

THE STRUCTURE OF Q-W ALGEBRAS

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Abstract. We suggest two explicit descriptions of the Poisson q-W algebras which are Poisson algebras of regular functions on certain algebraic group analogues of the Slodowy transversal slices to adjoint orbits in a complex semisimple Lie algebra \mathfrak{g} . To obtain the first description we introduce certain projection operators which are analogous to the quasi-classical versions of the so-called Zhelobenko and extremal projection operators. As a byproduct we obtain some new formulas for natural coordinates on Bruhat cells in algebraic groups.

1. Introduction

The purpose of this paper is to give two descriptions of the Poisson structure for Poisson q-W algebras which are group-like analogues of ordinary Poisson W-algebras. As algebras Poisson q-W algebras are algebras of regular functions on transversal slices to conjugacy slices in algebraic groups defined in [11] while their Poisson structures are obtained by Poisson reduction from Poisson structures of Poisson–Lie groups dual to quasi-triangular Poisson Lie groups. If G is a complex semi-simple quasi-triangular Poisson–Lie group used in the definition of q-W algebras then its dual group G^* is solvable and there is a smooth morphism of manifolds $q : G^* \rightarrow G$ the image of which is dense in G . Actually this morphism is a finite cover over its image, so that one can equip G with a Poisson structure such that q becomes a Poisson mapping. Denote by G_* the Poisson manifold obtained this way. In order to define the corresponding Poisson q-W-algebra one has to fix a coisotropic submanifold $\mathcal{C} \subset G_*$ and then factorize it over a free action of a subgroup $N \subset G$ on \mathcal{C} induced by the conjugation action of N on G . The reduced Poisson manifold $\Sigma = \mathcal{C}/N$ has an explicit description as a submanifold in \mathcal{C} and carries the reduced Poisson structure.

In fact G and N are closed algebraic groups and $\mathcal{C} \subset G_*$ is a closed subvariety, so that Σ is a closed subvariety in G_* , and the Poisson q-W algebra is the Poisson algebra of regular functions on Σ . Our first task is to obtain a projection operator $\Pi : \mathbb{C}[\mathcal{C}] \rightarrow \mathbb{C}[\mathcal{C}]^N = \mathbb{C}[\Sigma]$ with the help of which one can explicitly describe the

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reduced Poisson structure in terms of the Poisson bracket on G_* (see Proposition 3.4).

The projection operator Π is quite remarkable. It is an analogue of the quasi-classical versions of the so-called Zhelobenko and extremal projection operators introduced in [17] to describe subspaces of singular vectors in modules from the BGG category \mathcal{O} . In turn, as it was observed in [14], the quasi-classical versions of Zhelobenko and extremal projection operators are examples of the realization of the following simple construction.

Let U be an algebraic group, and M an algebraic variety equipped with a regular action of U . Assume that there exists a cross-section $X \subset M$ for this action in the sense that the map

$$U \times X \rightarrow M, (g, x) \mapsto gx, \quad g \in U, x \in X \tag{1.1}$$

is an isomorphism of varieties.

Denote by $\mathbb{C}[M]$ the algebra of regular functions on M . The group U naturally acts on $\mathbb{C}[M]$. Let $\mathbb{C}[M]^U$ be the subspace of U -invariant elements of $\mathbb{C}[M]$. Define the projection operator $\bar{P} : \mathbb{C}[M] \rightarrow \mathbb{C}[M]^U$ as follows

$$(\bar{P}f)(gx) = f(x) = f(g^{-1}gx) = (gf)(gx), \quad g \in U, x \in X. \tag{1.2}$$

If for $y \in M$ we denote by $g(y) \in U$ the unique element such that $y = g(y)x(y)$ for a unique $x(y) \in X$ then

$$(\bar{P}f)(y) = (g(y)f)(y). \tag{1.3}$$

The operator \bar{P} is called the projection operator corresponding to isomorphism (1.1).

If every element of U can be uniquely represented as a product of elements from subgroups U_1, \dots, U_n , i.e., $U = U_1 \cdots U_n$, then the operator \bar{P} can be expressed as a composition of operators \bar{P}_i ,

$$(\bar{P}_i f)(y) = (g_i(y)f)(y), \quad g(y) = g_1(y) \cdots g_n(y), \quad g_i(y) \in U_i, \tag{1.4}$$

$$(\bar{P}f)(y) = (\bar{P}_1 \cdots \bar{P}_n f)(y). \tag{1.5}$$

The projection operator $\Pi : \mathbb{C}[\mathcal{C}] \rightarrow \mathbb{C}[\mathcal{C}]^N = \mathbb{C}[\Sigma]$ defined in Proposition 2.12 is of the same type with $U = N$, $M = \mathcal{C}$ and U_i being one-parametric subgroups corresponding to some roots. Technically the construction is quite involved and as a preliminary auxiliary exercise we obtain some new formulas for natural coordinates on Bruhat cells in algebraic groups (see Proposition 2.9).

In Proposition 3.3 we give another explicit compact description of the reduced Poisson bracket on Σ . Formula (3.12) for the reduced Poisson structure obtained in that proposition is a simple consequence of the general definition of the reduced Poisson structure and of the explicit description of the reduced space Σ . In case of the usual W-algebras such description was obtained in [3]. But this description is much more complicated than in the case of q-W algebras since the corresponding reduced spaces in the W-algebra case are the Slodowy slices, and they do not have a nice geometric description compatible with the Poisson structure.

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2. Analogues of Zhelobenko operators for q-W algebras

In this section we construct analogues of the classical versions of Zhelobenko operators for q-W-algebras. First we recall the definition of the coisotropic submanifolds \mathcal{C} , of the subgroups N and of the slices Σ . They are associated to conjugacy classes of Weyl group elements and are defined in terms of certain systems of positive roots associated to conjugacy classes of Weyl group elements. We recall these definitions and the related properties of \mathcal{C} , N and Σ following [11], [12], [13], [15]. To define the projection operator Π we shall also need the normal orderings of the systems of positive roots associated to conjugacy classes of Weyl group elements in [12].

Let G be a complex semisimple connected simply-connected algebraic group, \mathfrak{g} its Lie algebra, H a maximal torus in G . Denote by \mathfrak{h} the Cartan subalgebra in \mathfrak{g} corresponding to H . Let Δ be the root system of the pair $(\mathfrak{g}, \mathfrak{l})$, W the Weyl group of the pair $(\mathfrak{g}, \mathfrak{h})$. For any root $\alpha \in \Delta$ we denote by $\alpha^\vee \in \mathfrak{l}$ the corresponding coroot and by $s_\alpha \in W$ the reflection with respect to α . Let $X_\alpha \in \mathfrak{g}$ be a non-zero root vector corresponding to α .

Let s be an element of the Weyl group W and $\mathfrak{l}_\mathbb{R}$ the real form of \mathfrak{h} , the real linear span of simple coroots in \mathfrak{h} . The set of roots Δ is a subset of the dual space $\mathfrak{h}_\mathbb{R}^*$. By Theorem C in [2] s can be represented as a product of two involutions,

$$s = s^1 s^2, \tag{2.1}$$

where $s^1 = s_{\gamma_1} \cdots s_{\gamma_n}$, $s^2 = s_{\gamma_{n+1}} \cdots s_{\gamma_{l'}}$, the roots in each of the sets $\gamma_1, \dots, \gamma_n$ and $\gamma_{n+1}, \dots, \gamma_{l'}$ are positive and mutually orthogonal.

The Weyl group element s naturally acts on $\mathfrak{h}_\mathbb{R}$ as an orthogonal transformation with respect to the scalar product induced by the Killing form of \mathfrak{g} . Using the spectral theory of orthogonal transformations we can decompose $\mathfrak{h}_\mathbb{R}$ into a direct orthogonal sum of s -invariant subspaces,

$$\mathfrak{h}_\mathbb{R} = \bigoplus_{i=0}^K \mathfrak{h}_i, \tag{2.2}$$

where we assume that \mathfrak{h}_0 is the linear subspace of $\mathfrak{h}_\mathbb{R}$ fixed by the action of s , and each of the other subspaces $\mathfrak{h}_i \subset \mathfrak{h}_\mathbb{R}$, $i = 1, \dots, K$, is either two-dimensional or one-dimensional and the Weyl group element s acts on it as rotation with angle θ_i , $0 < \theta_i < \pi$ or as reflection with respect to the origin, respectively. Note that since s has finite order $\theta_i = 2\pi n_i/m_i$, $n_i, m_i \in \{1, 2, 3, \dots\}$. By Proposition 3.1 in [13] the subspaces \mathfrak{h}_i can be chosen in such a way that each of them is invariant with respect to the involutions s^1 and s^2 , and if \mathfrak{h}_i is one-dimensional one of the involutions acts on it in the trivial way. We shall assume that the subspaces \mathfrak{h}_i are chosen in this way. We shall also assume that the one-dimensional subspaces \mathfrak{h}_i on which s^1 acts by multiplication by -1 are immediately following \mathfrak{h}_0 in sum (2.2).

