

BASIC SETS FOR THE DOUBLE COVERING GROUPS OF THE SYMMETRIC AND ALTERNATING GROUPS IN ODD CHARACTERISTIC

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ABSTRACT. In this paper, following the methods of [8], we show that the double covering groups of the symmetric and alternating groups have p -basic sets for any odd prime p .

1. INTRODUCTION

Let G be a finite group and \mathcal{C} be a union of conjugacy classes of G . For any class function α on G , define $\text{res}_{\mathcal{C}}(\alpha) = \widehat{\alpha}$ to be the class function on G that takes the same values as α on \mathcal{C} and 0 elsewhere. A \mathcal{C} -basic set of G is a subset b of the set $\text{Irr}(G)$ of complex irreducible characters of G such that $\widehat{b} = \{\widehat{\chi} \mid \chi \in b\}$ is a \mathbb{Z} -basis of the \mathbb{Z} -module generated by $\widehat{\text{Irr}}(G) = \{\widehat{\chi} \mid \chi \in \text{Irr}(G)\}$.

In [9, §2.1], the authors develop a generalized modular theory of G with respect to \mathcal{C} . In particular, there is a partition of $\text{Irr}(G)$ into \mathcal{C} -blocks with respect to this \mathcal{C} -structure. The construction goes as follows. Given any \mathbb{Z} -basis β of the \mathbb{Z} -module generated by $\widehat{\text{Irr}}(G)$, two characters $\chi, \chi' \in \text{Irr}(G)$ are called β -linked if $\widehat{\chi}$ and $\widehat{\chi}'$ have a common factor when decomposed in the basis β . The transitive closure of β -linking gives a partition of $\text{Irr}(G)$ into equivalence classes called \mathcal{C} -blocks. Note that the \mathcal{C} -blocks we obtain depend on the choice of β , but that β doesn't need to be a basic set for G . Furthermore, by [9, Prop 2.14], there always exists a choice of basis β such that the \mathcal{C} -blocks coincide with the generalized blocks constructed by Külshammer, Olsson and Robinson in [23, §1], and defined by orthogonality across \mathcal{C} . Throughout the paper, we will always assume that this specific choice of basis is made.

We can then analogously define the notion of \mathcal{C} -basic set of any \mathcal{C} -block B . Note that, if b is a \mathcal{C} -basic set of G and B is any \mathcal{C} -block of G , then $b \cap B$ is a \mathcal{C} -basic set of B . Conversely, given a \mathcal{C} -basic set for each \mathcal{C} -block of G , one recovers a \mathcal{C} -basic set of G .

Let p be a prime. When \mathcal{C} is the set of p -regular elements of G , that is, elements whose order is prime to p , the \mathcal{C} -modular theory of G is the classical p -modular character theory, and the notion of \mathcal{C} -basic set coincides with that of p -basic set. One of the main challenges of Brauer's theory is the determination of the p -decomposition matrix of G , of which a first approximation can be obtained when one has a p -basic set of G .

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While it is unclear how to exhibit a p -basic set of G in general, Hiss conjectured that all finite groups do have p -basic sets. This conjecture is true in the following cases: for p -soluble groups [11, X.2.1], for certain finite reductive groups under various assumptions ([17], [12], [15], [13], [14], [22], [10], [6]), and for symmetric and alternating groups ([21], [8], [7], [2]).

The covering groups of the symmetric groups have a 2-basic set (see [4, Thm 5.2]). For p odd, attempts have been made in [3] and [1] to generalize the method of James and Kerber, but these methods fail for $p \geq 7$. In this paper, we prove the following.

Theorem 1.1. *The covering groups of the symmetric and alternating groups have p -basic sets for any odd prime p .*

In [9] (see also [23]), the authors gave a notion of perfect isometries, which generalizes to \mathcal{C} the perfect isometries introduced by Broué in [5]. Such a perfect isometry I furthermore satisfies $I(\widehat{\chi}) = \widehat{I(\chi)}$ for all $\chi \in B$, whence in particular sends a basic set to a basic set.

To prove Theorem 1.1, we follow the strategy of [8]. In Section 3, we present groups \widetilde{G} and \widetilde{H} and unions of conjugacy classes \mathcal{C} of \widetilde{G} and \mathcal{D} of \widetilde{H} , and construct a \mathcal{C} -basic set of \widetilde{G} (see Theorem 3.3) and a \mathcal{D} -basic set of \widetilde{H} (see Theorem 3.4).

In Section 4, we adapt the perfect isometry constructed by Livesey in [24], where he proves Broué's Abelian Defect Perfect Isometry Conjecture, and we apply the methods and results of [9]. This produces, for any p -block B of a double covering group of the symmetric or alternating groups, a perfect isometry between B and a \mathcal{C} -block of \widetilde{G} or a \mathcal{D} -block of \widetilde{H} , from which Theorem 1.1 is deduced. Finally, Section 5 is devoted to some further remarks. In particular, we recover some results of Olsson on the Brauer characters in p -blocks of covering groups (p odd), and their number.

2. BACKGROUND

Let n be a positive integer. We write $\widetilde{\mathfrak{S}}_n$ for the double covering group of the symmetric group \mathfrak{S}_n defined by

$$\widetilde{\mathfrak{S}}_n = \langle z, t_i, 1 \leq i \leq n-1 \mid z^2 = 1, t_i^2 = z, (t_i t_{i+1})^3 = z, (t_i t_j)^2 = z \ (|i-j| \geq 2) \rangle.$$

The group $\widetilde{\mathfrak{S}}_n$ and its representation theory were first studied by I. Schur in [32], and, unless otherwise specified, we always refer to [32] for details or proofs.

Note that \mathfrak{S}_n has another, non-isomorphic, double covering group, usually denoted by $\widehat{\mathfrak{S}}_n$. However, from the fact that $\widetilde{\mathfrak{S}}_n$ has a p -basic set, we will deduce that $\widehat{\mathfrak{S}}_n$ does as well (see Remark 4.10).

We recall that we have the following exact sequence

$$1 \rightarrow \langle z \rangle \rightarrow \widetilde{\mathfrak{S}}_n \rightarrow \mathfrak{S}_n \rightarrow 1.$$

We denote by $\theta : \widetilde{\mathfrak{S}}_n \rightarrow \mathfrak{S}_n$ the natural projection. Note that for every $s \in \mathfrak{S}_n$, we have $\theta^{-1}(s) = \{\widetilde{s}, z\widetilde{s}\}$, where $\widetilde{s} \in \widetilde{\mathfrak{S}}_n$ is such that $\theta(\widetilde{s}) = s$. Whenever G is a subgroup of \mathfrak{S}_n , we set $\widetilde{G} = \theta^{-1}(G)$. Any irreducible (complex) character of \widetilde{G} with z in its kernel is simply lifted from an irreducible character of G . Any other irreducible character ξ of \widetilde{G} is called a *spin character*, and it satisfies $\xi(z) = -\xi(1)$. We denote by $\text{SI}(\widetilde{G})$ the set of irreducible spin characters of \widetilde{G} . Define $\varepsilon = \text{sgn} \circ \theta$,

where sgn is the sign character of \mathfrak{S}_n . In particular, if G is the alternating group \mathcal{A}_n , then $\tilde{\mathcal{A}}_n = \ker(\varepsilon)$.

Note that the restriction of ε to \tilde{G} (again denoted by ε) is a linear (irreducible) character of \tilde{G} , and for any spin character ξ of \tilde{G} , $\varepsilon \otimes \xi$ is a spin character (because $\varepsilon \otimes \chi(z) = -\varepsilon \otimes \chi(1)$). A spin character ξ is said to be *self-associate* if $\varepsilon\xi = \xi$. Otherwise, ξ and $\varepsilon\xi$ are called *associate characters*.

