MINIMAL PRIMAL IDEALS IN THE INNER CORONA ALGEBRA OF A
$C_0(X)$-ALGEBRA

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Abstract. Let $A = C(X) \otimes K(H)$, where $X$ is an infinite compact Hausdorff space and
$K(H)$ is the algebra of compact operators on a separable, infinite-dimensional Hilbert space.
Let $A^\ast$ be the norm-closed ideal of the multiplier algebra $M(A)$ consisting of all the strong*-continuous
functions from $X$ to $K(H)$. Then $A^\ast/A$ is the inner corona algebra of $A$. We identify the space MinPrimal($A^\ast$) of minimal closed primal ideals in $A^\ast$. If $A$ is separable then MinPrimal($A^\ast$) is compact and extremally disconnected. Using ultrapowers, we exhibit
a faithful family of irreducible representations of $A^\ast/A$ and hence show that if every point
of $X$ lies in the boundary of a zero set (i.e. if $X$ has no P-points) then the minimal closed
primal ideals of $A^\ast/A$ are precisely the images under the quotient map of the minimal closed
primal ideals of $A^\ast$. The map between MinPrimal($A^\ast$) and MinPrimal($A^\ast/A$) need not be
continuous, however, and MinPrimal($A^\ast/A$) is not weakly Lindelof. As an application, it is
shown that if $X = \beta N \setminus N$ then the relation of inseparability on Prim($A^\ast/A$) is an equivalence
relation but not an open equivalence relation.

2010 Mathematics Subject Classification: 46L05, 46L08, 46L45 (primary); 46E25,

1. Introduction

Let $A = C_0(X) \otimes K(H) \cong C_0(X,K(H))$, where $C_0(X)$ is the C*-algebra of complex-valued functions vanishing at infinity on an infinite locally compact Hausdorff space $X$ and
$K(H)$ is the algebra of compact operators on a separable infinite-dimensional Hilbert space
$H$. Then it is well known that $M(A)$, the multiplier algebra of $A$, is isomorphic to the
algebra of bounded strong*-continuous functions from $X$ to $B(H)$, the algebra of bounded
operators on $H$ [1]. In investigating the ideal structure of $M(A)$, and of the corona algebra
$C(A) = M(A)/A$, a natural ideal of $M(A)$ to consider is $A^\ast$, the algebra of bounded strong*-continuous functions from $X$ to $K(H)$. Since $X$ is infinite, $A^\ast \neq A$ (see [12, Theorems 3.3 and 3.7] for more general results).

A general study of the ideal structure of $A^\ast/A$ in [12] exposed something of the complexity;
and the purpose of this present paper is to continue this study by identifying the set of
minimal closed primal ideals in $A^\ast/A$ in the case when $A$ is $\sigma$-unital (equivalently, when $X$
is $\sigma$-compact). We also say a little about the $\tau_w$-topology (see below) on this set of ideals,
but this seems to be a difficult subject. The reason for trying to identify the minimal closed
primal ideals is that they tend to form an accessible family in the ideal lattice of a C*-algebra,
and they are closely related to the primitive ideals (in particular, the minimal closed primal
ideals are the minimal elements in the $\tau_w$-closure of the set Prim($B$) of primitive ideals of a
C*-algebra $B$ [3, Proposition 3.1]).

Recall that if $A$ is a ring and $I$ an ideal in $A$ then $I$ is primal if whenever $J_1, J_2, \ldots, J_n$ is a
finite family of ideals of $A$ with $J_1 J_2 \ldots J_n = \{0\}$ then $J_i \subseteq I$ for at least one $i$. If $I$ is a closed
ideal of a C*-algebra $A$ then $\text{Prim}(A/I)$ can be canonically identified with the closed subset \{\(P \in \text{Prim}(A) : P \supseteq I\)\} of $\text{Prim}(A)$ (with the hull-kernel topology) and it is well-known that $I$ is primal if and only if $\text{Prim}(A/I)$ is contained in a limit set in $\text{Prim}(A)$ (cf. [4, Proposition 3.2]). Every prime ideal (and hence every primitive ideal) of a C*-algebra $A$ is primal, and every ideal which contains a primal ideal is primal. A Zorn’s Lemma argument shows that every closed primal ideal contains a minimal closed primal ideal. Let $\text{MinPrimal}(A)$ denote the set of minimal closed primal ideals of $A$. The topology $\tau_w$ is defined on the set $\text{Id}(A)$ of all closed ideals of $A$ by taking sets of the form \{\(I \in \text{Id}(A) : a \notin I\)\} (\(a \in A\)) as sub-basic (see [3, p.525] where an equivalent definition is given). On $\text{Prim}(A)$, $\tau_w$ coincides with the hull-kernel topology, and on $\text{MinPrimal}(A)$, $\tau_w$ is a Hausdorff topology [3, Corollary 4.3].

Where possible we work in the context of a general $C_0(X)$-algebra $A$ (see the definition in Section 2), but for most of the results we have to impose some restrictions. The motivating example is $A = C_0(X) \otimes K(H)$, and we now describe the main results in this case.

In Section 3 we identify $\text{MinPrimal}(A^*)$ in the case where $A$ is $\sigma$-unital (Theorem 3.3), and we show furthermore that if $A$ is separable then $\text{MinPrimal}(A^*)$ is $\tau_w$-compact and extremely disconnected (Corollary 3.6). Our identification builds on that already established in [11] for $\text{MinPrimal}(M(A))$.

In Section 4 we use ultrapowers to exhibit a faithful family of irreducible representations of $A^*/A$ in the case where $A$ is $\sigma$-unital (Theorem 4.7). This enables us to determine, in Section 5, the set of minimal closed primal ideals of $A^*/A$. We show that if $X$ has no P-points (recall that a P-point is one which does not lie in the boundary of any zero set) then the minimal closed primal ideals of $A^*/A$ are precisely the images of the minimal closed primal ideals of $A^*$ under the quotient map (Theorem 5.6).

In Section 6 we investigate the topology on $\text{MinPrimal}(A^*/A)$, showing that while this can be described in some fairly simple cases it looks intractable in general. For example, if $A$ is separable and $X$ has no isolated points then $\text{MinPrimal}(A^*/A)$ is not weakly Lindelof, and hence is certainly not homeomorphic to $\text{MinPrimal}(A^*)$ (Theorem 6.5). Finally we study the case where $X$ is an F-space without isolated points, such as $\beta N \setminus N$ (recall that a completely regular topological space $Y$ is an F-space if disjoint cozero sets in $Y$ are contained in disjoint zero sets, see [17, 3.1, 14N4, 1.15 and 14.27]). We show, using $\text{MinPrimal}(A^*/A)$ and its topology, that the relation $\sim$ of inseparability by disjoint open sets on $\text{Prim}(A^*/A)$ is an equivalence relation on $\text{Prim}(A^*/A)$ (Theorem 6.8) but is not an open equivalence relation (Theorem 6.9).

We are grateful to the referee for several helpful comments.

2. Preliminaries

We begin by collecting some of the information that we need on $C_0(X)$-algebras. Recall that a C*-algebra $A$ is a $C_0(X)$-algebra if there is a continuous map $\phi$, called the base map, from $\text{Prim}(A)$ to the locally compact Hausdorff space $X$ [32, Proposition C.5]. We will use $X_\phi$ to denote the image of $\phi$ in $X$. Then $X_\phi$ is completely regular; and if $A$ is $\sigma$-unital, $X_\phi$ is $\sigma$-compact and hence normal [8, Section 1].

For $x \in X_\phi$, set $J_x = \bigcap \{P \in \text{Prim}(A) : \phi(P) = x\}$, and for $x \in X \setminus X_\phi$, set $J_x = A$. For $a \in A$, the function $x \to \|a + J_x\|$ ($x \in X$) is upper semi-continuous [32, Proposition C.10]. The $C_0(X)$-algebra $A$ is said to be continuous if, for all $a \in A$, the norm function
$x \to \|a + J_x\|$ ($x \in X$) is continuous. By Lee’s theorem [32, Proposition C.10 and Theorem C.26], this happens if and only if the base map $\phi$ is open.

One case of special importance (through which all other cases factor) is when the base map $\phi$ is the complete regularization map

$$\phi_A : \text{Prim}(A) \to \text{Glimm}(A) \subseteq \beta(\text{Glimm}(A))$$

(see [6, Section 2] for the identification of the complete regularization of $\text{Prim}(A)$ with the space $\text{Glimm}(A)$ of Glimm ideals of $A$). If $\text{Glimm}(A)$ is locally compact (for the complete regularization topology $\tau_{\phi}$) then one may take $X = X_\phi = \text{Glimm}(A)$; otherwise, one may take $X = \beta(\text{Glimm}(A))$.

In this setting, if $x \in X_\phi = \text{Glimm}(A)$ then the ideal $J_x$ coincides with the Glimm ideal $x$.

Let $J$ be a proper, closed, two-sided ideal of a $C^*$-algebra $A$. The quotient map $q_J : A \to A/J$ has a canonical extension $\hat{q}_J : M(A) \to M(A/J)$ such that $\hat{q}_J(b)q_J(a) = q_J(ba)$ and $q_J(a)\hat{q}_J(b) = q_J(ab)$ ($a \in A, b \in M(A)$). We define a proper, closed, two-sided ideal $\tilde{J}$ of $M(A)$ by

$$\tilde{J} = \ker \hat{q}_J = \{b \in M(A) : ba, ab \in J \text{ for all } a \in A\}.$$ 

The following proposition was proved in [7, Proposition 1.1].

**Proposition 2.1.** Let $J$ be a proper, closed, two-sided ideal of a $C^*$-algebra $A$. Then

(i) $\tilde{J}$ is the strict closure of $J$ in $M(A)$;

(ii) $\tilde{J} \cap A = J$;

(iii) if $P \in \text{Prim}(A)$ then $\tilde{P}$ is primitive (and hence is the unique ideal in $\text{Prim}(M(A))$ whose intersection with $A$ is $P$);

(iv) $\tilde{J} = \bigcap \{\tilde{P} : P \in \text{Prim}(A) \text{ and } P \supseteq J\}$ and for all $b \in M(A)$

$$\|b + \tilde{J}\| = \sup\{\|b + \tilde{P}\| : P \in \text{Prim}(A) \text{ and } P \supseteq J\};$$

(v) $(A + \tilde{J})/\tilde{J}$ is an essential ideal in $M(A)/\tilde{J}$.

Furthermore, the map $P \mapsto \tilde{P}$ ($P \in \text{Prim}(A)$) maps $\text{Prim}(A)$ homeomorphically onto a dense, open subset of $\text{Prim}(M(A))$ [26, 4.1.10].

The next proposition was proved in [7, Proposition 1.2].

**Proposition 2.2.** Let $A$ be a $C_0(X)$-algebra with base map $\phi$. Then $\phi$ has a unique extension to a continuous map $\overline{\phi} : \text{Prim}(M(A)) \to \beta X$ such that $\overline{\phi}(\tilde{P}) = \phi(P)$ for all $P \in \text{Prim}(A)$. Hence $M(A)$ is a $C(\beta X)$-algebra with base map $\overline{\phi}$ and $\text{Im}(\overline{\phi}) = \text{cl}_{\beta X}(X_\phi)$.

Now let $A$ be a $C_0(X)$-algebra with base map $\phi$ and let $\overline{\phi} : \text{Prim}(M(A)) \to \beta X$ be as in Proposition 2.2. For $x \in \beta X$, we define $H_x = \bigcap \{Q \in \text{Prim}(M(A)) : \overline{\phi}(Q) = x\}$, a closed two-sided ideal of $M(A)$. Thus $H_x$ is defined in relation to $(M(A), \beta X, \overline{\phi})$ in the same way that $J_x$ (for $x \in X$) is defined in relation to $(A, X, \phi)$. It follows that for each $b \in M(A)$, the function $x \to \|b + H_x\|$ ($x \in \beta X$) is upper semi-continuous.

The next proposition was proved in [8, Proposition 1.3].

**Proposition 2.3.** Let $A$ be a $C_0(X)$-algebra with base map $\phi$, and set $X_\phi = \text{Im}(\phi)$.

(i) For all $x \in X$, $J_x \subseteq H_x \subseteq \tilde{J}_x$ and $J_x = H_x \cap A$.

(ii) For all $x \in X$, $H_x$ is strictly closed if and only if $H_x = \tilde{J}_x$. 
(iii) For all \( b \in M(A) \), \( \|b\| = \sup\{\|b + \tilde{J}_x\| : x \in X_\phi\} = \sup\{\|b + H_x\| : x \in X_\phi\} \).

In the case where \( A = C_0(X) \otimes K(H) \) for a locally compact Hausdorff space \( X \), we shall assume that \( \phi : \text{Prim}(A) \to X \) is the canonical homeomorphism such that

\[
\phi^{-1}(x) = \{ f \in C_0(X) : f(x) = 0 \} \otimes K(H) \quad (x \in X).
\]

Then it follows from the definition of \( \tilde{J} \) above that

\[
\tilde{J}_x = \{ f \in M(A) : f(x) = 0 \},
\]

On the other hand, by [7, Lemma 1.5(ii)],

\[
H_x = \{ f \in M(A) : \|f(x_\alpha)\| \to 0 \text{ as } x_\alpha \to x \}.
\]

We now give the definition of the ideal \( A^* \) of \( M(A) \). Let \( A \) be a \( C_0(X) \)-algebra with base map \( \phi \), and for \( x \in X_\phi \) set \( N_x = A + \tilde{J}_x \). Define \( A^* = \bigcap_{x \in X_\phi} N_x \). Then \( A^* \) is a closed two-sided ideal in \( M(A) \) and \( A^* \supseteq A \). Clearly \( A^* \) depends on the particular way in which \( A \) is represented as a \( C_0(X) \)-algebra (there may be many continuous maps from \( \text{Prim}(A) \) to \( X \) in general). If \( A = C_0(X) \otimes K(H) \) for a locally compact Hausdorff space \( X \), then since \( \tilde{J}_x = \{ f \in M(A) : f(x) = 0 \} \), we have that \( N_x = \{ f \in M(A) : f(x) \in K(H) \} \). Hence \( A^* \) is precisely the algebra of bounded strong*-continuous functions from \( X \) to \( K(H) \) referred to in the introduction. In this case, \( A^* \) contains the algebra of bounded norm-continuous functions from \( X \) to \( K(H) \). A generalization of the latter algebra has been studied in [12, Section 3] and in [24].

The next lemma was proved in [12, Lemma 3.2].

**Lemma 2.4.** Let \( A \) be a \( C_0(X) \)-algebra with base map \( \phi \) and let \( b \in M(A) \). Then \( b \in A \) if and only if

(i) for all \( \epsilon > 0 \) the set \( \{ x \in X_\phi : \|b + H_x\| \geq \epsilon \} \) is compact;

(ii) for all \( x \in X_\phi \) there exists \( a \in A \) such that \( b - a \in H_x \).

We now recall from [8, Lemma 5.6] a means of constructing elements which will be of considerable importance in this paper.

**Lemma 2.5.** Let \( A \) be a \( \sigma \)-unital, continuous \( C_0(X) \)-algebra with base map \( \phi \) and suppose that \( A/J_x \) is non-unital for all \( x \in X_\phi \). Then for each zero set \( Z \) in \( X_\phi \) there exists a positive element \( c^Z \in A^* \) such that

(i) \( \|c^Z + \tilde{J}_x\| = 0 \) for \( x \in Z \);

(ii) \( \|c^Z + \tilde{J}_x\| = 1 \) for \( x \in X_\phi \setminus Z \);

(iii) for all \( x \in X_\phi \setminus Z \) there is a neighbourhood \( V \) of \( x \) in \( X_\phi \) and an element \( a \in A \) such that \( c^Z - a \in H_y \) for all \( y \in V \).