Since the number of roots in the root system Δ is finite one can always choose elements $h_i \in \mathfrak{h}_i$, $i = 0, \dots, K$, such that $h_i(\alpha) \neq 0$ for any root $\alpha \in \Delta$ which is not orthogonal to the s -invariant subspace \mathfrak{h}_i with respect to the natural pairing between $\mathfrak{h}_\mathbb{R}$ and $\mathfrak{h}_\mathbb{R}^*$.

Now we consider certain s -invariant subsets of roots $\Delta_i, i = 0, \dots, K$, defined as follows

$$\Delta_i = \{\alpha \in \Delta : h_j(\alpha) = 0, j > i, h_i(\alpha) \neq 0\}, \tag{2.3}$$

where we formally assume that $h_{K+1} = 0$. Note that for some indexes i the subsets Δ_i are empty, and that the definition of these subsets depends on the order of terms in direct sum (2.2).

Now consider the nonempty s -invariant subsets of roots $\Delta_{i_k}, k = 0, \dots, M$. For convenience we assume that indexes i_k are labeled in such a way that $i_j < i_k$ if and only if $j < k$. According to this definition $\Delta_0 = \{\alpha \in \Delta : s\alpha = \alpha\}$ is the set of roots fixed by the action of s . Observe also that the root system Δ is the disjoint union of the subsets Δ_{i_k} ,

$$\Delta = \bigcup_{k=0}^M \Delta_{i_k}.$$

Suppose that the direct sum $\bigoplus_{k=1}^r \mathfrak{h}_{i_k}$ of the one-dimensional subspaces \mathfrak{h}_{i_k} on which s^1 acts by multiplication by -1 and s^2 acts as the identity transformation is not trivial. Since the one-dimensional subspaces \mathfrak{h}_i on which s^1 acts by multiplication by -1 are immediately following \mathfrak{h}_0 in sum (2.2), the roots from the union $\bigcup_{k=1}^r \Delta_{i_k}$ must be orthogonal to all subspaces \mathfrak{h}_{i_k} on which s^1 does not act by multiplication by -1 and to all roots from the set $\gamma_{n+1}, \dots, \gamma_{l'}$ as s^2 acts trivially on $\bigoplus_{k=1}^r \mathfrak{h}_{i_k}$. Pick up a root $\gamma \in \bigcup_{k=1}^r \Delta_{i_k}$. Then $s^1\gamma = -\gamma$ and by our choice γ is orthogonal to the roots $\gamma_{n+1}, \dots, \gamma_{l'}$. Therefore $s_0^1 = s^1 s_\gamma$ is an involution the dimension of the fixed point space of which is equal to the dimension of the fixed point space of the involution s^1 plus one, and $s_0^2 = s_\gamma s^2$ is another involution the dimension of the fixed point space of which is equal to the dimension of the fixed point space of the involution s^1 minus one. We also have a decomposition $s = s_0^1 s_0^2$.

Now we can apply the above construction of the system of positive roots to the new decomposition of s . Iterating this procedure we shall eventually arrive at the situation when the direct sum $\bigoplus_{k=1}^r \mathfrak{h}_{i_k}$ of the subspaces \mathfrak{h}_{i_k} on which s^1 acts by multiplication by -1 is trivial. From now on we shall only consider decompositions $s = s^1 s^2$ which satisfy this property. This implies

$$s^2\alpha = \alpha \Rightarrow \alpha \in \Delta_0. \tag{2.4}$$

Now assume that

$$|h_{i_k}(\alpha)| > \left| \sum_{l \leq j < k} h_{i_j}(\alpha) \right| \quad \text{for any } \alpha \in \Delta_{i_k}, k = 0, \dots, M, l < k. \tag{2.5}$$

Condition (2.5) can be always fulfilled by suitable rescalings of the elements h_{i_k} .

Consider the element

$$\bar{h} = \sum_{k=0}^M h_{i_k} \in \mathfrak{h}_{\mathbb{R}}. \tag{2.6}$$

From definition (2.3) of the sets Δ_i we obtain that for $\alpha \in \Delta_{i_k}$

$$\bar{h}(\alpha) = \sum_{j \leq k} h_{i_j}(\alpha) = h_{i_k}(\alpha) + \sum_{j < k} h_{i_j}(\alpha). \tag{2.7}$$

Now condition (2.5), the previous identity and the inequality $|x + y| \geq ||x| - |y||$ imply that for $\alpha \in \Delta_{i_k}$ we have

$$|\bar{h}(\alpha)| \geq \left| |h_{i_k}(\alpha)| - \left| \sum_{j < k} h_{i_j}(\alpha) \right| \right| > 0.$$

Since Δ is the disjoint union of the subsets Δ_{i_k} , $\Delta = \bigcup_{k=0}^M \Delta_{i_k}$, the last inequality ensures that \bar{h} belongs to a Weyl chamber of the root system Δ , and one can define the subset of positive roots Δ_+^s with respect to that chamber. We call Δ_+^s a system of positive roots associated to s .

From condition (2.5) and formula (2.7) we also obtain that a root $\alpha \in \Delta_i \neq \emptyset$ is positive if and only if

$$h_i(\alpha) > 0. \tag{2.8}$$

Recall that an ordering of a set of positive roots Δ_+ is called normal if for any three roots α, β, γ such that $\gamma = \alpha + \beta$ we have either $\alpha < \gamma < \beta$ or $\beta < \gamma < \alpha$. Let $\alpha_1, \dots, \alpha_l$ be the simple roots in Δ_+ , s_1, \dots, s_l the corresponding simple reflections. Let \bar{w} be the element of W of maximal length with respect to the system s_1, \dots, s_l of simple reflections. For any reduced decomposition $\bar{w} = s_{i_1} \cdots s_{i_D}$ of \bar{w} the ordering

$$\beta_1 = \alpha_{i_1}, \beta_2 = s_{i_1} \alpha_{i_2}, \dots, \beta_D = s_{i_1} \cdots s_{i_{D-1}} \alpha_{i_D}$$

is a normal ordering in Δ_+ , and there is a one-to-one correspondence between normal orderings of Δ_+ and reduced decompositions of \bar{w} (see [16]).

From this fact and from properties of Coxeter groups it follows that any two normal orderings in Δ_+ can be reduced to each other by the so-called elementary transpositions (see [16, Thm. 1]). The elementary transpositions for rank 2 root systems are inversions of the following normal orderings (or the inverse normal orderings):

α, β	$A_1 + A_1,$	
$\alpha, \alpha + \beta, \beta$	$A_2,$	
$\alpha, \alpha + \beta, \alpha + 2\beta, \beta$	$B_2,$	(2.9)
$\alpha, \alpha + \beta, 2\alpha + 3\beta, \alpha + 2\beta, \alpha + 3\beta, \beta$	G_2	

where it is assumed that $(\alpha, \alpha) \geq (\beta, \beta)$. Moreover, any normal ordering in a rank 2 root system is one of orderings (2.9) or one of the inverse orderings.

In general an elementary inversion of a normal ordering in a set of positive roots Δ_+ is the inversion of an ordered segment of form (2.9) (or of a segment with the inverse ordering) in the ordered set Δ_+ , where $\alpha - \beta \notin \Delta$.

Let $\beta_1, \beta_2, \dots, \beta_D$ be a normal ordering of a positive root system Δ_+ . Then following [7] one can introduce the corresponding circular normal ordering of the

root system Δ where the roots in Δ are located on a circle in the following way

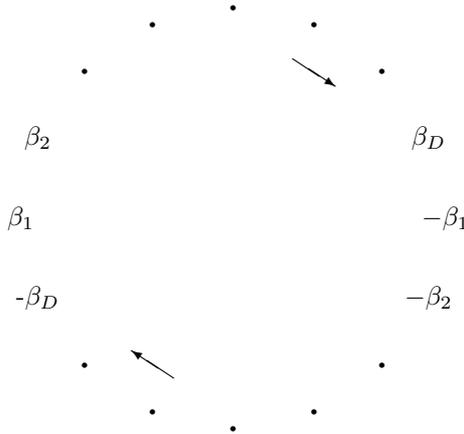


FIG. 1

Let $\alpha, \beta \in \Delta$. One says that the segment $[\alpha, \beta]$ of the circle is minimal if it does not contain the opposite roots $-\alpha$ and $-\beta$ and the root β follows after α on the circle above, the circle being oriented clockwise. In that case one also says that $\alpha < \beta$ in the sense of the circular normal ordering,

$$\alpha < \beta \Leftrightarrow \text{the segment } [\alpha, \beta] \text{ of the circle is minimal.} \tag{2.10}$$

Later we shall need the following property of minimal segments which is a direct consequence of Proposition 3.3 in [6].