If H_1 and H_2 are subgroups of \mathfrak{S}_n acting non-trivially on disjoint subsets of $\{1, \dots, n\}$ and $H = H_1 \times H_2 \leq \mathfrak{S}_n$, then Schur defined in [32] (see also [18]) the *twisted central product* denoted by $\hat{\times}$ and such that $\tilde{H} = \tilde{H}_1 \hat{\times} \tilde{H}_2 \leq \tilde{\mathfrak{S}}_n$. Schur proved that there is a surjective map $\hat{\otimes} : \text{SI}(\tilde{H}_1) \times \text{SI}(\tilde{H}_2) \rightarrow \text{SI}(\tilde{H})$. Schur explicitly described when two pairs of spin characters have the same image $\chi \hat{\otimes} \psi$ under this map, and how to detect whether $\chi \hat{\otimes} \psi$ is self-associate or not. Furthermore, we derive from case 1 of §2 and the proof of Theorem 2.4 of [18] that

Proposition 2.1. *Let \tilde{H}_1 and \tilde{H}_2 be as above and not contained in the kernel of ε . Let χ_1 and χ_2 be irreducible spin characters of \tilde{H}_1 and \tilde{H}_2 . Let $\tau_1 \in \tilde{H}_1$ and $\tau_2 \in \tilde{H}_2$ be such that $\varepsilon(\tau_1) = \varepsilon(\tau_2) = 1$. Then*

(1) *If χ_1 (or χ_2) is self-associate, then*

$$(\chi_1 \hat{\otimes} \chi_2)(\tau_1, \tau_2) = \chi_1(\tau_1)\chi_2(\tau_2).$$

(2) *If χ_1 and χ_2 are non-self-associate, then*

$$(\chi_1 \hat{\otimes} \chi_2)(\tau_1, \tau_2) = 2\chi_1(\tau_1)\chi_2(\tau_2).$$

Let \mathcal{D}_n be the set of *bar partitions* of n , i.e. the partitions of n with distinct parts. Denote by \mathcal{D}_n^+ (respectively \mathcal{D}_n^-) the subset of \mathcal{D}_n consisting of all partitions $\pi \in \mathcal{D}_n$ such that the number of even parts of π is even (respectively odd). In [32], Schur proved that the spin characters of $\tilde{\mathfrak{S}}_n$ are, up to association, labeled by \mathcal{D}_n . More precisely, he showed that every $\lambda \in \mathcal{D}_n^+$ indexes a self-associate spin character ξ_λ , and every $\lambda \in \mathcal{D}_n^-$ a pair $(\xi_\lambda^+, \xi_\lambda^-)$ of associate spin characters.

The irreducible spin characters of $\tilde{\mathcal{A}}_n$ are also labeled by the bar partitions of n . If $\lambda \in \mathcal{D}_n^+$, then $\text{Res}_{\tilde{\mathcal{A}}_n}^{\tilde{\mathfrak{S}}_n}(\xi_\lambda) = \zeta_\lambda^+ + \zeta_\lambda^-$ with $\zeta_\lambda^+, \zeta_\lambda^- \in \text{SI}(\tilde{\mathcal{A}}_n)$. If $\lambda \in \mathcal{D}_n^-$, then $\text{Res}_{\tilde{\mathcal{A}}_n}^{\tilde{\mathfrak{S}}_n}(\xi_\lambda^+) = \text{Res}_{\tilde{\mathcal{A}}_n}^{\tilde{\mathfrak{S}}_n}(\xi_\lambda^-) = \zeta_\lambda \in \text{SI}(\tilde{\mathcal{A}}_n)$.

For any bar partition $\lambda = (\lambda_1 > \dots > \lambda_k > 0)$ of n , we set, as in the case of partitions, $|\lambda| = \sum \lambda_i$ and we define the *length* $\ell(\lambda)$ of λ by $\ell(\lambda) = k$. Furthermore, we set $\sigma(\lambda) = (-1)^{|\lambda| - \ell(\lambda)}$. With this notation, we then have $\lambda \in \mathcal{D}_n^{\sigma(\lambda)}$ (see e.g. [31, p. 45]).

Let p be an odd integer. To any $\lambda \in \mathcal{D}_n$, one can associate in a canonical way its \bar{p} -core $\lambda_{(\bar{p})}$ and its \bar{p} -quotient $\lambda^{(\bar{p})}$; see [31, p. 28]. The \bar{p} -core $\lambda_{(\bar{p})}$ is a bar partition of $n - wp$, where w is a non-negative integer called the \bar{p} -weight of λ . The \bar{p} -quotient has the form $\lambda^{(\bar{p})} = (\lambda^0, \lambda^1, \dots, \lambda^{(p-1)/2})$, where λ^0 is a bar partition, the λ^i 's are partitions for $1 \leq i \leq (p-1)/2$, and the sizes of the λ_i 's ($0 \leq i \leq (p-1)/2$) add up to w . If we write, in analogy with bar partitions, $\sigma(\lambda^{(\bar{p})}) = (-1)^{w - \ell(\lambda^0)}$, then we obtain that

$$(1) \quad \sigma(\lambda) = \sigma(\lambda_{(\bar{p})})\sigma(\lambda^{(\bar{p})}).$$

3. SOME BASIC SETS

Let w be a positive integer and p be an odd prime. In this section, we will focus on a group \tilde{G} which, when $w < p$, is isomorphic to the normalizer of a Sylow p -subgroup of $\tilde{\mathfrak{S}}_{pw}$. We will recall the description of its irreducible characters, and describe a union \mathcal{C} of conjugacy classes and a \mathcal{C} -basic set for \tilde{G} .

Set $K = \mathbb{Z}/p\mathbb{Z} \rtimes \mathbb{Z}/(p-1)\mathbb{Z}$ and $G = K \wr \mathfrak{S}_w$. Recall that G is a natural subgroup of \mathfrak{S}_{pw} and that $G = N \rtimes \mathfrak{S}_w$, where $N = K^w$. Now, let $Q = (\mathbb{Z}/p\mathbb{Z})^w \triangleleft N$ and $L = \mathbb{Z}/(p-1)\mathbb{Z} \wr \mathfrak{S}_w$. Then $G = Q \rtimes L$. Note that $|\tilde{Q}| = 2p^w$. Thus, by Sylow's theorems, \tilde{Q} has a Sylow p -subgroup P of index 2, whence normal and unique in \tilde{Q} . Since $\langle z \rangle$ is normal in \tilde{Q} and $\langle z \rangle \cap P$ is trivial (as p is odd), we see that $\tilde{Q} = \langle z \rangle \times P$ and that $P \simeq Q$. Now take any $g \in \tilde{G}$. Then, since $\tilde{Q} \triangleleft \tilde{G}$, the group gPg^{-1} is a Sylow p -subgroup of \tilde{Q} . Hence, $gPg^{-1} = P$ and P is normal in \tilde{G} . Furthermore, as $Q \cap L = 1$, we have $P \cap \tilde{L} = 1$ (because $P \cap \tilde{L} = P \cap (\tilde{Q} \cap \tilde{L}) = P \cap \langle z \rangle = 1$). Hence

$$(2) \quad \tilde{G} = \tilde{N} \tilde{\mathfrak{S}}_w = P \rtimes \tilde{L}.$$

The irreducible spin characters of \tilde{G} were constructed by Michler and Olsson in [25] using Clifford theory (see for example [20, Chap 6]) of \tilde{G} with respect to the normal subgroup \tilde{N} . The parametrization they obtain is as follows.

Proposition 3.1. *The irreducible spin characters of \tilde{G} can be labeled explicitly by the \bar{p} -quotients of w . If $q = (q^0, q^1, \dots, q^{(p-1)/2})$ is a \bar{p} -quotient of w , then q labels two associate characters ψ_q^+ and ψ_q^- if and only if $\sigma(q) = -1$; otherwise, q labels a unique self-associate character ψ_q .*

For the second statement, an explanation can be found for example in [16, p. 422].

Proposition 3.2. *The spin characters of \tilde{G} which have P in their kernel are exactly those labeled by \bar{p} -quotients of the form $q = (\emptyset, q^1, \dots, q^{(p-1)/2})$.*

Proof. Take any $\psi \in \text{Irr}(\tilde{G})$. Since $P \leq \tilde{N}$, to study the restriction of ψ to P , we will look at the restriction of ψ to \tilde{N} . On the other hand, the restriction to \tilde{N} is given by Clifford's theorem [20, Thm 6.2] which goes as follows. There exist a unique \tilde{G} -orbit Ω_ψ of irreducible characters of \tilde{N} and a unique positive integer e_ψ such that

$$(3) \quad \text{Res}_{\tilde{N}}^{\tilde{G}}(\psi) = e_\psi \sum_{\chi \in \Omega_\psi} \chi.$$

It follows from (3), the fact that ψ is a spin character and that all $\chi \in \Omega_\psi$ take the same value on z that Ω_ψ consists only of spin characters. We will thus now recall the description given in [25] of the \tilde{G} -orbits of spin characters of \tilde{N} .

The irreducible spin characters of \tilde{N} are of the form $\chi_1 \hat{\otimes} \dots \hat{\otimes} \chi_w$, where $\chi_i \in \text{SI}(\tilde{K})$ for $1 \leq i \leq w$. As above, we have $\tilde{K} = \langle \tau \rangle \rtimes \theta^{-1}(\mathbb{Z}/(p-1)\mathbb{Z})$, where τ is an element of order p . In [25], Michler and Olsson show that \tilde{K} has exactly p spin characters: a unique self-associate η_0 , of degree $p-1$ and with $\eta_0(\tau) = -1$, and $(p-1)/2$ pairs of associate linear characters $\eta_1^+, \eta_1^-, \dots, \eta_{(p-1)/2}^+, \eta_{(p-1)/2}^-$, which are exactly those having $\langle \tau \rangle$ in their kernel.

