A PROOF OF THE FIRST KAC–WEISFEILER CONJECTURE IN LARGE CHARACTERISTICS

BENJAMIN MARTIN, DAVID STEWART AND LEWIS TOPLEY

ABSTRACT. In 1971, Kac and Weisfeiler made two influential conjectures describing the dimensions of simple modules of a restricted Lie algebra $\mathfrak{g}$. The first predicts the maximal dimension of simple $\mathfrak{g}$-modules and in this paper we apply the Lefschetz principle and classical techniques from Lie theory to prove this conjecture for all restricted Lie subalgebras of $\mathfrak{gl}_n(k)$ whenever k is an algebraically closed field of characteristic $p \gg n$. As a consequence we deduce that the conjecture holds for the the Lie algebra of a group scheme when specialised to an algebraically closed field of almost any characteristic.

1. Introduction

Since the pioneering work of Zassenhaus [Zas54], it has been known that the dimensions of simple modules of finite dimensional Lie algebras over a field k of characteristic $p > 0$ are bounded, and that the maximal dimension, which we denote $M(\mathfrak{g})$, is a power of p. Jacobson introduced the notion of a restricted Lie algebra with a view to developing a Galois theory for purely inseparable field extensions [Jac37]. Very briefly restricted Lie algebras are those which admit a p-power map $x \mapsto x^{[p]}$ satisfying axioms which are modelled on the properties of the map $\mathrm{Der}_k(A) \to \mathrm{Der}_k(A)$ given by $d \mapsto d^p$ where A is an associative k-algebra. Many of the modular Lie algebras arising in nature are restricted, for example when $\mathfrak{g}$ is the Lie algebra of an algebraic k-group G there is a natural G-equivariant restricted structure on $\mathfrak{g}$.

Now let k be algebraically closed. In [VslK71] Kac and Weisfeiler carried out the first intensive study of representations of restricted Lie algebras. The key property of the restricted structure on $\mathfrak{g}$ is that the elements $x^p - x^{[p]}$ are central in $U(\mathfrak{g})$ for $x \in \mathfrak{g}$, and the subalgebra $Z_p(\mathfrak{g}) \subseteq U(\mathfrak{g})$ generated these elements is known as *the p-centre*. One of the fundamental insights of [VslK71] is that the maximal ideals of $Z_p(\mathfrak{g})$ are parametrised by $\mathfrak{g}^*$. Since $U(\mathfrak{g})$ is finite over its p-centre it follows from Hilbert's Nullstellensatz that every simple $\mathfrak{g}$-module is annihilated by a unique maximal ideal of $Z_p(\mathfrak{g})$, and so to every simple $\mathfrak{g}$-module M we may assign some linear form $\chi \in \mathfrak{g}^*$ known as *the p-character of M*. This situation is reminiscent of Kirillov's orbit method, and so it is natural to hope that global properties of the module category $\mathfrak{g}$-mod will be controlled by geometric properties of the module $\mathfrak{g}^*$. These aspirations were formalised by Kac–Weisfeiler in the form of two conjectures: the first of these predicts the maximal dimension of simple $\mathfrak{g}$-modules, and in the current paper we apply techniques from model theory to confirm that conjecture for all restricted Lie subalgebras of $\mathfrak{gl}_n(k)$ when the characteristic of the field k is large. The second conjecture proposes lower bounds on powers of p dividing the dimensions of $\mathfrak{g}$-modules with p-character χ; for more detail see [PS99] and the references therein.

The coadjoint stabiliser of $\chi \in \mathfrak{g}^*$ is denoted $\mathfrak{g}^\chi$ and *the index of* $\mathfrak{g}$ is defined by

$$\mathrm{ind}(\mathfrak{g}) := \min_{\chi \in \mathfrak{g}^*} \dim \mathfrak{g}^\chi. \tag{1.1}$$

The first Kac–Weisfeiler conjecture (KW1) predicts that when $\mathfrak{g}$ is any restricted Lie algebra the maximal dimension of simple $\mathfrak{g}$-modules is

$$M(\mathfrak{g}) = p^{\frac{1}{2}(\dim \mathfrak{g} - \operatorname{ind} \mathfrak{g})}. \tag{1.2}$$

Theorem 1.1. *For all $d \in \mathbb{N}$ there exists a $p_0 \in \mathbb{N}$ such that if $k = \overline{k}$ is a field of characteristic $p > p_0$ and $\mathfrak{g} \subseteq \mathfrak{gl}_d(k)$ is a restricted Lie subalgebra, then the first Kac–Weisfeiler conjecture holds for $\mathfrak{g}$.*

We now outline the proof of the theorem. In [PS99] Premet and Skryabin studied deformations of reduced enveloping algebras to spectacular effect: one of their many results states that $M(\mathfrak{g}) \geqslant p^{\frac{1}{2}(\dim \mathfrak{g} - \operatorname{ind} \mathfrak{g})}$ for any restricted Lie algebra $\mathfrak{g}$ and so it remains to prove the opposite inequality. In Kirillov's thesis he introduced the notion of a *polarisation of a linear form* χ, which is a Lie subalgebra $\mathfrak{s} \subseteq \mathfrak{g}$ satisfying $\chi[\mathfrak{s}, \mathfrak{s}] = 0$ and $\dim \mathfrak{s} = \frac{1}{2}(\dim \mathfrak{g} + \dim \mathfrak{g}^\chi)$. These turn out to be central to the classification of primitive ideals in enveloping algebras of complex solvable Lie algebras [Dix96, Ch. 6], as well as the classification of simple modules over restricted solvable Lie algebras [VslK71, §2]. We say that $\mathfrak{s}$ *is a weak polarisation of* $\chi \in \mathfrak{g}^*$ if $\mathfrak{s} \subseteq \mathfrak{g}$ is a Lie subalgebra of dimension $\frac{1}{2}(\dim \mathfrak{g} + \operatorname{ind} \mathfrak{g})$ satisfying $\chi[\mathfrak{s}, \mathfrak{s}] = 0$. Solvable weak polarisations are known to exist for every linear form on every finite dimensional complex Lie algebra, and we deduce that the same holds for all modular Lie algebras in large characteristics. In order to make the aforementioned deduction we employ the Lefschetz principle in a rather novel way: we simultaneously quantify over all Lie algebras of a fixed dimension. Modulo some technical hurdles showing that the derived subalgebra of a weak polarisation is restricted, the proof concludes by observing that every simple module is a quotient of a module induced from a restricted Lie subalgebra containing a solvable polarisation (Theorem 3.3). This places the required upper bound on the dimension of simple modules. After providing an elementary introduction to the Lefschetz principle and the representation theory of restricted Lie algebras in §2, the proof of the main theorem is given in §3.