It was not stated in [8, Lemma 5.6] that \( c^Z \) could be chosen positive, but the proof shows that this is the case.

Now let \( A \) be a \( C_0(X) \)-algebra with base map \( \phi \). If \( X = X_\phi \) (that is, \( \phi \) is surjective) then we already have the canonical extension \( \overline{\phi} : \text{Prim}(M(A)) \to \beta X_\phi \). If \( X \neq X_\phi \) then we may replace \( X \) by the compact Hausdorff space \( \beta X_\phi \) (see the discussion in [12, p. 302]) and so we have again \( \overline{\phi} : \text{Prim}(M(A)) \to \beta X_\phi \). In either case, with the usual identifications, we may consider \( \overline{\phi}|_{\text{Prim}(A^*)} \) and \( \overline{\phi}|_{\text{Prim}(A^*/A)} \). Denoting the latter of these maps by \( \psi \), we have that
$A^*/A$ is a $C(\beta X_0)$-algebra with base map $\psi$. The next result explains why the notion of a P-point crops up frequently in this paper. The assumption that $X_0$ is infinite ensures that $A^*$ strictly contains $A$ [12, p. 306]. Note that if the non-empty Hausdorff space $X_0$ contains no isolated points (as will be assumed in several results in Section 6) then it is automatically infinite.

**Theorem 2.6.** [12, Theorem 5.2] Let $A$ be a $\sigma$-unital, continuous $C_0(X)$-algebra with base map $\phi$ and suppose that $X_0$ is infinite and that $A/J_x$ is non-unital for all $x \in X_0$. Then $A^*/A$ is a non-trivial $C(\beta X_0)$-algebra with base map $\psi$, and $X_\psi = \beta X_0 \setminus W$ where $W$ is the set of $P$-points in $X_0$.

### 3. Minimal primal ideals in $A^*$

In this section we determine the space of minimal closed primal ideals of $A^*$ where $A$ is a $\sigma$-unital, quasi-standard $C^*$-algebra. We exploit the identification of $\text{MinPrimal}(M(A))$ already established in [11].

Recall that a $C^*$-algebra $A$ is **quasi-standard** if the relation $\sim$ of inseparability by disjoint open sets is an open equivalence relation on $\text{Prim}(A)$ [6]. This condition is a wide generalization of the special case when $\text{Prim}(A)$ is Hausdorff. If $A$ is quasi-standard then the complete regularization map $\phi_A$ is open [6, Theorem 3.3], so $\text{Glimm}(A)$ is locally compact and $A$ is a continuous $C_0(X)$-algebra with $X = X_{\phi_A} = \text{Glimm}(A)$. Furthermore each Glimm ideal of $A$ is primal and the topological spaces $\text{Glimm}(A)$ and $\text{MinPrimal}(A)$ coincide [6, Theorem 3.3]. Examples of quasi-standard $C^*$-algebras include von Neumann algebras, $\text{AW}^*$-algebras, local multiplier algebras of $C^*$-algebras [29], the group $C^*$-algebras of amenable discrete groups (and many other groups) [19], [5], and algebras of the form $C_0(X) \otimes K(H)$ where $X$ is a locally compact Hausdorff space.

Let $X$ be a completely regular topological space [17, 3.1] and let $C_R(X)$ denote the ring of continuous real-valued functions on $X$. For $f \in C_R(X)$, let

$$Z(f) = \{x \in X : f(x) = 0\},$$

the zero set of $f$. Note that every zero set clearly arises as the zero set of a bounded continuous function. The set of all zero sets of $X$ is denoted $Z[X]$. A non-empty family $\mathcal{F}$ of zero sets of $X$ is called a z-filter if: (i) $\mathcal{F}$ is closed under finite intersections; (ii) $\emptyset \notin \mathcal{F}$; (iii) each zero set which contains a member of $\mathcal{F}$ belongs to $\mathcal{F}$. Each ideal $I \subseteq C_R(X)$ yields a z-filter $Z[I] = \{Z(f) : f \in I\}$. An ideal $I$ is called a z-ideal if $Z(f) \in Z[I]$ implies $f \in I$; and if $\mathcal{F}$ is a z-filter on $X$ then the ideal $I(\mathcal{F})$ defined by

$$I(\mathcal{F}) = \{f \in C_R(X) : Z(f) \in \mathcal{F}\}$$

is a z-ideal. There is a bijective correspondence between the set of z-ideals of $C_R(X)$ and the set of z-filters on $X$, given by $I = I(Z[I]) \leftrightarrow Z[I]$ (see [17, Chapter 2]).

A z-filter $\mathcal{F}$ on a completely regular space $X$ is said to be prime if $Z_1 \cup Z_2 \in \mathcal{F}$ implies that either $Z_1 \in \mathcal{F}$ or $Z_2 \in \mathcal{F}$, for zero sets $Z_1$ and $Z_2$. Let $PF(X)$ denote the set of prime z-filters, and let $PZ(X)$ be the set of prime z-ideals (recall that an ideal $P \subseteq C_R(X)$ is prime if $fg \in P$ implies $f \in P$ or $g \in P$). The bijective correspondence between z-ideals and z-filters restricts to a bijective correspondence $j : PZ(X) \rightarrow PF(X)$ given by $j(P) = \{Z(f) : f \in P\}$ (see [17, Chapter 2]). Every z-ideal of $C_R(X)$ is an intersection of
prime $z$-ideals and the minimal prime ideals of $C_{\mathbb{R}}(X)$ are $z$-ideals [17, 2.8, 14.7]. The prime ideals containing a given prime ideal form a chain [17, 14.8].

Now let $A$ be a $\sigma$-unital $C_0(X)$-algebra with base map $\phi$, and let $u \in A$ be a strictly positive element. For $a \in A$, set $Z(a) = \{x \in X_\phi : a \in J_x\}$. Unless norm functions of elements of $A$ are continuous on $X_\phi$, $Z(a)$ will not necessarily be a zero set of $X_\phi$. However, since $Z(u) = \emptyset$ and $A$ is closed under multiplication by $C^b(X_\phi)$, every zero set $Z(f)$ of $X_\phi$ arises as $Z(a)$ for the element $a = f \cdot u \in A$ ($f \in C^b_\mathbb{R}(X_\phi)$). For $b \in M(A)$, set $Z(b) = \{x \in X_\phi : b \in \tilde{J}_x\}$. Note that if $b \in A$ then this definition is consistent with the previous one because $\tilde{J}_x \cap A = J_x$ ($x \in X_\phi$). It is also useful to note that for $b \in M(A)$ and $x \in X_\phi$, $b \in \tilde{J}_x$ if and only if $bu \in \tilde{J}_x$ if and only if $bu \notin J_x$. Hence $Z(b) = Z(bu)$, and this is a zero set in $X_\phi$ if $A$ is continuous.

For a $z$-filter $\mathcal{F}$ on $X_\phi$ define $L^\text{alg}_\mathcal{F} = \{b \in M(A) : \exists Z \in \mathcal{F}, Z(b) \supseteq Z\}$, and let $L_\mathcal{F}$ be the norm-closure of $L^\text{alg}_\mathcal{F}$ in $M(A)$. Let $b \in L^\text{alg}_\mathcal{F}$. Then for $a \in M(A)$, $Z(ab) \supseteq Z(b)$ and $Z(ba) \supseteq Z(b)$, while for $a \in L^\text{alg}_\mathcal{F}$, $Z(a + b) \supseteq Z(a) \cap Z(b)$. Hence $L^\text{alg}_\mathcal{F}$ is an ideal of $M(A)$, so $L_\mathcal{F}$ is a closed ideal of $M(A)$.

**Theorem 3.1.** [8, Theorem 3.2] Let $A$ be a $\sigma$-unital $C_0(X)$-algebra with base map $\phi$. Suppose that $A/J_x$ is non-unital for all $x \in X_\phi$. Let $I$ and $J$ be $z$-ideals of $C_0(X_\phi)$ and suppose that there exists a zero set $Z$ of $X_\phi$ such that $Z \supseteq Z[I]$ but $Z \notin Z(J)$. Then $L_{Z[I]} \not\subseteq L_{Z[J]}$. Hence the assignment $I \mapsto L_{Z[I]}$ defines an order-preserving injective map $L$ from the lattice of $z$-ideals of $C_{\mathbb{R}}(X_\phi)$ into the lattice of closed ideals of $M(A)$.

For $x \in X_\phi$, let $M_x$ be the maximal ideal of $C_{\mathbb{R}}(X_\phi)$ given by $M_x = \{f \in C_{\mathbb{R}}(X_\phi) : f(x) = 0\}$, and let $O_x = \{f \in C_{\mathbb{R}}(X_\phi) : x \in \text{int}(Z(f))\}$ where $\text{int}(Z(f))$ denotes the interior of $Z(f)$ in $X_\phi$. Then $M_x$ and $O_x$ are $z$-ideals, and $O_x$ is the smallest ideal of $C_{\mathbb{R}}(X_\phi)$ which is not contained in any maximal ideal other than $M_x$. It is useful to extend the definitions just given as follows. Let $c_{\beta X}X_\phi$ denote the closure of $X_\phi$ in $\beta X$. For $p \in c_{\beta X}X_\phi$, let $M^p = \{f \in C_{\mathbb{R}}(X_\phi) : p \in c_{\beta X}Z(f)\}$ and define $O^p$ to be the set of all $f \in C_{\mathbb{R}}(X_\phi)$ for which $c_{\beta X}Z(f)$ is a neighbourhood of $p$ in $c_{\beta X}X_\phi$. Then for $x \in X_\phi$, $M^x = M_x$ and $O^x = O_x$.

In the next result and in several subsequent results in this paper, we shall take $X$ to be the locally compact Hausdorff space $\text{Glim}(A)$ associated with a $\sigma$-unital quasi-standard $C^*$-algebra $A$. In this case, it should be understood that $\phi : \text{Prim}(A) \to X$ is the complete regularization map $\phi_A$. Thus $X = X_\phi$, $c_{\beta X}X_\phi = \beta X$ and the sets $M^p$ and $O^p$ defined above coincide with those occurring in [17, 7.3 and 7.12]. Now suppose that $P \in PZ(X_\phi) = PZ(X)$. Then there exists $p \in \beta X$ such that $O^p \subseteq P \subseteq M^p$ [17, 7.15]. Hence, if $A/J_x$ is non-unital for all $x \in X$, it follows from Theorem 3.1 and [8, Theorem 4.3] that $H_p \not\subseteq L_{Z[P]}$ and, if $p \in X$, $L_{Z[P]} \not\subseteq \tilde{J}_p$.

For a ring $R$ let $\text{Min}(R)$ be the space of minimal (algebraic) prime ideals of $R$ with the lower topology generated by sub-basic sets of the form

$$\{P \in \text{Min}(R) : a \notin P\}$$

as $a$ varies through elements of $R$. If $R$ is a commutative ring then an argument of Krull’s shows that every minimal prime ideal of $R$ is prime, and $\text{Min}(R)$ is the usual space of minimal prime ideals of $R$ with the hull-kernel topology, see [28] and the references given there.
Theorem 3.2. [11, Theorem 3.4] Let $A$ be a $\sigma$-unital, quasi-standard $C^*$-algebra with $A/G$ non-unital for all $G \in \text{Glimm}(A)$ and set $X = \text{Glimm}(A)$. Then the assignment $P \mapsto L_{Z[P]}$ defines a homeomorphism from $\text{Min}(C_{\mathbb{R}}(X))$ onto $\text{MinPrimal}(M(A))$.

For the next theorem, we need the following family of functions which is useful for relating $L_{\mathcal{F}}$ and $L^\text{alg}_{\mathcal{F}}$. For $0 < \epsilon < 1/2$, define the continuous piecewise linear function $f_\epsilon : [0, \infty) \to [0, \infty) \setminus c$ by: (i) $f_\epsilon(x) = 0$ (0 $\leq x \leq \epsilon$); (ii) $f_\epsilon(x) = 2(x - \epsilon)$ ($\epsilon < x \leq 2\epsilon$); (iii) $f_\epsilon(x) = x$ ($2\epsilon < x$). Note that for $b \in M(A)^+$, if $b \in L_{\mathcal{F}}$ then $f_\epsilon(b)$ belongs to the Pedersen ideal of $L_{\mathcal{F}}$ for all $\epsilon$ [26, 5.6.1], and hence $f_\epsilon(b) \in L^\text{alg}_{\mathcal{F}}$. On the other hand, $\|b - f_\epsilon(b)\| \leq \epsilon$. Thus we have that $b \in L_{\mathcal{F}}$ if and only if $f_\epsilon(b) \in L^\text{alg}_{\mathcal{F}}$ for all $\epsilon \in (0, 1/2)$, which is Lemma 3.3 of [11].

We can now give the description of $\text{MinPrimal}(A^*)$. For a $C^*$-algebra $A$, let $\text{Primal}(A)$ denote the set of closed primal ideals of $A$ and $\text{Primal}'(A)$ the set of proper closed primal ideals of $A$.

Theorem 3.3. Let $A$ be a $\sigma$-unital, quasi-standard $C^*$-algebra and set $X = \text{Glimm}(A)$. Suppose that $A/J_x$ is non-unital for all $x \in X$. Then the map $P \mapsto P \cap A^*$ is a homeomorphism from $\text{MinPrimal}(M(A))$ onto $\text{MinPrimal}(A^*)$.

Proof. Let $P \in \text{Primal}(M(A))$ and let $I_1, \ldots, I_n \in \text{Id}(A^*)$ with $I_1 \ldots I_n = \{0\}$. Then $I_i \subseteq P$ for some $1 \leq i \leq n$, by the primality of $P$, so $I_i \subseteq P \cap A^*$. Hence $P \cap A^* \in \text{Primal}(A^*)$. Now let $P,Q \in \text{MinPrimal}(M(A))$ with $P \neq Q$. Then by [11, Theorem 2.4] there exist distinct minimal prime $z$-filters $\mathcal{F}$ and $\mathcal{G}$ on $X$ such that $P = L_{\mathcal{F}}$ and $Q = L_{\mathcal{G}}$. Let $Z \in X$ with $Z \in \mathcal{F} \setminus \mathcal{G}$ and let $c_1^Z$ be the $z$-filter on $X$ with the properties of Lemma 2.5. Let $Z \in X$ with $Z \in X$ with $Z \in \mathcal{F} \setminus \mathcal{G}$ and let $c_1^Z$ be the $z$-filter on $X$ with the properties of Lemma 2.5. Let $Z \in X$ with $Z \in \mathcal{F} \setminus \mathcal{G}$ and let $c_1^Z$ be the $z$-filter on $X$ with the properties of Lemma 2.5. Hence $c_1^Z \in L^\text{alg}_{\mathcal{F}} \subseteq L_{\mathcal{F}}$ and $f_\epsilon(c_1^Z) \notin L^\text{alg}_{\mathcal{G}}$ for $0 < \epsilon < 1/2$, and thus $c_1^Z \notin L_{\mathcal{G}}$ by [11, Lemma 3.3]. Hence $P \cap A^* \neq Q \cap A^*$, and similarly $Q \cap A^* \neq P \cap A^*$.