For any decomposition $t = (t_0, t_1, \dots, t_{(p-1)/2})$ of w , we set

$$(4) \quad \theta_t = \underbrace{\eta_0 \hat{\otimes} \dots \hat{\otimes} \eta_0}_{t_0 \text{ factors}} \underbrace{\hat{\otimes} \eta_1^+ \hat{\otimes} \dots \hat{\otimes} \eta_1^+}_{t_1 \text{ factors}} \hat{\otimes} \dots \hat{\otimes} \underbrace{\eta_{(p-1)/2}^+ \hat{\otimes} \dots \hat{\otimes} \eta_{(p-1)/2}^+}_{t_{(p-1)/2} \text{ factors}} \in \text{SI}(\tilde{N}).$$

Now suppose ψ is labeled by the \bar{p} -quotient $q = (q^0, \dots, q^{(p-1)/2})$ of w . From [25, Prop 3.12] and the construction of ψ , we get that, if $|q^0| > 1$, then $\theta_{(|q^0|, \dots, |q^{(p-1)/2}|)}$ is a representative for $\Omega_\psi = \Omega_{\varepsilon\psi}$ (even if $\psi \neq \varepsilon\psi$). If $|q^0| \leq 1$ and $w - |q^0|$ is odd, then $\psi \neq \varepsilon\psi$ and representatives for Ω_ψ and $\Omega_{\varepsilon\psi}$ are given by $\theta_{(|q^0|, \dots, |q^{(p-1)/2}|)}$ and $\varepsilon\theta_{(|q^0|, \dots, |q^{(p-1)/2}|)}$ (not necessarily in this order). In all cases, we write θ_ψ for the representative of Ω_ψ .

Note that $Q = \langle u_1 \rangle \times \dots \times \langle u_w \rangle$, where u_i has order p for $1 \leq i \leq w$. For $1 \leq i \leq w$, let τ_i be the unique element of order p in $\theta^{-1}(\{u_i\})$. Then

$$P = \langle \tau_1 \rangle \times \dots \times \langle \tau_w \rangle.$$

In particular, $P \leq \tilde{\mathcal{A}}_{pw}$.

Now, by (3), we see that, for any $(x_1, \dots, x_w) \in P$, we have

$$\psi(x_1, \dots, x_w) = e_\psi \sum_{\chi \in \Omega_\psi} \chi(x_1, \dots, x_w).$$

In particular, we have $\psi(1) = e_\psi \sum_{\chi \in \Omega_\psi} \chi(1) = e_\psi \theta_\psi(1) |\Omega_\psi|$. Furthermore, iterating Proposition 2.1 shows that, for any $\chi \in \Omega_\psi$, there is a non-negative integer α_χ such that $\chi(1) = 2^{\alpha_\chi} (p-1)^{|q^0|}$ and $\chi(\tau_1, \dots, \tau_w) = 2^{\alpha_\chi} (-1)^{|q^0|}$. Since $\chi(1)$ does not depend on the choice of χ in Ω_ψ , neither does α_χ which we thus denote by α . In particular, we have

$$\psi(1) = e_\psi 2^\alpha (p-1)^{|q^0|} |\Omega_\psi| \quad \text{and} \quad \psi(\tau_1 \dots \tau_w) = e_\psi 2^\alpha (-1)^{|q^0|} |\Omega_\psi|.$$

It follows that, if $q^0 \neq \emptyset$, then $\psi(\tau_1 \dots \tau_w) \neq \psi(1)$, whence P is not contained in $\ker(\psi)$. If $q^0 = \emptyset$, then a similar computation shows that, for any $(x_1, \dots, x_w) \in P$, we have $\psi(x_1, \dots, x_w) = e_\psi 2^\alpha |\Omega_\psi| = \psi(1)$, as required. \square

We define \mathcal{C} to be the union of conjugacy classes of \tilde{G} which have a representative in \tilde{L} .

Theorem 3.3. *The set of irreducible spin characters of \tilde{G} is a union of \mathcal{C} -blocks, with \mathcal{C} -basic set the set B of spin characters labeled by \bar{p} -quotients of the form $q = (\emptyset, q^1, \dots, q^{(p-1)/2})$.*

Proof. Let χ be a non-spin character of \tilde{G} and ψ be a spin character of \tilde{G} . For any $g \in \tilde{G}$, one has $\chi(g) = \chi(zg)$ and $\psi(g) = -\psi(zg)$. Also, since $z \in \tilde{L}$, we have that $\mathcal{C} = z\mathcal{C}$. Thus, writing, for any $A \subseteq \tilde{G}$, $\langle \psi, \chi \rangle_A = \frac{1}{|A|} \sum_{x \in A} \psi(x) \overline{\chi(x)}$, we have

$$2|\tilde{G}| \langle \psi, \chi \rangle_{\mathcal{C}} = |\tilde{G}| (\langle \psi, \chi \rangle_{\mathcal{C}} + \langle \psi, \chi \rangle_{z\mathcal{C}}) = \sum_{g \in \mathcal{C}} (\psi(g) + \psi(zg)) \overline{\chi(g)} = 0.$$

This proves that spin and non-spin characters are orthogonal across \mathcal{C} , so that the set of spin characters is a union of \mathcal{C} -blocks, as defined by Külshammer, Olsson and Robinson in [23, §1]. By [9, Prop 2.14], this in turn implies that it is a union of \mathcal{C} -blocks, as defined in Section 1.

By Lemma 4.1 and Proposition 4.2 of [8], the set of irreducible characters of \tilde{G} with P in their kernel is a \mathcal{C} -basic set of \tilde{G} . The intersection of this basic set with

the set of spin characters of \tilde{G} , which is given by Proposition 3.2, is thus a \mathcal{C} -basic set for the set of spin characters of \tilde{G} . \square

Let $H = G \cap \mathcal{A}_{pw}$. By applying Clifford theory to $\tilde{H} \triangleleft \tilde{G}$, the irreducible spin characters of \tilde{H} are again labeled by the \bar{p} -quotients of w . Any \bar{p} -quotient q of w labels two characters φ_q^+ and φ_q^- of \tilde{H} when $\sigma(q) = 1$, and one character φ_q otherwise. Set $\mathcal{D} = \mathcal{C} \cap \tilde{H}$.

Theorem 3.4. *The set of irreducible spin characters of \tilde{H} is a union of \mathcal{D} -blocks, with \mathcal{D} -basic set the set B' of spin characters labeled by \bar{p} -quotients of the form $q = (\emptyset, q^1, \dots, q^{(p-1)/2})$.*

Proof. Since z lies in \mathcal{D} , we conclude as in Theorem 3.3 that the set of irreducible spin characters of \tilde{H} is a union of \mathcal{D} -blocks. Furthermore, a simple counting argument shows that

$$\tilde{H} = P \rtimes (\tilde{L} \cap \tilde{\mathcal{A}}_{pw}).$$

Let $q = (q^0, q^1, \dots, q^{(p-1)/2})$ be a \bar{p} -quotient of w . If $\sigma(q) = -1$, then $\varphi_q = \text{Res}_{\tilde{H}}^{\tilde{G}}(\psi_q^+)$ has P in its kernel if and only if ψ_q^+ does. If $\sigma(q) = 1$, then $\text{Res}_{\tilde{H}}^{\tilde{G}}(\psi_q) = \varphi_q^+ + \varphi_q^-$, with $\varphi_q^+(1) = \varphi_q^-(1) = \psi_q(1)/2$. Let $x \in P$. We have $\psi_q(x) = \psi_q(1)$ if and only if $\varphi_q^+(x) + \varphi_q^-(x) = \psi_q(1)$; since, by [20, Lemma 2.15], $|\varphi_q^+(x)| \leq \psi_q(1)/2$ and $|\varphi_q^-(x)| \leq \psi_q(1)/2$, this holds if and only if $\varphi_q^+(x) = \varphi_q^-(x) = \psi_q(1)/2$.

This proves that any spin character of \tilde{H} labeled by q has P in its kernel if and only if $q^0 = \emptyset$, and we can conclude as in the proof of Theorem 3.3. \square

Remark 3.5. The same arguments as in the proofs of Theorems 3.3 and 3.4 show that the set of characters of G with Q in their kernel is a $\theta(\mathcal{C})$ -basic set of G (which is the same as that found in [8]), and that the set of characters of H with Q in their kernel is a $\theta(\mathcal{D})$ -basic set of H .

4. PERFECT ISOMETRIES AND BASIC SETS FOR THE COVERING GROUPS

We start by recalling the definitions of perfect isometry and of Broué isometry, and we prove some of their properties which will be useful later.