It is worth comparing the proof sketched above to the situation for Lie algebras of reductive groups. When $\mathfrak{g}$ is such a Lie algebra it is known that for every $\chi \in \mathfrak{g}^*$ there exists a Borel subalgebra $\mathfrak{b} \subseteq \mathfrak{g}$ such that $\chi[\mathfrak{b}, \mathfrak{b}] = 0$, $\dim \mathfrak{b} = \frac{1}{2}(\dim \mathfrak{g} + \operatorname{ind} \mathfrak{g})$ and $[\mathfrak{b}, \mathfrak{b}]$ is unipotent. It follows quickly that every simple $\mathfrak{g}$-module of p-character χ is a quotient of some baby Verma module. These are defined to be the $U_\chi(\mathfrak{g})$-modules induced from one dimensional $U_\chi(\mathfrak{b})$-modules. Hence the Borel subalgebras play the role of solvable weak polarisations in the reductive case.

Until recently it was considered to be possible that (1.2) might hold for non-restricted Lie algebras, however counterexamples to this hope were found by the third author, by presenting pairs of Lie algebras with isomorphic enveloping algebras and distinct indexes [Top17].

Let R be a commutative unital ring and say that k is an R-field if k is a field with an R-algebra structure. If G is an R-group scheme and k is an R-field then we write G_k for the base change of G from R to k. In our next theorem we describe a fruitful source of examples to which our main theorem can be applied.

Theorem 1.2. *Let G be a group scheme over R. There exists a $p_0 \in \mathbb{N}$ such that when $p > p_0$ is prime and $k = \overline{k}$ is an R-field of characteristic p, the first Kac–Weisfeiler conjecture holds for the Lie algebra* $\operatorname{Lie}(G_k)$.

Thus for a fixed group scheme, the KW1 conjecture holds in *almost all* characteristics. The proof, which is presented in §3.1, demonstrates that the Lie algebra $\operatorname{Lie}(G_k)$ admits a faithful restricted representation of dimension d independent of the choice of characteristic $p > 0$ of the field k, which allows us to apply the first main theorem. Finally we consider an interesting family of Lie algebras parameterised by the primes such that $\mathfrak{g}_p$ is restricted if and only if (1.2) holds, which is if and only if $p \equiv 1$ modulo 4.

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2. Preliminaries

2.1. **Model theory and the Lefschetz principle.** Since the main results and motivations of this paper come from representation theory, we expect that some of the readers will be unfamiliar with the model-theoretic methods which we use at several points. As such we include here a short recap of some of the elements of model theory; a more detailed introduction to the theory may be read in [Mar02]. Since our goal is to explain the Lefschetz principle, we work exclusively with the language of rings.

The *language of rings* $\mathcal{L}_{\text{ring}}$ is the collection of first-order formulas which can be built from the symbols $\{\forall, \exists, \vee, \wedge, \neg, +, -, \times, 0, 1, =\}$ along with arbitrary choice of variables. For example, for $n > 0$ fixed the following are formulas in $\mathcal{L}_{\text{ring}}$:

$$(\forall x)(\forall y)(x^n + y^n \neq z^n); \tag{2.1}$$

$$(\forall x)(\exists y)\left((xy = 1)) \vee (x = 0)\right); \tag{2.2}$$

$$(\forall x_0)(\forall x_1) \cdots (\forall x_{n-1})(\exists y)(y^n + x_{n-1}y^{n-1} + \cdots + x_0 = 0). \tag{2.3}$$

We say that a formula is a *sentence* if every variable is bound to a quantifier; for example for formula (2.1) is not a sentence because z is a free variable, whilst (2.2) and (2.3) are both sentences in $\mathcal{L}_{\text{ring}}$. If ϕ is a formula with free variables $x_1, ..., x_n$ then we might indicate this by writing $\phi = \phi(x_1, ..., x_n)$. In this case we can obtain a sentence from ϕ by binding the free variables to quantifiers. For example, if $\phi = \phi(z)$ is the formula from (2.1) then $(\forall z)\phi(z)$ is a sentence in the language of rings. In this way we may use formulas to build sentences.

For $p \geqslant 0$ we record one more first-order sentence ψ_p in $\mathcal{L}_{\text{ring}}$:

$$\psi_p \quad : \quad \underbrace{1 + \cdots + 1}_{p \text{ times}} = 0. \tag{2.4}$$

An $\mathcal{L}_{\text{ring}}$-*structure* is a set R together with elements $0_R, 1_R \in R$, binary operations $+_R, -_R, \times_R : R \times R \to R$, and the binary relation $=_R$ which is always taken to be the diagonal embedding $R \subseteq R \times R$. For example, every ring R gives rise to an $\mathcal{L}_{\text{ring}}$-structure in the obvious way.

Later in this paper we will need to express some statements about elements of vector spaces as formulas and sentences in the language $\mathcal{L}_{\text{ring}}$. To prepare for those arguments we now record a few examples which illustrate this procedure.

