SUPPLEMENT TO MAKING CORNISH–FISHER FIT FOR RISK MEASUREMENT

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INTRODUCTION

This supplement gives detailed derivations of results used in *Making Cornish–Fisher distributions fit for risk measurement.* Numbered cross-references to mathematical expressions, such as equation (1), without a letter prefix, refer to that article.

Supplement A

We show here that G, defined by equation (8) is invertible on the region R where the fourth order Cornish–Fisher expansion gives a distribution function. The results depend on results of Maillard (2012), which we derive here in two brief subsections so that they can be read independently of Maillard (2012), in corrected form and in the notation we use in *Making Cornish–Fisher distributions fit for risk measurement.*

The region R

Maillard (2012) derives the region R where the fourth-order Cornish–Fisher expansion gives a distribution function. We repeat the derivation for the variables s and k of Section 2.

The fourth order Cornish–Fisher expansion gives a distribution function when $\xi(u) = a_0 + a_1u + a_2u^2 + a_3u^3$ is strictly increasing: that is, when

$$\xi'(u) = 3a_3u^2 + 2a_2u + a_1 > 0.$$

This is a quadratic in u and must satisfy

$$k \geq 2q, \tag{A.1}$$

where $q = s^2$, and have no roots to be positive. That is, its discriminant,

$$4a_2^2 - 4a_1a_3 = 9k^2 - (3+33q)k + (30q^2 + 7q) \tag{A.2}$$

must be negative. The right side of equation (A.2) is a quadratic in k, which is negative between the roots when there are two roots. That is, the discriminant of the right side of equation (A.2),

$$(3-33q)^2 - 36(30q^2 + 7q) = 9(q^2 - 6q + 1) \tag{A.3}$$

must be positive and k must satisfy inequalities (5).

The right side of equation (A.3) is another quadratic, in q, which is positive when q is less that the lower root or greater than the upper root: that is, when

$$q < 3 - 2\sqrt{2} \quad \text{or} \quad q > 3 + 2\sqrt{2}.$$

Maillard (2012) suggests we can just choose the lower root. We need a little more explanation. First, $q^2 - 6q + 1 < q^2 - 6q + 9 = (q-3)^2$. So, if $q > 3 + 2\sqrt{2}$, $\sqrt{q^2 - 6q + 1} < q - 3$ and

$$k < \frac{1 + 11q + \sqrt{q^2 - 6q + 1}}{6} < \frac{1 + 11q + (q-3)}{6} = 2q - \frac{1}{3},$$

contradicting inequality (A.1). Second, $q^2 - 6q + 1 > q^2 - 2q + 1 = (q-1)^2$. If $q < 3 - 2\sqrt{2}$, then $q < 1$. So $\sqrt{q^2 - 6q + 1} > 1 - q$ and

$$k > \frac{1 + 11q - \sqrt{q^2 - 6q + 1}}{6} < \frac{1 + 11q + (1-q)}{6} = 2q,$$

making inequality (A.1) redundant. Thus R is defined by $q < 3 - 2\sqrt{2}$ and inequalities (5).

Moments of the fourth order Cornish–Fisher expansion

Maillard (2012) also derives the moments of the fourth-order Cornish–Fisher expansion. Again, we repeat the derivation here in terms of s and k.

First, note that a random variable whose quantile function is given by the fourth-order Cornish–Fisher expansion can be written as

$$C = a_0 + a_1 Z + a_2 Z^2 + a_3 Z^3,$$

where $Z \sim N(0,1)$. Thus, for a nonnegative integer n, C^n is a polynomial Z. And the nth moment of C is $\mathbb{E}\left[C^n\right]$. Thus, the first four moments of C are

$$\mu_1(s,k) = \sum_{i=1}^{3} a_i \mathbb{E}\left[Z^i\right], \tag{A.4}$$

$$\mu_2(s,k) = \sum_{i=1}^{3}\sum_{j=1}^{3} a_i a_j \mathbb{E}\left[Z^{i+j}\right], \tag{A.5}$$

$$\mu_3(s,k) = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} a_i a_j a_k \mathbb{E}\left[Z^{i+j+k}\right], \tag{A.6}$$

$$\mu_4(s,k) = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} a_i a_j a_k a_l \mathbb{E}\left[Z^{i+j+k+l}\right]. \tag{A.7}$$

A standard result is that

$$\mathbb{E}\left[Z^n\right] = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \prod_{i=1}^{n/2}(2i-1) & \text{if } n \text{ is even.} \end{cases}$$

In particular, $\mathbb{E}\left[Z^0\right] = \mathbb{E}\left[Z^2\right] = 1$, $\mathbb{E}\left[Z^4\right] = 3$, $\mathbb{E}\left[Z^6\right] = 15$, $\mathbb{E}\left[Z^8\right] = 105$, $\mathbb{E}\left[Z^{10}\right] = 945$ and $\mathbb{E}\left[Z^{12}\right] = 10395$.

Substituting $a_2 = -a_0 = s$, $a_1 = 1 + 5s^2 - 3k$, $a_3 = k - 2s^2$ in equations (A.4), evaluating each $\mathbb{E}\left[Z^n\right]$ and simplifying the resulting expressions, for example with Maxima, we get equations (6).

Invertibility of the Cornish–Fisher expansion on R

We now show G, defined by equation (8) is invertible on a region $S = S^+ \cup S^-$ containing R. R is the region where equation (A.3) is negative. Rearranging equation (A.3), we get

$$30q^2 + (7-33k)\,q + 9k^2 - 3k \leq 0,$$

a quadratic in $q \geq 0$. This is only negative between its roots and its discriminant is $9k^2 - 102k + 49$. This must be positive. So we have $k \leq (17-4\sqrt{15})/3 < 51/100$ when $q = 3 - 11/\sqrt{15} < 4/25$. Thus we can be sure that any (q,k) satisfying $(\sqrt{q},k) \in R$ lies below the line through $(0,1/3)$ and $(4/25, 51/100)$: that is, the line $k = 53q/48 + 1/3$. Define S^+ to be the region given by $q < 3 - 2\sqrt{2}$, $s \geq 0$ and

$$\frac{1 + 11q - \sqrt{q^2 - 6q + 1}}{6} < k < \frac{53q}{48} + \frac{1}{3}. \tag{A.8}$$

Define $S^- = \{(s,k) : (-s,k) \in S^+\}$. Then, for $(s,r) \in R$ with $s \geq 0$, $(s,r) \in S^+$ and $(-s,r) \in S^-$.

Notice that we can restrict search for (s,k) satisfying $G(s,k) = (\hat{s},\hat{k})$ to S^+ or S^-, because s and $\hat{s}$ have the same sign. Ideally we would like S^+ and S^- to be convex by replacing $k = 53q/48 + 1/3$ with $k = 53s/120 + 1/3$, but we find G is not invertible on such a reason. It is straightforward to restrict search to convex subsets of S^+ or S^- though we find in practice it is enough to restrict search to S^+ or S^-.

We have

$$\frac{\partial \hat{s}}{\partial s} = {\mu_2}^{-3/2}\frac{\partial \mu_3}{\partial s} - \frac{3}{2}\mu_3{\mu_2}^{-5/2}\frac{\partial \mu_2}{\partial s}, \qquad \frac{\partial \hat{s}}{\partial k} = {\mu_2}^{-3/2}\frac{\partial \mu_3}{\partial k} - \frac{3}{2}\mu_3{\mu_2}^{-5/2}\frac{\partial \mu_2}{\partial k},$$

$$\frac{\partial \hat{k}}{\partial s} = {\mu_2}^{-2}\frac{\partial \mu_4}{\partial s} - 2\mu_4{\mu_2}^{-3}\frac{\partial \mu_2}{\partial s}, \qquad \frac{\partial \hat{k}}{\partial k} = {\mu_2}^{-2}\frac{\partial \mu_4}{\partial k} - 2\mu_4{\mu_2}^{-3}\frac{\partial \mu_2}{\partial k}.$$

And

$$\begin{aligned}
\frac{\partial \mu_1}{\partial s} &= 0, \qquad \frac{\partial \mu_2}{\partial s} = 100s^3 - 48sk,\\
\frac{\partial \mu_3}{\partial s} &= 2550s^4 - 1404ks^2 - 228s^2 + 108k^2 + 36k + 6,\\
\frac{\partial \hat{\mu}_4}{\partial s} &= 519960s^7 - 742320ks^5 - 14400s^5 + 353520k^2s^3\\
&\quad + 32544ks^3 - 168s^3 - 56160k^3s - 12096k^2s - 1008ks.
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial \mu_1}{\partial k} &= 0, \qquad \frac{\partial \mu_2}{\partial k} = 12k - 24s^2, \qquad \frac{\partial \mu_3}{\partial k} = -468s^3 + 216ks + 36s,\\
\frac{\partial \mu_4}{\partial k} &= -123720s^6 + 176760ks^4 + 8136s^4 - 84240k^2s^2\\
&\quad - 12096ks^2 - 504s^2 + 13392k^3 + 3888k^2 + 504k + 24.
\end{aligned}$$

Hence

$$
\begin{aligned}
S(q,k) &= \frac{\mu_2^{9/2}|J|}{144} \\
&= -94775q^7 - 492005q^6 - (806490k^2 + 261735)q^5 - 1543770k^2q^4 \\
&\quad - (286740k^4 + 582660k^2 + 209520k + 19565)q^3 - 354780k^4q^2 \\
&\quad - (5832k^6 + 63180k^4 + 43200k^3 + 11250k^2 + 1236k + 53)q \\
&\quad + 477120kq^6 + 1333380kq^5 + (659340k^3 + 619680k + 114475)q^4 \\
&\quad + 976320k^3q^3 + (64152k^5 + 272160k^3 + 143100k^2 + 25800k + 1401)q^2 \\
&\quad + 69984k^5q - 5832k^6 + 5832k^5 + 4860k^4 + 1620k^3 + 270k^2 + 24k + 1.
\end{aligned}
$$

We have $k \geq 2q$ from inequality (A.1) and so $\partial\mu_2/\partial k = 12k - 24q \geq 24q - 24q = 0$. Thus μ_2 is maximum on R at $k = 0$, when $\mu_2 = 1 + 25q \geq 1$. Thus μ_2 is always positive for $k \geq 0$. So $|J| > 0$ on when $S(q,k) > 0$ and $k \geq 0$. For $q_u \geq q \geq q_l \geq 0$ and $k \geq 0$ we have

$$
\begin{aligned}
S(q,k) &\geq S(q_l, q_u, k) \\
&= -94775q_u^7 - 492005q_u^6 - (806490k^2 + 261735)q_u^5 - 1543770k^2q_u^4 \\
&\quad - (286740k^4 + 582660k^2 + 209520k + 19565)q_u^3 - 354780k^4q_u^2 \\
&\quad - (5832k^6 + 63180k^4 + 43200k^3 + 11250k^2 + 1236k + 53)q_u \\
&\quad + 477120kq_l^6 + 1333380kq_l^5 + (659340k^3 + 619680k + 114475)q_l^4 \\
&\quad + 976320k^3q_l^3 + (64152k^5 + 272160k^3 + 143100k^2 + 25800k + 1401)q_l^2 \\
&\quad + 69984k^5q_l - 5832k^6 + 5832k^5 + 4860k^4 + 1620k^3 + 270k^2 + 24k + 1.
\end{aligned}
$$

From inequality (A.8), if we can find intervals $[q_l, q_u]$ covering $[0, 3 - 2\sqrt{2}]$ such that $S(q_l, q_u, k)$ is positive for k satisfying

$$
k_{\min} = \frac{1 + 11q_l - \sqrt{q_l^2 - 6q_l + 1}}{6} \leq k \leq \frac{53q_u}{48} + \frac{1}{3} = k_{\max}, \tag{A.9}
$$

then we have $|J| > 0$ on S^+.

We find such intervals using computer search in Maxima. In practice we find the intervals are small, and we checked 1164 intervals. For each we compute $S(q_l, q_u, k)$ as a polynomial in k. This polynomial always has degree 6 with a negative coefficient for k^6. We use Maxima to check that there are precisely two real root r_l and r_u. The polynomial is positive between these roots. We check the lower exceeds $k_{\min}$, and the upper is less than $k_{\max}$, by at least 10^{-7}, the tolerance on the values of the roots computed by Maxima. Maxima computes the roots using Sturm sequences, and we can be sure of the number of real roots and of the tolerance of the solution because $S(q_l, q_k, k)$ is a polynomial in k with rational coefficients. Table A.1 shows the sequence of intervals. The roots r_l, r_u and values of $k_{\min}$ and $k_{\max}$ are rounded to 6 decimal places. Other values are exact. Note that $0.171573 > 3 - 2\sqrt{2}$.

Table A.1: Sequence of intervals for $S(q_l, q_k, k)$

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0	0.010724	0	0.379071	−0.115382	1.53688
0.010724	0.015752	0.025102	0.388766	−0.094667	1.584687
0.015752	0.020622	0.036928	0.396758	−0.06975	1.584958
0.020622	0.02534	0.048421	0.40364	0.026366	1.585221
0.02534	0.027626	0.059591	0.406743	−0.041464	1.611068
0.027626	0.029876	0.065016	0.409674	−0.019502	1.611494
0.029876	0.032091	0.070366	0.412453	0.012241	1.611911
0.032091	0.034271	0.075641	0.415097	0.036656	1.612331
0.034271	0.036417	0.080841	0.417618	0.055297	1.612742
0.036417	0.038529	0.085969	0.420027	0.071076	1.613156
0.038529	0.040609	0.091024	0.422337	0.085242	1.61355
0.040609	0.041633	0.096012	0.423452	0.018219	1.627202
0.041633	0.042649	0.09847	0.424545	0.026645	1.627455
0.042649	0.043657	0.100912	0.425616	0.035097	1.627706
0.043657	0.044657	0.103336	0.426667	0.043325	1.627957
0.044657	0.045649	0.105743	0.427698	0.051179	1.628207
0.045649	0.046633	0.108133	0.42871	0.058613	1.628457
0.046633	0.047611	0.110506	0.429705	0.065871	1.628679
0.047611	0.048581	0.112866	0.430682	0.072528	1.628927
0.048581	0.049543	0.11521	0.431641	0.078852	1.629175
0.049543	0.050497	0.117536	0.432583	0.084883	1.629422
0.050497	0.051443	0.119844	0.433508	0.090655	1.629669
0.051443	0.052383	0.122136	0.434419	0.096404	1.629887
0.052383	0.053315	0.124414	0.435314	0.101752	1.630133
0.053315	0.054239	0.126676	0.436194	0.106917	1.630378
0.054239	0.055157	0.12892	0.437061	0.11211	1.630594
0.055157	0.056067	0.131152	0.437913	0.116965	1.630839
0.056067	0.056971	0.133366	0.438753	0.121871	1.631053
0.056971	0.057867	0.135567	0.439579	0.126465	1.631297
0.057867	0.058757	0.137751	0.440393	0.131127	1.631511
0.058757	0.059639	0.139923	0.441193	0.135496	1.631755
0.059639	0.060515	0.142077	0.441982	0.139945	1.631967
0.060515	0.061383	0.144218	0.442759	0.144114	1.632211
0.061383	0.061815	0.146342	0.443143	0.084647	1.639367
0.061815	0.062245	0.147399	0.443525	0.087108	1.639491
0.062245	0.062673	0.148452	0.443903	0.089541	1.639614
0.062673	0.063099	0.149501	0.444278	0.091945	1.639737
0.063099	0.063523	0.150546	0.44465	0.094319	1.63986
0.063523	0.063947	0.151586	0.445021	0.097153	1.639949
0.063947	0.064369	0.152626	0.445389	0.099477	1.640071
0.064369	0.064789	0.153662	0.445754	0.101769	1.640193
0.064789	0.065207	0.154694	0.446116	0.104028	1.640315
0.065207	0.065623	0.155721	0.446475	0.106254	1.640436

