule, i.e., V is a left R-module, a right T-module, and*
$$\forall\, \lambda \in R, \mu \in T, v \in V : \ (\lambda v)\mu = \lambda(v\mu).$$
Let S be a subset of V. As before let $\widetilde{S}$ denote the halo of S in ${}_RV$, (= V as a left R-module). Then, for any $t \in T$
$$\widetilde{S}t \subset (St)^{\sim}.$$
If S is an additive spine of V then $\widetilde{S}t$ generates the left R-module Vt additively, and so St is an additive spine of Vt.

Proof. Apply Proposition 8.8 to the endomorphism $v \mapsto vt$ of ${}_RV$. □

Corollary 8.10. *If again V is an (R,T)-bimodule and t is a unit of T, then $\widetilde{S}t = (St)^{\sim}$, and S is an additive spine of V iff St is an additive spine of V.*

Proof. Let $u := t^{-1}$. Then by Corollary 8.9 $(\widetilde{S}t)u \subset (St)^{\sim}u \subset (Stu)^{\sim} = \widetilde{S}$. Multiplying by t, we obtain $\widetilde{S}t \subset (St)^{\sim} \subset \widetilde{S}t$, whence $\widetilde{S}t = (St)^{\sim}$, and then
$$\sum^{\infty}(St)^{\sim} = \left(\sum^{\infty} \widetilde{S}\right)t.$$
□

Example 8.11. *R is an (R,R)-bimodule in the obvious way. Thus, if M is an additive spine of R (as defined already in §7), and if u is a unit of R, then Mu is again an additive spine of R.*

Example 8.12. *Assume that C is a semiring which is a homomorphic image of $\mathbb{N}_0$, and $R := M_n(C)$. We have seen in Example 2.12 that $\{e_{11}, \dots, e_{nn}\}$ is an additive spine of R. Let $\sigma \in \Gamma_n$. Then $u := \sum_{i=1}^{n} e_{i,\sigma(i)}$ is a unit of R, namely u is the permutation matrix of σ^{-1}. We have $e_{ii}u = e_{i,\sigma(i)}$, and conclude that $\{e_{1,\sigma(1)}, \dots, e_{n,\sigma(n)}\}$ is an additive spine of $M_n(C)$.*

We can generalize Proposition 7.12 as follows:

Proposition 8.13. *Assume that R_1, R_2 are subsemirings of a semiring R with $R = \sum^{\infty} R_1 R_2$, and that V_1, V_2 are left modules over R_1 and R_2 respectively. Assume furthermore that there is given a composition $V_1 \times V_2 \xrightarrow{\bullet} V$ such that*

$$(\lambda_1\lambda_2)(v_1 \bullet v_2) = (\lambda_1 v_1) \bullet (\lambda_2 v_2)$$

for any $\lambda_i \in R$, $v_i \in V_i\,(i = 1,2)$. Assume finally that $V = \sum^{\infty} V_1 \bullet V_2$. Then, given subsets $S_i \subset V_i$ with halos $\widetilde{S}_i$ in the R_i-module $V_i\,(i = 1,2)$, the following holds.

a) *$\widetilde{S}_1 \bullet \widetilde{S}_2$ is contained in the halo $(S_1 \bullet S_2)^\sim$ of $S_1 \bullet S_2$ in V.*

b) *If S_i is an additive spine of $V_i\,(i = 1,2)$ then*

$$V = \sum^{\infty} \widetilde{S}_1 \bullet \widetilde{S}_2$$

and $S_1 \bullet S_2$ is an additive spine of V.

Proof. Let $v_i \in \widetilde{S}_i\,(i = 1,2)$. We have $\lambda_i, \mu_i \in R_i$ with $\lambda_i v_i = s_i \in S_i$, $\mu_i s_i = v_i$. Now

$$(\lambda_1\lambda_2)(v_1 \bullet v_2) = (\lambda_1 v_1) \bullet (\lambda_2 v_2) = s_1 \bullet s_2$$

and $(\mu_1\mu_2)(s_1 \bullet s_2) = (\mu_1 s_1) \bullet (\mu_2 s_2) = v_1 \bullet v_2$. This proves that $\widetilde{S}_1 \bullet \widetilde{S}_2 \subset (S_1 \bullet S_2)^\sim$. If now $\sum^{\infty} \widetilde{S}_i = V_i\,(i = 1,2)$, then