Lemma 2.1. *Let k be a field. The following statements can be formulated as sentences and formulas in $\mathcal{L}_{\text{ring}}$:*

1. *there exist elements $x_1, ..., x_m \in k^n$ which are linearly independent in k^n;*
2. *a given linear map $f : k^n \oplus k^n \to k^n$ defines a Lie bracket $[\cdot, \cdot]_f$ on k^n;*
3. *the Lie algebra $(k^n, [\cdot, \cdot]_f)$ is solvable.*

Proof. We view k^n as a set of tuples $(a_1, ..., a_n)$ of elements of k. Then (1) is equivalent to the following first-order sentence:

$$(\forall (i, j) \in \{1, \ldots n\} \times \{1, \ldots, m\})(\exists a_{i,j} \in k)(\forall b_1, ..., b_m \in k)$$
$$((\sum_{j=1}^{m} b_j a_{1,j} = \sum_{j=1}^{m} = b_j a_{2,j} = \cdots = \sum_{j=1}^{m} b_j a_{m,j} = 0) \Rightarrow (b_1 = b_2 = \cdots = b_m = 0)).$$

Let $v_1, ..., v_n \in k^n$ denote the standard basis. A linear map $f : k^n \oplus k^n \to k^n$ satisfies $f(v_i, v_j) = \sum_{l=1}^{n} f_{i,j;l} v_k$ for some scalars $f_{i,j;l}$ and we identify f with the array $(f_{i,j;l})_{1 \leqslant i,j,l \leqslant n} \in k^{n^3}$. Now the claim that f *defines a Lie bracket* can be encoded as a collection of linear and quadratic polynomial relations with integral coefficients in the variables

$$(f_{i,j;l})_{1\leqslant i,j,l\leqslant n}. \tag{2.5}$$

These relations correspond to skew-symmetry and the Jacobi identity. Since all integers can be defined using only the symbols $\{+,-,1,0\}$ it follows that f *defines a Lie bracket* is a first-order formula in $\mathcal{L}_{\text{ring}}$ with free variables (2.5). Similarly the statement *the Lie algebra* $(k^n,[\cdot,\cdot]_f)$ *is solvable* can be encoded in terms of the vanishing of all n-fold iterations of the Lie bracket, which is equivalent to the vanishing of a collection of homogeneous polynomials of degree 2^n-1 amongst the variables (2.5). Again this is a first-order formula in $\mathcal{L}_{\text{ring}}$ with free variables (2.5). □

An $\mathcal{L}_{\text{ring}}$-*theory* is a set T of first order sentences in $\mathcal{L}_{\text{ring}}$. If ϕ is a sentence and $M := (R,+_R,-_R,\times_R,0_R,1_R,=_R)$ is an $\mathcal{L}_{\text{ring}}$-structure then we say that M *is a model of* ϕ, and write $M\vDash\phi$, if the sentence ϕ is true when interpreted in M. If T is an $\mathcal{L}_{\text{ring}}$-theory then we say that M *is a model of* T and write $M\vDash T$ if $M\vDash\phi$ for all $\phi\in T$.

A theory should be thought of as the collection of sentences which are true for every model of a particular class of mathematical object. Since mathematical objects are usually determined by axioms, we shall briefly explain (by way of an example) how to pass from a set of axioms to a theory. Consider the set A of axioms of commutative rings, which are clearly first order sentences in $\mathcal{L}_{\text{ring}}$. We may then consider the set $\mathsf{CR}\subseteq\mathcal{L}_{\text{ring}}$ of sentences which are true for every model of A, ie. those which are true in every commutative ring. Thus CR denotes the theory of commutative rings. To illustrate the notation introduced above, observe that $(R,+_R,-_R,\times_R,0_R,1_R,=_R)\vDash\mathsf{CR}$ is equivalent to the statement that $(R,+_R,-_R,\times_R,0_R,1_R)$ is a commutative ring. As such we may slightly abuse terminology and identify the class of models of CR with the class of commutative rings.

In this paper we will be primarily interested in the theory of fields. The axioms of a field can obviously be written as first-order sentences in $\mathcal{L}_{\text{ring}}$; for instance (2.2) expresses the existence of multiplicative inverses. The axioms of the algebraically closed fields are obtained by including the sentences (2.3) for all $n>0$. If $p>0$ is prime then we may include the sentence ψ_p, defined in (2.4), to obtain the axioms of the algebraically closed fields of characteristic $p>0$; the corresponding theory is denoted AC_p. Alternatively we may include the sentences $\{\neg\psi_p\mid p>0\}$ to obtain the axioms of the algebraically closed fields of characteristic zero, and we denote their theory by AC_0. Since it will cause no confusion we identify the class of all models of AC_p with the class of all algebraically closed fields of characteristic p, whenever $p\geqslant 0$ is fixed.

The following result is Gödel's first completeness theorem in the context of $\mathcal{L}_{\text{ring}}$.

Lemma 2.2. *Let ϕ be a first-order sentence and T be any theory in $\mathcal{L}_{\text{ring}}$. Then ϕ is true when interpreted in every model of T if and only if ϕ can be deduced from T by means of a formal proof in $\mathcal{L}_{\text{ring}}$.*

We say that an $\mathcal{L}_{\text{ring}}$-theory T is *complete* if, for every first-order sentence ϕ in $\mathcal{L}_{\text{ring}}$, either ϕ is true when interpreted in every model of T, or $\neg\phi$ is true when interpreted in every model of T. By Lemma 2.2 this is equivalent to saying that for every sentence ϕ we can derive either ϕ or $\neg\phi$ from T by means of a formal proof. The following well-known result is proven by quantifier elimination [Mar02, Corollary 3.2.3].