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.065623	0.066037	0.156744	0.446831	0.108448	1.640557
0.066037	0.066451	0.157762	0.447187	0.111093	1.640644
0.066451	0.066863	0.158781	0.447539	0.113235	1.640765
0.066863	0.067273	0.159796	0.447889	0.115346	1.640885
0.067273	0.067681	0.160806	0.448235	0.117425	1.641006
0.067681	0.068087	0.161811	0.44858	0.119474	1.641125
0.068087	0.068493	0.162812	0.448923	0.121966	1.64121
0.068493	0.068897	0.163814	0.449263	0.123967	1.64133
0.068897	0.069299	0.164811	0.449601	0.125939	1.641449
0.069299	0.069699	0.165803	0.449936	0.127883	1.641568
0.069699	0.070097	0.166792	0.450268	0.129799	1.641687
0.070097	0.070495	0.167775	0.4506	0.132151	1.641769
0.070495	0.070891	0.168759	0.450929	0.134025	1.641888
0.070891	0.071285	0.169739	0.451255	0.135873	1.642006
0.071285	0.071677	0.170714	0.451579	0.137696	1.642124
0.071677	0.072069	0.171685	0.451902	0.139949	1.642205
0.072069	0.072459	0.172656	0.452222	0.141733	1.642322
0.072459	0.072847	0.173623	0.45254	0.143493	1.64244
0.072847	0.073233	0.174585	0.452855	0.14523	1.642557
0.073233	0.073619	0.175543	0.45317	0.147393	1.642636
0.073619	0.074003	0.176501	0.453482	0.149095	1.642753
0.074003	0.074385	0.177455	0.453792	0.150775	1.64287
0.074385	0.074765	0.178404	0.454099	0.152433	1.642986
0.074765	0.075145	0.179348	0.454406	0.154515	1.643064
0.075145	0.075523	0.180294	0.45471	0.156142	1.64318
0.075523	0.075899	0.181234	0.455011	0.157748	1.643295
0.075899	0.076273	0.18217	0.455311	0.159333	1.643411
0.076273	0.076647	0.183102	0.45561	0.161341	1.643488
0.076647	0.077019	0.184034	0.455906	0.162897	1.643603
0.077019	0.077389	0.184961	0.4562	0.164435	1.643718
0.077389	0.077757	0.185884	0.456492	0.165953	1.643832
0.077757	0.078125	0.186803	0.456783	0.167892	1.643908
0.078125	0.078491	0.187722	0.457072	0.169384	1.644022
0.078491	0.078855	0.188636	0.457358	0.170857	1.644136
0.078855	0.079219	0.189546	0.457644	0.172751	1.644211
0.079219	0.079581	0.190456	0.457928	0.174198	1.644325
0.079581	0.079941	0.191362	0.458209	0.175628	1.644438
0.079941	0.080299	0.192263	0.458489	0.177041	1.644551
0.080299	0.080657	0.19316	0.458767	0.178875	1.644625
0.080657	0.081013	0.194057	0.459044	0.180264	1.644738
0.081013	0.081367	0.194949	0.459318	0.181636	1.644851
0.081367	0.081721	0.195837	0.459592	0.18343	1.644923
0.081721	0.082073	0.196726	0.459864	0.18478	1.645036

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.082073	0.082423	0.19761	0.460133	0.186113	1.645148
0.082423	0.082773	0.198489	0.460402	0.187869	1.645219
0.082773	0.083121	0.199368	0.460669	0.18918	1.645332
0.083121	0.083467	0.200244	0.460934	0.190476	1.645443
0.083467	0.083813	0.201114	0.461198	0.192195	1.645514
0.083813	0.084157	0.201985	0.46146	0.19347	1.645626
0.084157	0.084499	0.202851	0.46172	0.194729	1.645737
0.084499	0.084841	0.203713	0.46198	0.196414	1.645807
0.084841	0.085181	0.204575	0.462237	0.197653	1.645918
0.085181	0.085519	0.205433	0.462493	0.198877	1.646029
0.085519	0.085857	0.206286	0.462748	0.200528	1.646098
0.085857	0.086193	0.207139	0.463001	0.201733	1.646208
0.086193	0.086527	0.207988	0.463252	0.202922	1.646319
0.086527	0.086861	0.208832	0.463502	0.204541	1.646387
0.086861	0.087193	0.209677	0.463751	0.205712	1.646497
0.087193	0.087523	0.210517	0.463997	0.206868	1.646608
0.087523	0.087853	0.211352	0.464243	0.208456	1.646674
0.087853	0.088181	0.212188	0.464488	0.209594	1.646784
0.088181	0.088507	0.213019	0.46473	0.210717	1.646894
0.088507	0.088833	0.213846	0.464971	0.212275	1.64696
0.088833	0.089157	0.214673	0.465211	0.21338	1.64707
0.089157	0.089479	0.215495	0.465449	0.214471	1.647179
0.089479	0.089801	0.216313	0.465687	0.216	1.647244
0.089801	0.090121	0.217131	0.465922	0.217074	1.647354
0.090121	0.090281	0.217945	0.46604	0.161351	1.650969
0.090281	0.090441	0.218351	0.466158	0.162258	1.651008
0.090441	0.090601	0.218758	0.466275	0.163164	1.651047
0.090601	0.090761	0.219166	0.466392	0.164068	1.651085
0.090761	0.090919	0.219573	0.466508	0.16374	1.651169
0.090919	0.091077	0.219975	0.466624	0.164633	1.651207
0.091077	0.091235	0.220377	0.466739	0.165524	1.651245
0.091235	0.091393	0.22078	0.466855	0.166414	1.651283
0.091393	0.091551	0.221182	0.46697	0.167302	1.651322
0.091551	0.091709	0.221585	0.467085	0.168189	1.65136
0.091709	0.091865	0.221988	0.467199	0.167836	1.651443
0.091865	0.092021	0.222385	0.467313	0.168711	1.651481
0.092021	0.092177	0.222783	0.467426	0.169585	1.651519
0.092177	0.092333	0.223181	0.46754	0.170456	1.651556
0.092333	0.092489	0.223579	0.467653	0.171326	1.651594
0.092489	0.092645	0.223978	0.467766	0.172195	1.651632
0.092645	0.092801	0.224376	0.467879	0.173062	1.651669
0.092801	0.092955	0.224774	0.467991	0.172687	1.651753
0.092955	0.093109	0.225168	0.468102	0.173542	1.65179

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.093109	0.093263	0.225561	0.468214	0.174396	1.651827
0.093263	0.093417	0.225955	0.468325	0.175248	1.651864
0.093417	0.093571	0.226349	0.468436	0.176098	1.651901
0.093571	0.093725	0.226742	0.468548	0.176946	1.651939
0.093725	0.093879	0.227136	0.468659	0.177793	1.651976
0.093879	0.094031	0.22753	0.468768	0.177397	1.652059
0.094031	0.094183	0.227919	0.468878	0.178233	1.652096
0.094183	0.094335	0.228308	0.468987	0.179067	1.652132
0.094335	0.094487	0.228698	0.469096	0.179899	1.652169
0.094487	0.094639	0.229087	0.469205	0.180729	1.652206
0.094639	0.094791	0.229476	0.469314	0.181558	1.652242
0.094791	0.094941	0.229866	0.469422	0.18114	1.652326
0.094941	0.095091	0.23025	0.469529	0.181958	1.652362
0.095091	0.095241	0.230635	0.469637	0.182774	1.652398
0.095241	0.095391	0.23102	0.469744	0.183588	1.652434
0.095391	0.095541	0.231405	0.469851	0.1844	1.65247
0.095541	0.095691	0.23179	0.469958	0.185211	1.652506
0.095691	0.095841	0.232175	0.470065	0.18602	1.652543
0.095841	0.095989	0.23256	0.470171	0.185582	1.652626
0.095989	0.096137	0.23294	0.470276	0.186381	1.652661
0.096137	0.096285	0.23332	0.470382	0.187177	1.652697
0.096285	0.096433	0.2337	0.470487	0.187972	1.652733
0.096433	0.096581	0.234081	0.470592	0.188765	1.652768
0.096581	0.096729	0.234461	0.470697	0.189557	1.652804
0.096729	0.096877	0.234842	0.470802	0.190346	1.65284
0.096877	0.097023	0.235223	0.470906	0.18989	1.652923
0.097023	0.097169	0.235598	0.471009	0.190669	1.652958
0.097169	0.097315	0.235974	0.471113	0.191447	1.652993
0.097315	0.097461	0.23635	0.471216	0.192223	1.653028
0.097461	0.097607	0.236726	0.471319	0.192997	1.653063
0.097607	0.097753	0.237102	0.471423	0.19377	1.653098
0.097753	0.097899	0.237478	0.471526	0.194541	1.653134
0.097899	0.098043	0.237855	0.471627	0.194065	1.653217
0.098043	0.098187	0.238226	0.471729	0.194826	1.653252
0.098187	0.098331	0.238597	0.47183	0.195585	1.653286
0.098331	0.098475	0.238969	0.471932	0.196343	1.653321
0.098475	0.098619	0.23934	0.472033	0.197099	1.653355
0.098619	0.098763	0.239712	0.472134	0.197853	1.65339
0.098763	0.098907	0.240084	0.472235	0.198606	1.653425
0.098907	0.099049	0.240455	0.472335	0.198111	1.653508
0.099049	0.099191	0.240822	0.472434	0.198854	1.653542
0.099191	0.099333	0.241189	0.472534	0.199595	1.653576
0.099333	0.099475	0.241556	0.472633	0.200335	1.65361

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.099475	0.099617	0.241923	0.472733	0.201073	1.653644
0.099617	0.099759	0.24229	0.472832	0.20181	1.653678
0.099759	0.099901	0.242658	0.472931	0.202545	1.653712
0.099901	0.100041	0.243025	0.473029	0.202031	1.653796
0.100041	0.100181	0.243387	0.473127	0.202757	1.65383
0.100181	0.100321	0.24375	0.473225	0.203481	1.653863
0.100321	0.100461	0.244112	0.473322	0.204203	1.653897
0.100461	0.100601	0.244475	0.47342	0.204924	1.65393
0.100601	0.100741	0.244838	0.473517	0.205644	1.653964
0.100741	0.100881	0.245201	0.473614	0.206362	1.653997
0.100881	0.101021	0.245563	0.473712	0.207078	1.654031
0.101021	0.101159	0.245926	0.473808	0.206547	1.654115
0.101159	0.101297	0.246284	0.473903	0.207255	1.654148
0.101297	0.101435	0.246642	0.473999	0.20796	1.654181
0.101435	0.101573	0.247001	0.474095	0.208665	1.654214
0.101573	0.101711	0.247359	0.47419	0.209368	1.654247
0.101711	0.101849	0.247717	0.474286	0.210069	1.65428
0.101849	0.101987	0.248076	0.474381	0.210769	1.654313
0.101987	0.102123	0.248434	0.474475	0.210218	1.654396
0.102123	0.102259	0.248788	0.474569	0.210909	1.654429
0.102259	0.102395	0.249141	0.474663	0.211599	1.654462
0.102395	0.102531	0.249495	0.474757	0.212287	1.654494
0.102531	0.102667	0.249848	0.474851	0.212974	1.654527
0.102667	0.102803	0.250202	0.474945	0.21366	1.654559
0.102803	0.102939	0.250556	0.475038	0.214344	1.654592
0.102939	0.103075	0.25091	0.475132	0.215027	1.654624
0.103075	0.103209	0.251264	0.475224	0.214458	1.654708
0.103209	0.103343	0.251613	0.475316	0.215132	1.65474
0.103343	0.103477	0.251963	0.475408	0.215804	1.654772
0.103477	0.103611	0.252312	0.4755	0.216476	1.654804
0.103611	0.103745	0.252661	0.475592	0.217146	1.654836
0.103745	0.103879	0.253011	0.475684	0.217815	1.654868
0.103879	0.104013	0.25336	0.475775	0.218482	1.6549
0.104013	0.104145	0.25371	0.475866	0.217892	1.654984
0.104145	0.104277	0.254054	0.475956	0.218551	1.655016
0.104277	0.104409	0.254399	0.476046	0.219209	1.655047
0.104409	0.104541	0.254743	0.476137	0.219865	1.655079
0.104541	0.104673	0.255088	0.476227	0.22052	1.65511
0.104673	0.104805	0.255433	0.476317	0.221174	1.655142
0.104805	0.104937	0.255778	0.476407	0.221827	1.655173
0.104937	0.105069	0.256123	0.476497	0.222479	1.655205
0.105069	0.105199	0.256468	0.476585	0.22187	1.655289
0.105199	0.105329	0.256808	0.476674	0.222513	1.65532

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.105329	0.105459	0.257149	0.476762	0.223154	1.655351
0.105459	0.105589	0.257489	0.476851	0.223795	1.655382
0.105589	0.105719	0.257829	0.476939	0.224435	1.655413
0.105719	0.105849	0.25817	0.477027	0.225073	1.655444
0.105849	0.105979	0.25851	0.477115	0.22571	1.655475
0.105979	0.106109	0.258851	0.477203	0.226346	1.655506
0.106109	0.106237	0.259192	0.47729	0.225718	1.65559
0.106237	0.106365	0.259527	0.477377	0.226345	1.655621
0.106365	0.106493	0.259863	0.477464	0.226972	1.655651
0.106493	0.106621	0.260199	0.47755	0.227597	1.655682
0.106621	0.106749	0.260534	0.477637	0.228222	1.655712
0.106749	0.106877	0.26087	0.477723	0.228845	1.655743
0.106877	0.107005	0.261206	0.47781	0.229468	1.655773
0.107005	0.107133	0.261543	0.477896	0.230089	1.655804
0.107133	0.107259	0.261879	0.477981	0.22944	1.655888
0.107259	0.107385	0.26221	0.478066	0.230052	1.655918
0.107385	0.107511	0.262541	0.478151	0.230664	1.655948
0.107511	0.107637	0.262872	0.478236	0.231275	1.655978
0.107637	0.107763	0.263204	0.47832	0.231885	1.656008
0.107763	0.107889	0.263535	0.478405	0.232494	1.656038
0.107889	0.108015	0.263867	0.47849	0.233102	1.656068
0.108015	0.108141	0.264199	0.478575	0.233709	1.656098
0.108141	0.108265	0.26453	0.478658	0.233038	1.656182
0.108265	0.108389	0.264857	0.478741	0.233637	1.656212
0.108389	0.108513	0.265184	0.478824	0.234235	1.656242
0.108513	0.108637	0.26551	0.478907	0.234831	1.656271
0.108637	0.108761	0.265837	0.47899	0.235427	1.656301
0.108761	0.108885	0.266164	0.479073	0.236022	1.65633
0.108885	0.109009	0.266491	0.479156	0.236616	1.65636
0.109009	0.109133	0.266818	0.479239	0.237209	1.656389
0.109133	0.109255	0.267146	0.479321	0.236516	1.656474
0.109255	0.109377	0.267468	0.479402	0.237101	1.656503
0.109377	0.109499	0.26779	0.479484	0.237685	1.656532
0.109499	0.109621	0.268112	0.479565	0.238268	1.656561
0.109621	0.109743	0.268435	0.479646	0.23885	1.65659
0.109743	0.109865	0.268757	0.479728	0.239432	1.656619
0.109865	0.109987	0.26908	0.479809	0.240012	1.656648
0.109987	0.110109	0.269402	0.47989	0.240592	1.656677
0.110109	0.110231	0.269725	0.479971	0.24117	1.656706
0.110231	0.110351	0.270048	0.480051	0.240457	1.65679
0.110351	0.110471	0.270365	0.480131	0.241028	1.656819
0.110471	0.110591	0.270683	0.480211	0.241598	1.656848
0.110591	0.110711	0.271001	0.48029	0.242166	1.656876