Now let $R \in \text{Primal}'(A^*)$ and define $R \in \text{Primal}'(M(A))$ as follows. Let $W$ be the hull of $R$ in $\text{Prim}(A^*)$ and let $i(W)$ be the image of $W$ in $\text{Prim}(M(A))$ under the canonical injection $i$ from $\text{Prim}(A^*)$ to $\text{Prim}(M(A))$. Let $V$ be the closure of $i(W)$ in $\text{Prim}(M(A))$ and set $R = \text{ker} V$. Clearly $R \cap A^* = R$. Since $W$ is a limit set in $\text{Prim}(A^*)$ it follows that $V$ is a limit set in $\text{Prim}(M(A))$ so $R$ is primal. Hence there exists $P \in \text{MinPrimal}(M(A))$ such that $R \supseteq P$, so $R \cap A^* \supseteq P \cap A^*$. Thus it follows from this paragraph and the previous one that

$$\text{MinPrimal}(A^*) = \{P \cap A^* : P \in \text{MinPrimal}(M(A))\}.$$ 

Next we show that the map $P \mapsto P \cap A^*$ is a homeomorphism. Sets of the form $\{P \in \text{MinPrimal}(A^*) : a \notin P\}$ ($a \in A^*$) are sub-basic for the $\tau_{\omega}$-topology on $\text{MinPrimal}(A^*)$ and their inverse images are open in $\text{MinPrimal}(M(A))$ since

$$\{P \in \text{MinPrimal}(M(A)) : a \notin P \cap A^*\} = \{P \in \text{MinPrimal}(M(A)) : a \notin P\}.$$ 

Thus the map is continuous.

Now let $b \in M(A)^+$ and set $V = \{P \in \text{MinPrimal}(M(A)) : b \notin P\}$. We aim to show that $U := \{P \cap A^* : P \in V\}$ is $\tau_{\omega}$-open in $\text{MinPrimal}(A^*)$. Since $\|f_{1/n}(b) - b\| \leq 1/n$, we may write $V = \bigcup_{n \geq 2} \{P = L_{\mathcal{F}} \in \text{MinPrimal}(M(A)) : f_{1/n}(b) \notin L^\text{alg}_{\mathcal{F}}\}$. Let $n \geq 2$ and set $Z_n = Z(f_{1/n}(b))$. Let $c_1^{Z_n} \in A^*$ with the properties of Lemma 2.5. Then for a (minimal prime) $z$-filter $\mathcal{F}$, $c_1^{Z_n} \in L^\text{alg}_{\mathcal{F}}$ if and only if $Z_n \in \mathcal{F}$ if and only if $f_{1/n}(b) \in L^\text{alg}_{\mathcal{F}}$. But
also $c^{Z^n} \in L^\text{alg}_X$ if and only if $c^{Z^n} \in L_X$ by [11, Lemma 3.3] (since $Z(f,c^{Z^n}) = Z_n$ for all $0 < \epsilon < 1/2$). Thus we have that $U = \bigcup_{n \geq 2} \{Q = P \cap A^* \in \text{MinPrimal}(M(A)) : c^{Z^n} \notin Q\}$, a union of $\tau_w$-open sets. Thus $U$ is open and the map is a homeomorphism. 

Theorem 3.3 has various immediate consequences. The next four corollaries follow at once from Theorem 3.3 and [11, Corollaries 4.1, 4.2, 4.3, and 4.4] respectively. Recall that a topological space $Y$ is countably compact if every countable open cover of $Y$ has a finite subcover. If $Y$ is a $T_1$-space then $Y$ is countably compact if and only if every infinite subset of $Y$ has a limit point in $Y$ [25, p. 181]. If $Y$ is countably compact then $Y$ is pseudocompact, that is, $Y$ does not admit any unbounded continuous real-valued functions [17, 1.4].

**Corollary 3.4.** Let $A$ be a σ-unital, quasi-standard $C^*$-algebra with $A/G$ non-unital for all $G \in \text{Glimm}(A)$. Then

(i) the Hausdorff space $\text{MinPrimal}(A^*)$ is totally disconnected and countably compact;

(ii) if $\text{MinPrimal}(A^*)$ is locally compact then it is basically disconnected.

**Corollary 3.5.** Let $A$ be a σ-unital, quasi-standard $C^*$-algebra and suppose that $A/G$ is non-unital for all $G \in \text{Glimm}(A)$. Then the following are equivalent:

(i) $\text{MinPrimal}(A^*)$ is compact;

(ii) $\text{Glimm}(A)$ is cozero-complemented; that is, for every cozero set $U$ in $\text{Glimm}(A)$ there exists a cozero set $V$ in $\text{Glimm}(A)$ such that $U \cap V = \emptyset$ and $U \cup V$ is dense in $\text{Glimm}(A)$.

Let $D$ be an infinite discrete space with one-point compactification $\alpha D = D \cup \{p\}$. For $f \in C(\alpha D)$, if $p \in Z(f)$ then $Z(f)$ is co-countable, and if $p \notin Z(f)$ then $Z(f)$ is finite. It follows from the ‘cozero-complemented’ criterion used in Corollary 3.5 that $\text{Min}(C_\mathbb{R}(\alpha D))$ is compact if and only if $D$ is countable. Applying this to $\text{MinPrimal}(A^*)$ where $A$ is as in Corollary 3.5 with $\text{Glimm}(A)$ homeomorphic to $\alpha D$, we have that $\text{MinPrimal}(A^*)$ is compact if and only if $D$ is countable.

If $A$ in Corollary 3.5 is separable then (ii) holds so $\text{MinPrimal}(A^*)$ is compact, but much more can be said. Recall that a regular closed set is one that is the closure of its interior. If $A$ is separable then $\text{Glimm}(A)$ is perfectly normal [9, Lemma 3.9] (i.e. every closed subset of $\text{Glimm}(A)$ is a zero set) so $A$ certainly satisfies condition (ii) of the next corollary.

**Corollary 3.6.** Let $A$ be a σ-unital, quasi-standard $C^*$-algebra. Suppose that $A/G$ is non-unital for $G \in \text{Glimm}(A)$. Then the following are equivalent:

(i) $\text{MinPrimal}(A^*)$ is compact and extremally disconnected;

(ii) every regular closed set in $\text{Glimm}(A)$ is the closure of a cozero set.

In particular, if $A$ is separable then $A$ satisfies these equivalent conditions.

**Corollary 3.7.** Set $A = C(\beta \mathbb{N} \setminus \mathbb{N}) \otimes K(H)$. Then $\text{MinPrimal}(A^*)$ is nowhere locally compact. If Martin’s Axiom holds then $\text{MinPrimal}(A^*)$ is not an $F$-space.

We conclude this section by describing the space of Glimm ideals of $A^*$ in the case where $A$ is a σ-unital, quasi-standard $C^*$-algebra with $A/G$ non-unital for all $G \in \text{Glimm}(A)$. Set $X = \text{Glimm}(A)$ and let $\phi = \phi_A : \text{Prim}(A) \to X$ be the complete regularization map. Let $\bar{\phi} : \text{Prim}(M(A)) \to \beta X$ be the canonical extension of $\phi$. Then the Glimm ideals of $M(A)$ are the ideals $H_x$ ($x \in \beta X$) and the assignment $x \to H_x$ defines a homeomorphism of $\beta X$ onto
Glimm$(M(A))$ (see the comment after [10, Proposition 4.4]). Since $A \subseteq A^* \subseteq M(A)$, it follows that $M(A)$ is the multiplier algebra of $A^*$, and hence that the ring of bounded continuous functions on Prim$(A^*)$ is isomorphic to the ring of bounded continuous functions on Prim$(A)$ by the restriction map (regarding Prim$(A)$ as an open subset of Prim$(A^*)$ in the usual way). Thus Glimm$(A^*)$ is homeomorphic to $\overline{\phi}(\text{Prim}(A^*))$. Furthermore, $\overline{\phi}(\text{Prim}(A^*)) = \beta X$. To see this, note first of all that if $X$ is finite then $A^* = A$ and $\beta X = X$. On the other hand, if $X$ is infinite then we may apply Theorem 2.6 to the continuous $C_0(X)$-algebra $A$ to obtain that $\overline{\phi}(\text{Prim}(A^*/A)) = \beta X \setminus W$, where $W$ is the set of P-points of $X$. But

$$W \subseteq X = \phi(\text{Prim}(A)) \subseteq \overline{\phi}(\text{Prim}(A^*))$$

and hence $\overline{\phi}(\text{Prim}(A^*)) = \beta X$, as required. Finally, the Glimm ideals of $A^*$ have the form $H_x \cap A^*$ ($x \in \beta X$), where $H_x$ is as above.

One consequence of this is worth recording.

**Corollary 3.8.** Let $A$ be a $\sigma$-unital, quasi-standard $C^*$-algebra and suppose that $A/G$ is non-unital for all $G \in \text{Glimm}(A)$. Then the following are equivalent:

(i) $A^*$ is quasi-standard;

(ii) $M(A)$ is quasi-standard;

(iii) Glimm$(A)$ is basically disconnected.

**Proof.** The equivalence of (ii) and (iii) was established in [7, Corollary 4.9].

(ii)⇒(i). This follows from the general fact that closed ideals of quasi-standard $C^*$-algebras are quasi-standard [6, p. 356].

(i)⇒(iii). If $A^*$ is quasi-standard then Glimm$(A^*)$ and MinPrimal$(A^*)$ coincide as sets and topological spaces [6, Theorem 3.3]. We have seen above that Glimm$(A^*)$ is compact and thus MinPrimal$(A^*)$ is also compact, and hence locally compact. Thus it follows from Corollary 3.4 that MinPrimal$(A^*)$ is basically disconnected, and hence that Glimm$(A^*)$ is basically disconnected. But we saw above that Glimm$(A^*)$ is homeomorphic to the Stone-Cech compactification of Glimm$(A)$, and thus Glimm$(A)$ is basically disconnected by [17, 6M].

### 4. A Faithful Family of Irreducible Representations of $A^*/A$

In this section we exhibit a family of irreducible representations of the corona algebra $M(A)/A$ where $A$ is either a stable, separable, quasi-standard $C^*$-algebra or an algebra of the form $C_0(X) \otimes K(H)$ with $X$ $\sigma$-compact. The representations are constructed as irreducible representations of $M(A)$ with kernels containing $A$. We show that these irreducible representations form a faithful family for $A^*/A$. The method is based on the ultraproduct construction [16]. We begin with irreducible representations ‘at infinity’.

**Theorem 4.1.** Let $A$ be a $\sigma$-unital $C_0(X)$-algebra with base map $\phi$. Let $W$ be a countably infinite, closed, relatively discrete subset of $X_\phi$ and suppose that $J_x$ is a primitive ideal of $A$ for each $x \in W$. Let $\mathcal{F}^*$ be a free ultrafilter on $W$ and let $\mathcal{F} = \{Z \in Z[X_\phi] : Z \cap W \in \mathcal{F}^*\}$, a $\mathcal{Z}$-filter on $X_\phi$. Then $L_\mathcal{F} \in \text{Prim}(M(A))$ and $L_\mathcal{F} \supseteq A$.

**Proof.** Set $Y = \phi^{-1}(W)$. Then $Y$ is a closed subset of Prim$(A)$. Set $I = \ker Y$ and $B = A/I$. Then $B$ is isomorphic to the $c_0$-direct sum $\sum_{x \in W} A/J_x$, so $M(B)$ is isomorphic to the $l_\infty$-direct sum $\prod_{x \in W} M(A)/J_x$ [2, Lemma 1.2.21] (recall that, since $A$ is $\sigma$-unital, $M(A/J_x) \cong$
$M(A)/\tilde{J}_x$ for $x \in X_\phi$, by definition of $\tilde{J}_x$). Let $\pi : A \to B$ be the quotient map. Since $A$ is $\sigma$-unital, $\pi$ extends to a surjective homomorphism $\tilde{\pi} : M(A) \to M(B)$.

Let $C = \prod_x M(A)/\tilde{J}_x$ be the ultraproduct, and let $\rho : M(B) \to C$ denote the quotient map. Then $\ker \rho = \{ c \in M(B) : \lim_{x \in \mathcal{F}} \| c_x \| = 0 \}$. Since each $M(A)/\tilde{J}_x$ is primitive, $C$ is a primitive $C^*$-algebra by [16, Theorem 5.4]. Hence $C$ is a primitive quotient of $M(A)$, and we now show that $J := \ker \rho \circ \tilde{\pi}$ is equal to $L_F$.

Suppose that $b \in L^\text{alg}_F$. Then $Z(b) \supseteq Z$ for some $Z \in \mathcal{F}$ and hence $Z(b) \cap W \supseteq Z \cap W \in \mathcal{F}$. Thus $\tilde{\pi}(b) \in \ker \rho$, so $b \in J$. Hence $L_F \subseteq J$. Conversely, let $b \in J$ with $0 \leq b \leq 1$. Then $\tilde{\pi}(b) \in \ker \rho$, and hence for $1/2 > \epsilon > 0$ there exists $Z' \in \mathcal{F}$ such that $f_x(b) + \tilde{J}_x = 0$ for all $x \in Z' \left(\text{where } f_x \text{ is as defined before Theorem 3.3}\right)$. Thus $Z' \subseteq Z(f_x(b)) \cap W$. Since $Z'$ is closed in $X_\phi$ it follows from [8, Lemma 4.1] that there is a zero set $Z$ in $X_\phi$ with $Z' \subseteq Z \subseteq Z(f_x(b))$. Hence $Z \cap W \in \mathcal{F}$, so $Z \in \mathcal{F}$. Since $Z \subseteq Z(f_x(b))$, it follows that $f_x(b) \in L^\text{alg}_F$, and hence $b \in L_F$ by [11, Lemma 3.3]. Thus $J \subseteq L_F$, so $J = L_F$.

Finally, for $a \in A$ with $0 \leq a \leq 1$ and $0 < \epsilon < 1/2$, the set $\{ x \in X_\phi : \| a + J_x \| \geq \epsilon \}$ is compact (it is the image under $\phi$ of the compact subset $\{ P \in \text{Prim}(A) : \| a + P \| \geq \epsilon \}$ of $\text{Prim}(A)$) and hence has finite intersection with $W$. Arguing as in the previous paragraph, we see that there exists $Z \in \mathcal{F}$ such that $\| f_x(a) + J_x \| = 0$ for all $x \in Z$, and hence $a \in L_F$. □

If $A$ is a $C_0(X)$-algebra with base map $\phi$ and $Z$ is a non-empty closed subset of $X_\phi$ then we define $J_Z = \bigcap_{x \in Z} J_x$ (cf. [9, p. 5]). It is straightforward to check that $J_Z = \{ b \in A : Z(b) \supseteq Z \}$ and that $\tilde{J}_Z = \{ b \in M(A) : Z(b) \supseteq Z \}$.

**Lemma 4.2.** Let $A$ be a $\sigma$-unital $C_0(X)$-algebra with base map $\phi$ and let $Z$ be a non-empty closed subset of $X_\phi$. Then $J_Z$ is $\sigma$-unital if and only if $Z$ is a $G_\delta$.

**Proof.** Suppose that $Z$ is a $G_\delta$ subset of $X_\phi$. Then since $X_\phi$ is normal there is a continuous function $f : X_\phi \to [0,1]$ such that $Z(f) = Z$. Set $z = \theta_A(f \circ \phi)$ where $\theta_A : C^b(\text{Prim}(A)) \to ZM(A)$ is the Dauns-Hofmann isomorphism. Let $v = zu$ where $u$ is a strictly positive element for $A$. Then $v$ is a strictly positive element for the $C^*$-algebra $J_Z$, so $J_Z$ is $\sigma$-unital.