Let G and G' be two finite groups, and \mathcal{C} and \mathcal{C}' unions of conjugacy classes of G and G' respectively. Let B (resp. B') be a union of \mathcal{C} -blocks of G (resp. of \mathcal{C}' -blocks of G'); see [9, p. 4]. An isometry $I : \mathbb{C}B \rightarrow \mathbb{C}B'$ such that $I(\mathbb{Z}B) = \mathbb{Z}B'$ is a perfect isometry if

$$(5) \quad I \circ \text{res}_{\mathcal{C}} = \text{res}_{\mathcal{C}'} \circ I.$$

Remark 4.1. Note the change of terminology. What we call “perfect isometry” in this paper is what is referred to as “generalized perfect isometry” in [9, §2.4].

Remark 4.2. If $I : \mathbb{C}B \rightarrow \mathbb{C}B'$ and $J : \mathbb{C}B' \rightarrow \mathbb{C}B''$ are perfect isometries, then it follows easily from (5) that $J \circ I : \mathbb{C}B \rightarrow \mathbb{C}B''$ is a perfect isometry. It also follows from (5) that $I^{-1} : \mathbb{C}B' \rightarrow \mathbb{C}B$ is a perfect isometry.

To I , we associate $\hat{I} \in \mathcal{C}(B \times B')$ as in [9, §2.3] which satisfies

$$(6) \quad \hat{I} = \sum_{\chi \in B} \bar{\chi} \otimes I(\chi).$$

Recall that (see [9, Eq (16)]), for any $\psi \in \mathbb{C} \text{Irr}(B)$ and $y \in G'$, we have

$$(7) \quad I(\psi)(y) = \frac{1}{|G|} \sum_{x \in G} \widehat{I}(x, y) \psi(x).$$

Using (6) and (7), an easy computation shows that, if B'' is a union of \mathcal{C}'' -blocks of G'' and $J : \mathbb{C}B' \rightarrow \mathbb{C}B''$ is a perfect isometry, then for any $(x, z) \in G \times G''$

$$(8) \quad \widehat{J \circ I}(x, z) = \frac{1}{|G'|} \sum_{y \in G'} \widehat{I}(x, y) \widehat{J}(y, z).$$

If p is a prime and (K, \mathcal{R}, k) is a splitting p -modular system for G and G' , and if \mathcal{C} and \mathcal{C}' are the sets of p -regular elements of G and G' , then I is a Broué isometry if the following two properties are satisfied.

- (i) For every $(x, x') \in G \times G'$, $\widehat{I}(x, x')$ lies in $|\mathbb{C}_G(x)|\mathcal{R} \cap |\mathbb{C}_{G'}(x')|\mathcal{R}$.
- (ii) If $\widehat{I}(x, x') \neq 0$, then x and x' are either both p -regular or both p -singular.

Note that, in this case, I is also a perfect isometry in the above sense since (ii) implies (5) by [9, Prop. 2.15].

Lemma 4.3. *If $I : \mathbb{C}B \rightarrow \mathbb{C}B'$ and $J : \mathbb{C}B' \rightarrow \mathbb{C}B''$ are Broué isometries, then $J \circ I : \mathbb{C}B \rightarrow \mathbb{C}B''$ is a Broué isometry.*

Proof. Take any $x \in G$ and $z \in G''$. First note that $y \mapsto \widehat{I}(x, y) \widehat{J}(y, z)$ is a class function of G' . Thus, by (8),

$$\frac{\widehat{J \circ I}(x, z)}{|\mathbb{C}_G(x)|} = \sum_{y \in G'/\sim} \frac{\widehat{I}(x, y)}{|\mathbb{C}_G(x)|} \frac{\widehat{J}(y, z)}{|\mathbb{C}_{G'}(y)|} \in \mathcal{R}.$$

A similar argument shows that $\widehat{J \circ I}(x, z) \in |\mathbb{C}_G(z)|\mathcal{R}$, whence $\widehat{J \circ I}$ satisfies (i). Now suppose $\widehat{J \circ I}(x, z) \neq 0$. Then, by (8), there exists $y \in G'$ such that $\widehat{I}(x, y) \neq 0$ and $\widehat{J}(y, z) \neq 0$; (ii) follows. \square

Remark 4.4. Given a Broué isometry I , a formula for $\widehat{I^{-1}}$ can be found in [9, Remark 2.12]. It easily implies that I^{-1} is a Broué isometry.

Theorem 4.5. *Let p be an odd prime and B be a p -block of $\widetilde{\mathfrak{S}}_n$ such that $\{\xi_\lambda^+, \xi_\lambda^-\} \subseteq B$ for some $\lambda \in \mathcal{D}_n^-$. Let $J : \mathbb{C}B \rightarrow \mathbb{C}B$ be the isometry given by $J(\xi_\lambda^+) = \xi_\lambda^-$, $J(\xi_\lambda^-) = \xi_\lambda^+$, and $J(\xi) = \xi$ if $\xi \notin \{\xi_\lambda^+, \xi_\lambda^-\}$. Then J is a Broué isometry.*

Proof. Recall (see [32]) that a set of conjugacy classes of $\widetilde{\mathfrak{S}}_n$ (the split classes) outside which all spin characters vanish can be labeled by the elements of $\mathcal{O}_n \cup \mathcal{D}_n^-$, where \mathcal{O}_n is the set of partitions of n in odd parts. Each $\pi \in \mathcal{O}_n \cup \mathcal{D}_n^-$ labels two conjugacy classes with representatives t_π and zt_π such that the following hold. For all $\pi \in \mathcal{O}_n$, $\xi_\lambda^+(t_\pi) = \xi_\lambda^-(t_\pi)$ and $\xi_\lambda^+(zt_\pi) = \xi_\lambda^-(zt_\pi)$. For all $\pi \in \mathcal{D}_n^-$,

$$\xi_\lambda^+(t_\pi) = \xi_\lambda^-(zt_\pi) = -\xi_\lambda^+(zt_\pi) = -\xi_\lambda^-(t_\pi) = \delta_{\pi\lambda} i^{(n-k+1)/2} \sqrt{z_\lambda/2},$$

where $\lambda = (\lambda_1 > \dots > \lambda_k > 0)$ and $z_\lambda = \lambda_1 \cdots \lambda_k$ (see [32]).

Note that, for $\pi \in \mathcal{O}_n \cup \mathcal{D}_n^-$, we have $\{t_\pi, zt_\pi\} = \theta^{-1}(\sigma_\pi)$, where $\theta : \widetilde{\mathfrak{S}}_n \rightarrow \mathfrak{S}_n$ is the natural projection, and σ_π has cycle type π . If $\pi = (1^{m_1}, 2^{m_2}, \dots)$, we thus have $|\mathbb{C}_{\widetilde{\mathfrak{S}}_n}(t_\pi)| = |\mathbb{C}_{\widetilde{\mathfrak{S}}_n}(zt_\pi)| = 2|\mathbb{C}_{\mathfrak{S}_n}(\sigma_\pi)| = 2\prod_{j \geq 1} j^{m_j} (m_j!)$. In particular, since $\lambda = (\lambda_1 > \dots > \lambda_k > 0)$, we have $|\mathbb{C}_{\widetilde{\mathfrak{S}}_n}(t_\lambda)| = |\mathbb{C}_{\widetilde{\mathfrak{S}}_n}(zt_\lambda)| = 2\lambda_1 \cdots \lambda_k = 2z_\lambda$.

Denote by I the identity on $\mathbb{C}B$. By [9, Theorem 4.21], I is a Broué isometry, whence satisfies properties (i) and (ii).

First note that $\widehat{J}(x, x')$ vanishes whenever x or x' doesn't belong to a split class of $\widetilde{\mathfrak{S}}_n$. Properties (i) and (ii) are thus immediate in this case, and it is enough to consider $x, x' \in \{t_\pi, zt_\pi \mid \pi \in \mathcal{O}_n \cup \mathcal{D}_n^-\}$. Also, because of the definition of J and the description of the values of ξ_λ^\pm given above, we see that, whenever x or x' belongs to $\{t_\pi, zt_\pi \mid \pi \in \mathcal{O}_n \cup \mathcal{D}_n^-\setminus \lambda\}$, we have $\widehat{J}(x, x') = \widehat{I}(x, x')$. Hence properties (i) and (ii) are also true for J in this case.

It is therefore enough to study properties (i) and (ii) for J in the case where $x, x' \in \{t_\lambda, zt_\lambda\}$. Note that, in this case, (ii) is automatically satisfied since t_λ and $z^\nu t_\lambda$ are both p -regular or both p -singular. It only remains to study property (i) in this case.

We will in fact consider $\widehat{J}(x, x') - \widehat{I}(x, x')$, where $x, x' \in \{t_\lambda, zt_\lambda\}$, and show that this lies in $|C_G(x)|\mathcal{R} \cap |C_G(x')|\mathcal{R}$. Since I does satisfy (i), this will show that J also does.