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.110711	0.110831	0.271319	0.48037	0.242735	1.656905
0.110831	0.110951	0.271637	0.480449	0.243302	1.656933
0.110951	0.111071	0.271955	0.480529	0.243868	1.656962
0.111071	0.111191	0.272273	0.480608	0.244434	1.65699
0.111191	0.111309	0.272591	0.480687	0.243697	1.657075
0.111309	0.111427	0.272905	0.480765	0.244255	1.657103
0.111427	0.111545	0.273218	0.480843	0.244812	1.657131
0.111545	0.111663	0.273531	0.480921	0.245368	1.657159
0.111663	0.111781	0.273844	0.480999	0.245923	1.657187
0.111781	0.111899	0.274158	0.481077	0.246478	1.657215
0.111899	0.112017	0.274471	0.481154	0.247031	1.657243
0.112017	0.112135	0.274785	0.481232	0.247584	1.657271
0.112135	0.112253	0.275099	0.48131	0.248137	1.657299
0.112253	0.112369	0.275412	0.481387	0.247377	1.657384
0.112369	0.112485	0.275721	0.481463	0.247921	1.657412
0.112485	0.112601	0.27603	0.481539	0.248465	1.657439
0.112601	0.112717	0.276339	0.481616	0.249008	1.657467
0.112717	0.112833	0.276647	0.481692	0.24955	1.657495
0.112833	0.112949	0.276956	0.481768	0.250092	1.657522
0.112949	0.113065	0.277265	0.481844	0.250632	1.65755
0.113065	0.113181	0.277575	0.481921	0.251172	1.657577
0.113181	0.113297	0.277884	0.481997	0.251712	1.657605
0.113297	0.113411	0.278193	0.482071	0.250929	1.65769
0.113411	0.113525	0.278497	0.482146	0.25146	1.657717
0.113525	0.113639	0.278801	0.482221	0.251991	1.657744
0.113639	0.113753	0.279106	0.482296	0.252521	1.657771
0.113753	0.113867	0.27941	0.48237	0.25305	1.657798
0.113867	0.113981	0.279714	0.482445	0.253579	1.657825
0.113981	0.114095	0.280019	0.482519	0.254107	1.657852
0.114095	0.114209	0.280324	0.482594	0.254634	1.657879
0.114209	0.114323	0.280628	0.482668	0.255161	1.657906
0.114323	0.114435	0.280933	0.482741	0.254353	1.657991
0.114435	0.114547	0.281233	0.482815	0.254872	1.658018
0.114547	0.114659	0.281532	0.482888	0.255391	1.658045
0.114659	0.114771	0.281832	0.482961	0.255908	1.658071
0.114771	0.114883	0.282132	0.483034	0.256425	1.658098
0.114883	0.114995	0.282432	0.483107	0.256942	1.658124
0.114995	0.115107	0.282732	0.483179	0.257457	1.658151
0.115107	0.115219	0.283032	0.483252	0.257973	1.658178
0.115219	0.115331	0.283332	0.483325	0.258487	1.658204
0.115331	0.115441	0.283632	0.483397	0.257654	1.65829
0.115441	0.115551	0.283927	0.483468	0.25816	1.658316
0.115551	0.115661	0.284222	0.48354	0.258667	1.658342

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.115661	0.115771	0.284518	0.483611	0.259172	1.658368
0.115771	0.115881	0.284813	0.483682	0.259677	1.658394
0.115881	0.115991	0.285108	0.483754	0.260181	1.65842
0.115991	0.116101	0.285404	0.483825	0.260685	1.658446
0.116101	0.116211	0.285699	0.483896	0.261188	1.658472
0.116211	0.116321	0.285995	0.483968	0.261691	1.658498
0.116321	0.116429	0.286291	0.484037	0.260831	1.658584
0.116429	0.116537	0.286581	0.484107	0.261326	1.658609
0.116537	0.116645	0.286872	0.484177	0.26182	1.658635
0.116645	0.116753	0.287162	0.484247	0.262314	1.658661
0.116753	0.116861	0.287453	0.484317	0.262807	1.658686
0.116861	0.116969	0.287744	0.484387	0.2633	1.658712
0.116969	0.117077	0.288035	0.484456	0.263792	1.658738
0.117077	0.117185	0.288326	0.484526	0.264284	1.658763
0.117185	0.117293	0.288617	0.484596	0.264775	1.658789
0.117293	0.117401	0.288908	0.484665	0.265265	1.658814
0.117401	0.117507	0.289199	0.484734	0.264379	1.6589
0.117507	0.117613	0.289485	0.484802	0.264862	1.658925
0.117613	0.117719	0.289771	0.48487	0.265344	1.65895
0.117719	0.117825	0.290057	0.484938	0.265826	1.658975
0.117825	0.117931	0.290343	0.485006	0.266307	1.659001
0.117931	0.118037	0.29063	0.485075	0.266788	1.659026
0.118037	0.118143	0.290916	0.485143	0.267268	1.659051
0.118143	0.118249	0.291202	0.485211	0.267748	1.659076
0.118249	0.118355	0.291489	0.485279	0.268227	1.659101
0.118355	0.118459	0.291776	0.485346	0.267312	1.659187
0.118459	0.118563	0.292057	0.485412	0.267784	1.659212
0.118563	0.118667	0.292338	0.485479	0.268255	1.659236
0.118667	0.118771	0.29262	0.485546	0.268725	1.659261
0.118771	0.118875	0.292901	0.485612	0.269195	1.659286
0.118875	0.118979	0.293183	0.485679	0.269665	1.65931
0.118979	0.119083	0.293465	0.485745	0.270134	1.659335
0.119083	0.119187	0.293747	0.485812	0.270603	1.659359
0.119187	0.119291	0.294029	0.485879	0.271071	1.659384
0.119291	0.119395	0.294311	0.485945	0.271539	1.659409
0.119395	0.119497	0.294593	0.48601	0.270595	1.659495
0.119497	0.119599	0.29487	0.486075	0.271055	1.659519
0.119599	0.119701	0.295146	0.48614	0.271515	1.659543
0.119701	0.119803	0.295423	0.486206	0.271974	1.659568
0.119803	0.119905	0.2957	0.486271	0.272433	1.659592
0.119905	0.120007	0.295977	0.486336	0.272891	1.659616
0.120007	0.120109	0.296255	0.486401	0.273349	1.65964
0.120109	0.120211	0.296532	0.486466	0.273807	1.659664

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.120211	0.120313	0.296809	0.486531	0.274264	1.659688
0.120313	0.120415	0.297087	0.486595	0.27472	1.659712
0.120415	0.120515	0.297364	0.486659	0.273747	1.659799
0.120515	0.120615	0.297636	0.486723	0.274196	1.659823
0.120615	0.120715	0.297909	0.486786	0.274644	1.659846
0.120715	0.120815	0.298181	0.48685	0.275092	1.65987
0.120815	0.120915	0.298454	0.486913	0.27554	1.659894
0.120915	0.121015	0.298726	0.486977	0.275987	1.659917
0.121015	0.121115	0.298999	0.48704	0.276434	1.659941
0.121115	0.121215	0.299271	0.487104	0.276881	1.659965
0.121215	0.121315	0.299544	0.487167	0.277327	1.659988
0.121315	0.121415	0.299817	0.487231	0.277772	1.660012
0.121415	0.121513	0.30009	0.487293	0.276767	1.660099
0.121513	0.121611	0.300358	0.487355	0.277205	1.660122
0.121611	0.121709	0.300625	0.487417	0.277643	1.660145
0.121709	0.121807	0.300893	0.487479	0.27808	1.660168
0.121807	0.121905	0.301161	0.487541	0.278517	1.660191
0.121905	0.122003	0.301429	0.487603	0.278954	1.660215
0.122003	0.122101	0.301697	0.487665	0.27939	1.660238
0.122101	0.122199	0.301965	0.487727	0.279826	1.660261
0.122199	0.122297	0.302233	0.487789	0.280261	1.660284
0.122297	0.122395	0.302501	0.48785	0.280696	1.660307
0.122395	0.122493	0.30277	0.487912	0.281131	1.660331
0.122493	0.122589	0.303038	0.487973	0.280095	1.660418
0.122589	0.122685	0.303301	0.488033	0.280522	1.66044
0.122685	0.122781	0.303564	0.488094	0.280948	1.660463
0.122781	0.122877	0.303828	0.488154	0.281375	1.660486
0.122877	0.122973	0.304091	0.488215	0.281801	1.660508
0.122973	0.123069	0.304354	0.488275	0.282226	1.660531
0.123069	0.123165	0.304618	0.488336	0.282651	1.660554
0.123165	0.123261	0.304881	0.488396	0.283076	1.660577
0.123261	0.123357	0.305145	0.488456	0.283501	1.660599
0.123357	0.123453	0.305409	0.488517	0.283925	1.660622
0.123453	0.123547	0.305672	0.488576	0.282855	1.660709
0.123547	0.123641	0.305931	0.488635	0.283271	1.660731
0.123641	0.123735	0.306189	0.488694	0.283688	1.660754
0.123735	0.123829	0.306448	0.488753	0.284103	1.660776
0.123829	0.123923	0.306706	0.488812	0.284519	1.660798
0.123923	0.124017	0.306965	0.488871	0.284934	1.66082
0.124017	0.124111	0.307224	0.48893	0.285349	1.660843
0.124111	0.124205	0.307483	0.488989	0.285764	1.660865
0.124205	0.124299	0.307742	0.489048	0.286178	1.660887
0.124299	0.124393	0.308001	0.489106	0.286592	1.660909

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.124393	0.124487	0.30826	0.489165	0.287006	1.660931
0.124487	0.124579	0.308519	0.489223	0.285901	1.661019
0.124579	0.124671	0.308773	0.48928	0.286307	1.661041
0.124671	0.124763	0.309027	0.489338	0.286713	1.661063
0.124763	0.124855	0.309281	0.489395	0.287118	1.661084
0.124855	0.124947	0.309535	0.489453	0.287524	1.661106
0.124947	0.125039	0.309789	0.48951	0.287928	1.661128
0.125039	0.125131	0.310043	0.489568	0.288333	1.66115
0.125131	0.125223	0.310297	0.489625	0.288737	1.661171
0.125223	0.125315	0.310552	0.489683	0.289141	1.661193
0.125315	0.125407	0.310806	0.48974	0.289544	1.661215
0.125407	0.125499	0.311061	0.489797	0.289948	1.661237
0.125499	0.125589	0.311315	0.489854	0.288807	1.661324
0.125589	0.125679	0.311564	0.48991	0.289203	1.661346
0.125679	0.125769	0.311814	0.489966	0.289599	1.661367
0.125769	0.125859	0.312063	0.490022	0.289994	1.661388
0.125859	0.125949	0.312312	0.490078	0.290389	1.66141
0.125949	0.126039	0.312562	0.490134	0.290783	1.661431
0.126039	0.126129	0.312811	0.49019	0.291178	1.661452
0.126129	0.126219	0.313061	0.490246	0.291572	1.661473
0.126219	0.126309	0.313311	0.490302	0.291965	1.661495
0.126309	0.126399	0.313561	0.490357	0.292359	1.661516
0.126399	0.126489	0.31381	0.490413	0.292752	1.661537
0.126489	0.126579	0.31406	0.490469	0.293145	1.661559
0.126579	0.126667	0.314311	0.490524	0.291968	1.661647
0.126667	0.126755	0.314555	0.490578	0.292353	1.661668
0.126755	0.126843	0.3148	0.490633	0.292739	1.661688
0.126843	0.126931	0.315044	0.490688	0.293123	1.661709
0.126931	0.127019	0.315289	0.490742	0.293508	1.66173
0.127019	0.127107	0.315534	0.490797	0.293892	1.661751
0.127107	0.127195	0.315779	0.490851	0.294276	1.661772
0.127195	0.127283	0.316024	0.490906	0.29466	1.661792
0.127283	0.127371	0.316269	0.49096	0.295043	1.661813
0.127371	0.127459	0.316515	0.491015	0.295427	1.661834
0.127459	0.127547	0.31676	0.491069	0.295809	1.661855
0.127547	0.127633	0.317005	0.491122	0.294593	1.661943
0.127633	0.127719	0.317245	0.491175	0.294969	1.661964
0.127719	0.127805	0.317485	0.491228	0.295344	1.661984
0.127805	0.127891	0.317725	0.491282	0.295719	1.662004
0.127891	0.127977	0.317965	0.491335	0.296094	1.662025
0.127977	0.128063	0.318206	0.491388	0.296468	1.662045
0.128063	0.128149	0.318446	0.491441	0.296842	1.662065
0.128149	0.128235	0.318686	0.491494	0.297216	1.662086

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.128235	0.128321	0.318927	0.491547	0.29759	1.662106
0.128321	0.128407	0.319167	0.4916	0.297963	1.662126
0.128407	0.128493	0.319408	0.491653	0.298337	1.662147
0.128493	0.128579	0.319648	0.491706	0.298709	1.662167
0.128579	0.128663	0.319889	0.491758	0.297453	1.662256
0.128663	0.128747	0.320124	0.491809	0.297818	1.662276
0.128747	0.128831	0.32036	0.491861	0.298184	1.662295
0.128831	0.128915	0.320595	0.491913	0.298549	1.662315
0.128915	0.128999	0.320831	0.491964	0.298914	1.662335
0.128999	0.129083	0.321066	0.492016	0.299278	1.662355
0.129083	0.129167	0.321302	0.492067	0.299643	1.662375
0.129167	0.129251	0.321538	0.492119	0.300007	1.662395
0.129251	0.129335	0.321773	0.492171	0.300371	1.662415
0.129335	0.129419	0.322009	0.492222	0.300734	1.662434
0.129419	0.129503	0.322245	0.492274	0.301097	1.662454
0.129503	0.129587	0.322481	0.492325	0.301461	1.662474
0.129587	0.129671	0.322717	0.492377	0.301823	1.662494
0.129671	0.129753	0.322954	0.492427	0.300526	1.662583
0.129753	0.129835	0.323184	0.492477	0.300882	1.662602
0.129835	0.129917	0.323415	0.492528	0.301237	1.662622
0.129917	0.129999	0.323646	0.492578	0.301592	1.662641
0.129999	0.130081	0.323877	0.492628	0.301947	1.662661
0.130081	0.130163	0.324108	0.492678	0.302302	1.66268
0.130163	0.130245	0.324339	0.492729	0.302656	1.662699
0.130245	0.130327	0.32457	0.492779	0.30301	1.662719
0.130327	0.130409	0.324801	0.492829	0.303364	1.662738
0.130409	0.130491	0.325032	0.492879	0.303718	1.662758
0.130491	0.130573	0.325263	0.492929	0.304071	1.662777
0.130573	0.130655	0.325495	0.492979	0.304424	1.662796
0.130655	0.130735	0.325726	0.493028	0.303082	1.662886
0.130735	0.130815	0.325952	0.493077	0.303428	1.662905
0.130815	0.130895	0.326178	0.493126	0.303774	1.662924
0.130895	0.130975	0.326404	0.493175	0.30412	1.662943
0.130975	0.131055	0.326631	0.493223	0.304465	1.662962
0.131055	0.131135	0.326857	0.493272	0.30481	1.66298
0.131135	0.131215	0.327083	0.493321	0.305155	1.662999
0.131215	0.131295	0.32731	0.49337	0.3055	1.663018
0.131295	0.131375	0.327536	0.493418	0.305844	1.663037
0.131375	0.131455	0.327763	0.493467	0.306188	1.663056
0.131455	0.131535	0.327989	0.493516	0.306532	1.663075
0.131535	0.131615	0.328216	0.493565	0.306876	1.663094
0.131615	0.131695	0.328443	0.493613	0.30722	1.663113
0.131695	0.131773	0.32867	0.493661	0.305831	1.663203