Lemma 2.1. *Let $[\alpha, \beta]$ be a minimal segment in a circular normal ordering of a root system Δ . Then if $\alpha + \beta$ is a root we have*

$$\alpha < \alpha + \beta < \beta.$$

The following proposition is a refinement of Proposition 5.1 in [12].

Proposition 2.2. *Let $s \in W$ be an element of the Weyl group W of the pair $(\mathfrak{g}, \mathfrak{h})$, Δ the root system of the pair $(\mathfrak{g}, \mathfrak{h})$, Δ_+^s a system of positive roots associated to s . For any Weyl group element $w \in W$ we denote $\Delta_w = \{\alpha \in \Delta_+^s : w\alpha \in \Delta_-^s\}$, where $\Delta_-^s = -\Delta_+^s$. Then the decomposition $s = s^1 s^2$ is reduced in the sense that $l(s) = l(s^2) + l(s^1)$, where $l(\cdot)$ is the length function in W with respect to the system of simple roots in Δ_+^s , and $\Delta_s = \Delta_{s^2} \cup s^2(\Delta_{s^1})$, $\Delta_{s^{-1}} = \Delta_{s^1} \cup s^1(\Delta_{s^2})$ (disjoint unions). Here s^1, s^2 are the involutions entering decomposition (2.1), $s^1 = s_{\gamma_1} \dots s_{\gamma_n}$, $s^2 = s_{\gamma_{n+1}} \dots s_{\gamma_{\nu}}$ and satisfying (2.4), the roots in each of the sets $\gamma_1, \dots, \gamma_n$ and $\gamma_{n+1}, \dots, \gamma_{\nu}$ are positive and mutually orthogonal.*

Moreover, there is a normal ordering of the root system Δ_+^s of the following form:

$$\begin{aligned}
 &\beta_1^1, \dots, \beta_t^1, \beta_{t+1}^1, \dots, \beta_{t+\frac{p-n}{2}}^1, \gamma_1, \beta_{t+\frac{p-n}{2}+2}^1, \dots, \beta_{t+\frac{p-n}{2}+n_1}^1, \gamma_2, \\
 &\beta_{t+\frac{p-n}{2}+n_1+2}^1 \dots, \beta_{t+\frac{p-n}{2}+n_2}^1, \gamma_3, \dots, \gamma_n, \beta_{t+p+1}^1, \dots, \beta_{l(s^1)}^1, \dots, \\
 &\beta_1^2, \dots, \beta_q^2, \gamma_{n+1}, \beta_{q+2}^2, \dots, \beta_{q+m_1}^2, \gamma_{n+2}, \beta_{q+m_1+2}^2, \dots, \beta_{q+m_2}^2, \gamma_{n+3}, \dots, \\
 &\gamma_{l'}, \beta_{q+m_{l(s^2)}+1}^2, \dots, \beta_{2q+2m_{l(s^2)}-(l'-n)}^2, \beta_{2q+2m_{l(s^2)}-(l'-n)+1}^2, \dots, \beta_{l(s^2)}^2, \\
 &\beta_1^0, \dots, \beta_{D_0}^0,
 \end{aligned} \tag{2.11}$$

where

$$\begin{aligned}
 &\left\{ \beta_1^1, \dots, \beta_t^1, \beta_{t+1}^1, \dots, \beta_{t+\frac{p-n}{2}}^1, \gamma_1, \beta_{t+\frac{p-n}{2}+2}^1, \dots, \beta_{t+\frac{p-n}{2}+n_1}^1, \gamma_2, \right. \\
 &\quad \left. \beta_{t+\frac{p-n}{2}+n_1+2}^1, \dots, \beta_{t+\frac{p-n}{2}+n_2}^1, \gamma_3, \dots, \gamma_n, \beta_{t+p+1}^1, \dots, \beta_{l(s^1)}^1 \right\} = \Delta_{s^1}, \\
 &\left\{ \beta_{t+1}^1, \dots, \beta_{t+\frac{p-n}{2}}^1, \gamma_1, \beta_{t+\frac{p-n}{2}+2}^1, \dots, \beta_{t+\frac{p-n}{2}+n_1}^1, \gamma_2, \right. \\
 &\quad \left. \beta_{t+\frac{p-n}{2}+n_1+2}^1, \dots, \beta_{t+\frac{p-n}{2}+n_2}^1, \gamma_3, \dots, \gamma_n \right\} = \{ \alpha \in \Delta_+^s \mid s^1(\alpha) = -\alpha \}, \\
 &\left\{ \beta_1^2, \dots, \beta_q^2, \gamma_{n+1}, \beta_{q+2}^2, \dots, \beta_{q+m_1}^2, \gamma_{n+2}, \beta_{q+m_1+2}^2, \dots, \beta_{q+m_2}^2, \gamma_{n+3}, \dots, \right. \\
 &\quad \left. \gamma_{l'}, \beta_{q+m_{l(s^2)}+1}^2, \dots, \beta_{2q+2m_{l(s^2)}-(l'-n)}^2, \beta_{2q+2m_{l(s^2)}-(l'-n)+1}^2, \dots, \beta_{l(s^2)}^2 \right\} = \Delta_{s^2}, \\
 &\left\{ \gamma_{n+1}, \beta_{q+2}^2, \dots, \beta_{q+m_1}^2, \gamma_{n+2}, \beta_{q+m_1+2}^2, \dots, \beta_{q+m_2}^2, \gamma_{n+3}, \dots, \right. \\
 &\quad \left. \gamma_{l'}, \beta_{q+m_{l(s^2)}+1}^2, \dots, \beta_{2q+2m_{l(s^2)}-(l'-n)}^2 \right\} = \{ \alpha \in \Delta_+^s \mid s^2(\alpha) = -\alpha \}, \\
 &\quad \{ \beta_1^0, \dots, \beta_{D_0}^0 \} = \{ \alpha \in \Delta_+^s \mid s(\alpha) = \alpha \}.
 \end{aligned}$$

The length of the ordered segment $\Delta_{m_+} \subset \Delta$ in normal ordering (2.11),

$$\begin{aligned}
 \Delta_{m_+} = &\gamma_1, \beta_{t+\frac{p-n}{2}+2}^1, \dots, \beta_{t+\frac{p-n}{2}+n_1}^1, \gamma_2, \beta_{t+\frac{p-n}{2}+n_1+2}^1, \dots, \beta_{t+\frac{p-n}{2}+n_2}^1, \\
 &\gamma_3, \dots, \gamma_n, \beta_{t+p+1}^1, \dots, \beta_{l(s^1)}^1, \dots, \beta_1^2, \dots, \beta_q^2, \\
 &\gamma_{n+1}, \beta_{q+2}^2, \dots, \beta_{q+m_1}^2, \gamma_{n+2}, \beta_{q+m_1+2}^2, \dots, \beta_{q+m_2}^2, \gamma_{n+3}, \dots, \gamma_{l'},
 \end{aligned} \tag{2.12}$$

is equal to

$$D - \left(\frac{l(s) - l'}{2} + D_0 \right), \tag{2.13}$$

where D is the number of roots in Δ_+^s , $l(s)$ is the length of s and D_0 is the number of positive roots fixed by the action of s .

For any two roots $\alpha, \beta \in \Delta_{m_+}$ such that $\alpha < \beta$ the sum $\alpha + \beta$ cannot be represented as a linear combination $\sum_{k=1}^q c_k \gamma_{i_k}$, where $c_k \in \{0, 1, 2, \dots\}$ and $\alpha < \gamma_{i_1} < \dots < \gamma_{i_k} < \beta$.

The roots from the set Δ_s form a minimal segment in Δ_+^s of the form $\gamma_1, \dots, \beta_{l(s^2)}^2$ which contains Δ_{s^2} .

Normal ordering (2.11) also satisfies the property that for any $\alpha \in (\Delta_{i_k})_+$ such that $s\alpha \in (\Delta_{i_k})_+$ one has $s\alpha > \alpha$, and if $\beta, \gamma \in \Delta_{i_j} \cup \{0\}$, $j < k$ and $s\alpha + \beta, \alpha + \gamma \in \Delta$ then $s\alpha + \beta, \alpha + \gamma \in \Delta_+^s$ and $s\alpha + \beta > \alpha + \gamma$.

In particular, for any $\alpha \in \Delta_+^s$, $\alpha \notin \Delta_0$ and any $\alpha_0 \in \Delta_0$ such that $s\alpha \in \Delta_+^s$ one has $s\alpha > \alpha$ and if $s\alpha + \alpha_0 \in \Delta$ then $s\alpha + \alpha_0 > \alpha$.

Proof. The proof is a refinement of the proof of Proposition 5.1 in [12] where a normal ordering associated to s and satisfying all properties listed in this proposition except for the last three is constructed.