Take $\nu, \nu' \in \{0, 1\}$. Then, by (6),

$$\begin{aligned} \widehat{J}(z^\nu t_\lambda, z^{\nu'} t_\lambda) - \widehat{I}(z^\nu t_\lambda, z^{\nu'} t_\lambda) &= \left(\overline{\xi_\lambda^+(z^\nu t_\lambda)} \xi_\lambda^-(z^{\nu'} t_\lambda) + \overline{\xi_\lambda^-(z^\nu t_\lambda)} \xi_\lambda^+(z^{\nu'} t_\lambda) \right) \\ &\quad - \left(\overline{\xi_\lambda^+(z^{\nu'} t_\lambda)} \xi_\lambda^+(z^\nu t_\lambda) + \overline{\xi_\lambda^-(z^{\nu'} t_\lambda)} \xi_\lambda^-(z^\nu t_\lambda) \right). \end{aligned}$$

If $(n-k+1)/2$ is odd, then, by the above description, $\xi_\lambda^\pm(z^\nu t_\lambda), \xi_\lambda^\pm(z^{\nu'} t_\lambda) \in i\mathbb{R}$, and we have $\overline{\xi_\lambda^+(z^\nu t_\lambda)} = \xi_\lambda^-(z^\nu t_\lambda)$ and $\overline{\xi_\lambda^-(z^\nu t_\lambda)} = \xi_\lambda^+(z^\nu t_\lambda)$.

We thus obtain

$$\begin{aligned} \widehat{J}(z^\nu t_\lambda, z^{\nu'} t_\lambda) - \widehat{I}(z^\nu t_\lambda, z^{\nu'} t_\lambda) &= \left(\xi_\lambda^-(z^\nu t_\lambda) \xi_\lambda^-(z^{\nu'} t_\lambda) + \xi_\lambda^+(z^\nu t_\lambda) \xi_\lambda^+(z^{\nu'} t_\lambda) \right) \\ &\quad - \left(\xi_\lambda^-(z^\nu t_\lambda) \xi_\lambda^+(z^{\nu'} t_\lambda) + \xi_\lambda^+(z^\nu t_\lambda) \xi_\lambda^-(z^{\nu'} t_\lambda) \right) \\ &= \left(\xi_\lambda^+(z^\nu t_\lambda) - \xi_\lambda^-(z^\nu t_\lambda) \right) \left(\xi_\lambda^+(z^{\nu'} t_\lambda) - \xi_\lambda^-(z^{\nu'} t_\lambda) \right) \\ &= 2\xi_\lambda^+(z^\nu t_\lambda) 2\xi_\lambda^+(z^{\nu'} t_\lambda) \\ &= \pm 4 \left(\sqrt{z_\lambda/2} \right)^2 \\ &= \pm 2z_\lambda. \end{aligned}$$

Since $2z_\lambda = |C_{\widetilde{\mathfrak{S}}_n}(z^\nu t_\lambda)| = |C_{\widetilde{\mathfrak{S}}_n}(z^{\nu'} t_\lambda)|$, this shows that, in this case, (i) holds for \widehat{J} (and $x = z^\nu t_\lambda, x' = z^{\nu'} t_\lambda$).

If, on the other hand, $(n-k+1)/2$ is even, then $\xi_\lambda^\pm(z^\nu t_\lambda), \xi_\lambda^\pm(z^{\nu'} t_\lambda) \in \mathbb{R}$, and we have $\overline{\xi_\lambda^+(z^\nu t_\lambda)} = \xi_\lambda^+(z^\nu t_\lambda)$ and $\overline{\xi_\lambda^-(z^\nu t_\lambda)} = \xi_\lambda^-(z^\nu t_\lambda)$.

We thus obtain

$$\begin{aligned}
\widehat{J}(z^\nu t_\lambda, z^{\nu'} t_\lambda) - \widehat{I}(z^\nu t_\lambda, z^{\nu'} t_\lambda) &= \left(\xi_\lambda^+(z^\nu t_\lambda) \xi_\lambda^-(z^{\nu'} t_\lambda) + \xi_\lambda^-(z^\nu t_\lambda) \xi_\lambda^+(z^{\nu'} t_\lambda) \right) \\
&\quad - \left(\xi_\lambda^+(z^\nu t_\lambda) \xi_\lambda^+(z^{\nu'} t_\lambda) + \xi_\lambda^-(z^\nu t_\lambda) \xi_\lambda^-(z^{\nu'} t_\lambda) \right) \\
&= (\xi_\lambda^+(z^\nu t_\lambda) - \xi_\lambda^-(z^\nu t_\lambda)) (\xi_\lambda^-(z^{\nu'} t_\lambda) - \xi_\lambda^+(z^{\nu'} t_\lambda)) \\
&= 2\xi_\lambda^+(z^\nu t_\lambda)(-2)\xi_\lambda^+(z^{\nu'} t_\lambda) \\
&= \pm 4 \left(\sqrt{z_\lambda/2} \right)^2 \\
&= \pm 2z_\lambda.
\end{aligned}$$

Since $2z_\lambda = |\mathbb{C}_{\widetilde{\mathfrak{S}}_n}(z^\nu t_\lambda)| = |\mathbb{C}_{\widetilde{\mathfrak{S}}_n}(z^{\nu'} t_\lambda)|$, this shows that, in this case as well, (i) holds for \widehat{J} (and $x = z^\nu t_\lambda$, $x' = z^{\nu'} t_\lambda$). This completes the proof. \square

From now on, we fix an integer $n \geq 2$ and an odd prime p . If B is any p -block of $\widetilde{\mathfrak{S}}_n$, then B contains either no or only spin characters (as can be seen by evaluating the central characters on the central element z ; see also [31]). In the former case, B coincides with a p -block of \mathfrak{S}_n ; in the latter, we say that B is a *spin block*. Similarly, any p -block of $\widetilde{\mathfrak{A}}_n$ contains either no spin character, and coincides with a p -block of \mathfrak{A}_n , or only spin characters, and is then called a spin block. The spin blocks of $\widetilde{\mathfrak{S}}_n$ and $\widetilde{\mathfrak{A}}_n$ are described by the following:

Theorem 4.6 (Morris [27], Humphreys [19]). *Let χ and ψ be two spin characters of $\widetilde{\mathfrak{S}}_n$, or two spin characters of $\widetilde{\mathfrak{A}}_n$, labeled by bar partitions λ and μ respectively. Then χ is of p -defect 0 (and thus alone in its p -block) if and only if λ is a \bar{p} -core. If λ is not a \bar{p} -core, then χ and ψ belong to the same p -block if and only if $\lambda_{(\bar{p})} = \mu_{(\bar{p})}$.*

One can therefore define the \bar{p} -core of a spin block B and its \bar{p} -weight, as well as its *sign* $\sigma(B) = \sigma(\lambda_{(\bar{p})})$ (for any bar partition λ labeling some character $\chi \in B$).

Theorem 4.7. *Let B be any spin p -block of $\widetilde{\mathfrak{S}}_n$ of \bar{p} -weight $w > 0$. Let \widetilde{G} , \widetilde{H} , \mathcal{C} and \mathcal{D} be as in §3, and let $\delta_{\bar{p}}(\lambda)$ be the relative sign of a bar partition λ introduced by Morris and Olsson in [28].*

(i) *If $\sigma(B) = 1$, then $I : \mathbb{C}B \rightarrow \mathbb{C} \text{SI}(\widetilde{G})$ given by*

$$\begin{aligned}
I(\xi_\lambda) &= \delta_{\bar{p}}(\lambda)(-1)^{|\lambda^0|} \psi_{\lambda_{(\bar{p})}} \quad \text{if } \sigma(\lambda) = 1, \\
I(\xi_\lambda^\pm) &= \delta_{\bar{p}}(\lambda)(-1)^{|\lambda^0|} \psi_{\lambda_{(\bar{p})}}^\pm \quad \text{if } \sigma(\lambda) = -1,
\end{aligned}$$

is a perfect isometry with respect to p -regular elements of $\widetilde{\mathfrak{S}}_n$ and \mathcal{C} .

(ii) *If $\sigma(B) = -1$, then $I : \mathbb{C}B \rightarrow \mathbb{C} \text{SI}(\widetilde{H})$ given by*

$$\begin{aligned}
I(\xi_\lambda) &= \delta_{\bar{p}}(\lambda)(-1)^{|\lambda^0|} \varphi_{\lambda_{(\bar{p})}} \quad \text{if } \sigma(\lambda) = -1, \\
I(\xi_\lambda^\pm) &= \delta_{\bar{p}}(\lambda)(-1)^{|\lambda^0|} \varphi_{\lambda_{(\bar{p})}}^\pm \quad \text{if } \sigma(\lambda) = 1,
\end{aligned}$$

is a perfect isometry with respect to p -regular elements of $\widetilde{\mathfrak{S}}_n$ and \mathcal{D} .