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.131773	0.131851	0.328891	0.493708	0.306168	1.663221
0.131851	0.131929	0.329112	0.493756	0.306504	1.66324
0.131929	0.132007	0.329334	0.493803	0.30684	1.663258
0.132007	0.132085	0.329555	0.49385	0.307176	1.663276
0.132085	0.132163	0.329777	0.493898	0.307511	1.663295
0.132163	0.132241	0.329998	0.493945	0.307847	1.663313
0.132241	0.132319	0.33022	0.493993	0.308182	1.663332
0.132319	0.132397	0.330442	0.49404	0.308517	1.66335
0.132397	0.132475	0.330664	0.494087	0.308852	1.663369
0.132475	0.132553	0.330886	0.494135	0.309186	1.663387
0.132553	0.132631	0.331108	0.494182	0.309521	1.663406
0.132631	0.132709	0.33133	0.494229	0.309855	1.663424
0.132709	0.132785	0.331552	0.494275	0.308418	1.663514
0.132785	0.132861	0.331769	0.494321	0.308745	1.663532
0.132861	0.132937	0.331985	0.494367	0.309071	1.66355
0.132937	0.133013	0.332202	0.494413	0.309398	1.663568
0.133013	0.133089	0.332419	0.494459	0.309724	1.663586
0.133089	0.133165	0.332636	0.494505	0.310051	1.663604
0.133165	0.133241	0.332853	0.494551	0.310377	1.663622
0.133241	0.133317	0.33307	0.494597	0.310703	1.66364
0.133317	0.133393	0.333287	0.494643	0.311028	1.663658
0.133393	0.133469	0.333504	0.494689	0.311354	1.663676
0.133469	0.133545	0.333721	0.494735	0.311679	1.663694
0.133545	0.133621	0.333938	0.494781	0.312004	1.663712
0.133621	0.133697	0.334156	0.494827	0.312329	1.66373
0.133697	0.133771	0.334373	0.494872	0.31084	1.66382
0.133771	0.133845	0.334585	0.494916	0.311157	1.663838
0.133845	0.133919	0.334797	0.494961	0.311475	1.663855
0.133919	0.133993	0.335009	0.495006	0.311792	1.663873
0.133993	0.134067	0.335221	0.49505	0.31211	1.66389
0.134067	0.134141	0.335433	0.495095	0.312427	1.663908
0.134141	0.134215	0.335645	0.49514	0.312743	1.663925
0.134215	0.134289	0.335858	0.495184	0.31306	1.663943
0.134289	0.134363	0.33607	0.495229	0.313376	1.66396
0.134363	0.134437	0.336282	0.495273	0.313693	1.663978
0.134437	0.134511	0.336495	0.495318	0.314009	1.663995
0.134511	0.134585	0.336707	0.495362	0.314325	1.664013
0.134585	0.134659	0.33692	0.495407	0.31464	1.66403
0.134659	0.134733	0.337133	0.495451	0.314956	1.664048
0.134733	0.134805	0.337345	0.495495	0.313413	1.664139
0.134805	0.134877	0.337553	0.495538	0.313722	1.664156
0.134877	0.134949	0.33776	0.495581	0.31403	1.664173
0.134949	0.135021	0.337967	0.495625	0.314338	1.66419

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.135021	0.135093	0.338174	0.495668	0.314646	1.664207
0.135093	0.135165	0.338382	0.495711	0.314954	1.664224
0.135165	0.135237	0.338589	0.495754	0.315262	1.664241
0.135237	0.135309	0.338797	0.495798	0.315569	1.664258
0.135309	0.135381	0.339004	0.495841	0.315876	1.664275
0.135381	0.135453	0.339212	0.495884	0.316183	1.664292
0.135453	0.135525	0.33942	0.495927	0.31649	1.664309
0.135525	0.135597	0.339628	0.49597	0.316797	1.664326
0.135597	0.135669	0.339836	0.496014	0.317104	1.664343
0.135669	0.135741	0.340044	0.496057	0.31741	1.66436
0.135741	0.135811	0.340252	0.496099	0.31581	1.664452
0.135811	0.135881	0.340454	0.496141	0.31611	1.664468
0.135881	0.135951	0.340656	0.496183	0.316409	1.664485
0.135951	0.136021	0.340859	0.496225	0.316708	1.664501
0.136021	0.136091	0.341062	0.496266	0.317007	1.664518
0.136091	0.136161	0.341264	0.496308	0.317306	1.664535
0.136161	0.136231	0.341467	0.49635	0.317604	1.664551
0.136231	0.136301	0.34167	0.496392	0.317902	1.664568
0.136301	0.136371	0.341873	0.496434	0.318201	1.664584
0.136371	0.136441	0.342075	0.496476	0.318499	1.664601
0.136441	0.136511	0.342278	0.496518	0.318797	1.664618
0.136511	0.136581	0.342482	0.496559	0.319094	1.664634
0.136581	0.136651	0.342685	0.496601	0.319392	1.664651
0.136651	0.136721	0.342888	0.496643	0.319689	1.664667
0.136721	0.136791	0.343091	0.496685	0.319987	1.664684
0.136791	0.136859	0.343295	0.496726	0.318327	1.664775
0.136859	0.136927	0.343492	0.496766	0.318617	1.664792
0.136927	0.136995	0.34369	0.496807	0.318908	1.664808
0.136995	0.137063	0.343888	0.496847	0.319198	1.664824
0.137063	0.137131	0.344086	0.496888	0.319487	1.66484
0.137131	0.137199	0.344283	0.496928	0.319777	1.664856
0.137199	0.137267	0.344481	0.496969	0.320067	1.664872
0.137267	0.137335	0.34468	0.497009	0.320356	1.664888
0.137335	0.137403	0.344878	0.49705	0.320645	1.664904
0.137403	0.137471	0.345076	0.49709	0.320934	1.66492
0.137471	0.137539	0.345274	0.497131	0.321223	1.664937
0.137539	0.137607	0.345472	0.497171	0.321512	1.664953
0.137607	0.137675	0.345671	0.497212	0.3218	1.664969
0.137675	0.137743	0.345869	0.497252	0.322089	1.664985
0.137743	0.137811	0.346068	0.497293	0.322377	1.665001
0.137811	0.137877	0.346267	0.497332	0.320654	1.665093
0.137877	0.137943	0.346459	0.497371	0.320935	1.665109
0.137943	0.138009	0.346652	0.497411	0.321216	1.665124

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.138009	0.138075	0.346845	0.49745	0.321497	1.66514
0.138075	0.138141	0.347038	0.497489	0.321778	1.665156
0.138141	0.138207	0.347232	0.497528	0.322059	1.665171
0.138207	0.138273	0.347425	0.497567	0.32234	1.665187
0.138273	0.138339	0.347618	0.497607	0.32262	1.665203
0.138339	0.138405	0.347811	0.497646	0.322901	1.665218
0.138405	0.138471	0.348005	0.497685	0.323181	1.665234
0.138471	0.138537	0.348198	0.497724	0.323461	1.66525
0.138537	0.138603	0.348392	0.497763	0.323741	1.665265
0.138603	0.138669	0.348586	0.497802	0.32402	1.665281
0.138669	0.138735	0.348779	0.497842	0.3243	1.665296
0.138735	0.138801	0.348973	0.497881	0.324579	1.665312
0.138801	0.138867	0.349167	0.49792	0.324859	1.665328
0.138867	0.138931	0.349361	0.497958	0.323068	1.66542
0.138931	0.138995	0.349549	0.497996	0.323341	1.665435
0.138995	0.139059	0.349737	0.498034	0.323613	1.665451
0.139059	0.139123	0.349925	0.498071	0.323885	1.665466
0.139123	0.139187	0.350114	0.498109	0.324157	1.665481
0.139187	0.139251	0.350302	0.498147	0.324429	1.665496
0.139251	0.139315	0.350491	0.498185	0.324701	1.665511
0.139315	0.139379	0.350679	0.498223	0.324972	1.665527
0.139379	0.139443	0.350868	0.498261	0.325244	1.665542
0.139443	0.139507	0.351056	0.498299	0.325515	1.665557
0.139507	0.139571	0.351245	0.498336	0.325786	1.665572
0.139571	0.139635	0.351434	0.498374	0.326058	1.665587
0.139635	0.139699	0.351623	0.498412	0.326328	1.665602
0.139699	0.139763	0.351812	0.49845	0.326599	1.665618
0.139763	0.139827	0.352001	0.498488	0.32687	1.665633
0.139827	0.139891	0.35219	0.498526	0.32714	1.665648
0.139891	0.139953	0.352379	0.498562	0.325277	1.665741
0.139953	0.140015	0.352562	0.498599	0.325541	1.665756
0.140015	0.140077	0.352746	0.498635	0.325805	1.66577
0.140077	0.140139	0.352929	0.498672	0.326068	1.665785
0.140139	0.140201	0.353113	0.498708	0.326331	1.6658
0.140201	0.140263	0.353296	0.498745	0.326594	1.665814
0.140263	0.140325	0.35348	0.498782	0.326857	1.665829
0.140325	0.140387	0.353664	0.498818	0.32712	1.665844
0.140387	0.140449	0.353847	0.498855	0.327383	1.665859
0.140449	0.140511	0.354031	0.498891	0.327646	1.665873
0.140511	0.140573	0.354215	0.498928	0.327908	1.665888
0.140573	0.140635	0.354399	0.498964	0.328171	1.665903
0.140635	0.140697	0.354583	0.499001	0.328433	1.665917
0.140697	0.140759	0.354767	0.499037	0.328695	1.665932

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.140759	0.140821	0.354952	0.499074	0.328957	1.665947
0.140821	0.140883	0.355136	0.49911	0.329219	1.665962
0.140883	0.140943	0.35532	0.499145	0.327277	1.666055
0.140943	0.141003	0.355499	0.499181	0.327532	1.666069
0.141003	0.141063	0.355677	0.499216	0.327787	1.666083
0.141063	0.141123	0.355856	0.499251	0.328042	1.666098
0.141123	0.141183	0.356034	0.499287	0.328297	1.666112
0.141183	0.141243	0.356213	0.499322	0.328552	1.666126
0.141243	0.141303	0.356392	0.499357	0.328806	1.66614
0.141303	0.141363	0.356571	0.499392	0.32906	1.666155
0.141363	0.141423	0.35675	0.499428	0.329315	1.666169
0.141423	0.141483	0.356929	0.499463	0.329569	1.666183
0.141483	0.141543	0.357108	0.499498	0.329823	1.666197
0.141543	0.141603	0.357287	0.499533	0.330077	1.666212
0.141603	0.141663	0.357466	0.499568	0.330331	1.666226
0.141663	0.141723	0.357645	0.499604	0.330584	1.66624
0.141723	0.141783	0.357825	0.499639	0.330838	1.666254
0.141783	0.141843	0.358004	0.499674	0.331091	1.666269
0.141843	0.141903	0.358184	0.499709	0.331344	1.666283
0.141903	0.141961	0.358363	0.499743	0.329319	1.666377
0.141961	0.142019	0.358537	0.499777	0.329565	1.666391
0.142019	0.142077	0.358711	0.499811	0.329812	1.666404
0.142077	0.142135	0.358884	0.499845	0.330058	1.666418
0.142135	0.142193	0.359058	0.499879	0.330304	1.666432
0.142193	0.142251	0.359232	0.499913	0.33055	1.666446
0.142251	0.142309	0.359406	0.499947	0.330796	1.666459
0.142309	0.142367	0.35958	0.499981	0.331042	1.666473
0.142367	0.142425	0.359754	0.500015	0.331288	1.666487
0.142425	0.142483	0.359928	0.500049	0.331534	1.666501
0.142483	0.142541	0.360102	0.500083	0.331779	1.666515
0.142541	0.142599	0.360277	0.500117	0.332024	1.666528
0.142599	0.142657	0.360451	0.500151	0.33227	1.666542
0.142657	0.142715	0.360626	0.500185	0.332515	1.666556
0.142715	0.142773	0.3608	0.500218	0.33276	1.66657
0.142773	0.142831	0.360975	0.500252	0.333005	1.666584
0.142831	0.142889	0.361149	0.500286	0.33325	1.666597
0.142889	0.142947	0.361324	0.50032	0.333495	1.666611
0.142947	0.143003	0.361499	0.500353	0.331379	1.666705
0.143003	0.143059	0.361667	0.500386	0.331617	1.666719
0.143059	0.143115	0.361836	0.500418	0.331855	1.666732
0.143115	0.143171	0.362005	0.500451	0.332093	1.666745
0.143171	0.143227	0.362174	0.500484	0.332331	1.666759
0.143227	0.143283	0.362343	0.500516	0.332568	1.666772

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.143283	0.143339	0.362512	0.500549	0.332806	1.666785
0.143339	0.143395	0.362682	0.500582	0.333043	1.666799
0.143395	0.143451	0.362851	0.500614	0.333281	1.666812
0.143451	0.143507	0.36302	0.500647	0.333518	1.666825
0.143507	0.143563	0.36319	0.50068	0.333755	1.666839
0.143563	0.143619	0.363359	0.500712	0.333992	1.666852
0.143619	0.143675	0.363528	0.500745	0.334229	1.666865
0.143675	0.143731	0.363698	0.500777	0.334465	1.666879
0.143731	0.143787	0.363868	0.50081	0.334702	1.666892
0.143787	0.143843	0.364037	0.500843	0.334939	1.666905
0.143843	0.143899	0.364207	0.500875	0.335175	1.666918
0.143899	0.143955	0.364377	0.500908	0.335411	1.666932
0.143955	0.144009	0.364547	0.500939	0.333197	1.667027
0.144009	0.144063	0.364711	0.500971	0.333427	1.667039
0.144063	0.144117	0.364875	0.501002	0.333657	1.667052
0.144117	0.144171	0.365039	0.501034	0.333886	1.667065
0.144171	0.144225	0.365203	0.501065	0.334116	1.667078
0.144225	0.144279	0.365367	0.501096	0.334345	1.667091
0.144279	0.144333	0.365531	0.501128	0.334574	1.667104
0.144333	0.144387	0.365696	0.501159	0.334803	1.667117
0.144387	0.144441	0.36586	0.50119	0.335032	1.667129
0.144441	0.144495	0.366024	0.501222	0.335261	1.667142
0.144495	0.144549	0.366189	0.501253	0.33549	1.667155
0.144549	0.144603	0.366353	0.501285	0.335719	1.667168
0.144603	0.144657	0.366518	0.501316	0.335947	1.667181
0.144657	0.144711	0.366683	0.501347	0.336176	1.667194
0.144711	0.144765	0.366847	0.501379	0.336404	1.667207
0.144765	0.144819	0.367012	0.50141	0.336633	1.667219
0.144819	0.144873	0.367177	0.501441	0.336861	1.667232
0.144873	0.144927	0.367342	0.501473	0.337089	1.667245
0.144927	0.144981	0.367507	0.501504	0.337317	1.667258
0.144981	0.145033	0.367672	0.501534	0.334996	1.667353
0.145033	0.145085	0.367831	0.501564	0.335218	1.667366
0.145085	0.145137	0.36799	0.501594	0.335439	1.667378
0.145137	0.145189	0.368149	0.501625	0.33566	1.66739
0.145189	0.145241	0.368308	0.501655	0.335881	1.667403
0.145241	0.145293	0.368468	0.501685	0.336103	1.667415
0.145293	0.145345	0.368627	0.501715	0.336323	1.667428
0.145345	0.145397	0.368786	0.501745	0.336544	1.66744
0.145397	0.145449	0.368946	0.501775	0.336765	1.667452
0.145449	0.145501	0.369105	0.501805	0.336986	1.667465
0.145501	0.145553	0.369265	0.501835	0.337206	1.667477
0.145553	0.145605	0.369425	0.501865	0.337427	1.66749