Conversely, suppose that $J_Z$ is $\sigma$-unital. Then the set $W = \{ P \in \text{Prim}(A) : P \not\supseteq J_Z \}$ is $\sigma$-compact, so $\phi(W) = X_\phi \setminus Z$ is also $\sigma$-compact. Hence $Z$ is a $G_\delta$. □

In particular, if $A$ in Lemma 4.2 is separable, then every ideal $J_Z$ is $\sigma$-unital so every non-empty closed subset $Z$ of $X_\phi$ is a $G_\delta$ (cf. [9, Lemma 3.9]).

**Lemma 4.3.** Let $A$ be a $C_0(X)$-algebra with base map $\phi$ and let $Z$ be a non-empty closed subset of $X_\phi$. Then $J_Z$ is an hereditary $C^*$-subalgebra of $M(J_Z)$.

**Proof.** We work in $A^{**}$, identifying $A$ with its canonical image in $A^{**}$. Hence we may identify $M(A)$ with the idealizer of $A$ in $A^{**}$ and $M(J_Z)$ with the idealizer of $J_Z$ in $J^{**}_Z$ where the latter is canonically embedded in $A^{**}$ [26, Proposition 3.12.3]. First note that if $b \in \tilde{J}_Z$ and $c \in M(J_Z)$ then $bc \tilde{b} \in \tilde{J}_Z$. To see this observe that $\tilde{J}_Z = \{ b \in M(A) : bA + Ab \subseteq J_Z \}$. Hence for $a \in A$, $bcba \in b^cJ_Z \subseteq J_Z$. Similarly $abcb \in J_Zc^b \subseteq J_Z$. Thus $bc$ belongs firstly to $M(A)$ and secondly to $\tilde{J}_Z$.

Now suppose that $b \in \tilde{J}_Z$ and $c \in M(J_Z)$ with $0 \leq c \leq b$. Then $c \in bM(J_Z)b$, the hereditary $C^*$-subalgebra of $M(J_Z)$ generated by $b$ [23, 1.5.9]. But $bM(J_Z)b \subseteq \tilde{J}_Z$ by the previous paragraph. Hence $c \in \tilde{J}_Z$ as required. □
We want now to apply Theorem 4.1 in the context of ideals of the form $J_Z$. Suppose that $A$ is a $\sigma$-unital, continuous $C_0(X)$-algebra with base map $\phi$ and that $A/J_x$ is non-unital for all $x \in X_\phi$. Let $Z$ be a zero set of $X_\phi$, so that the $C^*$-algebra $J_Z$ is $\sigma$-unital by Lemma 4.2. Let $\psi = \phi|_{\text{Prim}(J_Z)}$. Then $J_Z$ is a $C_0(X)$-algebra with base map $\psi$ and $X_\psi = X_\phi \setminus Z$, where $X_\psi = \text{Im}(\psi)$. Hence $M(J_Z)$ is a $C(\beta X)$-algebra with base map $\overline{\psi}$. Let $W$ be a countably infinite, relatively closed, relatively discrete subset of $X_\psi$ and suppose that $J_x \in \text{Prim}(A)$ for each $x \in W$ so that $J_x \cap J_Z$ is a primitive ideal of $J_Z$. Let $\mathcal{F}^z$ be a free $z$-ultrafilter on $W$ and let $\mathcal{F} = \{Z \in Z[X_\psi] : Z \cap W \in \mathcal{F}^z\}$. Using the superscript $M(J_Z)$ to indicate that we are applying Theorem 4.1 with $J_Z$ in place of $A$, we obtain an ideal $L_{\mathcal{F}}^M(J_Z)$ in $\text{Prim}(M(J_Z))$ such that $L_{\mathcal{F}}^M(J_Z) \supseteq J_Z$.

The next lemma compares $L_{\mathcal{F}}^M(J_Z)$ (an ideal of $M(J_Z)$) with $L_{\mathcal{F}}^M(A) := L_{\mathcal{F}}(\text{an ideal of } M(A))$, where $\mathcal{F} = \{Z \in Z[X_\psi] : Z \cap W \in \mathcal{F}^z\}$. From now on, it will be convenient for various reasons to assume that $A$ is a $\sigma$-unital, continuous $C_0(X)$-algebra. In particular, this will imply that $Z(b)$ is a zero set in $X_\phi$ for every $b \in M(A)$ (see Section 3).

**Lemma 4.4.** Let $A$ be a $\sigma$-unital, continuous $C_0(X)$-algebra with base map $\phi$ and suppose that $A/J_x$ is non-unital for all $x \in X_\phi$. Let $Z$ be a zero set of $X_\phi$ and let $Y$ be a countably infinite, relatively closed, relatively discrete subset of $X_\phi \setminus Z$ such that $J_x \in \text{Prim}(A)$ for $x \in Y$. Then with the notation above, $L_{\mathcal{F}}^M(J_Z) \cap M(A) \subseteq L_{\mathcal{F}}^M(A)$. If $J_Z$ is an essential ideal of $A$ then $L_{\mathcal{F}}^M(J_Z) \cap M(A) = L_{\mathcal{F}}^M(A)$.

**Proof.** Let $b \in L_{\mathcal{F}}^M(J_Z) \cap M(A)$ with $b \geq 0$, and let $0 < \epsilon < 1/2$. Then $f_\epsilon(b) \in (L_{\mathcal{F}}^z)^M(J_Z)$. Since $b \in M(A)$, $Z(f_\epsilon(b)) \cap X_\psi$ is a zero set of $X_\psi$ and hence belongs to $\mathcal{F}$. It follows that $Z(f_\epsilon(b)) \cap Y \in \mathcal{F}^z$ and hence that $b \in L_{\mathcal{F}}^M(A)$.

Now suppose that $J_Z$ is an essential ideal of $A$, that $0 \leq b \in L_{\mathcal{F}}^M(A)$, and that $0 < \epsilon < 1/2$. Then $f_\epsilon(b) \in (L_{\mathcal{F}}^z)^M(A)$ so $Z(f_\epsilon(b)) \in \mathcal{F}$. Hence $Z(f_\epsilon(b)) \cap Y \in \mathcal{F}^z$. Since $J_Z$ is essential, $b \in M(J_Z)$, and $(Z(f_\epsilon(b)) \cap X_\psi) \cap Y \in \mathcal{F}^z$. Thus $b \in L_{\mathcal{F}}^M(J_Z)$. \hfill \Box

Note that, under the hypotheses of Lemma 4.4, the base map $\phi$ is open, so the ideal $J_Z$ is essential in $A$ if and only if the zero set $Z$ has empty interior in $X_\phi$.

**Theorem 4.5.** Let $A$ be a $\sigma$-unital, continuous $C_0(X)$-algebra with base map $\phi$ and suppose that $A/J_x$ is non-unital for all $x \in X_\phi$. Let $Z$ be a zero set of $X_\phi$ and let $Y$ be a countably infinite, relatively closed, relatively discrete subset of $X_\phi \setminus Z$ such that $J_x \in \text{Prim}(A)$ for $x \in Y$. Let $\mathcal{F}^z$ be a free $z$-ultrafilter on $Y$ and set $\mathcal{F} = \{Z' \in Z[X_\psi] : Z' \cap Y \in \mathcal{F}^z\}$. Then there exists an irreducible representation $\pi_\mathcal{F}$ of $M(A)$ such that

(i) $\ker \pi_{\mathcal{F}} \supseteq A$;
(ii) $\ker \pi_{\mathcal{F}} \not\supseteq A^*$;
(iii) $\ker \pi_{\mathcal{F}} = \{b \in M(A) : b\tilde{J}_Z + \tilde{J}_Z b \subseteq L_{\mathcal{F}}^M(J_Z) \cap M(A)\}$;
(iv) $\ker \pi_{\mathcal{F}} \cap \tilde{J}_Z \subseteq L_{\mathcal{F}}^M(A)$.

**Proof.** Since $Z$ is a zero set of $X_\phi$, the $C^*$-algebra $J_Z$ is $\sigma$-unital by Lemma 4.2. Hence it follows from Theorem 4.1 applied to $J_Z$, as detailed above, that there is an irreducible representation $\rho$ of $M(J_Z)$ on a Hilbert space $K$ with $\ker \rho = L_{\mathcal{F}}^M(J_Z)$. In particular, $L_{\mathcal{F}}^M(J_Z) \supseteq J_Z$. 


Let \( c^Z \in A^s \) be as in Lemma 2.5 ([8, Lemma 5.6]) with \( c^Z \in \tilde{J}_x \) for \( x \in Z \) and \( \|c^Z + \tilde{J}_x\| = 1 \) for \( x \in X_\phi \setminus Z \). Then \( c^Z \in \tilde{J}_Z \), but by the construction of \( \rho \), \( c^Z \notin \ker \rho \), and hence \( \tilde{J}_Z \not\subset \ker \rho \). Since \( \tilde{J}_Z \) is an hereditary subalgebra of \( M(J_Z) \) by Lemma 4.3, it follows that the representation \( \pi = \rho|_{J_Z} \) is an irreducible representation of \( \tilde{J}_Z \) on the Hilbert space \( H = \rho(J_Z)K [26, 4.1.5] \) with kernel \( \tilde{J}_Z \cap L^M_{\pi}(J_Z) \). Let \( \pi_\mathcal{F} \) be the canonical extension of \( \pi \) to an irreducible representation of \( M(A) \) on \( H \). Then \( c^Z \notin \ker \pi_\mathcal{F} \) so \( \pi_\mathcal{F} \) does not annihilate \( A^s \), establishing (ii).

By the construction of \( \pi_\mathcal{F} \) we have that

\[
\ker \pi_\mathcal{F} = \{ b \in M(A) : b\tilde{J}_Z + \tilde{J}_Z b \subseteq \ker \pi = L^M_{\pi}(J_Z) \cap \tilde{J}_Z \} = \{ b \in M(A) : b\tilde{J}_Z + \tilde{J}_Z b \subseteq L^M_{\pi}(J_Z) \} = \{ b \in M(A) : b\tilde{J}_Z + \tilde{J}_Z b \subseteq L^M_{\pi}(J_Z) \cap M(A) \},
\]

establishing (iii). Furthermore, for \( a \in A \),

\[
a\tilde{J}_Z + \tilde{J}_Z a \subseteq J_Z \subseteq L^M_{\pi}(J_Z) \cap M(A),
\]

so \( A \subseteq \ker \pi_\mathcal{F} \), establishing (i).

Finally, suppose that \( b \geq 0 \) and that \( b \in \ker \pi_\mathcal{F} \cap \tilde{J}_Z \). Then \( 2b^2 \in L^M_{\pi}(J_Z) \cap M(A) \) by (iii), and hence \( 2b^2 \in L^M_{\pi}(A) \) by Lemma 4.4. Thus \( b \in L^M_{\pi}(A) \), establishing (iv). \( \square \)

**Corollary 4.6.** In the context of Theorem 4.5, if \( J_Z \) is an essential ideal of \( A \) then (iii) and (iv) may be replaced by

(iii)' \( \ker \pi_\mathcal{F} = \{ b \in M(A) : b\tilde{J}_Z + \tilde{J}_Z b \subseteq L^M_{\pi}(A) \} \);

(iv)' \( \ker \pi_\mathcal{F} \cap \tilde{J}_Z = L^M_{\pi}(A) \cap \tilde{J}_Z \).

If \( Z = \{ x \} \) is a singleton with \( x \) non-isolated in \( X_\phi \) then \( \ker \pi_\mathcal{F}|_{A^*} = (L^M_{\pi}(A) \cap A^*) + A \), so \( (L^M_{\pi}(A) \cap A^*) + A \in \text{Prim}(A^*) \).

**Proof.** If \( J_Z \) is an essential ideal of \( A \) then (iii)' follows from Theorem 4.5(iii) and Lemma 4.4. This then implies that \( L^M_{\pi}(A) \subseteq \ker \pi_\mathcal{F} \), and thus (iv)' follows from Theorem 4.5(iv).

Now suppose that \( Z = \{ x \} \) is a singleton. Then \( J_x = J_Z \) is an essential ideal in \( A \) since \( x \) is non-isolated in \( X_\phi \), so we know from the previous paragraph and Theorem 4.5(i) that \( \ker \pi_\mathcal{F}|_{A^*} \supseteq (L^M_{\pi}(A) \cap A^*) + A \). Suppose that \( b \in \ker \pi_\mathcal{F}|_{A^*} \). Let \( a \in A \) such that \( b - a \in \tilde{J}_x \). Then \( b - a \in \ker \pi_\mathcal{F}|_{A^*} \), so \( b - a \in L^M_{\pi}(A) \) by (iv)' above. Hence \( b \in (L^M_{\pi}(A) \cap A^*) + A \) as required. \( \square \)

We now define \( \mathcal{S} \) to be the family of irreducible representations of \( M(A) \) obtainable by the methods of Theorem 4.1 and Theorem 4.5. Obviously the size of \( \mathcal{S} \) depends on the size of the set \( \{ x \in X_\phi : J_x \in \text{Prim}(A) \} \).

**Theorem 4.7.** Let \( A \) be a \( \sigma \)-unital, continuous \( C_0(X) \)-algebra with base map \( \phi \) such that \( X_\phi \) is infinite. Suppose that \( A / J_x \) is non-unital for all \( x \in X_\phi \) and that the set \( \{ x \in X_\phi : J_x \in \text{Prim}(A) \} \) is dense in \( X_\phi \). Then the family \( \mathcal{S} \) of irreducible representations is faithful on \( A^* / A \).
Proof. Since $X_\phi$ is infinite, there exists $b \in A^* \setminus A$ [12, p. 306]. Then either (i) or (ii) of Lemma 2.4 fails.

Suppose first that (i) fails. Then there exists $\epsilon > 0$ such that the set

$$Y = \{x \in X_\phi : \|b + Hx\| \geq \epsilon\}$$

is non-compact. Since $Y$ is closed in $X_\phi$ by the upper semi-continuity of norm functions, it follows that $X_\phi$ is non-compact. Hence, since $A$ is continuous and $\sigma$-unital and $X_\phi$ is locally compact and $\sigma$-compact, we may write $X_\phi = \bigcup_{n=1}^{\infty} X_n$ where each $X_n$ is compact and is strictly contained in the interior of $X_{n+1}$. The non-compactness of $Y$ implies that for each $n \geq 1$, $Y \cap (X_\phi \setminus X_n)$ is non-empty so we may choose a sequence $(y_n)_{n \geq 1}$ with $y_n \in Y \cap (X_\phi \setminus X_n)$. Temporarily fix $n \geq 1$. Then $\|b + H y_n\| \geq \epsilon$, so by [7, Lemma 1.5] there exists $x_n$ in the open neighbourhood $X_\phi \setminus X_n$ of $y_n$ such that $\|b + J w_n\| > \epsilon/2$. By the lower semi-continuity of norm functions [12, Lemma 6.2(i)], and the density of the set $\{x \in X_\phi : J_x \in \text{Prim}(A)\}$ in $X_\phi$, it follows that there exists $w_n \in X_\phi \setminus X_n$ with $J w_n \in \text{Prim}(A)$ and $\|b + J w_n\| > \epsilon/2$. Set $W = \{w_n : n \geq 1\}$. Then $W$ is countably infinite, and at most finitely many elements of $W$ belong to each $X_n$, so $W$ is a closed, relatively discrete set in $X_\phi$. Let $\mathcal{F}$ be a free $z$-ultrafilter on $W$ and let $\mathcal{F} = \{Z \in Z[X_\phi] : Z \cap W \in \mathcal{F}\}$. Then it follows from Theorem 4.1 that there is an irreducible representation $\pi_F \in \mathcal{S}$ with kernel $L_F$. But $f_\delta(b) \notin L^{alg}_F$ for all $\delta$ in $(0, \epsilon/2)$, so $\pi_F(b) \neq 0$.