We describe first the set $(\Delta_{i_k})_+ = \Delta_{i_k} \cap \Delta_+^s$ for $i_k > 0$. Suppose that the corresponding s -invariant subspace \mathfrak{h}_{i_k} is a two-dimensional plane. The case when \mathfrak{h}_{i_k} is an invariant line on which s^2 acts by reflection and s^1 acts trivially can be treated in a similar way. The plane \mathfrak{h}_{i_k} is shown at Figure 2.

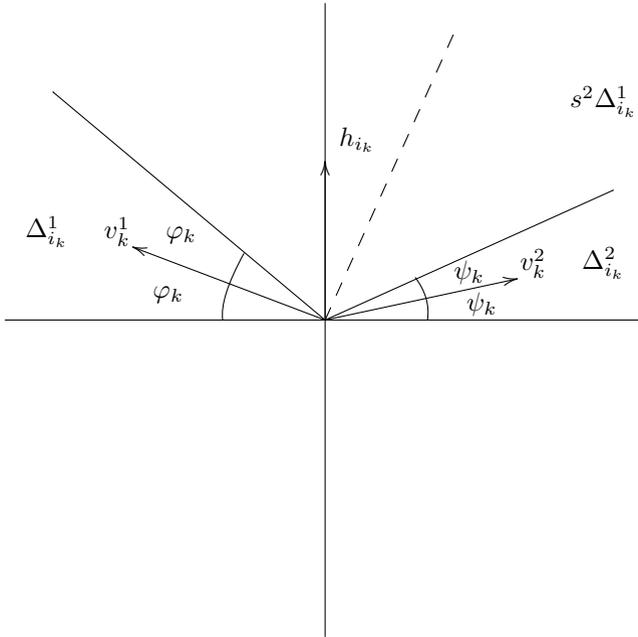


FIG. 2

The vector h_{i_k} is directed upwards at the picture. By (2.5) and (2.7) a root $\alpha \in \Delta_{i_k}$ belongs to the set $(\Delta_{i_k})_+$ if and only if $h_{i_k}(\alpha) > 0$. Identifying $\mathfrak{h}_{\mathbb{R}}$ and $\mathfrak{h}_{\mathbb{R}}^*$ with the help of the Killing form one can deduce that $\alpha \in \Delta_{i_k}$ is in Δ_+^s iff its orthogonal, with respect to the Killing form, projection onto \mathfrak{h}_{i_k} is contained in the upper-half plane shown at Figure 2.

The element s acts on \mathfrak{h}_{i_k} by clockwise rotation with the angle $\theta_{i_k} = 2(\varphi_k + \psi_k)$. Therefore the set $\Delta_s \cap \Delta_{i_k}$ consists of the roots the orthogonal projections of which onto \mathfrak{h}_{i_k} belong to the union of the sectors labeled $s^2 \Delta_{i_k}^1$ and $\Delta_{i_k}^2$ at Figure 2.

Define other s -invariant subsets of roots $\bar{\Delta}_{i_k}, k = 0, \dots, M,$

$$\bar{\Delta}_{i_k} = \bigcup_{i_j \leq i_k} \Delta_{i_j}. \tag{2.14}$$

According to this definition we have a chain of strict inclusions

$$\bar{\Delta}_{i_M} \supset \bar{\Delta}_{i_{M-1}} \supset \dots \supset \bar{\Delta}_{i_0}, \tag{2.15}$$

such that $\overline{\Delta}_{i_M} = \Delta$, and $\overline{\Delta}_{i_k} \setminus \overline{\Delta}_{i_{k-1}} = \Delta_{i_k}$. Note also that by construction each $\overline{\Delta}_{i_k}$ is additively closed and hence it is a root subsystem in Δ .

Let $i_k > 0$ and $\Delta_{i_k}^{(r)}$ the subset of roots in $(\Delta_{i_k})_+$ orthogonal projections of which onto \mathfrak{h}_{i_k} are directed along a ray $r \subset \mathfrak{h}_{i_k}$ starting at the origin. We call $\Delta_{i_k}^{(r)}$ the family corresponding to the ray r . Below we shall only consider rays which correspond to sets of the form $\Delta_{i_k}^{(r)}$.

Lemma 2.3. *Suppose that $i_k > 0$. Then each $\Delta_{i_k}^{(r)}$ is an additively closed set of roots.*

Let $\Delta_{i_k}^{(r_1)}$ and $\Delta_{i_k}^{(r_2)}$ be two families corresponding to rays r_1 and r_2 , and $\delta_1 \in \Delta_{i_k}^{(r_1)}$, $\delta_2 \in \Delta_{i_k}^{(r_2)}$ two roots such that $\delta_1 + \delta_2 = \delta_3 \in \Delta$. Then $\delta_3 \in \Delta_{i_k}^{(r_3)}$, where $\Delta_{i_k}^{(r_3)}$ is the family corresponding to a ray r_3 such that r_3 lies inside of the angle formed by r_1 and r_2 .

Proof. All statements are simple consequences of the fact that the sum of the orthogonal projections of any two roots onto \mathfrak{h}_{i_k} is equal to the orthogonal projection of the sum.

In the first case the orthogonal projections of any two roots α, β from $\Delta_{i_k}^{(r)}$ onto \mathfrak{h}_{i_k} have the same direction therefore the orthogonal projection of the sum $\alpha + \beta$ onto \mathfrak{h}_{i_k} has the same direction as the orthogonal projections of α and β , and hence $\alpha + \beta \in \Delta_{i_k}^{(r)}$.

In the second case it suffices to observe that the sum of the orthogonal projections of δ_1 and δ_2 onto \mathfrak{h}_{i_k} is equal to the orthogonal projection of the sum, and the sum of the orthogonal projections of δ_1 and δ_2 onto \mathfrak{h}_{i_k} lies inside of the angle formed by r_1 and r_2 . \square

Now we construct an auxiliary normal ordering on Δ_+^s by induction starting from the set $(\Delta_{i_0})_+$ as follows.

If $i_0 = 0$ or \mathfrak{h}_{i_0} is one-dimensional then we fix an arbitrary normal order on $(\Delta_{i_0})_+$.

If \mathfrak{h}_{i_0} is two-dimensional then we choose a normal ordering in $(\Delta_{i_0})_+$ in the following way. First fix an initial arbitrary normal ordering on $(\Delta_{i_0})_+$. Since by Lemma 2.3 each set $\Delta_{i_0}^{(r)}$ is additively closed we obtain an induced ordering for $\Delta_{i_0}^{(r)}$ which satisfies the defining property for the normal ordering.

Now using these induced orderings on the sets $\Delta_{i_0}^{(r)}$ we define an auxiliary normal ordering on $(\Delta_{i_0})_+$ such that on the sets $\Delta_{i_0}^{(r)}$ it coincides with the induced normal ordering defined above, and if $\Delta_{i_0}^{(r_1)}$ and $\Delta_{i_0}^{(r_2)}$ are two families corresponding to rays r_1 and r_2 such that r_2 lies on the right from r_1 in \mathfrak{h}_{i_0} then for any $\alpha \in \Delta_{i_0}^{(r_1)}$ and $\beta \in \Delta_{i_0}^{(r_2)}$ one has $\alpha < \beta$. By Lemma 2.3 the two conditions imposed on the auxiliary normal ordering in $(\Delta_{i_0})_+$ are compatible and define it in a unique way for the given initial normal ordering on $(\Delta_{i_0})_+$. Since s acts by a clockwise rotation on \mathfrak{h}_{i_0} we have $s(\Delta_{i_0}^{(r)}) = \Delta_{i_0}^{(s(r))}$ for $s(r)$ in the upper-half plane, and hence the new normal ordering satisfies the condition that for any $\alpha \in (\Delta_{i_0})_+$ such that $s\alpha \in (\Delta_{i_0})_+$ one has $s\alpha > \alpha$.

Now assume that an auxiliary normal ordering has already been constructed for the set $\overline{\Delta}_{i_{k-1}}$ and define it for the set $\overline{\Delta}_{i_k}$. Firstly we shall need the following property of these sets of roots.