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.145605	0.145657	0.369584	0.501896	0.337647	1.667502
0.145657	0.145709	0.369744	0.501926	0.337867	1.667514
0.145709	0.145761	0.369904	0.501956	0.338088	1.667527
0.145761	0.145813	0.370064	0.501986	0.338308	1.667539
0.145813	0.145865	0.370224	0.502016	0.338528	1.667551
0.145865	0.145917	0.370384	0.502046	0.338748	1.667564
0.145917	0.145969	0.370544	0.502076	0.338967	1.667576
0.145969	0.146021	0.370704	0.502106	0.339187	1.667589
0.146021	0.146071	0.370864	0.502135	0.33675	1.667684
0.146071	0.146121	0.371018	0.502164	0.336964	1.667696
0.146121	0.146171	0.371172	0.502193	0.337177	1.667708
0.146171	0.146221	0.371327	0.502222	0.33739	1.66772
0.146221	0.146271	0.371481	0.50225	0.337603	1.667732
0.146271	0.146321	0.371635	0.502279	0.337816	1.667744
0.146321	0.146371	0.37179	0.502308	0.338029	1.667756
0.146371	0.146421	0.371944	0.502337	0.338241	1.667768
0.146421	0.146471	0.372099	0.502366	0.338454	1.66778
0.146471	0.146521	0.372253	0.502395	0.338666	1.667792
0.146521	0.146571	0.372408	0.502424	0.338879	1.667804
0.146571	0.146621	0.372563	0.502452	0.339091	1.667816
0.146621	0.146671	0.372718	0.502481	0.339303	1.667828
0.146671	0.146721	0.372872	0.50251	0.339516	1.667839
0.146721	0.146771	0.373027	0.502539	0.339728	1.667851
0.146771	0.146821	0.373182	0.502568	0.33994	1.667863
0.146821	0.146871	0.373337	0.502597	0.340151	1.667875
0.146871	0.146921	0.373492	0.502625	0.340363	1.667887
0.146921	0.146971	0.373648	0.502654	0.340575	1.667899
0.146971	0.147021	0.373803	0.502683	0.340786	1.667911
0.147021	0.147069	0.373958	0.502711	0.33822	1.668007
0.147069	0.147117	0.374107	0.502738	0.338425	1.668019
0.147117	0.147165	0.374256	0.502766	0.33863	1.66803
0.147165	0.147213	0.374406	0.502794	0.338835	1.668042
0.147213	0.147261	0.374555	0.502821	0.33904	1.668053
0.147261	0.147309	0.374704	0.502849	0.339245	1.668065
0.147309	0.147357	0.374854	0.502876	0.33945	1.668076
0.147357	0.147405	0.375003	0.502904	0.339655	1.668088
0.147405	0.147453	0.375153	0.502932	0.339859	1.668099
0.147453	0.147501	0.375302	0.502959	0.340064	1.66811
0.147501	0.147549	0.375452	0.502987	0.340268	1.668122
0.147549	0.147597	0.375602	0.503014	0.340473	1.668133
0.147597	0.147645	0.375752	0.503042	0.340677	1.668145
0.147645	0.147693	0.375902	0.50307	0.340881	1.668156
0.147693	0.147741	0.376051	0.503097	0.341085	1.668168

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.147741	0.147789	0.376201	0.503125	0.341289	1.668179
0.147789	0.147837	0.376352	0.503152	0.341493	1.668191
0.147837	0.147885	0.376502	0.50318	0.341697	1.668202
0.147885	0.147933	0.376652	0.503207	0.341901	1.668214
0.147933	0.147981	0.376802	0.503235	0.342104	1.668225
0.147981	0.148029	0.376952	0.503263	0.342308	1.668236
0.148029	0.148075	0.377103	0.503289	0.339598	1.668333
0.148075	0.148121	0.377247	0.503315	0.339795	1.668344
0.148121	0.148167	0.377391	0.503342	0.339992	1.668355
0.148167	0.148213	0.377535	0.503368	0.340189	1.668366
0.148213	0.148259	0.37768	0.503395	0.340386	1.668377
0.148259	0.148305	0.377824	0.503421	0.340583	1.668388
0.148305	0.148351	0.377968	0.503447	0.34078	1.668399
0.148351	0.148397	0.378113	0.503474	0.340977	1.66841
0.148397	0.148443	0.378258	0.5035	0.341174	1.668421
0.148443	0.148489	0.378402	0.503526	0.341371	1.668432
0.148489	0.148535	0.378547	0.503553	0.341567	1.668443
0.148535	0.148581	0.378692	0.503579	0.341764	1.668454
0.148581	0.148627	0.378836	0.503605	0.34196	1.668465
0.148627	0.148673	0.378981	0.503632	0.342156	1.668476
0.148673	0.148719	0.379126	0.503658	0.342352	1.668487
0.148719	0.148765	0.379271	0.503684	0.342549	1.668498
0.148765	0.148811	0.379416	0.503711	0.342745	1.668509
0.148811	0.148857	0.379561	0.503737	0.342941	1.66852
0.148857	0.148903	0.379706	0.503763	0.343137	1.668531
0.148903	0.148949	0.379852	0.50379	0.343332	1.668542
0.148949	0.148995	0.379997	0.503816	0.343528	1.668553
0.148995	0.149041	0.380142	0.503842	0.343724	1.668564
0.149041	0.149087	0.380288	0.503869	0.343919	1.668575
0.149087	0.149131	0.380433	0.503894	0.341051	1.668672
0.149131	0.149175	0.380572	0.503919	0.34124	1.668683
0.149175	0.149219	0.380712	0.503944	0.34143	1.668694
0.149219	0.149263	0.380851	0.503969	0.341619	1.668704
0.149263	0.149307	0.38099	0.503994	0.341808	1.668715
0.149307	0.149351	0.38113	0.50402	0.341997	1.668725
0.149351	0.149395	0.381269	0.504045	0.342186	1.668736
0.149395	0.149439	0.381409	0.50407	0.342375	1.668746
0.149439	0.149483	0.381548	0.504095	0.342564	1.668757
0.149483	0.149527	0.381688	0.50412	0.342753	1.668767
0.149527	0.149571	0.381828	0.504145	0.342942	1.668778
0.149571	0.149615	0.381967	0.50417	0.34313	1.668788
0.149615	0.149659	0.382107	0.504196	0.343319	1.668799
0.149659	0.149703	0.382247	0.504221	0.343507	1.668809

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.149703	0.149747	0.382387	0.504246	0.343696	1.66882
0.149747	0.149791	0.382527	0.504271	0.343884	1.66883
0.149791	0.149835	0.382667	0.504296	0.344072	1.668841
0.149835	0.149879	0.382807	0.504321	0.34426	1.668852
0.149879	0.149923	0.382948	0.504346	0.344448	1.668862
0.149923	0.149967	0.383088	0.504371	0.344636	1.668873
0.149967	0.150011	0.383228	0.504396	0.344824	1.668883
0.150011	0.150055	0.383368	0.504421	0.345012	1.668894
0.150055	0.150099	0.383509	0.504447	0.3452	1.668904
0.150099	0.150141	0.383649	0.50447	0.342151	1.669002
0.150141	0.150183	0.383784	0.504494	0.342333	1.669012
0.150183	0.150225	0.383918	0.504518	0.342514	1.669022
0.150225	0.150267	0.384052	0.504542	0.342696	1.669032
0.150267	0.150309	0.384186	0.504566	0.342877	1.669042
0.150309	0.150351	0.384321	0.50459	0.343059	1.669052
0.150351	0.150393	0.384455	0.504614	0.34324	1.669062
0.150393	0.150435	0.38459	0.504638	0.343422	1.669072
0.150435	0.150477	0.384724	0.504662	0.343603	1.669083
0.150477	0.150519	0.384859	0.504686	0.343784	1.669093
0.150519	0.150561	0.384993	0.50471	0.343965	1.669103
0.150561	0.150603	0.385128	0.504734	0.344146	1.669113
0.150603	0.150645	0.385263	0.504757	0.344327	1.669123
0.150645	0.150687	0.385398	0.504781	0.344508	1.669133
0.150687	0.150729	0.385533	0.504805	0.344688	1.669143
0.150729	0.150771	0.385668	0.504829	0.344869	1.669153
0.150771	0.150813	0.385803	0.504853	0.34505	1.669163
0.150813	0.150855	0.385938	0.504877	0.34523	1.669173
0.150855	0.150897	0.386073	0.504901	0.345411	1.669183
0.150897	0.150939	0.386208	0.504925	0.345591	1.669193
0.150939	0.150981	0.386343	0.504949	0.345771	1.669203
0.150981	0.151023	0.386479	0.504972	0.345951	1.669213
0.151023	0.151065	0.386614	0.504996	0.346132	1.669223
0.151065	0.151107	0.386749	0.50502	0.346312	1.669233
0.151107	0.151187	0.386885	0.505066	0.386407	1.667561
0.151187	0.151267	0.387143	0.505111	0.386719	1.667578
0.151267	0.151347	0.387401	0.505156	0.387031	1.667596
0.151347	0.151427	0.38766	0.505202	0.387343	1.667613
0.151427	0.151507	0.387919	0.505247	0.387655	1.667631
0.151507	0.151587	0.388178	0.505293	0.387966	1.667648
0.151587	0.151667	0.388437	0.505338	0.388278	1.667666
0.151667	0.151745	0.388697	0.505382	0.387091	1.667773
0.151745	0.151823	0.38895	0.505426	0.387395	1.66779
0.151823	0.151901	0.389204	0.505471	0.387698	1.667807

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.151901	0.151979	0.389457	0.505515	0.388002	1.667824
0.151979	0.152057	0.389711	0.505559	0.388306	1.667842
0.152057	0.152135	0.389965	0.505603	0.38861	1.667859
0.152135	0.152211	0.39022	0.505646	0.387378	1.667966
0.152211	0.152287	0.390468	0.505689	0.387674	1.667982
0.152287	0.152363	0.390716	0.505732	0.38797	1.667999
0.152363	0.152439	0.390964	0.505775	0.388266	1.668016
0.152439	0.152515	0.391213	0.505818	0.388562	1.668033
0.152515	0.152591	0.391462	0.505861	0.388858	1.668049
0.152591	0.152667	0.391711	0.505904	0.389154	1.668066
0.152667	0.152741	0.39196	0.505946	0.387877	1.668173
0.152741	0.152815	0.392203	0.505988	0.388165	1.66819
0.152815	0.152889	0.392447	0.50603	0.388453	1.668206
0.152889	0.152963	0.39269	0.506071	0.388741	1.668222
0.152963	0.153037	0.392934	0.506113	0.389029	1.668239
0.153037	0.153111	0.393177	0.506155	0.389317	1.668255
0.153111	0.153185	0.393421	0.506197	0.389605	1.668272
0.153185	0.153257	0.393666	0.506237	0.38828	1.668379
0.153257	0.153329	0.393903	0.506278	0.38856	1.668395
0.153329	0.153401	0.394142	0.506318	0.38884	1.668411
0.153401	0.153473	0.39438	0.506359	0.389121	1.668427
0.153473	0.153545	0.394618	0.5064	0.389401	1.668443
0.153545	0.153617	0.394857	0.50644	0.389681	1.668459
0.153617	0.153689	0.395096	0.506481	0.389961	1.668475
0.153689	0.153759	0.395335	0.50652	0.388585	1.668582
0.153759	0.153829	0.395568	0.50656	0.388858	1.668598
0.153829	0.153899	0.395801	0.506599	0.38913	1.668613
0.153899	0.153969	0.396034	0.506638	0.389402	1.668629
0.153969	0.154039	0.396267	0.506678	0.389675	1.668644
0.154039	0.154109	0.3965	0.506717	0.389947	1.66866
0.154109	0.154179	0.396734	0.506757	0.390219	1.668675
0.154179	0.154247	0.396968	0.506795	0.38879	1.668783
0.154247	0.154315	0.397195	0.506833	0.389055	1.668798
0.154315	0.154383	0.397423	0.506871	0.38932	1.668813
0.154383	0.154451	0.397651	0.506909	0.389584	1.668828
0.154451	0.154519	0.397879	0.506948	0.389849	1.668844
0.154519	0.154587	0.398107	0.506986	0.390113	1.668859
0.154587	0.154655	0.398336	0.507024	0.390378	1.668874
0.154655	0.154723	0.398564	0.507062	0.390642	1.668889
0.154723	0.154789	0.398793	0.507099	0.389157	1.668997
0.154789	0.154855	0.399015	0.507136	0.389414	1.669011
0.154855	0.154921	0.399238	0.507173	0.389671	1.669026
0.154921	0.154987	0.399461	0.50721	0.389928	1.669041

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.154987	0.155053	0.399684	0.507247	0.390185	1.669056
0.155053	0.155119	0.399907	0.507284	0.390442	1.66907
0.155119	0.155185	0.40013	0.507321	0.390698	1.669085
0.155185	0.155251	0.400354	0.507358	0.390955	1.6691
0.155251	0.155315	0.400577	0.507394	0.38941	1.669208
0.155315	0.155379	0.400794	0.50743	0.38966	1.669222
0.155379	0.155443	0.401012	0.507466	0.389909	1.669236
0.155443	0.155507	0.401229	0.507502	0.390158	1.669251
0.155507	0.155571	0.401447	0.507538	0.390407	1.669265
0.155571	0.155635	0.401665	0.507573	0.390656	1.669279
0.155635	0.155699	0.401883	0.507609	0.390905	1.669294
0.155699	0.155763	0.402101	0.507645	0.391154	1.669308
0.155763	0.155825	0.40232	0.50768	0.389546	1.669416
0.155825	0.155887	0.402532	0.507714	0.389788	1.66943
0.155887	0.155949	0.402744	0.507749	0.39003	1.669444
0.155949	0.156011	0.402956	0.507784	0.390271	1.669458
0.156011	0.156073	0.403169	0.507819	0.390512	1.669472
0.156073	0.156135	0.403381	0.507853	0.390754	1.669486
0.156135	0.156197	0.403594	0.507888	0.390995	1.6695
0.156197	0.156259	0.403807	0.507922	0.391236	1.669514
0.156259	0.156319	0.404021	0.507956	0.389561	1.669622
0.156319	0.156379	0.404227	0.507989	0.389795	1.669635
0.156379	0.156439	0.404434	0.508023	0.390029	1.669649
0.156439	0.156499	0.404641	0.508056	0.390263	1.669662
0.156499	0.156559	0.404848	0.50809	0.390497	1.669676
0.156559	0.156619	0.405055	0.508123	0.390731	1.669689
0.156619	0.156679	0.405263	0.508157	0.390964	1.669703
0.156679	0.156739	0.405471	0.50819	0.391198	1.669716
0.156739	0.156797	0.405679	0.508223	0.389451	1.669825
0.156797	0.156855	0.40588	0.508255	0.389677	1.669838
0.156855	0.156913	0.406081	0.508287	0.389904	1.669851
0.156913	0.156971	0.406283	0.50832	0.39013	1.669864
0.156971	0.157029	0.406484	0.508352	0.390356	1.669877
0.157029	0.157087	0.406686	0.508384	0.390582	1.66989
0.157087	0.157145	0.406889	0.508417	0.390808	1.669903
0.157145	0.157203	0.407091	0.508449	0.391034	1.669916
0.157203	0.157261	0.407293	0.508481	0.39126	1.66993
0.157261	0.157317	0.407496	0.508512	0.389436	1.670038
0.157317	0.157373	0.407692	0.508544	0.389655	1.67005
0.157373	0.157429	0.407888	0.508575	0.389874	1.670063
0.157429	0.157485	0.408084	0.508606	0.390093	1.670076
0.157485	0.157541	0.408281	0.508637	0.390311	1.670089
0.157541	0.157597	0.408477	0.508668	0.39053	1.670101

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.157597	0.157653	0.408674	0.508699	0.390748	1.670114
0.157653	0.157709	0.408871	0.508731	0.390967	1.670127
0.157709	0.157765	0.409068	0.508762	0.391185	1.670139
0.157765	0.157819	0.409266	0.508792	0.389278	1.670248
0.157819	0.157873	0.409456	0.508822	0.38949	1.67026
0.157873	0.157927	0.409647	0.508852	0.389701	1.670272
0.157927	0.157981	0.409838	0.508882	0.389912	1.670285
0.157981	0.158035	0.410029	0.508912	0.390123	1.670297
0.158035	0.158089	0.41022	0.508942	0.390334	1.670309
0.158089	0.158143	0.410411	0.508972	0.390545	1.670322
0.158143	0.158197	0.410603	0.509002	0.390756	1.670334
0.158197	0.158251	0.410795	0.509032	0.390967	1.670346
0.158251	0.158305	0.410987	0.509062	0.391178	1.670358
0.158305	0.158357	0.411179	0.509091	0.389181	1.670467
0.158357	0.158409	0.411364	0.509119	0.389385	1.670479
0.158409	0.158461	0.411549	0.509148	0.389589	1.670491
0.158461	0.158513	0.411735	0.509177	0.389792	1.670503
0.158513	0.158565	0.411921	0.509206	0.389996	1.670514
0.158565	0.158617	0.412107	0.509235	0.390199	1.670526
0.158617	0.158669	0.412293	0.509264	0.390403	1.670538
0.158669	0.158721	0.412479	0.509292	0.390606	1.67055
0.158721	0.158773	0.412666	0.509321	0.39081	1.670562
0.158773	0.158823	0.412852	0.509349	0.388715	1.67067
0.158823	0.158873	0.413032	0.509377	0.388912	1.670682
0.158873	0.158923	0.413212	0.509404	0.389108	1.670693
0.158923	0.158973	0.413392	0.509432	0.389304	1.670705
0.158973	0.159023	0.413572	0.50946	0.3895	1.670716
0.159023	0.159073	0.413752	0.509487	0.389696	1.670728
0.159073	0.159123	0.413933	0.509515	0.389892	1.670739
0.159123	0.159173	0.414113	0.509543	0.390088	1.670751
0.159173	0.159223	0.414294	0.509571	0.390284	1.670762
0.159223	0.159273	0.414475	0.509598	0.39048	1.670774
0.159273	0.159323	0.414656	0.509626	0.390676	1.670785
0.159323	0.159371	0.414838	0.509652	0.388476	1.670894
0.159371	0.159419	0.415012	0.509679	0.388665	1.670905
0.159419	0.159467	0.415187	0.509705	0.388853	1.670916
0.159467	0.159515	0.415361	0.509732	0.389042	1.670927
0.159515	0.159563	0.415536	0.509759	0.389231	1.670938
0.159563	0.159611	0.415711	0.509785	0.38942	1.670949
0.159611	0.159659	0.415886	0.509812	0.389608	1.67096
0.159659	0.159707	0.416062	0.509838	0.389797	1.670971
0.159707	0.159755	0.416237	0.509865	0.389985	1.670982
0.159755	0.159803	0.416413	0.509891	0.390174	1.670993