Now suppose instead that (ii) fails. Then there exists $x \in X_\phi$ such that $b - a \notin H_x$ for all $a \in A$. Let $a \in A$ such that $c := b - a \in J_x$, and set $Z = Z(c)$. Then $Z$ is a zero set in $X_\phi$ and $x \in Z$. Since $c \notin H_x$ we see that $x$ is in the boundary of $Z$ by [7, Lemma 1.5]. We seek an irreducible representation $\pi_F \in \mathcal{S}$ such that $\pi_F(c) \neq 0$; and replacing by $c^* c$, we may assume that $c \geq 0$. Since $c \notin H_x$, there exists $\epsilon > 0$ such that $f_\epsilon(c) \notin L^{alg}_Z[O_x]$ [8, Theorem 4.3(ii)]. Thus for every open set $U$ containing $x$ we may find $y \in U \setminus Z$ such that $\|c + H_y\| > \epsilon$.

Since $X_\phi$ is locally compact and $\sigma$-compact, the cozero set $X_\phi \setminus Z$ is also locally compact and $\sigma$-compact; and as $x$ is a boundary point in $Z$, $X_\phi \setminus Z$ is non-compact. Thus we may write $X_n \setminus Z$ as a countable, strictly increasing union of compact sets $Y_n (n \geq 1)$ where each $Y_n$ is contained in the interior of $Y_{n+1}$.

For each $n \geq 1$, set $U_n = X_\phi \setminus Y_n$. Then $x \in U_n$. Temporarily fix $n$ and choose $y_n \in U_n \setminus Z$ such that $\|c + H y_n\| > \epsilon$. Then by [7, Lemma 1.5] there exists $x_n \in U_n \setminus Z$ such that $\|c + J x_n\| > \epsilon$. By the lower semi-continuity of norm functions [12, Lemma 6.2(i)], and the density of the set $\{x \in X_\phi : J_x \in \text{Prim}(A)\}$ in $X_\phi$, it follows that there exists $w_n \in U_n \setminus Z$ with $J w_n \in \text{Prim}(A)$ and $\|c + J w_n\| > \epsilon$.

Set $W = \{w_n : n \geq 1\}$. The sets $U_n \setminus Z$ are decreasing and $\bigcap_{n=1}^{\infty} (U_n \setminus Z) = \emptyset$, so $W$ is countably infinite. Let $y \in X_\phi$ be an accumulation point of $W$. Since $W$ has finite intersection with each $Y_{n+1}$, $y$ cannot be in the interior of $Y_{n+1}$, so in particular $y \notin Y_n$. Thus $y \in Z$. It follows that $W$ is relatively closed and relatively discrete in $X_\phi \setminus Z$. Let $\mathcal{F}$ be a free $z$-ultrafilter on $W$ and set $\mathcal{F} = \{Z \in Z[X_\phi] : Z \cap W \in \mathcal{F}\}$. Then by Theorem 4.5(iv) there exists an irreducible representation $\pi_F \in \mathcal{S}$ such that

$$\ker \pi_F \cap J Z \subseteq L_M^F(A).$$

Since $c \in J Z \setminus L_M^F(A)$ we have that $c \notin \ker \pi_F$. □
The question of whether this family $S$ of irreducible representations is faithful on the whole of $C(A) = M(A)/A$ is equivalent to the question whether $A^* / A$ is an essential ideal in $C(A)$, i.e. whether for each $b \in M(A) \setminus A$ there exists $a \in A^*$ such that either $ba \notin A$ or $ab \notin A$.

It was shown in [12, Theorem 4.1] that if $A = C_0(X) \otimes K(H)$ where $X$ is a locally compact Hausdorff space then $A^* / A$ is an essential ideal in $C(A)$ if and only if $X$ has no isolated points.

5. Primal ideals in $A^* / A$

In this section we use the irreducible representations just constructed to identify the set $\text{MinPrimal}(A^*/A)$ of minimal closed primal ideals of $A^*/A$ where $A$ is a $\sigma$-unital, quasi-standard C*-algebra with non-unital Glimm quotients such that $\text{Glimm}(A)$ is infinite and $\text{Prim}(A) \cap \text{Glimm}(A)$ is dense in $\text{Glimm}(A)$ (this latter condition is automatic if $A$ is separable and quasi-standard [6, Corollary 3.5]).

We have already mentioned that if $A$ is a C*-algebra and $I$ a closed ideal in $A$ then $I$ is primal if $\text{Prim}(A/I)$ is contained in a limit set in $\text{Prim}(A)$. Let $J$ be a closed ideal in $A$ and $\pi : A \to A/J$ the quotient map. If $K$ is a primal ideal in $A/J$ then $\pi^{-1}(K)$ is evidently primal in $A$. On the other hand there is no general reason why the image $\pi(I)$ of a primal ideal $I$ in $A$ should be primal in $A/J$. For example, let $A$ be the Kaplansky example of the C*-algebra of sequences $x = (x_n)_{n \geq 1}$ of $2 \times 2$ complex matrices converging at infinity to a diagonal matrix $\text{diag}(\lambda(x), \mu(x))$. Then the closed ideal $J = \ker \lambda \cap \ker \mu$ is primal in $A$, but $A/J \cong \mathbb{C} \oplus \mathbb{C}$ so the image of $J$ in the quotient (namely $\{0\}$) is not primal in $A/J$. It is therefore somewhat surprising to find that the primal ideals in $A^* / A$ in the context above are precisely the images of the primal ideals in $A^*$ (see Theorem 5.5).

The first point to clarify is when the image of a certain kind of ideal in $A^*$ consists of the whole of $A^* / A$. We shall need the following lemma.

Lemma 5.1. Let $F$ be a $z$-filter on a completely regular space $X$ and suppose that there is a non-empty compact subset $Z'$ of $X$ such that $Z' \in F$. Then

(i) $Y := \bigcap \{Z : Z \in F\}$ is compact and non-empty;

(ii) if $W$ is a zero-set neighbourhood of $Y$ in $X$ then $W \in F$.

Proof. (i) This follows from the finite intersection property.

(ii) Let $U$ be the interior of $W$. Then $Y \subseteq U$ so there exists $Z \in F$ such that $Z' \cap Z$ does not meet $X \setminus U$ (for otherwise $Y$ would meet the compact set $Z' \cap (X \setminus U)$). Hence $W \supseteq U \supseteq Z' \cap Z \in F$ and so $W \in F$. \hfill $\Box$

Now let $A$ be a $\sigma$-unital, continuous $C_0(X)$-algebra with base map $\phi$. Suppose that $X_\phi$ is infinite and that $A/J_x$ is non-unital for all $x \in X_\phi$. For a $z$-filter $F$ on $X_\phi$, we define

$$D_F = ((L_F \cap A^*) + A) / A.$$ 

Proposition 5.2. Let $A$ be a $\sigma$-unital, continuous $C_0(X)$-algebra with base map $\phi$. Suppose that $X_\phi$ is infinite and that $A/J_x$ is non-unital for all $x \in X_\phi$. Let $F$ be a $z$-filter on $X_\phi$. Then the following are equivalent:

(i) $D_F$ is a proper closed ideal of $A^* / A$;

(ii) there is no non-empty finite subset $W$ of $X_\phi$ such that $F = \{Z \in Z[X_\phi] : Z \supseteq W\}$.
\textbf{Proof.} First suppose that $\mathcal{F}$ does not satisfy (ii). Then there exists a non-empty finite subset $W$ of $X_\phi$ such that $\mathcal{F} = \{Z \in Z[X_\phi] : Z \supseteq W\}$. Let $b \in A^\ast$. Then since $W$ is finite it is easy to find $a \in A$ such that $b - a \in \mathcal{J}_x$ for all $x \in W$. Hence $Z(b - a) \supseteq W$, so $b - a \in L_\mathcal{F} \cap A^\ast$. Thus $b \in (L_\mathcal{F} \cap A^\ast) + A$ so $(L_\mathcal{F} \cap A^\ast) + A = A^\ast$, and hence $D_\mathcal{F} = A^\ast/A$. Thus (i) fails.

Now suppose that $\mathcal{F}$ satisfies (ii). We deal first with the case when every zero set in $\mathcal{F}$ is non-compact. Let $c^Z \in A^\ast$ be an element with the properties of Lemma 2.5 in the case when $Z$ is the empty set. Let $V \in \mathcal{F}$. Then $\|c^Z + \mathcal{J}_x\| = 1$ for all $x \in V$. For $a \in A$ and $\varepsilon > 0$, the set $\{x \in X_\phi : \|a + \mathcal{J}_x\| \geq \varepsilon\}$ is compact (being the image under $\phi$ of the compact subset $\{P \in \text{Prim}(A) : \|a + P\| \geq \varepsilon\}$ of $\text{Prim}(A)$). Therefore, since $V$ is closed and non-compact, $\inf_{x \in V}\|a + \mathcal{J}_x\| = 0$ and so $\sup_{x \in V}\|(c^Z - a) + \mathcal{J}_x\| \geq 1$. Hence there does not exist $a \in A$ such that $\|(c^Z - a) + L_\mathcal{F}\| < 1$. Thus $c^Z \notin L_\mathcal{F} + A$, so $c^Z + A \notin D_\mathcal{F}$.

Next suppose that $\mathcal{F}$ satisfies (ii) and contains a compact zero set. Then $Y := \{Z : Z \in \mathcal{F}\}$ is compact and is non-empty by Lemma 5.1(i). First we deal with the case when $Y$ is infinite. If $Y$ is infinite then $Y$ contains a non-$P$-point by [17, 4K.1], so there is a zero set $Z'$ in $Y$ with non-empty boundary in $Y$. The normality of $X_\phi$ implies that there is a zero set $Z$ of $X_\phi$ such that $Z \cap Y = Z'$ (in other words, $Y$ is ‘$z$-embedded’ in $X_\phi$). Let $c^Z \in A^\ast$ be an element with the properties of Lemma 2.5. Then $c^Z \in \mathcal{J}_x$ for $x \in Z$ but $\|c^Z + \mathcal{J}_x\| = 1$ for $x \in X_\phi \setminus Z$. Let $b \in L_{\mathcal{F}}^{ag}$ and let $a \in A$. Then $Z(b) \in \mathcal{F}$ so $Z(b) \supseteq Y$. Since $Z'$ has non-empty boundary in $Y$, there is a net $(x_\alpha)$ in $Y \setminus Z'$ converging to some $x \in Z'$. Then $\|(b + a) + \mathcal{J}_{x_\alpha}\| \to \|(b + a) + \mathcal{J}_x\|$, while $\|c^Z + \mathcal{J}_{x_\alpha}\| = 1$ and $\|c^Z + \mathcal{J}_x\| = 0$. Hence $\|(b + a) - c^Z\| \geq 1/2$, so $c^Z \notin L_\mathcal{F} + A$. Thus $c^Z + A \notin D_\mathcal{F}$.

Now suppose that $Y$ is finite. Then since $\mathcal{F}$ satisfies (ii), there is a zero set $Z$ containing $Y$ such that $Z \notin \mathcal{F}$. Let $V \in \mathcal{F}$. We claim that $Z \cap V$ is not clopen in $V$. Suppose to the contrary. Since $V \cap Z$ and $V \setminus (V \cap Z)$ are disjoint closed sets in the normal space $X_\phi$, they can be separated by disjoint closed neighbourhoods and hence there is a zero set neighbourhood $U$ of $Y$ containing $Z \cap V$ with $U$ disjoint from $V \setminus (Z \cap V)$. But $U \in \mathcal{F}$ by Lemma 5.1(ii) and $U \cap V = Z \cap V$, so $Z \cap V \in \mathcal{F}$. Hence $Z \notin \mathcal{F}$, a contradiction.

Let $c^Z \in A^\ast$ be an element with the properties of Lemma 2.5. Then $c^Z \in \mathcal{J}_x$ for $x \in Z$ but $\|c^Z + \mathcal{J}_x\| = 1$ for $x \in X_\phi \setminus Z$. Let $b \in L_{\mathcal{F}}^{ag}$ and let $a \in A$. Then $Z(b) \in \mathcal{F}$ so $Z \cap Z(b)$ is not clopen in $Z(b)$. Hence there is a net $(x_\alpha)$ in $Z(b) \setminus Z$ converging to some $x \in Z \cap Z(b)$. Then $\|(b + a) + \mathcal{J}_{x_\alpha}\| \to \|(b + a) + \mathcal{J}_x\|$, while $\|c^Z + \mathcal{J}_{x_\alpha}\| = 1$ and $\|c^Z + \mathcal{J}_x\| = 0$. Hence $\|(b + a) - c^Z\| \geq 1/2$, so $c^Z \notin L_\mathcal{F} + A$. Thus $c^Z + A \notin D_\mathcal{F}$. \hfill \Box

Suppose that $A$ is a $C_0(X)$-algebra with base map $\phi$, that $\mathcal{F}$ is a $z$-filter on $X_\phi$ and that $p \in \text{cl}_X X_\phi$. Then $\mathcal{F}$ is said to converge to $p$ if every neighbourhood (in $\text{cl}_X(X_\phi)$) of $p$ contains a member of $\mathcal{F}$ [17, 6.2]. In this case, $\mathcal{F} \supseteq Z[O^p]$. Indeed, if $f \in O^p$ then $\text{cl}_X Z(f)$ contains some member $Z$ of $\mathcal{F}$ and hence the closed subset $Z(f)$ of $X_\phi$ contains $Z$. Hence, in the context of Lemma 5.3 below, it follows from Theorem 3.1 and [8, Theorem 4.3] that $H_p \subseteq L_\mathcal{F}$ and, if $p \in X_\phi$, $L_\mathcal{F} \subseteq \mathcal{J}_p$.

\textbf{Lemma 5.3.} Let $A$ be a $\sigma$-unital $C_0(X)$-algebra with base map $\phi$ and suppose that $A/J_x$ is non-unital for all $x \in X_\phi$. Let $\mathcal{F}$ be a $z$-filter on $X_\phi$ that converges to some $y \in X_\phi$ and let $b \in M(A)$. Then $b \in L_\mathcal{F} + A$ if and only if
\begin{itemize}
\item[(a)] $b \in A + \mathcal{J}_y = N_y$ and
\item[(b)] for all $a \in A$ such that $b - a \in \mathcal{J}_y$, $b - a \in L_\mathcal{F}$.
\end{itemize}
Proposition 5.4. Let $A$ be a $\sigma$-unital, continuous $C_0(X)$-algebra with base map $\phi$. Suppose that $X_\phi$ is infinite and that $A/J_x$ is non-unital for all $x \in X_\phi$. Let $F_1$ and $F_2$ be $z$-filters on $X_\phi$ which converge to points in $\text{cl} \beta X_\phi$ and suppose that neither $F_1$ nor $F_2$ has the form $Z[M_y]$ ($y \in X_\phi$). If $F_1 \nsubseteq F_2$ then $D_{F_1} \nsubseteq D_{F_2}$. In particular, if $F_1 \neq F_2$ then $D_{F_1} \neq D_{F_2}$.

Proof. Suppose that $F_1 \nsubseteq F_2$. By assumption, there is a zero set $Z$ with $Z \in F_1 \setminus F_2$. Let $cZ \in A^+$ with the properties of Lemma 2.5. Then it follows from the proof of [8, Theorem 5.7] that $cZ \in (A^* \cap L_{F_2}^{\text{nr}}) \setminus L_{F_2}$.