$$\sum^{\infty}(\widetilde{S}_1 \bullet \widetilde{S}_2) \supset \left(\sum^{\infty} \widetilde{S}_1\right) \bullet \left(\sum^{\infty} \widetilde{S}_2\right) = V_1 \bullet V_2,$$

and so $\sum^{\infty}(\widetilde{S}_1 \bullet \widetilde{S}_2) \supset \sum^{\infty} V_1 \bullet V_2 = V$, whence $\sum^{\infty} \widetilde{S}_1 \bullet \widetilde{S}_2 = V$. A fortiori $\sum^{\infty}(S_1 \bullet S_2)^\sim = V$. □

Note that Proposition 7.12 is indeed a special case of this proposition: Given an R-module V, take $R_1 = R_2 = R$, $V_1 = R$, $V_2 = V$ and the scalar product $R \times V \to V$.

9. The posets $\mathrm{SA}(V)$, $\Sigma\,\mathrm{SA}(V)$ and $\Sigma_f\,\mathrm{SA}_f$ in good cases

Assume now that R has a finite additive spine M consisting of $m := |M|$ elements. We have seen in §7 that, when S is a set of generators of V, then every $W \in \mathrm{SA}(V)$ is generated by the set $W \cap (MS)$. Thus, if $s := |S|$ is finite, we see that the lattice $\mathrm{SA}(V)$ is finite, consisting of at most $2^{m|S|}$ elements. More generally we have the following fact.

Theorem 9.1. *Assume that V_0 is a submodule of an R-module V and S is a subset of V, such that V is generated over V_0 by S, i.e.,*

$$V = V_0 + \sum^{\infty} RS. \tag{9.1}$$

Let $W_0 \in \mathrm{SA}(V)$ be given with $W_0 \subset V_0$, and consider the set

$$\mathrm{SA}(V; W_0, V_0) = \{W \in \mathrm{SA}(V) \mid W \cap V_0 = W_0\}. \tag{9.2}$$

Then if $s := |S|$ is finite, this set $\mathrm{SA}(V; W_0, V_0)$ consists of at most 2^{ms} elements. Furthermore, any chain $W_0 \subsetneqq W_1 \subsetneqq \cdots \subsetneqq W_r$ in $\mathrm{SA}(V; W_0, V_0)$ has length $r \le ms$.

Proof. Let U denote the submodule of V generated by S. We have $V = V_0 + U$. If $W \in \mathrm{SA}(V; W_0, V_0)$ then by (1.1)

$$W = W \cap V_0 + W \cap U = W_0 + W \cap U, \tag{9.3}$$

and, of course, $W \cap U \in \mathrm{SA}(U)$. Since $|\mathrm{SA}(U)| \leq 2^{ms}$, as stated above, we infer that $|\mathrm{SA}(V; W_0, V_0)| \leq 2^{ms}$. Also, if $W_0 \subsetneqq W_1 \subsetneqq \cdots \subsetneqq W_r$ is a chain in $\mathrm{SA}(V; W_0, V_0)$, we conclude from (9.3) for $U_i := W_i \cap U$ that

$$U_0 \subsetneqq U_1 \subsetneqq \cdots \subsetneqq U_r.$$

Every U_i is generated by the set $U_i \cap (MS)$ and so

$$U_0 \cap (MS) \subsetneqq U_1 \cap (MS) \subsetneqq \cdots \subsetneqq U_r \cap (MS).$$

This implies that $r \leq |MS| = ms$. □

We return to an arbitrary semiring R.

Theorem 9.2. *Assume that T is an additive spine of the R-module V (cf. Def. 8.1).*

a) *Then any $U \in \Sigma\,\mathrm{SA}(V)$ is generated by the set $U \cap T$.*

b) *If T is finite, $|T| = t$, then $|\Sigma\,\mathrm{SA}(V)| \leq 2^t$, and any chain*

$$U_0 \subsetneqq U_1 \subsetneqq \cdots \subsetneqq U_r$$

in $\Sigma\,\mathrm{SA}(V)$ has length $r \leq t$.