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.159803	0.159849	0.416589	0.509917	0.387855	1.671102
0.159849	0.159895	0.416758	0.509942	0.388037	1.671113
0.159895	0.159941	0.416926	0.509967	0.388218	1.671123
0.159941	0.159987	0.417095	0.509993	0.3884	1.671134
0.159987	0.160033	0.417265	0.510018	0.388581	1.671144
0.160033	0.160079	0.417434	0.510044	0.388763	1.671155
0.160079	0.160125	0.417604	0.510069	0.388944	1.671166
0.160125	0.160171	0.417773	0.510094	0.389125	1.671176
0.160171	0.160217	0.417943	0.51012	0.389307	1.671187
0.160217	0.160263	0.418113	0.510145	0.389488	1.671197
0.160263	0.160309	0.418283	0.510171	0.389669	1.671208
0.160309	0.160353	0.418454	0.510195	0.38722	1.671317
0.160353	0.160397	0.418617	0.510219	0.387394	1.671327
0.160397	0.160441	0.41878	0.510243	0.387568	1.671337
0.160441	0.160485	0.418944	0.510268	0.387742	1.671348
0.160485	0.160529	0.419108	0.510292	0.387917	1.671358
0.160529	0.160573	0.419271	0.510316	0.388091	1.671368
0.160573	0.160617	0.419435	0.51034	0.388265	1.671378
0.160617	0.160661	0.419599	0.510365	0.388439	1.671388
0.160661	0.160705	0.419764	0.510389	0.388613	1.671398
0.160705	0.160749	0.419928	0.510413	0.388787	1.671409
0.160749	0.160793	0.420093	0.510437	0.388961	1.671419
0.160793	0.160837	0.420258	0.510461	0.389135	1.671429
0.160837	0.160879	0.420423	0.510485	0.38654	1.671538
0.160879	0.160921	0.42058	0.510508	0.386707	1.671548
0.160921	0.160963	0.420738	0.510531	0.386874	1.671558
0.160963	0.161005	0.420896	0.510554	0.387041	1.671567
0.161005	0.161047	0.421054	0.510577	0.387208	1.671577
0.161047	0.161089	0.421213	0.5106	0.387375	1.671587
0.161089	0.161131	0.421371	0.510623	0.387542	1.671597
0.161131	0.161173	0.42153	0.510646	0.387709	1.671606
0.161173	0.161215	0.421688	0.51067	0.387876	1.671616
0.161215	0.161257	0.421847	0.510693	0.388043	1.671626
0.161257	0.161299	0.422006	0.510716	0.388209	1.671636
0.161299	0.161341	0.422166	0.510739	0.388376	1.671645
0.161341	0.161381	0.422325	0.510761	0.385617	1.671755
0.161381	0.161421	0.422477	0.510783	0.385777	1.671764
0.161421	0.161461	0.422629	0.510805	0.385937	1.671773
0.161461	0.161501	0.422782	0.510827	0.386097	1.671783
0.161501	0.161541	0.422934	0.510849	0.386257	1.671792
0.161541	0.161581	0.423087	0.510871	0.386417	1.671801
0.161581	0.161621	0.423239	0.510893	0.386576	1.67181
0.161621	0.161661	0.423392	0.510915	0.386736	1.67182

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.161661	0.161701	0.423545	0.510937	0.386896	1.671829
0.161701	0.161741	0.423699	0.510959	0.387056	1.671838
0.161741	0.161781	0.423852	0.510981	0.387215	1.671848
0.161781	0.161821	0.424006	0.511002	0.387375	1.671857
0.161821	0.161861	0.424159	0.511024	0.387534	1.671866
0.161861	0.161899	0.424313	0.511045	0.384588	1.671976
0.161899	0.161937	0.42446	0.511066	0.384741	1.671985
0.161937	0.161975	0.424606	0.511087	0.384894	1.671994
0.161975	0.162013	0.424753	0.511108	0.385047	1.672002
0.162013	0.162051	0.4249	0.511129	0.3852	1.672011
0.162051	0.162089	0.425047	0.51115	0.385353	1.67202
0.162089	0.162127	0.425194	0.51117	0.385505	1.672029
0.162127	0.162201	0.425341	0.511211	0.424253	1.670227
0.162201	0.162275	0.425628	0.511252	0.424535	1.670243
0.162275	0.162349	0.425916	0.511292	0.424816	1.670259
0.162349	0.162423	0.426205	0.511333	0.425097	1.670274
0.162423	0.162495	0.426494	0.511372	0.423794	1.670391
0.162495	0.162567	0.426775	0.511412	0.424068	1.670406
0.162567	0.162639	0.427058	0.511451	0.424342	1.670422
0.162639	0.162709	0.427341	0.511489	0.422992	1.670539
0.162709	0.162779	0.427617	0.511528	0.423258	1.670554
0.162779	0.162849	0.427893	0.511566	0.423524	1.670569
0.162849	0.162919	0.42817	0.511604	0.42379	1.670584
0.162919	0.162987	0.428447	0.511641	0.42239	1.6707
0.162987	0.163055	0.428718	0.511679	0.422648	1.670715
0.163055	0.163123	0.428988	0.511716	0.422907	1.67073
0.163123	0.163191	0.42926	0.511753	0.423165	1.670744
0.163191	0.163257	0.429532	0.511789	0.421713	1.670861
0.163257	0.163323	0.429796	0.511825	0.421964	1.670875
0.163323	0.163389	0.430061	0.511861	0.422215	1.670889
0.163389	0.163453	0.430327	0.511896	0.420707	1.671006
0.163453	0.163517	0.430585	0.511931	0.420951	1.67102
0.163517	0.163581	0.430844	0.511966	0.421194	1.671034
0.163581	0.163645	0.431103	0.512001	0.421437	1.671048
0.163645	0.163707	0.431363	0.512035	0.419871	1.671164
0.163707	0.163769	0.431615	0.512069	0.420107	1.671178
0.163769	0.163831	0.431868	0.512103	0.420343	1.671191
0.163831	0.163893	0.432121	0.512136	0.420579	1.671205
0.163893	0.163953	0.432375	0.512169	0.41895	1.671321
0.163953	0.164013	0.432621	0.512202	0.419178	1.671334
0.164013	0.164073	0.432868	0.512235	0.419407	1.671347
0.164073	0.164133	0.433115	0.512267	0.419635	1.67136
0.164133	0.164193	0.433363	0.5123	0.419863	1.671373

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.164193	0.164251	0.433612	0.512332	0.418168	1.67149
0.164251	0.164309	0.433852	0.512363	0.418389	1.671502
0.164309	0.164367	0.434093	0.512395	0.41861	1.671515
0.164367	0.164425	0.434335	0.512426	0.418831	1.671528
0.164425	0.164481	0.434577	0.512457	0.417064	1.671644
0.164481	0.164537	0.434812	0.512487	0.417278	1.671656
0.164537	0.164593	0.435047	0.512518	0.417491	1.671669
0.164593	0.164649	0.435282	0.512548	0.417704	1.671681
0.164649	0.164705	0.435518	0.512579	0.417918	1.671694
0.164705	0.164759	0.435755	0.512608	0.416075	1.67181
0.164759	0.164813	0.435984	0.512637	0.416281	1.671822
0.164813	0.164867	0.436213	0.512667	0.416487	1.671834
0.164867	0.164921	0.436442	0.512696	0.416693	1.671846
0.164921	0.164973	0.436673	0.512725	0.414767	1.671961
0.164973	0.165025	0.436895	0.512753	0.414965	1.671973
0.165025	0.165077	0.437117	0.512781	0.415164	1.671985
0.165077	0.165129	0.437341	0.512809	0.415363	1.671996
0.165129	0.165181	0.437564	0.512838	0.415561	1.672008
0.165181	0.165231	0.437789	0.512865	0.413545	1.672124
0.165231	0.165281	0.438005	0.512892	0.413736	1.672135
0.165281	0.165331	0.438221	0.512919	0.413928	1.672146
0.165331	0.165381	0.438438	0.512946	0.414119	1.672157
0.165381	0.165431	0.438656	0.512973	0.41431	1.672168
0.165431	0.165479	0.438874	0.512999	0.412196	1.672284
0.165479	0.165527	0.439084	0.513025	0.41238	1.672295
0.165527	0.165575	0.439294	0.513052	0.412564	1.672306
0.165575	0.165623	0.439505	0.513078	0.412748	1.672317
0.165623	0.165671	0.439716	0.513104	0.412932	1.672327
0.165671	0.165719	0.439928	0.51313	0.413116	1.672338
0.165719	0.165765	0.44014	0.513155	0.410894	1.672454
0.165765	0.165811	0.440344	0.51318	0.411071	1.672464
0.165811	0.165857	0.440548	0.513204	0.411248	1.672474
0.165857	0.165903	0.440753	0.513229	0.411425	1.672485
0.165903	0.165949	0.440959	0.513254	0.411601	1.672495
0.165949	0.165993	0.441165	0.513278	0.40926	1.672611
0.165993	0.166037	0.441362	0.513302	0.409429	1.672621
0.166037	0.166081	0.44156	0.513326	0.409599	1.672631
0.166081	0.166125	0.441758	0.51335	0.409769	1.672641
0.166125	0.166169	0.441957	0.513374	0.409938	1.672651
0.166169	0.166213	0.442156	0.513397	0.410108	1.672661
0.166213	0.166255	0.442356	0.51342	0.407633	1.672776
0.166255	0.166297	0.442547	0.513443	0.407796	1.672786
0.166297	0.166339	0.442738	0.513466	0.407958	1.672795

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.166339	0.166381	0.44293	0.513488	0.408121	1.672805
0.166381	0.166423	0.443123	0.513511	0.408284	1.672814
0.166423	0.166465	0.443316	0.513534	0.408446	1.672824
0.166465	0.166505	0.443509	0.513556	0.405821	1.672939
0.166505	0.166545	0.443694	0.513577	0.405977	1.672948
0.166545	0.166585	0.443879	0.513599	0.406133	1.672957
0.166585	0.166625	0.444064	0.51362	0.406288	1.672967
0.166625	0.166665	0.44425	0.513642	0.406444	1.672976
0.166665	0.166705	0.444437	0.513664	0.406599	1.672985
0.166705	0.166745	0.444624	0.513685	0.406755	1.672994
0.166745	0.166821	0.444811	0.513726	0.443278	1.671092
0.166821	0.166897	0.445168	0.513768	0.443564	1.671107
0.166897	0.166971	0.445527	0.513808	0.442305	1.67123
0.166971	0.167043	0.445878	0.513846	0.441002	1.671352
0.167043	0.167115	0.446222	0.513885	0.441273	1.671367
0.167115	0.167185	0.446567	0.513923	0.439925	1.671489
0.167185	0.167255	0.446904	0.513961	0.440189	1.671503
0.167255	0.167323	0.447243	0.513998	0.438793	1.671625
0.167323	0.167391	0.447574	0.514034	0.439049	1.671639
0.167391	0.167457	0.447906	0.51407	0.437602	1.671761
0.167457	0.167523	0.448231	0.514106	0.437851	1.671775
0.167523	0.167587	0.448557	0.51414	0.43635	1.671896
0.167587	0.167651	0.448874	0.514175	0.436591	1.67191
0.167651	0.167713	0.449194	0.514208	0.435035	1.672031
0.167713	0.167775	0.449504	0.514242	0.435268	1.672044
0.167775	0.167835	0.449817	0.514274	0.433651	1.672165
0.167835	0.167895	0.450121	0.514306	0.433877	1.672178
0.167895	0.167953	0.450426	0.514337	0.432196	1.672299
0.167953	0.168011	0.450723	0.514369	0.432415	1.672311
0.168011	0.168067	0.451022	0.514399	0.430665	1.672431
0.168067	0.168123	0.451311	0.514429	0.430876	1.672444
0.168123	0.168177	0.451602	0.514458	0.429052	1.672564
0.168177	0.168231	0.451884	0.514487	0.429256	1.672576
0.168231	0.168285	0.452168	0.514516	0.42946	1.672587
0.168285	0.168337	0.452453	0.514544	0.427556	1.672707
0.168337	0.168389	0.452729	0.514572	0.427753	1.672719
0.168389	0.168439	0.453006	0.514599	0.425763	1.672838
0.168439	0.168489	0.453274	0.514626	0.425952	1.672849
0.168489	0.168539	0.453544	0.514653	0.426141	1.67286
0.168539	0.168587	0.453815	0.514679	0.424056	1.67298
0.168587	0.168635	0.454076	0.514705	0.424239	1.672991
0.168635	0.168681	0.454339	0.514729	0.42205	1.67311
0.168681	0.168727	0.454593	0.514754	0.422225	1.67312

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.168727	0.168773	0.454847	0.514779	0.4224	1.67313
0.168773	0.168817	0.455103	0.514802	0.420096	1.673249
0.168817	0.168861	0.45535	0.514826	0.420264	1.673259
0.168861	0.168905	0.455597	0.51485	0.420432	1.673269
0.168905	0.168947	0.455846	0.514872	0.418001	1.673387
0.168947	0.168989	0.456085	0.514895	0.418162	1.673397
0.168989	0.169071	0.456325	0.514939	0.45617	1.671225
0.169071	0.169151	0.456798	0.514982	0.455025	1.671351
0.169151	0.169227	0.457265	0.515023	0.452329	1.671586
0.169227	0.169301	0.457713	0.515062	0.451067	1.671711
0.169301	0.169373	0.458154	0.515101	0.449763	1.671836
0.169373	0.169443	0.458588	0.515139	0.448413	1.671961
0.169443	0.169511	0.459015	0.515175	0.447017	1.672085
0.169511	0.169577	0.459434	0.51521	0.44557	1.672209
0.169577	0.169641	0.459845	0.515245	0.444072	1.672333
0.169641	0.169703	0.460249	0.515278	0.442517	1.672456
0.169703	0.169763	0.460645	0.51531	0.440903	1.672579
0.169763	0.169821	0.461032	0.515341	0.439227	1.672702
0.169821	0.169877	0.461411	0.515371	0.437482	1.672825
0.169877	0.169931	0.461781	0.5154	0.435665	1.672947
0.169931	0.169983	0.462142	0.515428	0.43377	1.673069
0.169983	0.170033	0.462494	0.515455	0.431789	1.67319
0.170033	0.170083	0.462836	0.515482	0.431978	1.673201
0.170083	0.170131	0.463183	0.515507	0.429905	1.673322
0.170131	0.170177	0.463519	0.515532	0.42773	1.673443
0.170177	0.170221	0.463846	0.515555	0.425442	1.673564
0.170221	0.170307	0.464162	0.515602	0.463626	1.671249
0.170307	0.170387	0.464791	0.515644	0.459667	1.671599
0.170387	0.170463	0.465391	0.515685	0.456967	1.671837
0.170463	0.170533	0.465975	0.515722	0.452499	1.672186
0.170533	0.170599	0.466526	0.515758	0.449401	1.672423
0.170599	0.170661	0.46706	0.515791	0.446108	1.672659
0.170661	0.170719	0.467574	0.515822	0.442596	1.672895
0.170719	0.170773	0.468067	0.515851	0.43883	1.67313
0.170773	0.170823	0.468539	0.515877	0.434766	1.673364
0.170823	0.170871	0.468986	0.515903	0.432698	1.673486
0.170871	0.170959	0.469428	0.51595	0.467445	1.671264
0.170959	0.171037	0.47027	0.515992	0.460627	1.671841
0.171037	0.171105	0.471058	0.516028	0.452988	1.672417
0.171105	0.171165	0.471784	0.51606	0.446159	1.672878
0.171165	0.171217	0.472463	0.516088	0.438411	1.673338
0.171217	0.171307	0.473086	0.516136	0.470099	1.671219
0.171307	0.171375	0.474266	0.516172	0.453998	1.672473

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.171375	0.171425	0.475278	0.516199	0.437031	1.673496
0.171425	0.171499	0.476125	0.516238	0.459305	1.67216
0.171499	0.171573	0.477674	0.516278	0.459582	1.672175

It follows that $|J| > 0$ on S^+ and so, by symmetry, on S^-.