Lemma 2.4. *Let Γ be the set of simple roots in Δ_+^s . Then $\Gamma \cap \overline{\Delta}_{i_k}$ is a set of simple roots in $\overline{\Delta}_{i_k}$.*

Proof. Indeed, let $\alpha \in \overline{\Delta}_{i_k} \cap \Delta_+^s$, $\alpha = \sum_{i=1}^l n_i \alpha_i$, where $n_i \in \{0, 1, 2, \dots\}$ and $\Gamma = \{\alpha_1, \dots, \alpha_l\}$. Assume that α does not belong to the linear span of roots from $\Gamma \cap \overline{\Delta}_{i_k}$ and $t > i_k$ is maximal possible such that for some $\alpha_q \in \Delta_t$ one has $n_q > 0$. Then by (2.3) and (2.8) $h_t(\alpha) = \sum_{i=1}^l n_i h_t(\alpha_i) = \sum_{\alpha_i \in \Delta_t} n_i h_t(\alpha_i) > 0$, and by the choice of t $h_r(\alpha) = 0$ for $r > t$. Therefore $\alpha \in \Delta_t$, and hence $\alpha \notin \overline{\Delta}_{i_k}$. Thus we arrive at a contradiction. \square

By Lemma 2.4 $\overline{\Delta}_{i_{k-1}} \subset \overline{\Delta}_{i_k}$ is the root system of a standard Levi subalgebra $\mathfrak{g}_{i_{k-1}}$ inside of the standard Levi subalgebra \mathfrak{g}_{i_k} of \mathfrak{g} with the root system $\overline{\Delta}_{i_k}$. In particular, $\overline{\Delta}_{i_{k-1}}$ is generated by some subset of simple roots of the set of simple roots of $(\overline{\Delta}_{i_k})_+$. Therefore there exists an initial normal ordering on $(\overline{\Delta}_{i_k})_+$ in which the roots from the set $(\overline{\Delta}_{i_k})_+ \setminus (\overline{\Delta}_{i_{k-1}})_+ = (\Delta_{i_k})_+$ form an initial segment and the remaining roots from $(\overline{\Delta}_{i_{k-1}})_+$ are ordered according to the previously defined auxiliary normal ordering. As in the case of the induction base this initial normal ordering gives rise to an induced ordering on each set $\Delta_{i_k}^{(r)}$.

Now using these induced orderings on the sets $\Delta_{i_k}^{(r)}$ we define an auxiliary normal ordering on $(\overline{\Delta}_{i_k})_+$. We impose the following conditions on it. Firstly we require that the roots from the set $(\Delta_{i_k})_+$ form an initial segment and the remaining roots from $(\overline{\Delta}_{i_{k-1}})_+$ are ordered according to the previously defined auxiliary normal ordering. Secondly, on the sets $\Delta_{i_k}^{(r)}$ the auxiliary normal ordering coincides with the induced normal ordering defined above, and if $\Delta_{i_k}^{(r_1)}$ and $\Delta_{i_k}^{(r_2)}$ are two families corresponding to rays r_1 and r_2 such that r_2 lies on the right from r_1 in \mathfrak{h}_{i_k} then for any $\alpha \in \Delta_{i_k}^{(r_1)}$ and $\beta \in \Delta_{i_k}^{(r_2)}$ one has $\alpha < \beta$. By Lemma 2.3 the conditions imposed on the auxiliary normal ordering in $(\Delta_{i_k})_+$ are compatible and define it in a unique way. Since s acts by a clockwise rotation on \mathfrak{h}_{i_k} we have $s(\Delta_{i_k}^{(r)}) = \Delta_{i_k}^{(s(r))}$ for $s(r)$ in the upper-half plane, and hence the new normal ordering satisfies the condition that for any $\alpha \in (\Delta_{i_k})_+$ such that $s\alpha \in (\Delta_{i_k})_+$ one has $s\alpha > \alpha$.

Note also that the roots from $\overline{\Delta}_{i_{k-1}}$ have zero orthogonal projections onto \mathfrak{h}_{i_k} . Therefore if $\alpha \in (\Delta_{i_k})_+$, $\beta, \gamma \in \overline{\Delta}_{i_{k-1}}$ are such that $s\alpha \in (\Delta_{i_k})_+$, $s\alpha + \beta, \alpha + \gamma \in \Delta$ then by (2.3) and (2.8) $s\alpha + \beta, \alpha + \gamma \in \Delta_+^s$ and $s\alpha + \beta > \alpha + \gamma$ as $s(\Delta_{i_k}^{(r)}) = \Delta_{i_k}^{(s(r))}$ for $s(r)$ in the upper-half plane.

These properties of the new normal ordering are summarized in the following lemma.

Lemma 2.5. *Suppose that $i_k > 0$. Then for any $\alpha \in (\Delta_{i_k})_+$ such that $s\alpha \in (\Delta_{i_k})_+$ one has $s\alpha > \alpha$ and if $\beta, \gamma \in \overline{\Delta}_{i_{k-1}} \cup \{0\}$, $s\alpha + \beta, \alpha + \gamma \in \Delta$ then $s\alpha + \beta, \alpha + \gamma \in \Delta_+^s$ and $s\alpha + \beta > \alpha + \gamma$.*

Now we proceed by induction and obtain an auxiliary normal ordering on Δ_+^s .

Observe that, according to the definition of the auxiliary normal ordering of Δ_+^s constructed above we have the following properties of this normal ordering.

Lemma 2.6. *Let $i_k > 0$. Then for any $\alpha \in \Delta_{i_k}$ and $\beta \in \Delta_{i_{k+1}}$ we have $\alpha > \beta$, and if $\Delta_{i_k}^{(r_1)}$ and $\Delta_{i_k}^{(r_2)}$ are two families corresponding to rays r_1 and r_2 such that r_2 lies on the right from r_1 in \mathfrak{h}_{i_k} then for any $\alpha \in \Delta_{i_k}^{(r_1)}$ and $\beta \in \Delta_{i_k}^{(r_2)}$ one has $\alpha < \beta$. Moreover, the roots from the sets $\Delta_{i_k}^{(r)}$ form minimal segments, and the roots from the set $(\Delta_0)_+$ form a final minimal segment.*

The involutions s^1 and s^2 act in \mathfrak{h}_{i_k} as reflections with respect to the lines orthogonal to the vectors labeled by v_k^1 and v_k^2 , respectively, at Figure 2, the angle between v_k^1 and v_k^2 being equal to $\pi - (\varphi_{i_k} + \psi_{i_k})$. The nonzero projections of the roots from the set $\{\gamma_1, \dots, \gamma_n\} \cap \Delta_{i_k}$ onto the plane \mathfrak{h}_{i_k} have the same (or the opposite) direction as the vector v_k^1 , and the nonzero projections of the roots from the set $\{\gamma_{n+1}, \dots, \gamma_l\} \cap \Delta_{i_k}$ onto the plane \mathfrak{h}_{i_k} have the same (or the opposite) direction as the vector v_k^2 .

For each of the involutions s^1 and s^2 we obviously have decompositions $\Delta_{s^{1,2}} = \bigcup_{k=0}^M \Delta_{i_k}^{1,2}$, $\Delta_s = \bigcup_{k=0}^M \Delta_{i_k}^s$, where $\Delta_{i_k}^{1,2} = \Delta_{i_k} \cap \Delta_{s^{1,2}}$, $\Delta_{s^{1,2}} = \{\alpha \in \Delta_+^s : s^{1,2}\alpha \in -\Delta_+^s\}$, $\Delta_{i_k}^s = \Delta_{i_k} \cap \Delta_s$, $\Delta_{i_k}^s = \Delta_{i_k}^2 \cup s^2 \Delta_{i_k}^1$. In the plane \mathfrak{h}_{i_k} , the elements from the sets $\Delta_{i_k}^{1,2}$ are projected onto the interiors of the sectors labeled by $\Delta_{i_k}^{1,2}$ and the elements from the set $\Delta_{i_k}^s$ are projected onto the interior of the union of the sectors labeled by $\Delta_{i_k}^2$ and $s^2 \Delta_{i_k}^1$. Therefore the sets $\Delta_{i_k}^1$ and $\Delta_{i_k}^2$ have empty intersection and are the unions of the sets $\Delta_{i_k}^{(r)}$ with r belonging to the sector $\Delta_{i_k}^{1,2}$, and the sets $\Delta_{i_k}^s$ have empty intersection and are the unions of the sets $\Delta_{i_k}^{(r)}$ with r belonging to the union of the sectors labeled by $\Delta_{i_k}^2$ and $s^2 \Delta_{i_k}^1$.

Let $\alpha \in \Delta_{s^1}$. By Theorem C in [2] the roots $\gamma_1, \dots, \gamma_l$ form a linear basis in the annihilator $\mathfrak{h}_{\mathbb{R}}^*$ of \mathfrak{h}_0 with respect to the pairing between $\mathfrak{h}_{\mathbb{R}}$ and $\mathfrak{h}_{\mathbb{R}}^*$. Therefore $s^1 = s_{\gamma_1} \cdots s_{\gamma_n}$ fixes all roots from Δ_0 , and hence $\alpha \notin \Delta_0$. Also one obviously has $\alpha \in \Delta_{i_k}^1$, where \mathfrak{h}_{i_k} is a two-dimensional plane, as by the assumption imposed before (2.4) there are no one-dimensional subspaces \mathfrak{h}_{i_k} on which s^1 acts by multiplication by -1 . Thus in case if \mathfrak{h}_{i_k} is an invariant line on which s acts by multiplication by -1 the set $\Delta_{i_k}^1$ is empty and hence $\Delta_{i_k}^s = \Delta_{i_k}^2$. This set is the set $\Delta_{i_k}^{(r)} = (\Delta_{i_k})_+$, where r is the positive semi-axis in \mathfrak{h}_{i_k} . From the last two observations we deduce that the sets Δ_{s^1} and Δ_{s^2} have always empty intersection. In particular, by the results of [16, §3] the decomposition $s = s^1 s^2$ is reduced in the sense that $l(s) = l(s^2) + l(s^1)$, and $\Delta_s = \Delta_{s^2} \cup s^2(\Delta_{s^1})$ (disjoint union).