Proof. a): Write $U = \sum_{i \in I} W_i$ with $W_i \in \mathrm{SA}(V)$. We know by Theorem 8.3 that every W_i is generated by $W_i \cap S$. Thus U is generated by the set

$$\bigcup_{i \in I} (W_i \cap S) = \left(\bigcup_{i \in I} W_i \right) \cap S.$$

A fortiori U is generated by $U \cap S$.

b): Every $U \in \Sigma\,\mathrm{SA}(V)$ is generated by the set $U \cap T \subset T$. We have at most 2^t possibilities for this set, and so $|\Sigma\,\mathrm{SA}(V)| \leq 2^t$. Furthermore, if $U_0 \subsetneqq \cdots \subsetneqq U_r$ is a chain in $\Sigma\,\mathrm{SA}(V)$, then

$$U_0 \cap T \subsetneqq U_1 \cap T \subsetneqq \cdots \subsetneqq U_r \cap T,$$

since each U_i generated by $U_i \cap T$, and so $r \leq t$. □

By a variation of our previous arguments we obtain

Theorem 9.3. *Assume that R has a finite additive spine M, furthermore that $U \in \Sigma_f\,\mathrm{SA}_f(V)$. Let S be a finite set of generators of U. Then every $W \in \mathrm{SA}_f(U)$ is generated by the finite set $W \cap (MS)$ and every chain*

$$W \supsetneqq W_1 \supsetneqq W_2 \supsetneqq \ \ldots \ \supsetneqq W_r$$

in $\mathrm{SA}(U)$, hence in $\mathrm{SA}_f(U)$, has length $r \leq |M| \cdot |S|$. A fortiori this holds if W and all W_i are in $\mathrm{SA}_f(V)$.

Proof. Every $W \in \mathrm{SA}_f(U)$ is generated by the finite set $W \cap (MS)$, cf. Theorem 7.8. Furthermore by the same theorem every W_i is generated by the subset $W_i \cap (MS)$ of $W \cap (MS)$. It follows that

$$W \cap (MS) \supsetneqq W_1 \cap (MS) \supsetneqq \ \ldots \ \supsetneqq W_r \cap (MS)$$

and so $r \leq |W \cap (MS)| \leq |M| \cdot |S|$. It is obvious that every SA-submodule of V contained in U is SA in U. □

Example 9.4. *We read off from Theorem 9.3 that, if R has a finite additive spine, then every module $U \in \Sigma_f\,\mathrm{SA}_f(V)$ is SAF-accessible.*

Our final result in this section refers to modules with additive spines which are not necessarily finite.

Theorem 9.5. *Assume that $T \subset V$ is an additive spine of the R-module V, and $U \in \Sigma\,\mathrm{SA}(V)$.*

a) *Then U is generated by the set $U \cap T$.*

b) *If U is an SA_f-sum in V, and $(W_i \mid i \in I)$ is a family of finitely generated SA_f-submodules of V with $U = \sum_{i\in I} W_i$, then every W_i is generated by a finite subset T_i of $W_i \cap T$, and so U is generated by the subset $\bigcup_{i\in I} T_i = T'$ of T. This subset T' is an additive spine of U.*

c) *If $U \in \Sigma_f\,\mathrm{SA}_f(V)$ then U is generated by a finite subset of $U \cap T$, and this is an additive spine of U.*

Proof. We choose a family $(W_i \mid i \in I)$ in $\mathrm{SA}_f(V)$ with $U = \sum_{i\in I} W_i$.

a): Done before (Theorem 9.2).

b): We assume now that all W_i are finitely generated. Every W_i is generated by $W_i \cap T$ (Theorem 8.3). It follows that W_i is generated by a finite subset T_i of $W_i \cap T$. Indeed, given generators $s_1, \dots, s_r$ of W_i for i fixed, write every s_j as a linear combination of a finite subset T_{ij} of $W_i \cap T$. Then $T_i := \bigcup_{j=1}^{r} T_{ij}$ does it. It follows by Theorem 8.3 that T_i is an additive spine of W_i. It now is clear that $T' := \bigcup_{i\in I} T_i$ generates $U = \sum_{i\in I} W_i$, and it follows by Proposition 8.6 that T' is an additive spine of U.

c): Now evident, since the index set I can be assumed to be finite, and so $T' = \bigcup_{i\in I} T_i$ is a finite additive spine of U. □

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