We note that $\partial\hat{s}/\partial s$ can take both positive and negative values on S^+.

We can use the fact that μ_2 is positive further to find conditions under which the derivatives of $\hat{s}$ and $\hat{k}$ with respect to s and k are positive or negative.

We have

$$\frac{\partial\hat{s}}{\partial k} = \mu_2^{-5/2}\left(\mu_2\frac{\partial\mu_3}{\partial k} - \frac{3}{2}\mu_3\frac{\partial\mu_2}{\partial k}\right) = 36\mu_2^{-5/2} s S_{sk}(q,k),$$

where

$$\begin{aligned} S_{sk}(q,k) = {} & 185q^3 + (120k^2 + 50k)q \\ & - (261k + 51)q^2 - 7q - 18k^3 - 12k^2 + 3k + 1. \end{aligned}$$

At $q = 0$ we need only consider $k \leq 1/3$, because of equation (5). And $S_{sk}(0,1/3) = 0$. So we cannot have $S_{sk}(q,k) > 0$ on S^+. But we can have $S_{sk}(q,k) \geq 0$. Define

$$S'_{sk}(q,k) = \frac{\partial}{\partial k}S_{sk}(q,k) = 240kq + 50q - 261q^2 - 54k^2 - 24k + 3.$$

At $(0,1/3)$ we have $S'_{sk}(q,k) = -11 < 0$. So we can reasonably hope that in some region of S^+ close to $(0,1/3)$ we have $S_{sk}(q,k) \geq 0$.

Put $S'_{sk}(q_l,q_u,k) = 240kq_u + 50q_u - 261q_l^2 - 54k^2 - 24k + 3$. Then $S'_{sk}(0,1/50,k) = -54k^2 - 96k/5 + 4$, which is a quadratic in k whose larger root is $(\sqrt{214} - 8)/45 < 3/20$. So $S'_{sk} < 0$ for $q \leq 1/50$ and $k > 3/20$.

On the line $k = 53q/48 + 1/3$, $S_{sk}(q,k) = q(116019q^2 - 190672q + 66688)/6144$, which has three roots, 0, $-(88\sqrt{174571} - 95336)/116019 \approx 0.5048$ and $(88\sqrt{174571} + 95336)/116019 \approx 1.1386$. Hence $S_{sk}(q,k) \geq 0$ for points on $k = 53q/48 + 1/3$ between $q = 0$ and $q = 3 - 2\sqrt{2} < 0.5048$. It follows that for $q \leq 1/50$ and $3/20 \leq k \leq 53q/48 + 1/3$ we have $S_{sk}(q,k) \geq 0$ with equality only at $(0,1/3)$.

Put

$$\begin{aligned} S_{sk}(q_l,q_u,k) = {} & 185q_l^3 + 120k^2 q_l + 50kq_l + 3k + 1 \\ & - (261k + 51)q_u^2 - 7q_u - 18k^3 - 12k^2. \end{aligned}$$

Then, as before, we seek intervals q_l, q_u such that $S_{sk}(q_l,q_u,k)$ is positive between the lower bound $k_{\min}$ and upper bound $k_{\max}$ of expression (A.9). $S_{sk}(q_l,q_u,l)$ is cubic in k with a negative coefficient of k^3. So, when there are three real roots, it is positive between the second root r_l and the third r_u. Table A.2 shows these roots and the values of $k_{\min}$ and $k_{\max}$ to six decimal places together with the exact values of q_u, q_l for intervals covering $[0, 3 - 2\sqrt{2}]$. As before we compute values with Maxima. Note that we choose $q_l = 0$, $q_u = 1/50$ and $k_{\max} = 3/20$ in the first line because we have already shown $S_{sk}(q,k) \geq 0$ for $q \in [0,1/50]$ and $k \geq 3/20$. Since the second and third roots all fall outside $[k_{\min}, k_{\max}]$ we conclude that, for $(s,k) \in S^+$, $\partial\hat{s}/\partial k \geq 0$, with equality only only at $(0,1/3)$.

Table A.2: Sequence of intervals for $S(q_l, q_k, k)$

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0	0.02	0	0.15	−0.795199	0.314841
0.02	0.024738	0.046951	0.360648	−0.161921	0.369249
0.024738	0.033916	0.058163	0.370782	−0.145165	0.371486
0.033916	0.04252	0.079994	0.380282	−0.125118	0.389093
0.04252	0.050586	0.100601	0.389189	−0.106186	0.405802
0.050586	0.058148	0.12006	0.397538	−0.08836	0.421671
0.058148	0.072327	0.138437	0.413194	−0.05526	0.414638
0.072327	0.084733	0.173296	0.426893	−0.023018	0.442433
0.084733	0.095589	0.204303	0.43888	0.004662	0.468202
0.095589	0.105087	0.231913	0.449367	0.028191	0.492037
0.105087	0.113399	0.256515	0.458545	0.048083	0.513956
0.113399	0.127943	0.278465	0.474604	0.105492	0.48453
0.127943	0.138851	0.318111	0.486648	0.12612	0.531085
0.138851	0.147033	0.349314	0.495682	0.138362	0.569463
0.147033	0.159303	0.373995	0.50923	0.198417	0.540556
0.159303	0.171573	0.414765	0.481217	0.241581	0.551643

We have

$$\frac{\partial \hat{k}}{\partial k} = \mu_2^{-3}\left(\mu_2 \frac{\partial \mu_4}{\partial k} - 2\mu_4 \frac{\partial \mu_2}{\partial k}\right) = 24\mu_2^{-3} S_{kk}(q,k),$$

where

$$\begin{aligned} S_{kk}(q,k) = {} & 1115q^5 + 3675q^4 + 5040k^2q^3 + (7428k+364)q^2 \\ & + (324k^4 + 1728k^3)q \\ & - 4590kq^4 - (2064k+5764)q^3 - (2160k^3+2052k^2)q^2 \\ & - (3132k^2+480k+15)q - 324k^4 + 432k^3 + 144k^2 + 18k + 1. \end{aligned}$$

Write

$$\begin{aligned} S_{kk}(q_l,q_u,k) = {} & 1115q_l^5 + 3675q_l^4 + 5040k^2q_l^3 + (7428k+364)q_l^2 \\ & + (324k^4 + 1728k^3)q_l \\ & - 4590kq_u^4 - (2064k+5764)q_u^3 - (2160k^3+2052k^2)q_u^2 \\ & - (3132k^2+480k+15)q_u - 324k^4 + 432k^3 + 144k^2 + 18k + 1. \end{aligned}$$

Putting $q_l = 0$ and $q_u = 0.171753 > 3 - 2\sqrt{2}$, we get a polynomial in k:

$$\begin{aligned} & \frac{171753(324k^4+1728k^3)}{1000000} - 324k^4 + 432k^3 + \frac{2119193136459210951k^2}{12500000000000000} \\ & + \frac{29499093009(7428k+364)}{1000000000000} + 18k + \frac{8729237441544369438552647804439}{2000000000000000000000000000000}. \end{aligned}$$

This has degree four and the coefficient of k^4 is negative. Using Maxima we find two real roots (to within 10^{-7}): -0.0658944 and 3.0234489. The polynomial must be positive between the roots. It follows that $\partial\hat{k}/\partial k > 0$ on S^+.

We have

$$\frac{\partial\hat{k}}{\partial s} = \mu_2^{-3}\left(\mu_2\frac{\partial\mu_4}{\partial s} - 2\mu_4\frac{\partial\mu_2}{\partial s}\right) = -48s\mu_2^{-3}S_{ks}(q,k),$$

where

$$\begin{aligned}S_{ks}(q,k) = {} & 1115kq^4 + 14550kq^3 + (5040k^3 + 13890k + 300)q^2 + 7380k^3q \\ & - 2500q^4 - (4590k^2 + 10920)q^3 - 17100k^2q^2 \\ & - (2160k^4 + 5790k^2 - 578k + 16)q \\ & + 324k^5 - 1080k^4 + 792k^3 + 204k^2 + 15k.\end{aligned}$$

When $q = 0$, $S_{ks}(q,k) = 3k(108k^4 - 360k^3 + 264k^2 + 68k + 5)$, which has one real root at $k = 0$. So we cannot have $S_{ks}(q,k) > 0$. But we may hope to show $S_{ks}(q,k) \geq 0$.

Put

$$\begin{aligned}S'_{ks}(q,k) &= \frac{\partial}{\partial k}S_{ks}(k,s) \\ &= 1620k^4 + (15120q^2 + 22140q)k^2 + 2376k^2 + 408k \\ &\quad - (8640q + 4320)k^3 - (9180q^3 + 34200q^2 + 11580q)k \\ &\quad + 1115q^4 + 14550q^3 + 13890q^2 + 578q + 15\end{aligned}$$

and

$$\begin{aligned}S'_{ks}(q,k_l,k_u) = {} & 1620k_l^4 + (15120q^2 + 22140q)k_l^2 + 2376k_l^2 + 408k_l \\ & - (8640q + 4320)k_u^3 - (9180q^3 + 34200q^2 + 11580q)k_u \\ & + 1115q^4 + 14550q^3 + 13890q^2 + 578q + 15.\end{aligned}$$

Then, for $k \in [k_l, k_u]$ $(0 \leq k_l < k_u)$ and $q \geq 0$, $S'_{ks}(q,s) > S'_{ks}(q,k_l,k_u)$.

$$\begin{aligned}S'_{ks}(q,0,51/100) = {} & 1115q^4 + 14550q^3 \\ & + \frac{51(-9180q^3 - 34200q^2 - 11580q)}{100} \\ & + 13890q^2 + 578q + \frac{132651(-8640q - 4320)}{1000000} + 15.\end{aligned}$$

We use Maxima to find the real roots to within 10^{-7}. They are approximately -9.1303203, -0.0920217 and 0.9798701. Since $S'_{ks}(q,0,51/100)$ is a polynomial of degree four in q in which q^4 has positive coefficient and $k < 51/100$ for $(s,k) \in S^+$, it follows that $S'_{ks}(q,k) > 0$ on S^+.

We have $\sqrt{q^2 - 6q + 1} \leq 1$. So on S^+ we must have

$$k \geq \frac{1 + 11q - \sqrt{q^2 - 6q + 1}}{6} \geq \frac{11q}{6}.$$

We have

$$S_{ks}(q, 11q/6) = -\frac{143q^5}{8} - 25q^4 - \frac{71q^3}{2} + \frac{6136q^2}{3} + \frac{23q}{2}.$$

This has a root at 0. We estimate the remaining real roots with Maxima: these are approximately -0.0056220 and 4.3095737. Thus $S_{ks}(q, 11q/6) \geq 0$ for $q \in [0, 3-2\sqrt{2}]$. Since $S'_{ks}(q,k) \geq 0$ on S^+ it follows that $S_{ks}(s,k) \geq 0$ on S^+ also. Thus $\partial\hat{k}/\partial s \leq 0$ on S^+ with equality only when $s = 0$.

For $(s,k) \in S^+$ with $s > 0$ we have

$$\frac{\partial \hat{k}}{\partial k} > 0, \qquad \frac{\partial \hat{k}}{\partial s} < 0, \quad \text{and} \quad \frac{\partial \hat{s}}{\partial k} > 0. \tag{A.10}$$

Since

$$|J| = \frac{\partial \hat{s}}{\partial s}\frac{\partial \hat{k}}{\partial k} - \frac{\partial \hat{s}}{\partial k}\frac{\partial \hat{k}}{\partial s} > 0$$

we have

$$\frac{\partial \hat{s}}{\partial s}\frac{\partial \hat{k}}{\partial k} > \frac{\partial \hat{s}}{\partial k}\frac{\partial \hat{k}}{\partial s}.$$

Combining this with inequalities (A.10), we get

$$\frac{\partial \hat{s}}{\partial s}\bigg/\frac{\partial \hat{k}}{\partial s} < \frac{\partial \hat{s}}{\partial k}\bigg/\frac{\partial \hat{k}}{\partial k}.$$

But

$$\frac{\partial \hat{s}}{\partial k}\bigg/\frac{\partial \hat{k}}{\partial k} \quad \text{and} \quad \frac{\partial \hat{s}}{\partial s}\bigg/\frac{\partial \hat{k}}{\partial s}$$

are the gradients in the $\hat{k}$–$\hat{s}$ plane of the lines given by $(\hat{k}(s,k), \hat{s}(s,k))$ for constant k and s respectively. Since the first gradient exceeds the second, it follows that the lines have at most one point in common. That is, for $(s,k) \in S^+$ with $s > 0$ there is a unique $(\hat{s},\hat{k}) \in G(S^+)$ such that $G((s,k)) = (\hat{s},\hat{k})$. The case $s = 0$ is straightforward. From equations (6) and (7), we find that $\hat{s} = 0 \Leftrightarrow s = 0$ and that $\hat{k}$ is an increasing function of k when $s = 0$. So there is also a unique $(0,\hat{k}) \in G(S^+)$ such that $G((0,k)) = (0,\hat{k})$.

It follows by symmetry that $G:\ S \to G(S)$ has a unique inverse.

Supplement B

We show here some properties of the Cornish–Fisher distribution.

Proposition 3.1 *Let $Y \sim \mathscr{F}(\mathbf{p})$. The density function of Y is*

$$f(x) = \frac{\mu_2^{1/2}\phi(v)}{\sigma\xi'(v)}, \tag{A.11}$$

with $v = \xi^{-1}\left(\mu_2^{1/2}(x-\mu)/\sigma\right)$. The distribution function is smooth, and $f(x)$ is unimodal and satisfies $f(x) > 0$ for $x \in \mathbb{R}$.

Proof: First note that

$$\xi'(u) = a_1 + 2a_2 u + 3a_3 u^2, \qquad \xi^{(2)}(u) = 2a_2 + 6a_3 u, \qquad \xi^{(3)}(u) = 6a_3, \tag{A.12}$$

and higher order derivatives are zero. And

$$v'(x) = \frac{1}{\xi'(v)} \frac{\mathrm{d}}{\mathrm{d}x} \frac{\mu_2^{1/2}(x-\mu)}{\sigma}(x) = \frac{\mu_2^{1/2}}{\sigma \xi'(v)}.$$

Thus, differentiating $F(\Phi(v))$, we get equation (A.11). Since $\phi(v) > 0$, $\sigma > 0$ and $\xi'(v) > 0$ for $v \in \mathbb{R}$, we have $f(x) > 0$ for $x \in \mathbb{R}$.