Lemma 2.7. *Assume that Δ_+^s is equipped with an arbitrary normal ordering such that for some ray $r \subset \mathfrak{h}_{i_k}$ the roots from the set $\Delta_{i_k}^{(r)} = \{\delta_1, \dots, \delta_a\}$ form a minimal segment $\delta_1, \dots, \delta_a$. Suppose also that for some natural p such that $0 \leq p < k$, $i_p \neq 0$ and some ray $t \subset \mathfrak{h}_{i_p}$ the roots from the set $\Delta_{i_p}^{(t)} = \{\xi_1, \dots, \xi_b\}$ form a minimal segment ξ_1, \dots, ξ_b and that the segment $\delta_1, \dots, \delta_a, \xi_1, \dots, \xi_b$ is also minimal. Then applying elementary transpositions one can reduce the last segment to the form $\xi_{i_1}, \dots, \xi_{i_b}, \delta_{j_1}, \dots, \delta_{j_a}$.*

Proof. The proof is by induction. First consider the minimal segment $\delta_1, \dots, \delta_a, \xi_1$.

Since the orthogonal projection of the roots from the set Δ_{i_p} onto \mathfrak{h}_{i_k} are equal to zero, for any $\alpha \in \Delta_{i_p}^{(t)}$ and $\beta \in \Delta_{i_k}^{(r)}$ such that $\alpha + \beta \in \Delta$ we have $\alpha + \beta \in \Delta_{i_k}^{(r)}$. Assume now that α and β are contained in an ordered segment of form (2.9) or in a segment with the inverse ordering. By the above observation this segment contains no other roots from $\Delta_{i_p}^{(t)}$, and α is the first or the last element in that segment. For the same reason the other roots in that segment must also belong to $\Delta_{i_k}^{(r)}$. Therefore applying an elementary transposition, if necessary, one can move α to the first position in that segment.

Applying this procedure iteratively to the segment $\delta_1, \dots, \delta_a, \xi_1$ we can reduce it to the form $\xi_1, \delta_{k_1}, \dots, \delta_{k_a}$.

Now we can apply the same procedure to the segment $\delta_{k_1}, \dots, \delta_{k_a}, \xi_2$ to reduce the segment $\xi_1, \delta_{k_1}, \dots, \delta_{k_a}, \xi_2$ to the form $\xi_1, \xi_2, \delta_{l_1}, \dots, \delta_{l_a}$.

Iterating this procedure we obtain the statement of the lemma. \square

Now observe that according to Lemma 2.6 the roots from each of the sets $(\Delta_{i_k})_+$ form a minimal segment in the auxiliary normal ordering of Δ_+^s , and the roots from the sets $\Delta_{i_k}^{(r)}$ form minimal segments inside $(\Delta_{i_k})_+$. As we observed above the sets $\Delta_{i_k}^{1,2}$ are the unions of the sets $\Delta_{i_k}^{(r)}$ with r belonging to the sectors $\Delta_{i_k}^{1,2}$ and hence by Lemma 2.6 the roots from the sets $\Delta_{i_k}^{1,2}$ form an initial and a final segment inside $(\Delta_{i_k})_+$.

Therefore we can apply Lemmas 2.6 and 2.7 to move all roots from the segments $\Delta_{i_k}^1, k = 0, \dots, M$ to the left and to move all roots from the segments $\Delta_{i_k}^2, k = 0, \dots, M$ to the right to positions preceding the final segment formed by the roots from $(\Delta_0)_+$.

Now using similar arguments the roots from the sets $s^2\Delta_{i_k}^1, k = 0, \dots, M$ forming minimal segments by Lemma 2.6 as well can be moved to the right to positions preceding the final segment formed by the roots from the set $\Delta_{s^2} \cup (\Delta_0)_+$.

Note that according to the algorithm given in Lemma 2.7 for each fixed k the order of the minimal segments formed by the roots from the sets $\Delta_{i_k}^{(r)}$ is preserved after applying that lemma. Therefore the new normal ordering obtained this way still satisfies the second property mentioned in Lemma 2.6, i.e., if $\Delta_{i_k}^{(r_1)}$ and $\Delta_{i_k}^{(r_2)}$ are two families corresponding to rays r_1 and r_2 such that r_2 lies on the right from r_1 in \mathfrak{h}_{i_k} then for any $\alpha \in \Delta_{i_k}^{(r_1)}$ and $\beta \in \Delta_{i_k}^{(r_2)}$ one has $\alpha < \beta$.

Now we can apply elementary transpositions used in the proof Proposition 5.1 in [12] to bring the initial segment formed by the roots from Δ_{s^1} and the segment formed by the roots from Δ_{s^2} and preceding the final segment $(\Delta_0)_+$ to the form described in (2.11). These elementary transpositions do not affect positions of other roots. We claim that for $\alpha \in \Delta_{i_k}^{(r)}$ we still have $s\alpha > \alpha$ if $s\alpha \in \Delta_+^s$.

Indeed, if $\alpha \in (\Delta_{i_k})_+, \alpha \notin \Delta_{i_k}^1, s\alpha \notin \Delta_{i_k}^2, s\alpha \in (\Delta_{i_k})_+$ this follows from the second property mentioned in Lemma 2.6.

If $\alpha \in \Delta_{i_k}^1$ and $s\alpha \in \Delta_+^s$ then $s\alpha \notin \Delta_{s^1}$ as $s^1(s\Delta_{s^1}) = s^2(\Delta_{s^1}) \subset \Delta_+^s$ since the decomposition $s = s^1s^2$ is reduced. Therefore $s\alpha > \alpha$ as the roots from the set Δ_{s^1} form an initial segment in the normal ordering of Δ_+^s .

If $\alpha \in (\Delta_{i_k})_+, \alpha \notin \Delta_{i_k}^1, s\alpha \in \Delta_{i_k}^2$ then $\alpha \notin \Delta_s$ and $s\alpha \in \Delta_s$ as $\Delta_{i_k}^2 \subset \Delta_s$. Therefore $s\alpha > \alpha$ as the roots from the set $\Delta_s \cup (\Delta_0)_+$ form a final segment in the

normal ordering of Δ_+^s and α does not belong to that segment.

Finally if $\alpha \in \Delta_{i_k}^2$ then $s\alpha \in -\Delta_+^s$.

Moreover, similar arguments together with the fact that all roots from $\overline{\Delta}_{i_{k-1}}$ have zero orthogonal projections onto \mathfrak{h}_{i_k} show that the new normal ordering still satisfies the property of Lemma 2.5. Note that for one-dimensional \mathfrak{h}_{i_k} this property is void.

Since Δ_+^s is the disjoint union of the sets $(\Delta_{i_k})_+$, the last statement of the proposition is a particular case of the property proved above.

By construction the roots from the set Δ_s form a minimal segment in Δ_+^s of the form $\gamma, \dots, \beta_{l(s^2)}^2$ which contains Δ_{s^2} . This completes the proof of the proposition. \square

We call normal ordering (2.11) a normal ordering associated to s .

Later we shall use the circular normal ordering of Δ corresponding to the system of positive roots Δ_+^s and to its normal ordering introduced in Proposition 2.2.

Let \mathfrak{b}_+^s be the Borel subalgebra associated to Δ_+^s and \mathfrak{b}_-^s the opposite Borel subalgebra. Let \mathfrak{n}_\pm^s be the nilradicals of \mathfrak{b}_\pm^s . Denote by B_\pm^s the corresponding Borel subgroups and by N_\pm^s their unipotent radicals.

We shall need the parabolic subalgebra $\mathfrak{p} \supset \mathfrak{b}_+^s$ of \mathfrak{g} and the parabolic subgroup $P \supset B_+^s$ associated to the subset $\Gamma_0 = \Gamma \cap \Delta_0$ of simple roots. Let \mathfrak{n} and \mathfrak{l} be the nilradical and the Levi factor of \mathfrak{p} , N and L the unipotent radical and the Levi factor of P , respectively.

Note that we have natural inclusions of Lie algebras $\mathfrak{p} \supset \mathfrak{n}$, and by Lemma 2.4 Δ_0 is the root system of the reductive Lie algebra \mathfrak{l} . We also denote by $\overline{\mathfrak{n}}$ the nilpotent subalgebra opposite to \mathfrak{n} and by \overline{N} the subgroup in G corresponding to $\overline{\mathfrak{n}}$.