Proof. (i) We use the same convention as [24, p. 796] to label the cyclic structure of elements of G . In particular, if $(\pi_0, \dots, \pi_{p-1})$ is the cyclic structure associated to an element of G (see also [21, §4.2]), then π_0 corresponds to the part with cyclic

product (see [21, 4.2.1]) of order p . Note that, with this convention, $\theta(\mathcal{C})$ is the set of elements with cyclic structure $(\pi_0, \dots, \pi_{p-1})$ such that $\pi_0 = \emptyset$.

Let B_0 be the principal spin p -block of $\tilde{\mathfrak{S}}_{pw}$ (that is, the spin p -block of $\tilde{\mathfrak{S}}_{pw}$ corresponding to the empty \bar{p} -core), and consider the isometry $I_0 : \mathbb{C}B_0 \rightarrow \mathbb{C}\text{SI}(\tilde{G})$ given by

$$\begin{aligned} I_0(\xi_\mu) &= \delta_{\bar{p}}(\mu)(-1)^{w(p^2-1)/8+w+|\mu^0|} \psi_{\mu(\bar{p})} & \text{if } \sigma(\mu) = 1, \\ I_0(\xi_\mu^\pm) &= \delta_{\bar{p}}(\mu)(-1)^{w(p^2-1)/8+w+|\mu^0|} \psi_{\mu(\bar{p})}^{\eta_{\bar{p}}(\mu)} & \text{if } \sigma(\mu) = -1, \end{aligned}$$

where

$$\eta_{\bar{p}}(\mu) = \begin{cases} \pm \delta_{\bar{p}}(\mu)(-1)^{w+\frac{p-1}{4}(|\mu^0|-\ell(\mu^0))} & \text{if } \sigma(\mu^0) = 1, \\ \pm \delta_{\bar{p}}(\mu)(-1)^{w+\frac{p-1}{4}(|\mu^0|-\ell(\mu^0)+1)} & \text{if } \sigma(\mu^0) = -1. \end{cases}$$

We first show that I_0 is a perfect isometry with respect to p -regular elements of $\tilde{\mathfrak{S}}_n$ and \mathcal{C} . By [9, Prop. 2.15], all we have to prove is that $\widehat{I}_0(x, x') = 0$ whenever x is a p -regular element of $\tilde{\mathfrak{S}}_n$ and $x' \notin \mathcal{C}$, or x is p -singular and $x' \in \mathcal{C}$. In order to prove this, we follow the strategy of the proof of [24, Thm 10.1]. First, we introduce the set C of elements $t \in \tilde{\mathfrak{S}}_n$ such that $\theta(t)$ has no cycles of length an odd multiple of p , as in [9, Eq. (60)], and the set C' of elements $t' \in \tilde{G}$ such that the cyclic structure of $\theta(t')$ is $(\pi_0, \dots, \pi_{p-1})$, with π_0 having only even parts, as in [24, §10]. In the proof of [24, Thm 10.1], Livesey shows that I_0 is a perfect isometry with respect to C and C' . Note that, although Livesey is assuming in [24] that $w < p$, this part of the proof runs through fine for general w . Hence [9, Prop. 2.15] implies that $\widehat{I}_0(x, x') = 0$ whenever $x \in C$ and $x' \notin C'$, or $x \notin C$ and $x' \in C$. To prove that I_0 is a perfect isometry with respect to the set of p -regular elements of $\tilde{\mathfrak{S}}_n$ and \mathcal{C} , it remains to show that $\widehat{I}_0(x, x') = 0$ for any p -regular element $x \in \tilde{\mathfrak{S}}_n$ and $x' \in C' \setminus \mathcal{C}$, and for any p -singular element $x \in C$ and $x' \in \mathcal{C}$. This last property is in fact already proved in the proof of [24, Thm 10.1]. Indeed, to show that I_0 satisfies Property (ii) of a Broué isometry when $p > w$, Livesey uses repeatedly the sole argument that the cycle structure of x' is $(\emptyset, \pi_1, \dots, \pi_{p-1})$ (which, by [24, Lemma 7.2], coincides, when $p > w$, with the fact that x' is p -regular). Hence I_0 is a perfect isometry with respect to p -regular elements of $\tilde{\mathfrak{S}}_n$ and \mathcal{C} , as claimed.

Using the fact that, if J is perfect isometry, then so is $-J$, we deduce that $I_1 = (-1)^{w(p^2-1)/8+w} I_0$ is again a perfect isometry with respect to p -regular elements of $\tilde{\mathfrak{S}}_n$ and \mathcal{C} . By [9, Thm 4.21], there exists a perfect isometry I_2 with respect to p -regular elements between B_0 and B that associates characters of B_0 and B labeled by bar partitions with the same \bar{p} -quotient. More precisely, if $\lambda(\bar{p}) = \mu(\bar{p})$, then $I_2(\xi_\mu) = \delta_{\bar{p}}(\mu) \delta_{\bar{p}}(\lambda) \xi_\lambda$ if $\sigma(\mu) = 1$, and $I_2(\{\xi_\mu^+, \xi_\mu^-\}) = \{\delta_{\bar{p}}(\mu) \delta_{\bar{p}}(\lambda) \xi_\lambda^+, \delta_{\bar{p}}(\mu) \delta_{\bar{p}}(\lambda) \xi_\lambda^-\}$ if $\sigma(\mu) = -1$. Note that, since $\sigma(B) = \sigma(B_0) = 1$, if $\lambda(\bar{p}) = \mu(\bar{p})$, then $\sigma(\lambda) = \sigma(\mu)$.

It follows from Remark 4.2 that I_1^{-1} is a perfect isometry, and that the composition $I_2 \circ I_1^{-1} : \text{SI}(\tilde{G}) \rightarrow \mathbb{C}B$ is a perfect isometry with respect to \mathcal{C} and p -regular elements of $\tilde{\mathfrak{S}}_n$. If $\sigma(\lambda) = 1$, then $I_2 \circ I_1^{-1}(\psi_{\lambda(\bar{p})}) = \delta_{\bar{p}}(\lambda)(-1)^{|\lambda^0|} \xi_\lambda$, while, if $\sigma(\lambda) = -1$, then $I_2 \circ I_1^{-1}(\{\psi_{\lambda(\bar{p})}^+, \psi_{\lambda(\bar{p})}^-\}) = \{\delta_{\bar{p}}(\lambda)(-1)^{|\lambda^0|} \xi_\lambda^+, \delta_{\bar{p}}(\lambda)(-1)^{|\lambda^0|} \xi_\lambda^-\}$.

Denote by Λ the set of partitions λ such that $\sigma(\lambda) = -1$ and $I_2 \circ I_1^{-1}(\psi_{\lambda(\bar{p})}^+) = \delta_{\bar{p}}(\lambda)(-1)^{|\lambda^0|} \xi_\lambda^-$. Then, for each $\lambda \in \Lambda$, we can define a Broué isometry $I_\lambda : \mathbb{C}B \rightarrow \mathbb{C}B$ as in Theorem 4.5. Composing all the I_λ 's ($\lambda \in \Lambda$) gives a perfect isometry

$I_3 : \mathbb{C}B \rightarrow \mathbb{C}B$ with respect to p -regular elements (by Remark 4.2), which is such that $I_3 \circ I_2 \circ I_1^{-1} : \text{SI}(\tilde{G}) \rightarrow \mathbb{C}B$ is given by $I_3 \circ I_2 \circ I_1^{-1}(\psi_{\lambda(\bar{p})}) = \delta_{\bar{p}}(\lambda)(-1)^{|\lambda^0|}\xi_\lambda$ if $\sigma(\lambda) = 1$ and $I_3 \circ I_2 \circ I_1^{-1}(\psi_{\lambda(\bar{p})}^\pm) = \delta_{\bar{p}}(\lambda)(-1)^{|\lambda^0|}\xi_\lambda^\pm$ if $\sigma(\lambda) = -1$. And, by Remark 4.2, $I_3 \circ I_2 \circ I_1^{-1}$ is a perfect isometry with respect to \mathcal{C} and p -regular elements of $\tilde{\mathfrak{S}}_n$. The result now follows from the fact that

$$(9) \quad I = (I_3 \circ I_2 \circ I_1^{-1})^{-1}.$$

(ii) Let B_0^* be the principal spin p -block of $\tilde{\mathcal{A}}_{pw}$, and define the isometry $I_A : \mathbb{C}B_0^* \rightarrow \mathbb{C}\text{SI}(\tilde{H})$

$$\begin{aligned} I_A(\zeta_\lambda) &= \delta_{\bar{p}}(\lambda)(-1)^{|\lambda^0|}\varphi_{\lambda(\bar{p})} \quad \text{if } \sigma(\lambda) = -1, \\ I_A(\zeta_\lambda^\pm) &= \delta_{\bar{p}}(\lambda)(-1)^{|\lambda^0|}\varphi_{\lambda(\bar{p})}^\pm \quad \text{if } \sigma(\lambda) = 1. \end{aligned}$$