Theorem 2.3. *For $p=0$ or p prime, the theory AC_p is complete.*

As an immediate consequence we obtain:

Corollary 2.4. *(Lefschetz principle) If ϕ is a sentence in $\mathcal{L}_{\text{ring}}$ then:*

(1) *If ϕ is true in some model of AC_p where $p\geqslant 0$ then ϕ is true in every model of AC_p.*

(2) *If ϕ is true in some model of AC_0 then there exists a $p_0 \in \mathbb{N}$ such that ϕ is true in any model of AC_p for $p > p_0$.*

Proof. Part (1) is precisely Theorem 2.3. For part (2) suppose that ϕ is true over some field satisfying the axioms of AC_0. Then by part (1) it is true for every such field, and by Lemma 2.2 we conclude that there exists a formal proof for ϕ in $\mathcal{L}_{\text{ring}}$ using only the axioms of AC_0. Since the proof of ϕ can be written as a finite sequence of sentences in $\mathcal{L}_{\text{ring}}$ joined by logical connectives, it follows that the set of primes

$$P_\phi := \{p \mid \neg\psi_p \text{ occurs in the proof of } \phi\}$$

is finite, where ψ_p is defined in (2.4). Hence for $p > \max(P_\phi)$ there is a formal proof of ϕ using the axioms of AC_p. Using Lemma 2.2 once more we see that ϕ is true for $\mathcal{L}_{\text{ring}}$-structure satisfying the axioms of AC_p. ☐

2.2. **Restricted Lie algebras and reduced enveloping algebras.** Fix a field k of characteristic $p > 0$ and let $\mathfrak{g}$ be a Lie algebra over k. As usual we write $U(\mathfrak{g})$ for the enveloping algebra and $Z(\mathfrak{g})$ for the centre of $U(\mathfrak{g})$. Then $\mathfrak{g}$ is said to be a *restricted Lie algebra over* k if it comes equipped with a p-map $\mathfrak{g} \to \mathfrak{g}$, written $x \mapsto x^{[p]}$, which satisfies two axioms: if we write $\xi : \mathfrak{g} \to U(\mathfrak{g})$ for the map $x \mapsto x^p - x^{[p]}$, then $(-)^{[p]}$ must satisfy

(1) $\xi(\mathfrak{g}) \subseteq Z(\mathfrak{g})$;

(2) ξ is p-semilinear in the sense of [Jan98, Lemma 2.1].

It follows from the PBW theorem that the vector space $\xi(\mathfrak{g})$ generates a polynomial algebra of rank equal to $\dim(\mathfrak{g})$ inside $U(\mathfrak{g})$. This algebra is referred to as the p-centre of $U(\mathfrak{g})$, and is denoted $Z_p(\mathfrak{g})$. If $\{x_i \mid i \in I\}$ is a basis for $\mathfrak{g}$ then the PBW theorem for $U(\mathfrak{g})$ implies that $Z_p(\mathfrak{g})$ is isomorphic to a polynomial ring generated by $\{\xi(x_i) \mid i \in I\}$. Hence $Z_p(\mathfrak{g})$ can be naturally identified with the coordinate ring $k[(\mathfrak{g}^*)^{(1)}]$ on the Frobenius twist of $\mathfrak{g}^*$.

When k is an algebraically closed field we have $\mathfrak{g}^* = (\mathfrak{g}^*)^{(1)}$ as sets and so for every $\chi \in \mathfrak{g}^*$ there is a maximal ideal $I_\chi \in \operatorname{Spec} Z_p(\mathfrak{g})$. Explicitly we have $I_\chi := (x^p - x^{[p]} - \chi(x)^p \mid x \in \mathfrak{g})$ and *the reduced enveloping algebra with p-character* χ is defined to be

$$U_\chi(\mathfrak{g}) := U(\mathfrak{g})/U(\mathfrak{g})I_\chi.$$

If $\mathfrak{g}_0 \subseteq \mathfrak{g}$ is a restricted subalgebra and $\chi \in \mathfrak{g}^*$ then we might abuse notation by identifying $U_{\chi|_{\mathfrak{g}_0}}(\mathfrak{g}_0)$ with the subalgebra of $U_\chi(\mathfrak{g})$ generated by $\mathfrak{g}_0$, and denote this subalgebra by $U_\chi(\mathfrak{g}_0)$. We say that a $\mathfrak{g}$-module M has p-character χ if the corresponding representation $U(\mathfrak{g}) \to \operatorname{End}_k(M)$ factors through the quotient $U(\mathfrak{g}) \to U_\chi(\mathfrak{g})$. If $\mathfrak{g}_0 \subseteq \mathfrak{g}$ are restricted Lie algebras, $\chi \in \mathfrak{g}^*$ and M_0 is a $U_\chi(\mathfrak{g}_0)$-module then we may define the induced module

$$\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g},\chi}(M_0) := U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{g}_0)} M_0. \tag{2.6}$$

We have

$$\dim \operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g},\chi}(M_0) = p^{\dim \mathfrak{g} - \dim \mathfrak{g}_0} \dim(M_0) \tag{2.7}$$

and this induced module is universal amongst $U_\chi(\mathfrak{g})$-modules such that the restriction to $U_\chi(\mathfrak{g}_0)$ contains a submodule isomorphic to M_0.

The coadjoint $\mathfrak{g}$-module is the vector space $\mathfrak{g}^*$ with module structure given by $(x \cdot \chi)(y) := \chi([y, x])$ where $x, y \in \mathfrak{g}$ and $\chi \in \mathfrak{g}^*$. The stabiliser of $\chi \in \mathfrak{g}^*$ is then defined to be

$$\mathfrak{g}^\chi := \{x \in \mathfrak{g} \mid x \cdot \chi = 0\} = \{x \in \mathfrak{g} \mid \chi[x, \mathfrak{g}] = 0\}$$

and the index of $\mathfrak{g}$ is the minimal dimension of $\mathfrak{g}^\chi$ as χ varies over all elements of $\mathfrak{g}^*$, commonly denoted $\operatorname{ind}(\mathfrak{g})$.