Differentiating equation (A.11) and substituting $\phi'(v) = -v\phi(v)$, we get

$$f'(x) = -\frac{\mu_2 \phi(v)}{\sigma^2 [\xi'(v)]^3} \left(v\xi'(v) + \xi^{(2)}(v) \right),$$

which is zero whenever $p(v) = v\xi'(v) + \xi^{(2)}(v) = 0$. So f is unimodal whenever $p(v)$ has only one root. Since $p(v)$ is a cubic polynomial it has either one or three roots and has only one root whenever $p'(v) > 0$ for $v \in \mathbb{R}$. And

$$\begin{aligned} p'(v) &= \xi'(v) + v\xi^{(2)}(v) + \xi^{(3)}(v) \\ &= 9a_3 v^2 + 4a_2 v + a_1 + 6a_3. \end{aligned}$$

This has discriminant $16a_2^2 - 36a_3(a_1 + 6a_3) = -4d(q,k)$ with

$$d(q,k) = 126q^2 - 153kq - 22q + 45k^2 + 9k.$$

We solve

$$\frac{\partial d}{\partial q} = 252q - 153k - 22 = 0, \qquad \frac{\partial d}{\partial k} = 90k - 153q + 9 = 0$$

to show $d(q,k)$ has a minimum, $188/81$, at $q = -67/81$, $k = -122/81$. Thus $d(q,k) > 0$ and so the discriminant is always negative. It follows that $p'(v)$ has no real roots. Since $p'(v) > 0$ for large enough v, we have $p'(v) > 0$ for $v \in \mathbb{R}$. It follows that f is unimodal.

It remains to show F is smooth. It is clearly continuous. So we need to show f and all its derivatives are also continuous. Define $g_0(v) = \mu_2^{1/2}\phi(v)/\sigma$ and

$$g_{n+1}(v) = \frac{\mu_2^{1/2}}{\sigma} \left[g_n'(v)\xi'(v) - (2n+1) g_n(v) \xi^{(2)}(v) \right]$$

for $n = 0, 1, 2, \ldots$. Then g_{n+1} is smooth if g_n is smooth. And g_0 is smooth. So g_n is smooth for all $n \geq 0$. And, since $\xi'(v) > 0$ and $v = \xi^{-1}\left(\mu_2^{1/2}(x-\mu)/\sigma \right)$ is smooth,

$$h_n(x) = \frac{g_n(v)}{[\xi'(v)]^{2n+1}}$$

is also smooth for all $n \geq 0$. We have defined $h_n(x)$ so that $h_0(x) = f^{(0)}(x) = f(x)$. Suppose it is not true that $f^{(n)}(x) = h_n(x)$ for all $n > 0$ and choose k minimal such that $h_{k+1}(x) \neq f^{(k+1)}(x)$. Then

$$\begin{aligned}
f^{(k+1)}(x) &= \left(f^{(k)}\right)'(x) = (h_k)'(x) \\
&= \frac{g_k'(v)\,[\xi'(v)]^{2k+1} - (2k+1)g_k(v)\,[\xi'(v)]^{2k}\,\xi^{(2)}(v)}{[\xi'(v)]^{4k+2}}\,\frac{\mu_2^{1/2}}{\sigma\xi'(v)} \\
&= \frac{\mu_2^{1/2}}{\sigma}\,\frac{g_k'(v)\xi'(v) - (2k+1)g_k(v)\xi^{(2)}(v)}{[\xi'(v)]^{2k+3}} \\
&= \frac{g_{k+1}(v)}{[\xi'(v)]^{2(k+1)+1}} \\
&= h_{k+1}(x),
\end{aligned}$$

contradicting the minimality of k. It follows that $f^{(n)}(x) = h_n(x)$ for $n \geq 0$ and so F is smooth. □

Proposition 3.2 *F^{-1} is twice continuously differentiable with respect to $\mathbf{p} = (\mu, \sigma, \kappa_3, \kappa_4)$.*

Proof: This follows easily from equation (12) when we note that μ_2 and ξ are twice continuously differentiable. □

Supplement C

We derive expressions we can use to calculate the first and second partial derivatives of $F^{-1}(u;\mathbf{p})$ with respect to κ_3 and κ_4.

Let $p, q \in \{\mu, \sigma\}$. Then

$$\frac{\partial}{\partial p}F^{-1}(u;\mathbf{p}) = \frac{\partial \mu}{\partial p} + \mu_2^{-1/2}\xi\left(\Phi^{-1}(u)\right)\frac{\partial \sigma}{\partial p} - \frac{1}{2}\sigma\mu_2^{-3/2}\xi\left(\Phi^{-1}(u)\right)\frac{\partial \mu_2}{\partial p} + \sigma\mu_2^{-1/2}\frac{\partial}{\partial p}\xi\left(\Phi^{-1}(u)\right).$$

Since $\partial\mu/\partial p = 1$ whenever $p = \mu$ and $\partial\sigma/\partial p = 1$ whenever $p = \sigma$, we have $\partial^2\mu/(\partial p\partial q) = \partial^2\sigma/(\partial p\partial q) = 0$ and

$$\begin{aligned}
\frac{\partial^2}{\partial p\partial q}F^{-1}(u;\mathbf{p}) = &-\frac{1}{2}\mu_2^{-3/2}\xi\left(\Phi^{-1}(u)\right)\left(\frac{\partial \sigma}{\partial p}\frac{\partial \mu_2}{\partial q} + \frac{\partial \sigma}{\partial q}\frac{\partial \mu_2}{\partial p}\right) \\
&- \frac{1}{2}\sigma\mu_2^{-3/2}\left(\frac{\partial \mu_2}{\partial p}\frac{\partial}{\partial q}\xi\left(\Phi^{-1}(u)\right) + \frac{\partial \mu_2}{\partial q}\frac{\partial}{\partial p}\xi\left(\Phi^{-1}(u)\right)\right) \\
&+ \frac{3}{4}\sigma\mu_2^{-5/2}\xi\left(\Phi^{-1}(u)\right)\frac{\partial^2 \mu_2}{\partial p\partial q} \\
&+ \sigma\mu_2^{-1/2}\frac{\partial^2}{\partial p\partial q}\xi\left(\Phi^{-1}(u)\right).
\end{aligned}$$

The partial derivatives of μ_2 and ξ with respect to μ and σ are zero. Otherwise we have, for $\nu \in \{\mu_2, \xi\}$ and $p, q \in \{\kappa_3, \kappa_4\}$

$$\frac{\partial \nu}{\partial p} = \frac{\partial \nu}{\partial s}\frac{\partial s}{\partial p} + \frac{\partial \nu}{\partial k}\frac{\partial k}{\partial p} \quad \text{and}$$

$$\frac{\partial^2 \nu}{\partial p\partial q} = \frac{\partial^2 \nu}{\partial s^2}\frac{\partial s}{\partial p}\frac{\partial s}{\partial q} + \frac{\partial^2 \nu}{\partial s\partial k}\left(\frac{\partial s}{\partial p}\frac{\partial k}{\partial q} + \frac{\partial k}{\partial p}\frac{\partial s}{\partial q}\right) + \frac{\partial^2 \nu}{\partial k^2}\frac{\partial k}{\partial p}\frac{\partial k}{\partial q} + \frac{\partial \nu}{\partial s}\frac{\partial^2 s}{\partial p\partial q} + \frac{\partial \nu}{\partial k}\frac{\partial^2 k}{\partial p\partial q}.$$

The partial derivatives of μ_2 with respect to s and k are

$$\frac{\partial \mu_2}{\partial s} = 100s^3 - 48sk, \qquad \frac{\partial \mu_2}{\partial k} = \frac{k}{3} - 24s^2,$$

$$\frac{\partial^2 \mu_2}{\partial s^2} = 300s^2 - 48k, \qquad \frac{\partial^2 \mu_2}{\partial s \partial k} = -48s, \qquad \frac{\partial^2 \mu_2}{\partial k^2} = \frac{1}{3}.$$

And those of ξ are

$$\frac{\partial \xi(u)}{\partial s} = -1 + 10su + u^2 - 4su^3, \qquad \frac{\partial \xi(u)}{\partial k} = u^3 - u,$$

$$\frac{\partial^2 \xi(u)}{\partial s^2} = 10u - 12u^2, \qquad \frac{\partial^2 \xi(u)}{\partial s \partial k} = \frac{\partial^2 \xi(u)}{\partial k^2} = 0.$$

Since $\hat{s} = \kappa_3/6$ and $\hat{k} = \kappa_4/24$, for $t \in \{s,k\}$ we have

$$\frac{\partial t}{\partial \kappa_3} = \frac{1}{6}\frac{\partial t}{\partial \hat{s}}, \qquad \frac{\partial t}{\partial \kappa_4} = \frac{1}{24}\frac{\partial t}{\partial \hat{k}},$$

$$\frac{\partial^2 t}{\partial \kappa_3^2} = \frac{1}{36}\frac{\partial^2 t}{\partial \hat{s}^2}, \qquad \frac{\partial^2 t}{\partial \kappa_3 \partial \kappa_4} = \frac{1}{144}\frac{\partial^2 t}{\partial \hat{s}\partial \hat{k}}, \qquad \frac{\partial^2 t}{\partial \kappa_4^2} = \frac{1}{576}\frac{\partial^2 t}{\partial \hat{k}^2}.$$

So we need the partial derivatives of s and k with respect to $\hat{s}$ and $\hat{k}$. Given

$$\hat{s} = \hat{s}(s,k) \quad \text{and} \quad \hat{k} = \hat{k}(s,k),$$

we have

$$\begin{aligned} \frac{\partial \hat{s}}{\partial \hat{s}} &= \frac{\partial \hat{s}}{\partial s}\frac{\partial s}{\partial \hat{s}} + \frac{\partial \hat{s}}{\partial k}\frac{\partial k}{\partial \hat{s}} = 1, \qquad & \frac{\partial \hat{k}}{\partial \hat{k}} &= \frac{\partial \hat{k}}{\partial s}\frac{\partial s}{\partial \hat{k}} + \frac{\partial \hat{k}}{\partial k}\frac{\partial k}{\partial \hat{k}} = 1, \\ \frac{\partial \hat{s}}{\partial \hat{k}} &= \frac{\partial \hat{s}}{\partial s}\frac{\partial s}{\partial \hat{k}} + \frac{\partial \hat{s}}{\partial k}\frac{\partial k}{\partial \hat{k}} = 0, \qquad & \frac{\partial \hat{k}}{\partial \hat{s}} &= \frac{\partial \hat{k}}{\partial s}\frac{\partial s}{\partial \hat{s}} + \frac{\partial \hat{k}}{\partial k}\frac{\partial k}{\partial \hat{s}} = 0. \end{aligned} \tag{B.13}$$

Putting

$$\hat{J} = \begin{bmatrix} \frac{\partial \hat{s}}{\partial s} & \frac{\partial \hat{s}}{\partial k} \\ \frac{\partial \hat{k}}{\partial s} & \frac{\partial \hat{k}}{\partial k} \end{bmatrix} \quad \text{and} \quad J = \begin{bmatrix} \frac{\partial s}{\partial \hat{s}} & \frac{\partial s}{\partial \hat{k}} \\ \frac{\partial k}{\partial \hat{s}} & \frac{\partial k}{\partial \hat{k}} \end{bmatrix},$$

we can solve

$$J\hat{J} = I$$

to find the partial derivatives of the Jacobian matrix J. Put

$$A = \begin{bmatrix} \left(\frac{\partial s}{\partial \hat{s}}\right)^2 & 2\frac{\partial s}{\partial \hat{s}}\frac{\partial k}{\partial \hat{s}} & \left(\frac{\partial k}{\partial \hat{s}}\right)^2 \\ \frac{\partial s}{\partial \hat{s}}\frac{\partial s}{\partial \hat{k}} & \frac{\partial s}{\partial \hat{s}}\frac{\partial k}{\partial \hat{k}} + \frac{\partial s}{\partial \hat{k}}\frac{\partial k}{\partial \hat{s}} & \frac{\partial k}{\partial \hat{s}}\frac{\partial k}{\partial \hat{k}} \\ \left(\frac{\partial s}{\partial \hat{s}}\right)^2 & 2\frac{\partial s}{\partial \hat{k}}\frac{\partial k}{\partial \hat{k}} & \left(\frac{\partial k}{\partial \hat{k}}\right)^2 \end{bmatrix},$$

$$B = \begin{bmatrix} \frac{\partial^2 s}{\partial \hat{s}^2} & \frac{\partial^2 k}{\partial \hat{s}^2} \\ \frac{\partial^2 s}{\partial \hat{s} \partial \hat{k}} & \frac{\partial^2 k}{\partial \hat{s} \partial \hat{k}} \\ \frac{\partial^2 s}{\partial \hat{k}^2} & \frac{\partial^2 k}{\partial \hat{k}^2} \end{bmatrix} \quad \text{and} \quad \hat{B} = \begin{bmatrix} \frac{\partial^2 \hat{s}}{\partial s^2} & \frac{\partial^2 \hat{k}}{\partial s^2} \\ \frac{\partial^2 \hat{s}}{\partial s \partial k} & \frac{\partial^2 \hat{k}}{\partial s \partial k} \\ \frac{\partial^2 \hat{s}}{\partial k^2} & \frac{\partial^2 \hat{k}}{\partial k^2} \end{bmatrix}.$$

Differentiating equations (B.13) again and rearranging the resulting equations, we get

$$A\hat{B} = -B\hat{J}.$$

Since A, $\hat{B}$ and J are known, we can use this to find B and hence the second partial derivatives of s and k with respect to $\hat{s}$ and $\hat{k}$.

Supplement D

We summarise here the shrinkage estimators of Ledoit and Wolf (2004) for the covariance matrix and of Jorion (1986) for the mean of a multivariate sample. Herold and Maurer (2006) discuss further shrinkage estimators and estimation risk.

Assume we have a $n \times T$ matrix R of T observations for n variables. Write $r_{t,i}$ for the (t,i)th entry of R and define

$$\bar{r}_i = \frac{1}{T} \sum_{t=1}^{T} r_{t,i} \qquad (i = 1, \ldots, n).$$

Define

$$\bar{r}_0 = \frac{1}{n} \sum_{i=1}^{n} r_i \quad \text{and} \quad \bar{\mathbf{r}} = (\bar{r}_1, \ldots, \bar{r}_n)$$

Define $x_{t,i} = r_{t,i} - \bar{r}_i$ for $i = 1, \ldots, n$, $t = 1, \ldots, T$. Then define X to be the $T \times n$ matrix with (t,i)th entry $x_{t,i}$.

Let

$$S_n = \frac{1}{T} X^\top X$$

be the usual $n \times n$ estimate of the covariance matrix between the n columns of X (or W). Write X_t for the tth row of X so that $X_t^\top X_t$ is a symmetric $n \times n$ matrix.

For $n \times n$ square matrices A, A_1 and A_2, define

$$\langle A_1, A_2 \rangle = \frac{1}{n} \operatorname{tr}(A_1 A_2^\top) \quad \text{and} \quad \|A\|^2 = \langle A, A \rangle.$$

Write I_n for the $n \times n$ identity matrix and $\mathbb{1}_n$ the vector of 1s of length n.

The Ledoit and Wolf (2004) covariance shrinkage estimator is as follows. Define

$$m = \langle S_n, I_n \rangle;$$

$$d^2 = \|S_n - m I_n\|^2;$$

$$\bar{b}^2 = \frac{1}{T^2} \sum_{t=1}^{T} \|X_t X_t^\top - S_n\|^2 \quad \text{and} \quad b^2 = \min(\bar{b}^2, d^2);$$

$$a^2 = d^2 - b^2$$

This gives the *bona fide* estimate of the covariance matrix

$$S_n^* = \frac{b^2}{d^2} m I_n + \frac{a^2}{d^2} S_n.$$

This is a shrinkage estimator that shrinks S_n towards mI_n. Note that m is an average variance and $b^2/d^2 = 1 - a^2/d^2$ with both in $[0,1]$.

The shrinkage estimator of Jorion (1986) (see also Herold and Maurer (2006)) is as follows:

$$\bar{\mathbf{r}}* = \frac{\phi}{\phi + T} \mathbb{1}_n \bar{r}_0 + \frac{T}{\phi + T} \bar{\mathbf{r}},$$

where

$$\phi = \frac{n+2}{\left(\bar{\mathbf{r}} - \bar{r}_0 \mathbb{1}_n\right)^\top \Sigma^{-1} \left(\bar{\mathbf{r}} - \bar{r}_0 \mathbb{1}_n\right)}$$

and Σ is the covariance matrix. In practice we must estimate Σ. We estimate it using S_n^*. Note that $\phi/(\phi + T) = 1 - T/(\phi + T)$ with both in $[0,1]$. So the estimator shrinks $\bar{\mathbf{r}}$ towards $\bar{r}_0 \mathbb{1}_n$.

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