For all $x \in \tilde{\mathcal{A}}_n$ and $x' \in \tilde{H}$, we set $F(x, x') = \hat{I}_A(x, x') - \frac{1}{2}\hat{J}(x, x')$, where $J = (I_3 \circ I_2 \circ I_1^{-1})^{-1} : \mathbb{C}B_0 \rightarrow \mathbb{C}\text{SI}(\tilde{G})$ is the perfect isometry with respect to the set of p -regular elements of $\tilde{\mathfrak{S}}_n$ and \mathcal{C} given in Equation (9). For all $x \in \tilde{\mathcal{A}}_n$ and $x' \in \tilde{H}$, we have

$$F(x, x') = A(x, x') + B(x, x'),$$

where

$$A(x, x') = \sum_{\lambda \in \mathcal{D}_n^-} \delta_{\bar{p}}(\lambda)(-1)^{|\lambda^0|} \left(\overline{\zeta_\lambda(x)}\varphi_{\lambda(\bar{p})}(x') - \frac{1}{2} \left(\overline{\xi_\lambda^+(x)}\psi_{\lambda(\bar{p})}^+(x') + \overline{\xi_\lambda^-(x)}\psi_{\lambda(\bar{p})}^-(x') \right) \right)$$

and

$$B(x, x') = \sum_{\lambda \in \mathcal{D}_n^+} \delta_{\bar{p}}(\lambda)(-1)^{|\lambda^0|} \left(\left(\overline{\zeta_\lambda^+(x)}\varphi_{\lambda(\bar{p})}^+(x') + \overline{\zeta_\lambda^-(x)}\varphi_{\lambda(\bar{p})}^-(x') \right) - \frac{1}{2}\overline{\xi_\lambda(x)}\psi_{\lambda(\bar{p})}(x') \right).$$

Furthermore, for any $\lambda \in \mathcal{D}_n^-$ and x, x' as above, $\xi_\lambda^+(x) = \xi_\lambda^-(x) = \zeta_\lambda(x)$ and $\psi_{\lambda(\bar{p})}^+(x') = \psi_{\lambda(\bar{p})}^-(x') = \varphi_{\lambda(\bar{p})}(x')$. Hence, $\overline{\xi_\lambda^+(x)}\psi_{\lambda(\bar{p})}^+(x') + \overline{\xi_\lambda^-(x)}\psi_{\lambda(\bar{p})}^-(x') = 2\overline{\zeta_\lambda(x)}\varphi_{\lambda(\bar{p})}(x')$, and $A(x, x') = 0$.

Suppose now that $\lambda \in \mathcal{D}_n^+$ and that the cyclic structure of x is not of type λ . Then by [9, Equation (51)] $\zeta_\lambda^+(x) = \zeta_\lambda^-(x) = \frac{1}{2}\xi_\lambda(x)$, and for any $x' \in \tilde{H}$,

$$\overline{\zeta_\lambda^+(x)}\varphi_{\lambda(\bar{p})}^+(x') + \overline{\zeta_\lambda^-(x)}\varphi_{\lambda(\bar{p})}^-(x') = \frac{1}{2}\overline{\xi_\lambda(x)}(\varphi_{\lambda(\bar{p})}^+(x') + \varphi_{\lambda(\bar{p})}^-(x')) = \frac{1}{2}\overline{\xi_\lambda(x)}\psi_{\lambda(\bar{p})}(x').$$

Thus, $B(x, x') = 0$ except if x has cycle type λ for some $\lambda \in \mathcal{D}_n^+$. In this case, for all $x' \in \tilde{H}$,

$$\begin{aligned} B(x, x') &= \delta_{\bar{p}}(\lambda)(-1)^{|\lambda^0|} \left(\left(\overline{\zeta_\lambda^+(x)}\varphi_{\lambda(\bar{p})}^+(x') + \overline{\zeta_\lambda^-(x)}\varphi_{\lambda(\bar{p})}^-(x') \right) - \frac{1}{2}\overline{\xi_\lambda(x)}\psi_{\lambda(\bar{p})}(x') \right) \\ &= \frac{1}{2}\delta_{\bar{p}}(\lambda)(-1)^{|\lambda^0|} \left(\overline{\zeta_\lambda^+(x)} - \overline{\zeta_\lambda^-(x)} \right) (\varphi_{\lambda(\bar{p})}^+(x') - \varphi_{\lambda(\bar{p})}^-(x')), \end{aligned}$$

because $\xi_\lambda(x) = \zeta_\lambda^+(x) + \zeta_\lambda^-(x)$ and $\psi_{\lambda(\bar{p})}(x') = \varphi_{\lambda(\bar{p})}^+(x') + \varphi_{\lambda(\bar{p})}^-(x')$ by Clifford theory.

Denote by $\pi = (\pi_0, \dots, \pi_{p-1})$ the cyclic structure of x' . Assume that x is a p -regular element and $x' \notin \mathcal{D}$. Since x has type λ , it follows that $\lambda_0^{(\bar{p})} = \emptyset$ and $\ell(\lambda_0^{(\bar{p})}) = 0$, and $\pi_0 \neq \emptyset$. Furthermore, any conjugate of x also has type λ , thus we deduce from [24, Lemma 8.6] and the formula for character induction [20, (5.1)] that

$\varphi_{\lambda(\bar{p})}^+(x') - \varphi_{\lambda(\bar{p})}^-(x') = 0$ and $B(x, x') = 0$. Assume now x is p -singular and $x' \in \mathcal{D}$, that is $\pi_0 = \emptyset$. Then $\ell(\lambda_0(\bar{p})) > 0$. Again, the formula for character induction and the proof of [24, Lemma 8.9] give $\varphi_{\lambda(\bar{p})}^+(x') - \varphi_{\lambda(\bar{p})}^-(x') = 0$. Hence, $B(x, x') = 0$.

Finally, we proved that if $F(x, x') \neq 0$, then x and x' are either both p -regular elements and in \mathcal{D} , or both p -singular and not in \mathcal{D} . Since \widehat{J} has the same property by part (i) of the theorem, we deduce that this is also the case for \widehat{I}_A . By [9, Prop. 2.15], we deduce that I_A is a perfect isometry with respect to the set of p -regular elements of $\widetilde{\mathcal{A}}_n$ and \mathcal{D} .

The result now follows from the same process as in part (i). The analogue of the perfect isometry I_2 of case (i) is the perfect isometry between B_0^* and B given in [9, Thm 4.21]. \square

Remark 4.8. If $w < p$, then Livesey shows that the isometry I_0 of the proof is a Broué isometry. Hence using [9, Thm 4.21], Lemma 4.3, Remark 4.4 and Theorem 4.5, shows that the isometry given in Theorem 4.7(i) is a Broué isometry. Furthermore, still in the case where $w < p$, the argument given by Livesey in the proof of [24, Thm. 10.2] carries forward to show that F , and thus \widehat{I}_A , satisfy Property (i) of a Broué isometry. Since \widehat{I}_A also satisfies Property (ii), this similarly implies that the isometry given in Theorem 4.7(ii) is a Broué isometry.

Corollary 4.9. *If B is any spin p -block of $\widetilde{\mathfrak{S}}_n$ or of $\widetilde{\mathcal{A}}_n$ of \bar{p} -weight w , then the set of characters of B labeled by bar partitions λ such that $\lambda^0 = \emptyset$ is a p -basic set of B .*

Proof. In the case of $\widetilde{\mathfrak{S}}_n$, this follows from Theorem 4.7, Theorems 3.3 and 3.4, and [8, Prop 2.2] when $w > 0$. If $w = 0$, then B contains a single character and the result is obvious. If B is spin p -block of $\widetilde{\mathcal{A}}_n$, then [9, Thm 4.21] provides a perfect isometry from B to a spin p -block of same \bar{p} -weight w of some $\widetilde{\mathfrak{S}}_m$, and which preserves the labeling of characters. The result then follows from the first part of the proof and [8, Prop 2.2]. \square

Any spin p -block of $\widetilde{\mathfrak{S}}_n$ or of $\widetilde{\mathcal{A}}_n$ has a p -basic set by Corollary 4.9, and any other block has a p -basic set by [8]. Hence, $\widetilde{\mathfrak{S}}_n$ and $\widetilde{\mathcal{A}}_n$ have p -basic sets, which proves Theorem 1.1.