3. The maximal dimensions of simple modules

In this section we prove the main theorem. In order to do so we recall a few pieces of terminology. If $\mathfrak{g}$ is a Lie algebra over a field k and $\chi \in \mathfrak{g}^*$ we say that a Lie subalgebra $\mathfrak{s} \subseteq \mathfrak{g}$ is *subordinate to* χ if $\mathfrak{s}$ is an isotropic subspace with respect the skew-symmetric form

$$\begin{array}{rcl} B_\chi : \mathfrak{g} \times \mathfrak{g} & \to & k \\ (x, y) & \mapsto & \chi([x, y]). \end{array} \tag{3.1}$$

In other words, $\chi([\mathfrak{s}, \mathfrak{s}]) = 0$. We say that a subalgebra $\mathfrak{s} \subseteq \mathfrak{g}$ is *a polarisation of* χ if $\mathfrak{s}$ is a Lagrangian for B_χ, ie. $\mathfrak{s}$ is a maximal isotropic subspace of $\mathfrak{g}$. Since the stabiliser $\mathfrak{g}^\chi$ coincides with the radical of B_χ it follows from [Dix96, 1.12.1] that

$$\dim(\mathfrak{s}) \leqslant \frac{1}{2}(\dim(\mathfrak{g}) + \dim(\mathfrak{g}^\chi)) \tag{3.2}$$

if $\mathfrak{s}$ is subordinate to χ. Furthermore equality holds if and only if $\mathfrak{s}$ is a polarisation of χ. Finally we say that a Lie subalgebra $\mathfrak{s} \subseteq \mathfrak{g}$ is a *weak polarisation of* χ if $\mathfrak{s}$ is isotropic for the form (3.1) and

$$\dim(\mathfrak{s}) = \frac{1}{2}(\dim(\mathfrak{g}) + \mathrm{ind}(\mathfrak{g})). \tag{3.3}$$

The proof of the main theorem rests on the existence of solvable weak polarisations for linear forms, and the following result is the key step.

Proposition 3.1. *For all $n, d \in \mathbb{N}$, there exists $p_1 = p_1(n, d) \in \mathbb{N}$ such that if:*

(1) *k is an algebraically closed field of characteristic $p > p_1$;*
(2) *$\mathfrak{g}$ is a Lie algebra of dimension n over k;*
(3) *there exists a faithful representation $\rho : \mathfrak{g} \to \mathfrak{gl}_d(k)$.*

Then for every $\chi \in \mathfrak{g}^*$ there is a solvable weak polarisation $\mathfrak{s} \subseteq \mathfrak{g}$, such that $\rho(\mathfrak{s})$ is upper-triangularisable in $\mathfrak{gl}_d(k)$.

Proof. Fix $n, d \in \mathbb{N}$, $r \in \{0, ..., n\}$ and let k be an algebraically closed field of characteristic $p \geqslant 0$. Let $\{v_1, ..., v_n\}$ denote the standard basis for k^n. If $f \in \mathrm{Hom}_k(k^n \oplus k^n, k^n)$ then $f(v_i, v_j) = \sum_{l=1}^n f_{i,j;l} v_k$ and so we identify $\mathrm{Hom}_k(k^n \oplus k^n, k^n)$ with k^{n^3} and identify f with $(f_{i,j;l})_{1 \leqslant i,j,l \leqslant n} \in k^{n^3}$. For $i = 1, ..., n$ we write A_i for an element of $\mathrm{Mat}_d(k) \cong k^{d^2}$, so that the n-tuple $A = (A_1, ..., A_n)$ is an element of k^{nd^2}. Finally write $\chi = (\chi_1, ..., \chi_n) \in k^n$ and we view χ as an element of $(k^n)^* = \mathrm{Hom}_k(k^n, k)$ via $v_i \mapsto \chi_i$.

Fix $r \in \{0, ..., n\}$ and for any $(f, A, \chi) \in k^{n^3+nd^2+n}$ we consider the following four claims:

(i) $f = (f_{i,j;l})$ are the structure constants of a Lie bracket $[\cdot, \cdot]_f$ on k^n;
(ii) the Lie algebra $(k^n, [\cdot, \cdot]_f)$ has index equal to r;
(iii) the linear map $k^n \to \mathrm{Mat}_d(k)$ given by $v_i \mapsto A_i$ is a faithful Lie algebra representation $\rho : k^n \to \mathfrak{gl}_d(k)$;
(iv) There exist elements $x_1, ..., x_s \in k^n$ where $s = \frac{1}{2}(n + r)$ which are linearly independent and span a solvable Lie subalgebra $\mathfrak{s}$ of $(k^n, [\cdot, \cdot]_f)$ such that $\chi([\mathfrak{s}, \mathfrak{s}]_f) = 0$ and $\rho(\mathfrak{s})$ is upper-triangularisable inside $\mathfrak{gl}_d(k)$.

Now consider the following statements indexed by r:

$$\Phi_r \quad : \quad \forall (f, A, \chi) \in k^{n^3+d^2n+n}((\text{i}) \wedge (\text{ii}) \wedge (\text{iii})) \Rightarrow (\text{iv})) \tag{3.4}$$

Claim: Each statement Φ_r can be formulated as a first-order sentence in the language $\mathcal{L}_{\text{ring}}$ of rings (in the notation of §2.1).

In Lemma 2.1(2) we showed that (i) is a formula in the language $\mathcal{L}_{\text{ring}}$ with free variables $(f_{i,j;l})$. If $(f_{i,j;l})$ are the structure constants of a Lie bracket $[\cdot,\cdot]_f$ then the structure constants of the coadjoint representation $\text{ad}_f^* : k^n \to \text{Mat}_n(k)$ are $(-f_{i,l;j})_{1\leqslant i,j,l\leqslant n}$. It follows that for $x = (a_1, ..., a_n) \in k^n$ the statement $\text{ad}_f^*(x)\chi = 0$ can be expressed by the vanishing of certain polynomial functions, with integral coefficients, in the variables $(f_{i,j;l})$ and $a_1, ..., a_n, \chi_1, ..., \chi_n$. The statement (ii) can be phrased in the following way: *there exists* $\psi = (\psi_1, ..., \psi_n) \in k^n \cong \text{Hom}_k(k^n, k)$ *and linearly independent elements* $x_1, ..., x_r \in k^n$ *which satisfy* $\text{ad}_f^*(x_i)\psi = 0$*, and there does not exist* $\varphi \in \text{Hom}_k(k^n, k)$ *such that if* $x_1, ..., x_r \in k^n$ *satisfy* $\text{ad}_f^*(x_i)\varphi = 0$ *then* $x_1, ..., x_r$ *are linearly dependent.* This is a first-order formula in $\mathcal{L}_{\text{ring}}$ with free variables $(f_{i,j;l})$ thanks to part (1) of Lemma 2.1 and the previous remarks. The fact that (iii) is a first-order formula in $\mathcal{L}_{\text{ring}}$ with free variables (f, A) follows similarly. Statement (iv) asserts the existence of $x_1, ..., x_s$ spanning a solvable Lie subalgebra of $(k^n, [\cdot,\cdot]_f)$. The existence of elements which satisfy $\chi([x_i, x_j]_f) = 0$ and span a solvable Lie algebra is a first-order formula, indeed this follows quickly from Lemma 2.1(3). The fact that the solvable subalgebra can be upper-triangularised in $\mathfrak{gl}_d(k)$ can be expressed by the existence of a basis of k^d satisfying special properties which can also be expressed as first-order formulas in $\mathcal{L}_{\text{ring}}$. Since the last remark is proven in a manner almost identical to the previous parts, we leave the details to the reader. The only free variables in (iv) are (f, A, χ) and so we have shown that all of the variables in the formulas (i), (ii), (iii), (iv) are bound to quantifiers in Φ_r. Hence Φ_r is a first-order sentence in $\mathcal{L}_{\text{ring}}$, and this proves the claim.