Suppose first that $F_1$ and $F_2$ have the same limit $x \in X_\phi$. Since $x \in Z$, $cZ \in \tilde{J}_x$. Thus Lemma 5.3 implies that $cZ \notin L_{F_2} + A$. Hence $cZ + A \in D_{F_1} \setminus D_{F_2}$.

Suppose next that $F_1$ and $F_2$ have the same limit $p \in \text{cl} \beta X_\phi \setminus X_\phi$. Towards a contradiction, suppose that $cZ + a \in L_{F_2}$ for some $a \in A$. Then there exists $b \in M(A)$ such that $Z(b) \in F_2$ and $\|cZ + a - b\| < 1/4$. Let $C := \{t \in X_\phi : \|a + J_t\| \geq 1/4\}$, a compact subset of $X_\phi$. For $t \in X_\phi \setminus (C \cup Z)$, $\|(cZ + a) + J_t\| \geq 3/4$ and hence $\|b + J_t\| \geq 1/2$. In particular, $Z(b) \subseteq C \cup Z$. Since $F_2$ converges to $p$ and $\text{cl} \beta X_\phi \setminus C$ is a neighbourhood of $p$ in $\text{cl} \beta X_\phi$, there exists $z' \in F_2$ such that $Z' \cap C = \emptyset$. It follows that $Z$ contains $Z(b) \cap Z'$ and so $Z \in F_2$, a contradiction. Hence $cZ \notin L_{F_2} + A$ and so $cZ + A \in D_{F_1} \setminus D_{F_2}$.

Finally, suppose that $F_1$ and $F_2$ have different limits $p_1$ and $p_2$ (respectively) in $\text{cl} \beta X_\phi$. Then there exists $f_1 \in C(\beta X)$ with $0 \leq f_1 \leq 1$ such that $f_1(p_1) = 0$ and $f_1(p_2) = 1$. Set $f_2 = 1 - f_1$, and let $\overline{\mu} : C(\beta X) \to ZM(A)$ be the extension of the structure map $\mu$ of the $C_0(X)$-algebra $A$ [7, Proposition 1.2]. Then for $b \in A^*$, $b = \overline{\mu}(f_1)b + \overline{\mu}(f_2)b$. Now

$$\overline{\mu}(f_i)b \in H_{p_i} \cap A^* \subseteq L_{F_i} \cap A^* \quad (i = 1, 2)$$

(see [7, p. 77] for the first inclusion and the remarks preceding Lemma 5.3 for the second). Thus $A^* = (L_{F_1} \cap A^*) + (L_{F_2} \cap A^*)$ and so $D_{F_1} + D_{F_2} = A^*/A$. Suppose that $D_{F_1} = A^*/A$. By Proposition 5.2, there is a non-empty finite subset $W$ of $X_\phi$ such that $F_1 = \{Z \in Z[X_\phi] : Z \supseteq W\}$. Since $F_1$ converges to $p_1$, no point of $X_\phi \setminus \{p_1\}$ belongs to $W$. Thus $W = \{p_1\}$ and $F_1 = Z[M_{p_1}]$, a contradiction. Thus $D_{F_1}$ is a proper ideal of $A^*/A$ and a similar argument applies to $D_{F_2}$. Since their sum is $A^*/A$, neither can contain the other. \hfill $\Box$

Theorem 5.5. Let $A$ be a $\sigma$-unital, quasi-standard $C^*$-algebra. Suppose that $X := \text{Glimm}(A)$ is infinite, that $X \cap \text{Prim}(A)$ is dense in $X$ and that $A/J_x$ is non-unital for all $x \in X$. Let $P \in \text{MinPrim}(A^*)$. Then $(P + A)/A$ is a primal ideal of $A^*/A$.

Proof. Set $P' = P + A$. By [11, Theorem 3.4] and Theorem 3.3, there is a minimal prime $z$-filter $F$ on $X$ such that $P = L_F \cap A^*$. Suppose first that there exists $x \in X$ such that $Z[O_x] \subseteq F \subseteq Z[M_x]$ (see [17, Theorem 7.15]) and hence such that $H_x \subseteq L_F \subseteq \tilde{J}_x$, [8, Theorem 4.3]. If $x$ is a $P$-point of $X$ then $H_x = \tilde{J}_x$, so $P = \tilde{J}_x \cap A^*$. Hence $P' = (\tilde{J}_x \cap A^*) + A = A^*$, so $P'/A$ is trivially primal in $A^*/A$. Thus we may assume that $x$ is a non-$P$-point of $X$. This implies
that \( L_\mathcal{F} \) is strictly contained in \( \tilde{J}_\mathcal{F} \) [17, 14.12] and hence that \( P \) is strictly contained in \( \tilde{J}_\mathcal{F} \cap A^s \) [8, Theorem 5.7]. Thus \( P' \) is strictly contained in \( A^s \) by Lemma 5.3 (let \( b \in (\tilde{J}_\mathcal{F} \cap A^s) \setminus P \) if \( b \in P + A \) then we get a contradiction in Lemma 5.3(b) by taking \( a = 0 \)).

Let \( b_1 \in A^s \) (1 ≤ \( i \) ≤ \( n \)) with \( \|b_i\| = \|b_1 + P\| = 1 \). For the primality of \( P'/A \) it is enough to show that \( b_1A^s b_2 \ldots A^s b_n \not\subseteq A \). For 1 ≤ \( i \) ≤ \( n \), let \( a_i \in A \) such that \( b_1 - a_i \in \tilde{J}_\mathcal{F} \cap A^s \) and set \( c_i = b_1 - a_i \). Then it will be enough to show that \( c_1A^s c_2 \ldots A^s c_n \not\subseteq A \). We have that \( c_i \in \tilde{J}_\mathcal{F} \cap A^s \), but \( \|c_i + P\| \geq 1 \) (for if there exists \( p \in P \) such that \( ||c_i - p|| < 1 \) then \( ||b_i - (p + a_i)|| = ||c_i - p|| < 1 \), and \( p + a_i \in P + A = P' \)). Set \( Z = \bigcap_{i=1}^n Z(c_i) \), so that \( c_i \in \tilde{J}_Z \) for 1 ≤ \( i \) ≤ \( n \). Then \( Z \) is a zero set in \( X \) and \( x \in Z(c_i) \) for each \( i \) so \( x \in Z \).

Since \( c_i \notin P \), for each 1 ≤ \( i \) ≤ \( n \), \( c_i c_i^* \notin P \), so there exists 0 < \( \epsilon_i < 1/2 \) such that \( f_{c_i}(c_i c_i^*) \notin P \) (for otherwise \( c_i c_i^* = \lim_{\epsilon \to 0} f_{c_i}(c_i c_i^*) \in P \)). Hence \( Z(f_{c_i}(c_i c_i^*)) \notin \mathcal{F} \), for otherwise \( f_{c_i}(c_i c_i^*) \in L^\text{alg}_\mathcal{F} \cap A^s \subseteq L_\mathcal{F} \cap A^s = P \). Set \( W = \bigcup_{i=1}^n Z(f_{c_i}(c_i c_i^*)) \) and note that \( W \) is a zero set in \( X \) and that \( Z(c_i) = Z(c_i c_i^*) \subseteq Z(f_{c_i}(c_i c_i^*)) \) for each \( i \), so \( Z \subseteq W \). Hence, in particular, \( x \in W \). Since \( \mathcal{F} \) is a prime \( z \)-filter, \( W \notin \mathcal{F} \). This implies that \( W \neq X \), and also that \( x \) does not lie in the interior of \( W \), for otherwise \( W \subseteq Z[O_x] \subseteq F \). Set \( U = X \setminus W \). Then \( U \) is a non-empty cozero set in \( X \) and \( x \) lies in the closure of \( U \).

Let \( f \in C_\mathcal{R}(X) \) with 0 ≤ \( f \) ≤ 1 such that \( Z(f) = Z \). Then \( f(x) = 0 \), and \( x \) lies in the boundary of \( U \) so we may inductively construct a sequence \( (x_n)_{n \geq 1} \) in \( U \cap \text{Prim}(A) \) such that \( f(x_n) < 1 \) and \( f(x_n) < \min \{f(x_{n-1}), 1/n \} \) for all \( n \geq 2 \). Thus \( f(x_n) \to 0 \) as \( n \to \infty \). Hence the set \( Y = \{x_n : n \geq 1\} \) is contained in \( U \) and is relatively closed and relatively discrete in \( X \setminus Z \).

Let \( G^* \) be a free ultrafilter on \( Y \) and set \( G = \{Z' \subseteq Z[X] : Z' \cap Y \in G^* \} \). With \( Z \) and \( Y \) as above, let \( \pi_G \) be an irreducible representation of \( M(A) \) constructed as in the proof of Theorem 4.5. Note that for \( y \in Y \), \( f_{c_i}(c_i c_i^*) \not\in \mathcal{F} \) and hence \( \|c_i + \tilde{J}_y\|^2 > \epsilon_i \). Thus \( c_i \notin L^M_G(J) \) (where \( G' = \{Z' \subseteq Z[X \setminus Z] : Z' \cap Y \in G^* \} \)). On the other hand, \( c_i \in \tilde{J}_Z \); so \( c_i \notin \ker \pi_G \) using Theorem 4.5(iii). It follows that \( \pi_G(c_1A^s c_2 \ldots A^s c_n) \neq \{0\} \) since \( \pi_G \) is an irreducible representation. Thus \( c_1A^s c_2 \ldots A^s c_n \not\subseteq A \), by Theorem 4.5(i), so \( b_1A^s b_2 \ldots A^s b_n \not\subseteq A \), as required.

Now suppose that there exists \( x \in \beta X \setminus X \) such that \( Z[O^x] \subseteq F \subseteq Z[M^\mathcal{F}] \) (see [17, Theorem 7.15] again). Let \( a \) belong to the Pedersen ideal of \( A \) [26, 5.6.1]. Then the set \( \{P \in \text{Prim}(A) : a \notin P \} \) is contained in a compact subset \( C \) of \( \text{Prim}(A) \), so the complement of \( Z(a) = \{y \in X : a \in \tilde{J}_y \} \) is contained in the compact set \( \phi_A(C) \). Thus \( c_1aX(Z) \supseteq \beta X \setminus \phi_A(C) \) which is an open subset of \( \beta X \) containing \( x \). Hence \( Z(a) \in Z[O^x] \subseteq F \), and thus \( L^\text{alg}_\mathcal{F} \) contains the Pedersen ideal of \( A \). Hence \( P \supseteq A \), so \( P' = P = P' \). Set \( W = \bigcup_{i=1}^n Z(f_{c_i}(b_i b_i^*)) \) and note that \( W \) is a zero set in \( X \). Since \( F \) is a prime \( z \)-filter, \( W \notin \mathcal{F} \). This implies that \( W \neq X \), and also that \( x \) does not lie in the interior of \( W \), for otherwise \( W \subseteq Z[O^x] \subseteq F \). Set \( U = X \setminus W \). Then \( U \) is a non-empty cozero set in \( X \) and \( x \) lies in \( c_1aX(U) \). Let \( \gamma \in U \). Then \( f_{c_i}(b_i b_i^*) + \tilde{J}_y \neq 0 \) and hence \( \|b_i + \tilde{J}_y\|^2 > \epsilon_i \).

Since \( A \) is \( \sigma \)-unital and quasi-standard, \( X \) is a \( \sigma \)-compact open subset of \( \beta X \), and hence is a cozero set in \( \beta X \). Let \( f \in C_\mathcal{R}(\beta X) \) with 0 ≤ \( f \) ≤ 1 and \( Z(f) = \beta X \setminus X \). Then \( f(x) = 0 \),
and \( x \) lies in the closure of \( U \) in \( \beta X \) so we may inductively construct a sequence \( (x_n)_{n \geq 1} \) in \( U \cap \text{Prim}(A) \) such that \( f(x_1) < 1 \) and \( f(x_n) < \min\{f(x_{n-1}), 1/n\} \) for all \( n \geq 2 \). Thus \( f(x_n) \to 0 \) as \( n \to \infty \). Hence the set \( Y = \{x_n : n \geq 1\} \) is a closed, relatively discrete subset of \( X \). Let \( \pi \) be an irreducible representation of \( M(A) \) obtained from \( Y \) as in the proof of Theorem 4.1. Then \( A \subseteq \ker \pi \) but \( \|\pi(b_i)\|^2 \geq \epsilon_i > 0 \) for \( 1 \leq i \leq n \). Hence \( b_1 A^* b_2 \cdots A^* b_n \not\subseteq A \), as required. 

\[ \square \]

**Theorem 5.6.** Let \( A \) be a \( \sigma \)-unital, quasi-standard \( C^* \)-algebra. Suppose that \( X := \text{Glimm}(A) \) is infinite, that \( X \cap \text{Prim}(A) \) is dense in \( X \) and that \( A/J_x \) is non-unital for all \( x \in X \). Let \( Y \) be the set of \( P \)-points of \( X \) and set \( S = \{\tilde{J}_y \cap A^* : y \in Y\} \). Let \( \sigma : A^* \to A^*/A \) be the quotient map. Then

\[
\text{MinPrimal}(A^*/A) = \{\sigma(P) : P \in \text{MinPrimal}(A^*) \setminus S\}.
\]

Furthermore, the map induced by \( \sigma \) from \( \text{MinPrimal}(A^*) \setminus S \) onto \( \text{MinPrimal}(A^*/A) \) is bijective.

**Proof.** Let \( P \in \text{MinPrimal}(A^*) \). Then by Theorem 3.3, \( P = Q \cap A^* \) for some \( Q \in \text{MinPrimal}(M(A)) \); and by [11, Theorem 2.4], \( Q = L_F \) for some minimal prime \( z \)-filter \( F \) on \( X \). Hence \( \sigma(P) = D_F \). Suppose that \( \sigma(P) = A^*/A \). Then, by Proposition 5.2, there is a non-empty finite subset \( W \) of \( X \) such that \( F = \{Z \in Z[X] : Z \supseteq W\} \). Since \( F \) is prime, \( W = \{y\} \) for some \( y \in X \) and so \( F = Z[M_y] \). Then \( M_y \) is a minimal prime \( z \)-ideal and hence \( M_y = O_y \) [17, 14.12] and so \( y \) is a \( P \)-point of \( X \) [17, 4L]. We have \( L_F = L_{Z[M_y]} = H_y = \tilde{J}_y \) [8, Theorem 4.5] and so \( P = \tilde{J}_y \cap A^* \).

Conversely, if \( y \in Y \) then, again, \( \tilde{J}_y = H_y \). Since \( H_y \) is a Glimm ideal and \( \tilde{J}_y \) is primal [7, Lemma 4.5], it follows that \( \tilde{J}_y \) is a minimal closed primal ideal of \( M(A) \), so \( \tilde{J}_y \cap A^* \) is a minimal closed primal ideal by Theorem 3.3. But \( (\tilde{J}_y \cap A^*) + A = A^* \), so we see that \( \sigma(\tilde{J}_y \cap A^*) \) is not a proper ideal in \( A^*/A \) and hence is not a minimal closed primal ideal in \( A^*/A \).

Let \( R \) be any proper closed primal ideal of \( A^*/A \). Then \( T := \sigma^{-1}(R) \) is a proper closed primal ideal of \( A^* \), so there exists \( P' \in \text{MinPrimal}(A^*) \) such that \( T \supseteq P' \). Hence \( \sigma(T) = R \supseteq \sigma(P') \), which is primal by Theorem 5.5. This shows that \( P' \not\in S \), and also that every minimal closed primal ideal of \( A^*/A \) has the form \( \sigma(P') \) for some \( P' \in \text{MinPrimal}(A^*) \setminus S \).