Remark 4.10. Recall that \mathfrak{S}_n has another, non-isomorphic, double covering group, denoted by $\widehat{\mathfrak{S}}_n$. In fact, we can derive from [26] that $\widehat{\mathfrak{S}}_n \subset \mathbb{C}\widehat{\mathfrak{S}}_n$. More precisely, there is a bijection $\varphi : \widetilde{\mathfrak{S}}_n \rightarrow \widehat{\mathfrak{S}}_n$, $t \mapsto r(t)t$ where $r(t)$ is some 4 th-root of unity depending on the sign of the permutation $\theta(t)$. The bijection φ yields the inclusion $\widehat{\mathfrak{S}}_n \subset \mathbb{C}\widehat{\mathfrak{S}}_n$, which in turn implies that the two group algebras coincide, that is $\mathcal{A} = \mathbb{C}\widehat{\mathfrak{S}}_n = \mathbb{C}\widetilde{\mathfrak{S}}_n$. In particular, to any irreducible representation $\rho : \mathcal{A} \rightarrow \text{End}(V)$, we can associate irreducible representations $\rho|_{\widetilde{\mathfrak{S}}_n}$ and $\rho|_{\widehat{\mathfrak{S}}_n}$ of $\widetilde{\mathfrak{S}}_n$ and $\widehat{\mathfrak{S}}_n$ with characters $\chi_{\widetilde{\mathfrak{S}}_n}$ and $\chi_{\widehat{\mathfrak{S}}_n}$, respectively. Consider $\Psi : \mathbb{C}\text{Irr}(\widetilde{\mathfrak{S}}_n) \rightarrow \mathbb{C}\text{Irr}(\widehat{\mathfrak{S}}_n)$ defined by $\chi_{\widetilde{\mathfrak{S}}_n} \mapsto \chi_{\widehat{\mathfrak{S}}_n}$. Then for all $t \in \widetilde{\mathfrak{S}}_n$, we have

$$\Psi(\chi_{\widetilde{\mathfrak{S}}_n})(\varphi(t)) = \chi_{\widehat{\mathfrak{S}}_n}(\varphi(t)) = \text{tr}(\rho(\varphi(t)), V) = r(t) \text{tr}(\rho(t), V) = r(t)\chi_{\widetilde{\mathfrak{S}}_n}(t).$$

By linearity, we obtain that, for all $\chi \in \mathbb{C}\text{Irr}(\widetilde{\mathfrak{S}}_n)$ and all $t \in \widetilde{\mathfrak{S}}_n$,

$$\Psi(\chi)(\varphi(t)) = r(t)\chi(t).$$

Let p be an odd prime number. Let $t \in \widetilde{\mathfrak{S}}_n$. Since $r(t)$ is a 2-element, t is p -regular if and only if $\varphi(t)$ is, and we obtain from this last equality and the definition of $\chi \mapsto \widehat{\chi}$ that, for all $\chi \in \text{Irr}(\widetilde{\mathfrak{S}}_n)$,

$$\widehat{\Psi(\chi)} = \Psi(\widehat{\chi}).$$

Hence Ψ is a perfect isometry with respect to the p -regular elements of $\widetilde{\mathfrak{S}}_n$ and $\widehat{\mathfrak{S}}_n$, and by [8, Prop 2.2], $\widetilde{\mathfrak{S}}_n$ has a p -basic set if and only if $\widehat{\mathfrak{S}}_n$ has one.

5. SOME FURTHER REMARKS

First notice that, by construction, the p -basic set b_S of $\widetilde{\mathfrak{S}}_n$ we produce is stable under tensor product by ε . Furthermore, when restricted to $\widetilde{\mathcal{A}}_n$, it gives our p -basic set b_A of $\widetilde{\mathcal{A}}_n$, that is, the elements of b_A are exactly the irreducible constituents of the restrictions to $\widetilde{\mathcal{A}}_n$ of the elements of b_S .

By Remark 3.5, and using the perfect isometry (constructed in [9, Thm 5.12]) between a p -block of \mathcal{A}_n and $\text{Irr}(H)$ with respect to p -regular elements of \mathcal{A}_n and $\theta(\mathcal{D})$, one can directly obtain the p -basic set of \mathcal{A}_n constructed in [8, Thm 5.2]. The advantage of this new method is that we need not worry about Property (2) in [8, Thm 1.1] which is automatically satisfied.

If B is any spin p -block of $\widetilde{\mathfrak{S}}_n$ or $\widetilde{\mathcal{A}}_n$, then the characters in the p -basic set b of B obtained in Corollary 4.9 are labeled by bar partitions λ such that $\lambda^{(\bar{p})} = (\lambda^0 = \emptyset, \lambda^1, \dots, \lambda^{(p-1)/2})$, or, equivalently, by the $(p-1)/2$ -quotients of w . Furthermore, for each such λ , one has

$$\sigma(\lambda^{(\bar{p})}) = (-1)^{w-\ell(\lambda^0)} = (-1)^w,$$

so that all quotients label two spin characters of $\widetilde{\mathfrak{S}}_n$ and one of $\widetilde{\mathcal{A}}_n$ if w is odd and $\sigma(B) = 1$, or if w is even and $\sigma(B) = -1$, and one spin character of $\widetilde{\mathfrak{S}}_n$ and two of $\widetilde{\mathcal{A}}_n$ if w is odd and $\sigma(B) = -1$, or if w is even and $\sigma(B) = 1$. In particular, all the characters in b are self-associate, or none is. Since b is ε -stable, the proof of [8, Prop 6.1] implies that the same is true for the set of irreducible Brauer characters in B , that is, all are self-associate or none is. As a consequence, we also obtain a formula for the number of irreducible Brauer characters in B . We therefore recover results of Olsson (see [30, Prop (3.21) and (3.22)]) obtained by sophisticated methods.

We end this paper with some considerations about what we like to call *Olsson's Principle*. Roughly speaking, Olsson's Principle predicts that, for any statement which holds for \mathfrak{S}_n , an analogous spin statement must hold for $\widetilde{\mathfrak{S}}_n$, even though one might have to be careful with signs along the proof.

We now present a few illustrations of this principle which appear in our work. Recall that the irreducible characters of \mathfrak{S}_n are parametrized by the set of partitions of n . For any prime p , a partition λ of n is uniquely determined by its p -core $\lambda_{(p)}$ and its p -quotient $\lambda^{(p)} = (\lambda^1, \dots, \lambda^p)$ which is a p -tuple of partitions whose sizes add up to w , the p -weight of λ . The celebrated *Nakayama Conjecture* states that two irreducible characters χ_λ and χ_μ of \mathfrak{S}_n belong to the same p -block if and only if $\lambda_{(p)} = \mu_{(p)}$. The spin analogue is given by the *Morris Conjecture* (Theorem 4.6). Now, the p -basic set of \mathfrak{S}_n constructed in [8] consists of those characters which are labeled by partitions λ with $\lambda^{(p+1)/2} = \emptyset$, *i.e.* without symmetric diagonal (p) -hooks. The spin analogue of this is the set of bar partitions λ of n such that $\lambda^0 = \emptyset$, *i.e.* without parts divisible by p . The link between these two properties is

enlightened by the *doubling* construction presented in [29]. Via this process, a bar partition λ of n is associated to a partition $D(\lambda)$ of $2n$ such that $\lambda^0 = \emptyset$ if and only if $D(\lambda)^{(p+1)/2} = \emptyset$.

The way the irreducible spin characters of \tilde{G} can be labeled by \bar{p} -quotients of w , just like the irreducible characters of G are labeled by p -quotients of w is another example of Olsson's Principle. It should therefore not come as a surprise that the perfect isometry of Theorem 4.7 (which associates spin characters of $\tilde{\mathfrak{S}}_n$ and \tilde{G} corresponding to the same \bar{p} -quotient of w), is precisely the spin analogue of the perfect isometry of [8, Thm 3.6] (which associates characters of \mathfrak{S}_n and G corresponding to the same p -quotient of w), including the sign $\delta_{\bar{p}}(\lambda)(-1)^{|\lambda^0|}$ which is the spin analogue of $\delta_p(\lambda)(-1)^{|\lambda^{(p+1)/2}|}$.

By the same token, the basic set of $\text{SI}(\tilde{G})$ we construct in Theorem 3.3 is the spin analogue of that of G given in [8, Thm 4.3].

It should be noted that, while Olsson's Principle can be an inspiration for results about $\tilde{\mathfrak{S}}_n$, it is by no means a proof. However, it can be a useful guide to indicate what the proof should be. This is indeed the narrative behind the sequence given by [9], [24] and the present paper to generalize to the spin case the proof of [8], whose result's spin analogue was inspired to the authors by Olsson's Principle several years ago.

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