Keep $n, d \in \mathbb{N}$, $r \in \{0, ..., n\}$ fixed, and now suppose that k is algebraically closed of characteristic zero. By Lie's theorem [Dix96, Theorem 1.3.12] we know that every solvable Lie subalgebra of $\mathfrak{gl}_d(k)$ is upper-triangularisable and thanks to [Dix96, Corollary 1.12.17] we know that when $\mathfrak{g}$ is a Lie algebra over k of dimension n and index r, for all $\chi \in \mathfrak{g}^*$ there exists a solvable subalgebra of $\mathfrak{g}$ subordinate to χ. Hence every algebraically closed field of characteristic zero is a model for Φ_r. It follows by the Lefschetz principle (Corollary 2.4) that there is a number $p_1^r = p_1^r(n, d) \in \mathbb{N}$ such that Φ_r is also true when interpreted in any algebraically closed field of characteristic $p > p_1^r$. If we set $p_1 := \max\{p_1^0, p_1^1, ..., p_1^n\}$ then it follows from the above remarks that for all for all $r = 0, ..., n$, Φ_r is true for every algebraically closed field of characteristic $p > p_1$. This completes the proof of the current Proposition. □

Lemma 3.2. *Let* $\mathfrak{s}$ *be a restricted solvable Lie algebra over* k *with a faithful restricted representation* $\mathfrak{s} \to \mathfrak{gl}_d(k)$*. If* $\text{char}(k) > d$ *and* $\mathfrak{s}$ *is upper-triangularisable then* $[\mathfrak{s}, \mathfrak{s}]$ *is a restricted unipotent ideal of* $\mathfrak{s}$*.*

Proof. We may suppose that $\mathfrak{s} \subseteq \mathfrak{gl}_d(k)$ is an upper-triangular restricted subalgebra. Then $[\mathfrak{s}, \mathfrak{s}]$ is strictly upper-triangular, and $\text{char}(k) > d$ forces the p-power map to vanish identically on $[\mathfrak{s}, \mathfrak{s}]$, which implies the claim. □

Theorem 3.3. *For all* $d \in \mathbb{N}$ *there exists a* $p_0 = p_0(d) \in \mathbb{N}$ *such that if* $k = \overline{k}$ *is a field of characteristic* $p > p_0$*, if* $\mathfrak{g} \subseteq \mathfrak{gl}_d(k)$ *and* M *is a simple* $\mathfrak{g}$*-module with* p*-character* χ *then* M *is a quotient of a module of the form* $\text{Ind}_{\bar{\mathfrak{s}}}^{\mathfrak{g},\chi}(M_0)$ *where* $\bar{\mathfrak{s}}$ *is a restricted Lie subalgebra of* $\mathfrak{g}$ *which contains a solvable weak polarisation of* χ*, and* M_0 *is a one dimensional* $U_\chi(\bar{\mathfrak{s}})$*-module.*

Proof. Fix $d \in \mathbb{N}$, let $p_0 := \max\{d, p_1(1, d), p_1(2, d), ..., p_1(d^2, d)\}$ where $p_1(n, d)$ was defined in Proposition 3.1, and let k be an algebraically closed field of characteristic $p > p_0$. Let $\mathfrak{g} \subseteq \mathfrak{gl}_d(k)$ be restricted and let $\chi \in \mathfrak{g}^*$. Thanks to Proposition 3.1 we know that there exists a solvable weak polarisation $\mathfrak{s} \subseteq \mathfrak{g}$ of χ with $\mathfrak{s}$ upper-triangularisable. Let $\bar{\mathfrak{s}}$ denote the restricted closure of $\mathfrak{s}$ in $\mathfrak{gl}_d(k)$, which is the smallest restricted Lie subalgebra of $\mathfrak{gl}_d(k)$ containing $\mathfrak{s}$. Evidently we have $\mathfrak{s} \subseteq \bar{\mathfrak{s}} \subseteq \mathfrak{g}$. Since the upper-trinagular matrices form a restricted Lie subalgebra of $\mathfrak{gl}_d(k)$ it follows that $\bar{\mathfrak{s}}$ is upper-triangularisable and solvable, and so $[\bar{\mathfrak{s}}, \bar{\mathfrak{s}}]$ is a restricted unipotent ideal of $\bar{\mathfrak{s}}$.

It follows from [Jan98, Proposition 3.2] that $U_\chi([\bar{\mathfrak{s}},\bar{\mathfrak{s}}]) = U_0([\bar{\mathfrak{s}},\bar{\mathfrak{s}}])$ is a local ring with a unique one dimensional simple module. If M is a simple $\mathfrak{g}$-module with p-character χ then we may consider the restriction $M|_{U_\chi([\bar{\mathfrak{s}},\bar{\mathfrak{s}}])}$ of M to $U_\chi([\bar{\mathfrak{s}},\bar{\mathfrak{s}}])$. By our previous observation we may find a one dimensional $[\bar{\mathfrak{s}},\bar{\mathfrak{s}}]$-submodule M_0 in the socle of $M|_{U_\chi([\bar{\mathfrak{s}},\bar{\mathfrak{s}}])}$. Since $[\bar{\mathfrak{s}},\bar{\mathfrak{s}}]$ is an ideal of $\bar{\mathfrak{s}}$ it follows that M_0 is a $U_\chi(\bar{\mathfrak{s}})$-module. Now the existence of a surjection $\mathrm{Ind}_{\bar{\mathfrak{s}}}^{\mathfrak{g},\chi}(M_0) \twoheadrightarrow M$ follows from the universal property of induced modules. □ □

Now the proof of the main theorem follows from the previous theorem, combined with (2.7).