Let \( P \in \text{MinPrimal}(A^*) \setminus S \). By Theorem 5.5, \( \sigma(P) \) is a closed primal ideal of \( A^*/A \) and hence contains a minimal closed primal ideal of \( A^*/A \). As seen above, the latter has the form \( \sigma(P') \) for some \( P' \in \text{MinPrimal}(A^*) \setminus S \). Since \( \sigma(P') \supseteq \sigma(P) \), \( P + A \supseteq P' + A \). We have \( P = L_F \cap A^* \) and \( P' = L_{F'} \cap A^* \) where \( F \) and \( F' \) are minimal prime \( z \)-filters on \( X \). Note that \( F \) and \( F' \) are convergent to points of \( \beta X \) by [17, 7.15 and 10H1]. Suppose that \( F = Z[M_y] \) for some \( y \in X \). Then \( M_y \) is a minimal prime \( z \)-ideal and hence \( M_y = O_y \) [17, 14.12] and so \( y \in Y \) [17, 4L], contradicting the fact that \( P \not\in S \). Similarly, \( F' \) does not have the form \( Z[M_y] \) (\( y \in X \)). We have that \( D_F \supseteq D_{F'} \) and hence \( F \supseteq F' \) by Proposition 5.4. By minimality, \( F = F' \) and so \( \sigma(P) = \sigma(P') \) and hence \( \sigma(P) \) is a minimal closed primal ideal of \( A^*/A \) as required.

Finally, the injectivity of the map induced by \( \sigma \) follows from Proposition 5.4. \( \square \)

Note that Theorem 5.6 shows that the minimal closed primal ideals of \( A^*/A \) are those proper ideals of \( A^*/A \) which are obtained by intersecting with \( A^*/A \) those ideals of the corona algebra.
C(A) which lie in the image of Min(C_R(X)) under the second embedding map of [8, Theorem 3.3].

6. SOME EXAMPLES AND APPLICATIONS

In this section we investigate some examples, and study the topology on MinPrimal(A^*/A). We show that if X is locally compact, \( \sigma \)-compact and without isolated points and \( A = C_0(X) \otimes K(H) \) then MinPrimal(A^*/A) is not weakly Lindelof (recall that a topological space \( Y \) is weakly Lindelof if for any open cover \( U \) of \( Y \), there is a countable \( V \subseteq U \) such that \( \bigcup V \) is dense in \( Y \)). It follows that if \( X \) is also second countable then the induced bijective map in Theorem 5.6 is not continuous (Corollary 6.6). On the other hand, if \( X \) is an infinite F-space without isolated points, such as \( \beta \mathbb{N} \setminus \mathbb{N} \), then we are able to show that MinPrimal(A^*/A) and Glimm(A^*/A) coincide as sets (Theorem 6.8) but not as topological spaces (Theorem 6.9).

As well as the topology \( \tau_w \), we shall use another topology \( \tau_s \), defined on the set Id(A) of closed ideals of a C*-algebra A as the weakest topology with regard to which all the norm functions \( I \mapsto \|a + I\| \) (\( I \in \text{Id}(A) \), \( a \in A \)) are continuous [15], [3]. It is known that \( \tau_s \) coincides with \( \tau_w \) when restricted to MinPrimal(A) [3, Corollary 4.3]. We shall also make use of the following lemma in Example 6.3, Theorem 6.8 and Theorem 6.9. Recall the definition of \( D_F \) that was given before Proposition 5.2.

**Lemma 6.1.** Let \( A \) be a \( \sigma \)-unital \( C_0(X) \)-algebra. Suppose that \( X_\phi \) is infinite and that \( A/J_x \) is non-unital for all \( x \in X_\phi \). Let \( \sigma : A^* \to A^*/A \) be the quotient map. Let \( F \) be a z-filter on \( X_\phi \) that converges to some \( y \in X_\phi \) and let \( b \in A^* \cap \tilde{J}_y \). Then
\[
\|\sigma(b) + D_F\| = \|b + L_F\|.
\]

**Proof.** Let \( \nu : A^* \cap \tilde{J}_y \to (A^*/A)/D_F \) be the *-homomorphism given by \( \nu(c) = \sigma(c) + D_F \).

By Lemma 5.3, \( \ker \nu = (A^* \cap \tilde{J}_y) \cap L_F = A^* \cap L_F \) (recall that \( L_F \subseteq \tilde{J}_y \)). Thus
\[
\|\sigma(b) + D_F\| = \|b + \ker \nu\| = \|b + L_F\|
\]
(by two standard isomorphisms).

**Example 6.2.** Let \( X := \mathbb{N} \) and set \( A = C_0(X) \otimes K(H) \). Then the (minimal) prime z-filters on \( X \) are precisely the fixed and the free ultrafilters. The minimal prime z-filters corresponding to the points \( p \in \mathbb{N} \) have the form \( Z[O_\nu] = \{ Z \in Z[X] : p \in Z \} \) while those corresponding to the points of \( \beta \mathbb{N} \setminus \mathbb{N} \) have the form \( Z[O^p] = \{ Z \in Z[X] : Z \in \mathcal{F}_p \} \) where \( \mathcal{F}_p \) is the free ultrafilter on \( \mathbb{N} \) associated with \( p \in \beta \mathbb{N} \setminus \mathbb{N} \). It follows from Theorem 3.3 and [11, Theorem 3.4] that MinPrimal(A^*) is homeomorphic to \( \beta \mathbb{N} \). Furthermore, the ideals \( L_{\mathcal{F}_p} \cap A^* \) are primitive by Theorem 4.1, and contain \( A \). Thus Theorem 5.6 implies that MinPrimal(A^*/A) = \{ D_{\mathcal{F}_p} : p \in \beta \mathbb{N} \setminus \mathbb{N} \}. Since each \( L_{\mathcal{F}_p} \cap A^* \) contains \( A \), the bijective map of Theorem 5.6 from MinPrimal(A^*) \( \setminus S \) onto MinPrimal(A^*/A) is bi-continuous for the \( \tau_s \) topologies and so MinPrimal(A^*/A) is canonically homeomorphic to \( \beta \mathbb{N} \setminus \mathbb{N} \). Hence MinPrimal(A^*/A) is a compact F-space and every minimal closed primal ideal of A^*/A is primitive.

Since the minimal closed primal ideals of A^* coincide with the Glimm ideals of A^* (see the discussion preceding Corollary 3.8 and [8, Theorem 4.3(ii)]) and since continuous functions on Prim(A^*) restrict to continuous functions on Prim(A^*/A), the minimal closed primal ideals
of \( A^*/A \) are not only primitive ideals but also Glimm ideals. Using [6, Lemma 3.1(iii)], we see that the identity map from the compact space \( \text{MinPrimal}(A^*/A), \tau_s \) to the Hausdorff space \( (\text{Glimm}(A^*/A), \tau_{sr}) \) is continuous and hence is a homeomorphism. Thus \( A^*/A \) is quasi-standard (as are \( A^* \) and \( M(A) \) by Corollary 3.8).

Example 6.3. Let \( X := \mathbb{N} \cup \{\infty\} \) be the one-point compactification of \( \mathbb{N} \), and set \( A = C(X) \otimes K(H) \). Then it was shown by Kohls that \( \text{Min}(\mathbb{C}_R(X)) \) is homeomorphic to \( \beta\mathbb{N} \) [21, [18, p. 110], [17, 14G]. Again the minimal prime \( z \)-filters corresponding to the points \( p \in \mathbb{N} \) have the form \( Z[O_p] = \{ Z \in Z[X] : p \in Z \} \) while those corresponding to \( p \in \beta\mathbb{N} \setminus \mathbb{N} \) have the form \( \mathcal{F}_p = \{ Z \in Z[X] : Z \setminus \{\infty\} \in \mathcal{F}_p^Z \} \) where \( \mathcal{F}_p^Z \) is the free ultrafilter on \( \mathbb{N} \) associated with \( p \). Once again it follows from Theorem 3.3 and [11, Theorem 3.4] that \( \text{MinPrimal}(A^*) \) is homeomorphic to \( \beta\mathbb{N} \), and the ideals \( (L_{\mathcal{F}_p} \cap A^*) + A \) (\( p \in \beta\mathbb{N} \setminus \mathbb{N} \)) are primitive ideals of \( A^* \) by Corollary 4.6 (with \( Z = \{\infty\} \) and \( Y = \mathbb{N} \)). Thus again Theorem 5.6 implies that \( \text{MinPrimal}(A^*/A) = \{ D_{\mathcal{F}_p} : p \in \beta\mathbb{N} \setminus \mathbb{N} \} \).

For the topology on \( \text{MinPrimal}(A^*/A) \), let \( b \in A^* \) and let \( \sigma : A^* \to A^*/A \) denote the quotient map. Let \( a \in A \) such that \( c := b - a \in \tilde{J}_\infty \). For \( p \in \beta\mathbb{N} \setminus \mathbb{N} \), \( \mathcal{F}_p \) is convergent in \( X \) to the point \( \infty \) and so by Lemma 6.1

\[
\|\sigma(b) + D_{\mathcal{F}_p}\| = \|\sigma(c) + D_{\mathcal{F}_p}\| = \|c + L_{\mathcal{F}_p}\| = \|c + (L_{\mathcal{F}_p} \cap A^*)\|.
\]

It follows that the bijective map of Theorem 5.6 from the compact space \( \text{MinPrimal}(A^*) \setminus S \) onto the Hausdorff space \( \text{MinPrimal}(A^*/A) \) is continuous for the \( \tau_s \) topologies and therefore is a homeomorphism. Hence again \( \text{MinPrimal}(A^*/A) \) is a compact F-space and every minimal closed primal ideal of \( A^*/A \) is primitive.

Example 6.2 and Example 6.3 are unusual in the prevalence of isolated points in \( X \), and we shall now see that if \( A \) is separable and \( X \) has no isolated points then the topologies on \( \text{Min}(\mathbb{C}_R(X)) \) and \( \text{MinPrimal}(A^*/A) \) are very different from each other.

Proposition 6.4. Let \( B \) be a \( C^* \)-algebra such that \( \text{MinPrimal}(B) \) is weakly Lindelof. Then \( \text{Prim}(B) \) is weakly Lindelof.

Proof. Let \( \{U_\alpha\} \) be an open cover of \( \text{Prim}(B) \). Then for each \( \alpha \) there is a closed ideal \( I_\alpha \) of \( B \) such \( U_\alpha = \{ P \in \text{Prim}(B) : P \not\supseteq I_\alpha \} \). Set \( V_\alpha = \{ R \in \text{Prim}(B) : R \not\supseteq I_\alpha \} \). Since each proper primal ideal of \( B \) is contained in a primitive ideal it follows that \( \{V_\alpha\} \) is an open cover of \( \text{Prim}(B) \). By assumption there is a countable family \( \{V_i\}_{i \geq 1} \) of the sets \( V_\alpha \) such that \( W := \text{MinPrimal}(B) \cap \bigcup_{i \geq 1} V_i \) is dense in \( \text{MinPrimal}(B) \). We claim that \( Y := \bigcup_{i \geq 1} U_i \) is dense in \( \text{Prim}(B) \). Let \( U = \{ P \in \text{Prim}(B) : P \not\supseteq I \} \) be any non-empty open subset of \( \text{Prim}(B) \) and set \( V = \{ R \in \text{Prim}(B) : R \not\supseteq I \} \). Then \( W \cap V \) is a non-empty open subset of \( \text{MinPrimal}(B) \) so there exists \( S \in W \cap V \). Then \( S \in V_i \) for some \( i \geq 1 \), so \( V \cap V_i \) is an open neighbourhood of \( S \) in \( \text{Prim}(B) \). Hence by [3, Proposition 3.1] there exists \( P \in \text{Prim}(B) \) such that \( P \in V_i \cap V \). But \( V_i \cap V \cap \text{Prim}(B) = U_i \cap U \), so \( Y \cap U \) is non-empty as required. \( \square \)

Theorem 6.5. Let \( A \) be a continuous \( C_0(X) \)-algebra with base map \( \phi \). Suppose either that \( A \) is separable or that \( A = C_0(X) \otimes K(H) \) where \( X \) is a \( \sigma \)-compact, locally compact Hausdorff space. If \( A/J_x \) is non-unital for all \( x \in X_\phi \) and if \( X_\phi \) has no isolated points then \( \text{MinPrimal}(A^*/A) \) is not weakly Lindelof.
Proof. This follows from [12, Theorem 6.7] and Proposition 6.4.

Corollary 6.6. Let $A$ be a separable quasi-standard $C^*$-algebra and set $X = \text{Glimm}(A)$. Suppose that $A/J_x$ is non-unital for all $x \in X$ and that $X$ has no isolated points. Then the bijective map from $\text{MinPrimal}(A^*)$ to $\text{MinPrimal}(A^*/A)$ described in Theorem 5.6 is not continuous.

Proof. Since $A$ is separable and quasi-standard, the set $X \cap \text{Prim}(A)$ is dense in $X$ and so Theorem 5.6 applies. Since every singleton subset of $X$ is a zero set [9, Lemma 3.9], every $P$-point in $X$ is an isolated point, and thus the sets $Y$ and $S$ of Theorem 5.6 are empty. The map $P \mapsto \sigma(P)$ ($P \in \text{MinPrimal}(A^*)$) is therefore a bijection between $\text{MinPrimal}(A^*)$ and $\text{MinPrimal}(A^*/A)$. But $\text{MinPrimal}(A^*)$ is compact and extremally disconnected by Corollary 3.6, whereas $\text{MinPrimal}(A^*/A)$ is not weakly Lindelof by Theorem 6.5. Thus the map $P \mapsto \sigma(P)$ is not continuous.

We are far from having a complete description of the topology on $\text{MinPrimal}(A^*/A)$ but the next theorem sheds further light on the failure of the weak Lindelof property.

Theorem 6.7. Let $A$ be a $\sigma$-unital quasi-standard $C^*$-algebra and set $X = \text{Glimm}(A)$. Suppose that $A/J_x$ is non-unital for all $x \in X$ and that $X$ is first countable. Let $p$ be a non-isolated point in $X$. Then the set of minimal closed primal ideals of $A^*/A$ which contain $D_{Z[O_p]}$ is a non-empty clopen subset of $\text{MinPrimal}(A^*/A)$.

Proof. Set $Z = \{p\}$. Then $Z$ is a zero set in $X$ by the first countability of $X$. Let $c^Z \in A^*$ with the properties of Lemma 2.5. In particular, $c^Z \in \mathcal{J}_p$. Since $X$ is locally compact, there exists a continuous function $f : X \to [0,1]$ such that $f(V) = \{1\}$ for some neighbourhood $V$ of $p$ in $X$ and $C := \text{supp}(f)$ is compact. Furthermore, $V$ contains a zero set neighbourhood $N$ of $p$. Set $z := \mu(f) \in ZM(A)$, where $\mu : C_0(X) \to ZM(A)$ is the structure map. Let $F$ be a minimal prime $z$-filter on $X$.

Suppose first of all that there exists $F \in \mathcal{F}$ such that $C \cap F = \emptyset$. Then

$$F \subseteq X \setminus C \subseteq Z(z) \subseteq Z(zc^Z)$$

and so $Z(zc^Z) \in \mathcal{F}$, $zc^Z \in L_F \cap A^*$ and $\sigma(zc^Z) \in D_F$.