Introduce the element $\bar{h}_0 = \sum_{k=1}^M h_{i_k} \in \mathfrak{h}_\mathbb{R}$. Let $\mathfrak{h}_0^* \subset \mathfrak{h}_\mathbb{R}^*$ be the image of \mathfrak{h}_0 in $\mathfrak{h}_\mathbb{R}^*$ under the isomorphism $\mathfrak{h}_\mathbb{R}^* \simeq \mathfrak{h}_\mathbb{R}$ induced by the Killing form. By the definition of Δ_+^s for any $x \in \mathfrak{h}_0^*$ one has $\bar{h}_0(x) = 0$ and a root $\alpha \in \Delta \setminus \Delta_0$ is positive iff $\bar{h}_0(\alpha) > 0$.

Denote a representative for the Weyl group element s in G by the same letter. Let Z be the subgroup of G generated by the semisimple part of the Levi subgroup L and by the centralizer of s in H . The level surface \mathcal{C} is the variety $NZs^{-1}N$.

The following proposition is a modification of [11, Props. 2.1 and 2.2].

Proposition 2.8. *Let $N_s = \{v \in N \mid sv s^{-1} \in \overline{N}\}$. Then the conjugation map*

$$N \times N_s Z s^{-1} \rightarrow N Z s^{-1} N \tag{2.16}$$

is an isomorphism of varieties. Moreover, the variety $N_s Z s^{-1}$ is a transversal slice to the set of conjugacy classes in G .

The variety $\Sigma = N_s Z s^{-1} \simeq N Z s^{-1} N / N$ is a subvariety in G . It is an analogue of the Slodowy slices in algebraic group theory.

The operator Π will be defined in terms of certain functions on $N Z s^{-1} N$. These functions are related to some natural coordinates on Bruhat cells in G . First, as a warm up exercise, we obtain explicit formulas for these coordinates.

If $w \in W$ is any Weyl group element we shall denote a representative of w in G by the same letter. Let Δ_+ be any system of positive roots, $\Delta_- = -\Delta_+$, B_\pm and N_\pm the corresponding Borel subgroups in G and their unipotent radicals, \mathfrak{b}_\pm and \mathfrak{n}_\pm their Lie algebras, respectively. Denote $N_{w^{-1}} = \{n \in N_+ : w^{-1}nw \in N_-\}$. $N_{w^{-1}}$ is a subgroup in N_+ generated by one-parametric subgroups corresponding to the roots from the set $\Delta_{w^{-1}} = \{\alpha \in \Delta_+ : w^{-1}\alpha \in \Delta_-\}$. Denote by s_i the reflection with respect to a simple root $\alpha_i \in \Delta_+$, $i = 1, \dots, l$ and let $w = s_{i_1} \cdots s_{i_k}$ be a reduced decomposition of w . Then $\Delta_{w^{-1}} = \{\beta_1, \dots, \beta_k\}$, and $\beta_j = s_{i_1} \cdots s_{i_{j-1}}\alpha_j$. Note that the elements $w_j = s_{i_1} \cdots s_{i_j}$ can also be represented in the form $w_j = s_{\beta_j} \cdots s_{\beta_1}$, where for any root $\alpha \in \Delta$ we denote by s_α the reflection with respect to α . Observe also that $\Delta_{w_j^{-1}} = \{\beta_1, \dots, \beta_j\} = \{\alpha \in \Delta_+ : w_j^{-1}\alpha \in \Delta_-\}$.

Let $X_\alpha \in \mathfrak{g}$ be a non-zero root vector corresponding to root α , $X_\alpha(t) = \exp(tX_\alpha)$, $t \in \mathbb{C}$ and denote by N_α the one parametric subgroup corresponding to root α , so $X_\alpha(t) \in N_\alpha$. Any element of the Bruhat cell $B_+w^{-1}B_+$ can be uniquely represented in the form $nw^{-1}hn_{w^{-1}}$, $n \in N_+$, $h \in H$, $n_{w^{-1}} = X_{\beta_k}(q_k) \cdots X_{\beta_1}(q_1)$.

Fix a normal ordering β_1, \dots, β_D of the system of positive roots Δ_+ such that β_1, \dots, β_k is its initial segment and equip Δ with the corresponding circular normal ordering. This is always possible by the results of §3 in [16].

Let ω_i , $i = 1, \dots, l$ be the fundamental weights of \mathfrak{g} corresponding to Δ_+ , V_{ω_i} the irreducible representation of G with highest weight ω_i , $v_{\omega_i} \in V_{\omega_i}$ a non-zero highest weight vector. Denote by (\cdot, \cdot) the contravariant bilinear non-degenerate form on V_{ω_i} such that $(v_{\omega_i}, v_{\omega_i}) = 1$ and $(\omega(g)v, w) = (v, gw)$, where ω is the Chevalley anti-involution on G induced by the Chevalley anti-involution on \mathfrak{g} which is equal to minus the Chevalley involution as defined, e.g., in [5, (1.3.4)]. For any reflection s_α one can fix a representative s_α in G such that $\omega(s_\alpha) = s_\alpha^{-1}$. It suffices to do that for simple reflections and one can put $s_i = \exp(f_i) \exp(-e_i) \exp(f_i)$, where $e_i \in \mathfrak{n}_+$, $f_i \in \mathfrak{n}_-$, $h_i \in \mathfrak{h}$ are the Chevalley generators of \mathfrak{g} on which ω acts as follows $\omega(f_i) = e_i, \omega(e_i) = f_i, \omega(h_i) = h_i$.

If $\alpha, \beta \in \Delta$ are such that the segment $[\alpha, \beta]$ is minimal then we denote by $N_{[\alpha, \beta]}$ the subgroup in G generated by the one-parametric subgroups corresponding to the roots from $[\alpha, \beta]$.

Proposition 2.9. *Fix a natural number p , $p \in \{1, \dots, k\}$. Let $g = nw^{-1}hn_{w^{-1}} \in B_+w^{-1}B_+$, $n \in N_+$, $h \in H$, $n_{w^{-1}} = X_{\beta_k}(q_k) \cdots X_{\beta_p}(q_p)$ be an element of the Bruhat cell $B_+w^{-1}B_+$. Then*

$$q_p = c_p \frac{(w_{p-1}v_{\omega_{i_p}}, wgw_p v_{\omega_{i_p}})}{(w_{p-1}v_{\omega_{i_p}}, wgw_{p-1}v_{\omega_{i_p}})}, \tag{2.17}$$

where $w_p = s_{\beta_p} \cdots s_{\beta_1}$, $w_{p-1} = s_{\beta_{p-1}} \cdots s_{\beta_1}$, c_p is a non-zero constant only depending on the choice of the representative $s_{\beta_p} \in G$ and on the choice of the root vector X_{β_p} , and it is assumed that $w_0 = 1$.

Proof. In the proof we shall frequently use the following lemma.

Lemma 2.10. *Let $[\alpha, \beta] \subset \Delta$ be a minimal segment and assume that $[\alpha, \beta] = [\alpha, \gamma] \cup [\delta, \beta]$, where the segments $[\alpha, \gamma]$ and $[\delta, \beta]$ are disjoint and minimal as well. Then any element $m \in N_{[\alpha, \beta]}$ can be uniquely factorized as $m = g_1g_2 = g'_1g'_2$,*

$g_1, g'_1 \in N_{[\alpha, \gamma]}$, $g_2, g'_2 \in N_{[\delta, \beta]}$. Moreover, if $\delta = \beta$ then for any $m' \in N_{[\alpha, \gamma]}$ and any $t \in \mathbb{C}$ one has $m'X_\beta(t) = X_\beta(t)m''$, where $m'' \in N_{[\alpha, \gamma]}$.

Proof. The proof is obtained by straightforward application of Chevalley's commutation relations between one-parametric subgroups (see [1, Thm. 5.2.2]) and Lemma 2.1. \square

Now by Lemma 2.10 we can write $n_{w^{-1}} = X_{\beta_k}(q_k) \cdots X_{\beta_p}(q_p) = X_{\beta_p}(q_p)m_1$, $m_1 \in N_{[\beta_{p+1}, \beta_k]}$. Since $\Delta_{w_p^{-1}} = \{\beta_1, \dots, \beta_p\}$ we have $w_p^{-1}N_{[\beta_{p+1}, \beta_k]}w_p \subset N_+$, and hence

$$n_{w^{-1}}w_p v_{\omega_{i_p}} = X_{\beta_p}(q_p)m_1 w_p v_{\omega_{i_p}} = X_{\beta_p}(q_p)w_p w_p^{-1}m_1 w_p v_{\omega_{i_p}} = X_{\beta_p}(q_p)w_p v_{\omega_{i_p}}$$

as $v_{\omega_{i_p}}$ is a highest weight vector. We deduce that

$$(w_{p-1}v_{\omega_{i_p}}, wgw_p v_{\omega_{i_p}}) = (w_{p-1}v_{\omega_{i_p}}, wnw^{-1}hX_{\beta_p}(q_p)w_p v_{\omega_{i_p}}).$$

Now observe that by the choice of the roots β_1, \dots, β_k we have

$$w^{-1}(\{\beta_{k+1}, \dots, \beta_D, -\beta_1, \dots, -\beta_k\}) \subset \Delta_+.$$