Remark 3.4. It has been conjectured that for a restricted Lie algebra $\mathfrak{g}$ the following are equivalent:

(i) $\mathfrak{g}$ is Frobenius, meaning $\mathrm{ind}(\mathfrak{g}) = 0$;

(ii) there exists a non-empty open subset $\mathcal{O} \subseteq \mathfrak{g}^*$ such that $U_\chi(\mathfrak{g})$ is simple for all $\chi \in \mathcal{O}$.

Our main theorem implies that the conjecture holds for Lie subalgebra of $\mathfrak{gl}_d(k)$ provided $\mathrm{char}(k) > p_0(d)$. In fact (i) ⇒ (ii) holds in general thanks to [PS99, Theorem 4.4]. Conversely, supposing $U_\chi(\mathfrak{g})$ is simple, there is a simple $\mathfrak{g}$-module of the maximal possible dimension $\sqrt{\dim U_\chi(\mathfrak{g})} = p^{\frac{1}{2}\dim\mathfrak{g}}$. If the KW1 conjecture holds for $\mathfrak{g}$ then $M(\mathfrak{g}) = p^{\frac{1}{2}\dim\mathfrak{g}} = p^{\frac{1}{2}(\dim\mathfrak{g} - \mathrm{ind}\,\mathfrak{g})}$ and so $\mathfrak{g}$ is Frobenius.

3.1. **Example: the Lie algebras of group schemes.** In this final subsection we draw attention to families of important Lie algebras to which our first theorem can be applied, proving the second theorem from the introduction. We recall some of the elements of the theory of algebraic group schemes, following [DG70], [Jan03]. Throughout the subsection we fix a commutative unital ring R, we write R-alg for the category of R-algebras, and we say that k is an *R-field* if k is both a field and an object of R-alg. An affine algebraic group scheme G over R is a functor from R-algebras to groups, naturally equivalent to one of the form $\mathrm{Spec}_R\, R[G] := \mathrm{Hom}_{R\text{-alg}}(R[G], -)$ where $R[G]$ is some finitely presented R-algebra. The archetypal example of an algebraic group scheme is GL_d.

When G is a group scheme and k is any R-algebra we can consider the base change G_k, which is a group scheme over k obtained by viewing k-algebras as R-algebras. If $G \cong \mathrm{Spec}_R\, R[X]$ is algebraic and $R[X] \cong R[x_1, ..., x_n]/(g_1, ..., g_m)$ then we obtain a map $\omega : R[x_1, ..., x_n] \to k[x_1, ..., x_n]$ and we have

$$G_k \cong \mathrm{Spec}_k\, k[x_1, ..., x_n]/(\omega(g_1), ..., \omega(g_m)). \tag{3.5}$$

and we write $\mathfrak{g}_k = \mathrm{Lie}(G_k)$.

Lemma 3.5. *Let G be an affine algebraic group scheme over R. There exists $d \in \mathbb{N}$ depending only on G such that for each R-field k there exists a representation $\rho : G_k \to (\mathrm{GL}_d)_k$, with $d\rho : \mathfrak{g}_k \to (\mathfrak{gl}_d)_k$ faithful.*

Proof. Suppose that G corresponds to the Hopf algebra $(R[G], \Delta, \sigma, \epsilon)$, with finite presentation $R[G] = R[x_1, ..., x_n]/(g_1, ..., g_m)$. Fix $i \in \{1, ..., n\}$, write $\Delta(x_i) = \sum_{j=1}^{r(i)} f_{i,j}^{(1)} \otimes f_{i,j}^{(2)}$ and define $d := \sum_{i=1}^n r(i)$. Choose any R-field k and let M be the R-module generated by elements $\{f_{i,j}^{(1)} \mid i = 1, ..., n, j = 1, ..., r(i)\}$. Write $\omega : R[G] \to k[G_k]$ for the natural homomorphism.

Thanks to (3.5) there is a surjection $M \otimes_R k \twoheadrightarrow \omega(M)$ and so $\omega(M)$ identifies with a subspace of $k[G_k]$ of dimension $\leqslant d$. We observe that the coproduct $\Delta(\omega(x_i)) = \sum_{i=1}^{r(i)} \omega(f_{i,j}^{(1)}) \otimes \omega(f_{i,j}^{(2)})$ can be rewritten in the form $\Delta(\omega(x_i)) = \sum_{i=1}^{r_k(i)} h_{i,j} \otimes \omega(f_{i,j}^{(2)})$ for some $r_k(i) \leqslant r(i)$ and certain elements $h_{i,j} \in \omega(M)$, such that $\omega(f_{i,1}^{(2)}), ..., \omega(f_{i,r_k(i)}^{(2)})$ are k-linearly independent. According to [Jan03, I.2.13(4)] the space $N_i := \sum_{j=1}^{r_k(i)} k h_{i,j}$ is a G_k-submodule of $\omega(M)$ containing x_i. Furthermore it follows from [Mil17, Proof of Prop. 4.7] that $N = \sum_{i=1}^n N_i$ is a faithful G_k-submodule of $k[G_k]$ of dimension $\leqslant d$. Therefore $N \oplus k^{\oplus(d - \dim N)}$ is a faithful module of dimension d. Finally observe that

N is a faithful $\mathfrak{g}_k$-module. To see this note, for example, that by faithfulness $1 \to G(k[\epsilon]/(\epsilon^2)) \to \mathrm{GL}_N(k[\epsilon]/(\epsilon^2))$ is exact; also $\mathrm{Lie}(G) \cong \ker(G(k[\epsilon]/(\epsilon^2)) \to G(k))$ for the map $\epsilon \mapsto 0$, and similarly for GL_N. Hence the claim follows from the commutative diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathfrak{g}_k & \longrightarrow & G_k(k[\epsilon]/(\epsilon^2)) \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathfrak{gl}_N & \longrightarrow & \mathrm{GL}_N(k[\epsilon]/(\epsilon^2)). \end{array}$$

□

Remark 3.6. The question of when there is a faithful R-representation for G is rather subtle and is not known even when R is the ring of dual numbers over a field k and G is flat of finite type over R. However the existence of such a representation is known in the case where G is flat and of finite type over a Dedekind domain R, such as $\mathbb{Z}$, or indeed if R is any field.