On the other hand, suppose that $C \cap F$ is non-empty for all $F \in \mathcal{F}$. Since $C$ is compact, there exists $q \in C$ such that, for all $F \in \mathcal{F}$, $q \in C \cap F$. Thus $I(\mathcal{F}) \subseteq M_q$ and so, since $I(\mathcal{F})$ is a prime $z$-ideal, $O_q \subseteq I(\mathcal{F})$ [17, 41]. Hence $Z[O_q] \subseteq \mathcal{F}$, that is, $\mathcal{F}$ is convergent to $q$. Suppose that $q \neq p$. Then by Lemma 2.5(iii) there exists $a \in A$ such that $c^Z - a \in H_q = L_{Z[O_q]} \subseteq L_F$. Thus $c^Z - a \in L_F \cap A^*$ and again $\sigma(zc^Z) \in D_F$.

Finally, suppose that $q = p$ and note that this case will occur since $Z[O_p]$ is the intersection of the minimal prime $z$-filters on $X$ that contain it [17, 14.12]. Let $b \in L_{\mathcal{F}}^\text{alg}$. Then $Z(b) \in \mathcal{F}$ and so $Z(b) \cap N \in \mathcal{F}$. Since $p$ is not an isolated point in $X$, $Z(b) \cap N \neq \{p\}$ and so the non-empty set $Z(b) \cap N$ contains some point $x \in V \setminus \{p\}$. Then

$$\|zc^Z - b\| \geq \|(zc^Z - b) + \tilde{J}_x\| = \|zc^Z + \tilde{J}_x\| = f(x)\|c^Z + \tilde{J}_x\| = 1.$$ 

Since $zc^Z \in \mathcal{J}_p$, it follows from Lemma 6.1 that

$$1 \geq \|\sigma(zc^Z) + D_F\| = \|zc^Z + L_F\| \geq 1$$

and hence $\|\sigma(zc^Z) + D_F\| = 1$. 

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It now follows that the set of minimal closed primal ideals of \(A^s/A\) which contain \(D_{\beta X}[O]\) is a non-empty \(\tau_s\)-clopen subset of \(\text{MinPrimal}(A^s/A)\).

\[\Box\]

Theorem 6.7 can be used to give an alternative proof of Theorem 6.5 in the case \(A = C_0(X) \otimes K(H)\) where the locally compact Hausdorff space \(X\) is \(\sigma\)-compact, first countable, and without isolated points. Theorem 6.7 shows that in this case \(\text{MinPrimal}(A^s/A)\) can be covered by an uncountable family of disjoint clopen subsets, and hence is not weakly Lindelof.

We conclude with an application to a very different class of \(C^*\)-algebras.

**Theorem 6.8.** Let \(A\) be a \(\sigma\)-unital, quasi-standard \(C^*\)-algebra. Set \(X = \text{Glimm}(A)\) and suppose that

(i) \(X \cap \text{Prim}(A)\) is dense in \(X\) and \(A/J_x\) is non-unital for all \(x \in X\),

(ii) \(X\) is an infinite F-space.

Then every Glimm ideal of \(A^s/A\) is primal. Hence the relation \(\sim\) of inseparability by disjoint open sets is an equivalence relation on \(\text{Prim}(A^s/A)\).

**Proof.** Since \(X\) is an F-space, the minimal prime \(z\)-filters on \(X\) are of the form \(Z[O^x]\) for \(x \in \beta X\) [17, 7.15, 14.25]. Hence the minimal closed primal ideals of \(M(A)\) are of the form \(H_x = L_{Z[O^x]}(x \in \beta X)\) by [11, Proposition 2.5 and Theorem 3.4]. Thus by Theorem 5.6, the minimal closed primal ideals of \(A^s/A\) have the form \(D_{Z[O^x]} = ((H_x \cap A^s) + A)/A\) for \(x \in \beta X \setminus Y\) (where \(Y\) is the set of P-points in \(X\)).

For distinct \(x, y \in \beta X \setminus Y\), let \(h \in C(\beta X)\) with \(0 = h(x) \neq h(y)\). Set \(f := h \circ \overline{\phi_A} \in C(\text{Prim}(M(A)))\) and let \(g = f|_{\text{Prim}(A^s/A)}\). Then \(g \in \text{ch}(\text{Prim}(A^s/A))\) and \(g(R) = 0\) for \(R \in \text{hull}(\sigma(H_x \cap A^s))\) but \(g(R) \neq 0\) for \(R \in \text{hull}(\sigma(H_y \cap A^s))\). Thus distinct minimal closed primal ideals of \(A^s/A\) contain distinct Glimm ideals. Every closed primal ideal in a \(C^*\)-algebra \(B\) contains a unique Glimm ideal [6, Lemma 2.2], from which it follows that every Glimm ideal of \(B\) is the intersection of the minimal closed primal ideals containing it. Thus we have shown that every Glimm ideal of \(A^s/A\) is a minimal closed primal ideal; and from this it follows that the relation \(\sim\) is an equivalence relation on \(\text{Prim}(A^s/A)\) [6, Lemma 3.1(ii)].

\[\Box\]

Although the sets \(\text{MinPrimal}(A^s/A)\) and \(\text{Glimm}(A^s/A)\) coincide in the context of Theorem 6.8, the topologies on the two spaces are problematic. Various set-theoretic considerations come into play, such as the presence or absence of P-points in \(X\). Although we may not be able to identify the topologies \(\tau_s\) and \(\tau_{cr}\), we do have enough information about them to show that if \(X\) has no isolated points then they are not equal.

In preparation for this, we assume that \(A\) satisfies the hypotheses of Theorem 6.8. Firstly, it follows from [6, Lemma 3.1(iii)] that the \(\tau_s\)-topology on \(\text{MinPrimal}(A^s/A)\) is finer than the complete regularization topology \(\tau_{cr}\) on \(\text{Glimm}(A^s/A)\). Secondly, set \(Y^c := \beta X \setminus Y\) where \(Y\) is the set of P-points in \(X\). It follows from [12, Theorem 5.2] that \(Y^c = \psi(\text{Prim}(A^s/A))\), where \(\psi\) is the restriction to \(\text{Prim}(A^s/A)\) of the continuous map \(\overline{\phi_A} : \text{Prim}(M(A)) \to \beta X\). On the other hand, it follows from the proof of Theorem 6.8 that there is a bijection \(\rho : Y^c \to \text{Glimm}(A^s/A)\), given by

\[\rho(x) = D_{Z[O^x]} = \frac{(H_x \cap A^s) + A}{A} \quad (x \in Y^c),\]
such that \( \rho \circ \psi = \phi_{(A^*/A)} \), the complete regularisation map for the \( C^* \)-algebra \( A^*/A \). As a subspace of the compact Hausdorff space \( \beta X \), \( Y^c \) is completely regular. It follows from the universal property for the complete regularisation that \( \rho^{-1} \) is \( \tau_{\sigma} \)-continuous. Thirdly, the bijection \( \rho \) induces from the topologies \( \tau_s \) and \( \tau_{\sigma} \) on \( \text{Glimm}(A^*/A) \) two topologies on \( Y^c \) which we denote again by \( \tau_s \) and \( \tau_{\sigma} \) respectively. Thus, on \( Y^c \), \( \tau_s \) is finer than \( \tau_{\sigma} \) which in turn is finer than the relative topology on \( Y^c \) from \( \beta X \).

**Theorem 6.9.** Let \( A \) be a \( \sigma \)-unital, quasi-standard \( C^* \)-algebra. Set \( X = \text{Glimm}(A) \) and suppose that

(i) \( X \cap \text{Prim}(A) \) is dense in \( X \) and \( A/J_x \) is non-unital for all \( x \in X \),

(ii) \( X \) is an \( F \)-space without isolated points.

Then \( A^*/A \) is not quasi-standard.

**Proof.** Let \( V := X \setminus Y \) be the set of non-P-points of the locally compact space \( X \). Thus \( V \) is the union of the boundaries of the zero sets in \( X \) and \( V \) is contained in the subset \( Y^c \) of \( \beta X \) discussed above. Since a compact \( P \)-space is finite and \( X \) has no isolated points, \( V \) is dense in \( X \). Let \( U \) be an arbitrary non-empty, proper cozero set in \( X \), let \( Z := X \setminus U \) and let \( c^Z \in A^* \) with the properties of Lemma 2.5. Then \( \| c^Z \| = 1 \) and, for \( x \in Z \), \( c^Z \in \tilde{J}_x \) and so by Lemma 6.1

\[
\| \sigma(c^Z) + D_{Z[A_0]} \| = \| c^Z + H_x \|.
\]

On the other hand, if \( y \in U \) then

\[
1 \geq \| c^Z + H_y \| \geq \| c^Z + \tilde{J}_y \| = 1,
\]

so that \( \| c^Z + H_y \| = 1 \) and hence, by upper semi-continuity, \( \| c^Z + H_x \| = 1 \) for all \( x \) in the boundary (in \( X \)) of \( Z \). Thus, for \( x \) in the boundary of \( Z \), \( \| \sigma(c^Z) + D_{Z[A_0]} \| = 1 \).

Suppose next that \( x \in \text{int}(Z) \cap V \). Then \( \| c^Z + \tilde{J}_y \| = 0 \) for all \( y \) in a neighbourhood of \( x \) and hence \( \| c^Z + H_x \| = 0 \) by [7, Lemma 1.5]. Thus \( c^Z \in H_x \cap A^* \) and hence \( \sigma(c^Z) \in D_{Z[A_0]} \).

Now suppose that \( x \in U \cap V \). By Lemma 2.5(iii) there exists \( a \in A \) such that \( c^Z - a \in H_x \cap A^* \) and hence \( \sigma(c^Z) \in D_{Z[A_0]} \). Bearing in mind that the \( \tau_s \)-topology on \( \text{MinPrimal}(A^*/A) \) is finer than the relative topology on \( V \) from \( X \), we see that the three sets consisting of \( Z \cap V \), the boundary of \( Z \), and \( U \cap V \) are all \( \tau_s \)-clopen in \( V \).

Now suppose that we can find subsets \( U \) and \( W \) of \( X \) with the following properties: \( U \) is a non-empty cozero set of \( X \) such that \( U \) is non-closed in \( X \) but has compact closure, and \( W \) is a zero set of \( U \) such that the boundary (in \( U \)) of \( W \) is non-compact. Set \( T = (X \setminus U) \cup W \). Then \( T \) is easily seen to be a zero set of \( X \). Let \( c^T \in A^* \) with the properties of Lemma 2.5. Since \( X \) is locally compact, \( X \) is open in \( \beta X \). We have \( U \cap V = U \cap Y^c \), which is relatively open in \( Y^c \) (for the topology from \( \beta X \)) and hence \( \tau_{\sigma} \)-open in \( Y^c \). On the other hand, since the closure of \( U \) in \( X \) is compact and hence closed in \( \beta X \), it follows that the \( \tau_{\sigma} \)-closure of \( U \cap V \) in \( Y^c \) is contained in \( V \). We now suppose, for a contradiction, that \( U \cap V \) is \( \tau_{\sigma} \)-closed in \( V \). Then \( U \cap V \) is clopen in the \( \tau_{\sigma} \) topology on \( Y^c \). The characteristic function \( \chi \) of the \( \tau_{\sigma} \)-clopen subset \( \rho(U \cap V) \) of \( \text{Glimm}(A^*/A) \) induces a central projection \( z \in M(A^*/A) \) such that

\[
z d + D_{Z[A_0]} = \chi(\rho(x))(d + D_{Z[A_0]}) \quad (d \in A^*/A, x \in Y^c).
\]

Set \( b := z\sigma(c^T) \in A^*/A \). Then the function \( x \mapsto \| b + D_{Z[A_0]} \| (x \in V) \) is upper semi-continuous for \( \tau_{\sigma} \) on \( V \). If \( x \) is in the boundary of \( W \) in \( U \), then \( x \) is in the boundary of \( T \) in \( X \) and hence \( \| c^T + D_{Z[A_0]} \| = 1 \) (as for \( c^Z \) in the first part of the proof). Since
$x \in U \cap V$, $\|b + D_{Z(O_s)}\| = 1$. On the other hand, $\|b + D_{Z(O_s)}\| = 0$ for $x \in V \setminus U$. As the boundary of $W$ in $U$ is non-compact, its closure in $X$ meets $V \setminus U$, contradicting the $\tau_{cr}$ upper semi-continuity of the norm function. Thus if such $U$ and $W$ exist, we can conclude that $U \cap V$ is not closed in the $\tau_{cr}$ topology on $V$, whereas $U \cap V$ is closed in the $\tau_s$ topology on $V$ by the first part of the proof.

To complete the proof it remains to show that such sets $U$ and $W$ can be found. Let $x$ be any point in $V$. There exists a continuous function $f_1 : X \to [0, 1]$ such that $x$ lies in the boundary of $Z(f_1)$. Let $f_2 : X \to [0, 1]$ be a continuous function with compact support which is identically 1 on some neighbourhood of $x$. Set $f = f_1f_2$ and $U = X \setminus Z(f)$. Then $U$ is compact but $x \in U \setminus U$. Since $U$ is open and $V$ is dense in $X$, there is a net $(v_\alpha)$ in $U \cap V$ with limit $x$. Then $f(v_\alpha) \to f(x) = 0$ and so there exists a sequence $(x_i)_{i \geq 1}$ of distinct points of $U \cap V$ such that $f(x_i) \to 0$. Let $C = \{x_i : i \geq 1\}$, a relatively discrete subset of $U$.

For each $i \geq 1$, let $U_i$ be a neighbourhood of $x_i$ in $U$ disjoint from $C \setminus \{x_i\}$ and let $g_i : U \to [0, 1]$ be a continuous function supported in $U_i$ such that $x_i$ lies in the boundary (in $U$) of the zero set of $g_i$. Since $x_i \in V$ there is a continuous function $f_i : X \to [0, 1]$ such that $x_i$ lies in the boundary of $Z(f_i)$. Let $h_i = f_i|_U$. Since $U$ is open, $x_i$ lies in the boundary of $U$ of $Z(h_i)$. Then we may obtain $g_i$ by multiplying $h_i$ by a continuous function from $U$ to $[0, 1]$ which is identically 1 on a neighbourhood of $x_i$ and supported in $U_i$. Set $g = \sum_{i=1}^{\infty} g_i/2^i$. Then $g \in C_\mathbb{R}(U)$ and $C$ lies in the boundary in $U$ of the zero set of $g$. Suppose, for a contradiction, that the boundary in $U$ of $Z(g)$ is compact. Then there is a subnet of the sequence $(x_i)$ convergent to some point $u \in U$. Since $f$ is continuous, $f(u) = 0$, contradicting the fact that $u \in U$. Hence taking $W = Z(g)$ gives the required set. It follows that $U \cap V$ is not closed in the $\tau_{cr}$ topology on $V$, so the $\tau_s$ and the $\tau_{cr}$ topologies do not coincide on $V$, and $A^*/A$ is not quasi-standard [6, Theorem 3.3].

Corollary 6.10. Under the hypothesis of Theorem 6.9, the corona algebra $M(A)/A$ is not quasi-standard and also $M(A^*/A)$ is not quasi-standard.

Proof. If $B$ is a quasi-standard $C^*$-algebra then every closed ideal of $B$ is quasi-standard [6, p. 356]. Hence it follows from Theorem 6.9 that neither $M(A)/A$ nor $M(A^*/A)$ is quasi-standard.

It was shown in [12, Theorem 5.5] that if $A = C(\beta \mathbb{N}\setminus \mathbb{N}) \otimes K(H)$ then, under the Continuum Hypothesis, $M(A)/A$ is canonically isomorphic to a proper $C^*$-subalgebra of $M(A^*/A)$.

References

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