As an immediate corollary we obtain the second main theorem from the first.

Corollary 3.7. *If G is an affine algebraic group scheme over a commutative ring R then there exists a $p_0 \in \mathbb{N}$ such that if $k = \bar{k}$ is an R-field of characteristic $p > p_0$ then the first Kac–Weisfeiler conjecture holds for* $\mathrm{Lie}(G_k)$. □

For example, if G is a reductive group scheme then provided $\mathrm{char}(k) = p \gg 0$ the first Kac–Weisfeiler conjecture holds for all parabolic subalgebras of $\mathrm{Lie}(G_k)$, as well as for all centralisers in $\mathrm{Lie}(G_k)$.

3.2. **Example: families of non-algebraic Lie algebras.** In this final subsection we construct a family of Lie algebras $\{\mathfrak{g}_p \mid p \text{ prime}\}$ which exhibits a different kind of behaviour to that seen for Lie algebras of groups. Since the example is elementary enough we can actually describe the representation theory without recourse to our main theorem, but the example is illustrative of some interesting phenomena and so we have included it anyway. For each $p > 0$ prime pick an algebraically closed field k_p of characteristic p, and pick an element $i \in k_p$ satisfying $i^2 + 1 = 0$. Let $\mathfrak{g}_p$ be the Lie algebra spanned by $\{h, x, y\}$ over k_p and with Lie brackets $[h, x] = x, [h, y] = iy, [x, y] = 0$.

Proposition 3.8. *Suppose that $p > 2$. The following are equivalent:*

1. $M(\mathfrak{g}_p) = p^{\frac{1}{2}(\dim(\mathfrak{g}_p) - \mathrm{ind}(\mathfrak{g}_p))}$*;*
2. $\mathfrak{g}_p$ *is a restricted Lie algebra;*
3. $p \equiv 1$ *modulo 4.*

Proof. By Gauss' law of quadratic reciprocity we know that $i \in \mathbb{F}_p \subseteq k_p$ if and only if $p \equiv 1$ modulo 4. If this is the case then $\mathrm{ad}(h)^p = \mathrm{ad}(h)$ and it follows quickly that $\mathrm{ad}(h)^p, \mathrm{ad}(x)^p, \mathrm{ad}(y)^p \in \mathrm{ad}(\mathfrak{g})$. By Jacobson's theorem [SF88, Theorem 2.2.3] we deduce that $\mathfrak{g}_p$ is restricted. Conversely, if $p \equiv 3$ modulo 4 then $i \notin \mathbb{F}_p$ and so $\mathrm{ad}(h)^p \notin \mathrm{ad}(\mathfrak{g})$. Therefore by *loc. cit.* the algebra $\mathfrak{g}_p$ is not restricted. Thus we see (2) $\Leftrightarrow$ (3).

Since $\dim(\mathfrak{g}) - \mathrm{ind}(\mathfrak{g}_p)$ is even we conclude that $\mathrm{ind}(\mathfrak{g}_p) \in \{1, 3\}$. Clearly $\chi[z, \mathfrak{g}] = 0$ for some $0 \neq z \in \mathfrak{g}_p$ implies $\chi = 0$, and so $\mathrm{ind}(\mathfrak{g}_p) = 1$ for all $p > 0$. Let D_p denote the division ring of fractions of $U(\mathfrak{g}_p)$ and let Q_p denote the division ring of the centre $Z(\mathfrak{g}_p)$ of $U(\mathfrak{g}_p)$. Then D_p is a Q_p-vector space and, thanks to [Zas54], we have $M(\mathfrak{g})^2 = [D_p : Q_p]$. Supposing $p \equiv 1$ modulo 4, so that $i \in \mathbb{F}_p$ we have central elements $\{h^p - h, x^k y^j \mid k + ij \in p\mathbb{Z}, 1 \leqslant j, k \leqslant p-1\}$ and it follows that $U(\mathfrak{g}_p)$ is generated as a $Z(\mathfrak{g}_p)$-module by $\leqslant p^2$ elements. We deduce that $M(\mathfrak{g}_p)^2 = [D_p : Q_p] \leqslant p^2$ and so $M(\mathfrak{g}_p) \leqslant p$. By [PS99, Remark 5.4, (1)] it follows that $M(\mathfrak{g}_p) = p$, since $\mathfrak{g}_p$ is restricted. From $\frac{1}{2}(\dim(\mathfrak{g}_p) - \mathrm{ind}(\mathfrak{g}_p)) = 1$ it follows that (3) $\Rightarrow$ (1). Now suppose that $p \equiv 3$ modulo 4. A very explicit calculation will show that $Z(\mathfrak{g}_p)$ is a polynomial algebra generated by $\{x^p, y^p, h^{p^2} - h\}$

and so $U(\mathfrak{g}_p)$ is a free $Z(\mathfrak{g}_p)$-module of rank p^4. It follows that $M(\mathfrak{g}_p)^2 = [D_p : Q_p] = p^4$ and so $M(\mathfrak{g}_p) \neq p^{\frac{1}{2}(\dim(\mathfrak{g}_p)-\mathrm{ind}(\mathfrak{g}_p))} = p$ in this case, whence (1) $\Rightarrow$ (3). This concludes the proof. $\square$

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