Since the number of positive roots is equal to the number of roots in the set $\{\beta_{k+1}, \dots, \beta_D, -\beta_1, \dots, -\beta_k\}$ we deduce that $w^{-1}(\{\beta_{k+1}, \dots, \beta_D, -\beta_1, \dots, -\beta_k\}) = \Delta_+$ and hence

$$w(\Delta_+) = \{\beta_{k+1}, \dots, \beta_D, -\beta_1, \dots, -\beta_k\}.$$

This implies that $wnw^{-1} \in N_{[\beta_{k+1}, -\beta_k]}$. Recall also that H normalizes each one-parametric subgroup N_α . Therefore using Lemma 2.10 one can uniquely factorize the element $wnw^{-1}h$ as $wnw^{-1}h = m_2hm_3$, $m_2 \in N_{[-\beta_p, -\beta_k]}$, $m_3 \in N_{[\beta_{k+1}, -\beta_{p-1}]}$. Since $\omega(m_2) \in N_{[\beta_p, \beta_k]}$, $w_{p-1}^{-1}N_{[\beta_p, \beta_k]}w_{p-1} \subset N_+$ and $v_{\omega_{i_p}}$ is a highest weight vector we obtain

$$\begin{aligned} (w_{p-1}v_{\omega_{i_p}}, wgw_p v_{\omega_{i_p}}) &= (\omega(m_2)w_{p-1}v_{\omega_{i_p}}, hm_3X_{\beta_p}(q_p)w_p v_{\omega_{i_p}}) \\ &= (w_{p-1}v_{\omega_{i_p}}, hm_3X_{\beta_p}(q_p)w_p v_{\omega_{i_p}}). \end{aligned}$$

Observe also that using Lemma 2.10 we can uniquely factorize the element $m_3X_{\beta_p}(q_p) \in N_{[\beta_p, -\beta_{p-1}]}$ as $m_3X_{\beta_p}(q_p) = X_{\beta_p}(q_p)m_4m_5$, $m_4 \in N_{[\beta_{p+1}, \beta_D]}$, $m_5 \in N_{[-\beta_1, -\beta_{p-1}]}$. Finally remark that $w_p^{-1}N_{[-\beta_1, -\beta_{p-1}]}w_p \subset N_+$ and $w_p^{-1}N_{[\beta_{p+1}, \beta_D]}w_p \subset N_+$ as $\Delta_{w_p^{-1}} = \{\beta_1, \dots, \beta_p\}$ and hence

$$\begin{aligned} (w_{p-1}v_{\omega_{i_p}}, wgw_p v_{\omega_{i_p}}) &= (w_{p-1}v_{\omega_{i_p}}, hX_{\beta_p}(q_p)m_4m_5w_p v_{\omega_{i_p}}) \\ &= (w_{p-1}v_{\omega_{i_p}}, hX_{\beta_p}(q_p)w_p v_{\omega_{i_p}}) \end{aligned}$$

as $v_{\omega_{i_p}}$ is a highest weight vector.

Note that the vector $w_{p-1}v_{\omega_{i_p}}$ has weight $w_{p-1}\omega_{i_p}$. Since weight spaces corresponding to different weights are orthogonal with respect to the contravariant

bilinear non-degenerate form on $V_{\omega_{i_p}}$, only terms of weight $w_{p-1}\omega_{i_p}$ in the element $hX_{\beta_p}(q_p)w_p v_{\omega_{i_p}}$ will give non-trivial contributions to the scalar product

$$(w_{p-1}v_{\omega_{i_p}}, hX_{\beta_p}(q_p)w_p v_{\omega_{i_p}}).$$

To find these terms we observe that each weight space is an eigenspace for the action of H and that $X_{\beta_p}(q_p)w_p v_{\omega_{i_p}} = X_{\beta_p}(q_p)s_{\beta_p}w_{p-1}v_{\omega_{i_p}}$. Since $\beta_p \notin \Delta_{w_{p-1}^{-1}}$ we infer $X_{\beta_p}w_{p-1}v_{\omega_{i_p}} = 0$, and hence $w_{p-1}v_{\omega_{i_p}}$ is a highest weight vector for the \mathfrak{sl}_2 -triple generated by the elements $X_{\pm\beta_p}$. Moreover, since $w_{p-1}^{-1}(-\beta_p) = -\alpha_{i_p}$ the vector $X_{-\alpha_{i_p}} = w_{p-1}^{-1}X_{-\beta_p}w_{p-1}$ is a root vector corresponding to $-\alpha_{i_p}$, and hence $X_{-\beta_p}^2 w_{p-1}v_{\omega_{i_p}} = w_{p-1}X_{-\alpha_{i_p}}^2 v_{\omega_{i_p}} = 0$ by the definition of $v_{\omega_{i_p}}$. Therefore $w_{p-1}v_{\omega_{i_p}}$ is a highest weight vector for the two-dimensional irreducible representation of the \mathfrak{sl}_2 -triple generated by the elements $X_{\pm\beta_p}$, and $s_{\beta_p}w_{p-1}v_{\omega_{i_p}}$ is a non-zero lowest weight vector for that representation. Recalling the standard \mathfrak{sl}_2 -representation theory we deduce that

$$\begin{aligned} X_{\beta_p}(q_p)w_p v_{\omega_{i_p}} &= X_{\beta_p}(q_p)s_{\beta_p}w_{p-1}v_{\omega_{i_p}} = s_{\beta_p}w_{p-1}v_{\omega_{i_p}} + q_p X_{\beta_p}s_{\beta_p}w_{p-1}v_{\omega_{i_p}} \\ &= s_{\beta_p}w_{p-1}v_{\omega_{i_p}} + \frac{q_p}{c_p}w_{p-1}v_{\omega_{i_p}}, \end{aligned}$$

where c_p is a non-zero constant only depending on the choice of the representative $s_{\beta_p} \in G$ and on the choice of the root vector $X_{\pm\beta_p}$.

The only term of weight $w_{p-1}\omega_{i_p}$ in the right-hand side of the last identity is $q_p/c_p w_{p-1}v_{\omega_{i_p}}$, and hence

$$(w_{p-1}v_{\omega_{i_p}}, wg w_p v_{\omega_{i_p}}) = (w_{p-1}v_{\omega_{i_p}}, hX_{\beta_p}(q_p)w_p v_{\omega_{i_p}}) = \left(w_{p-1}v_{\omega_{i_p}}, h \frac{q_p}{c_p} w_{p-1}v_{\omega_{i_p}} \right),$$

and

$$q_p = c_p \frac{(w_{p-1}v_{\omega_{i_p}}, wg w_p v_{\omega_{i_p}})}{(w_{p-1}v_{\omega_{i_p}}, h w_{p-1}v_{\omega_{i_p}})}.$$

Similar arguments show that

$$(w_{p-1}v_{\omega_{i_p}}, h w_{p-1}v_{\omega_{i_p}}) = (w_{p-1}v_{\omega_{i_p}}, wg w_{p-1}v_{\omega_{i_p}}).$$

Combining the last two identities we obtain formula (2.17). \square

Let $g = nw^{-1}hn_{w^{-1}} \in B_+w^{-1}B_+$, $n \in N_+$, $h \in H$, $n_{w^{-1}} = X_{\beta_k}(q_k) \dots X_{\beta_1}(q_1)$ be an arbitrary element. Using formula (2.17) one can find the numbers q_p inductively. Namely,

$$q_1 = c_1 \frac{(v_{\omega_{i_1}}, gw_1 v_{\omega_{i_1}})}{(v_{\omega_{i_1}}, gw v_{\omega_{i_1}})},$$

and if q_1, \dots, q_{p-1} are already found then

$$q_p = c_p \frac{(w_{p-1}v_{\omega_p}, wg X_{\beta_1}(-q_1) \dots X_{\beta_{p-1}}(-q_{p-1})w_p v_{\omega_p})}{(w_{p-1}v_{\omega_p}, wg X_{\beta_1}(-q_1) \dots X_{\beta_{p-1}}(-q_{p-1})w_{p-1}v_{\omega_p})}. \tag{2.18}$$

Note also that once the numbers q_p are found, and hence the element $n_{w^{-1}}$ is determined, one can also find n using [4, Prop. 2.11].

Indeed, let $\bar{w} = s_{i_1} \cdots s_{i_D}$ be the reduced decomposition of the longest element of the Weyl group corresponding to a normal ordering β_1, \dots, β_D of the system of positive roots Δ_+ . Then $\beta_j = s_{i_1} \cdots s_{i_{j-1}} \alpha_j$. Consider the elements $w_j = s_{i_1} \cdots s_{i_j} = s_{\beta_j} \cdots s_{\beta_1}$. Observe that $\Delta_{w_j^{-1}} = \{\beta_1, \dots, \beta_j\} = \{\alpha \in \Delta_+ : w_j^{-1} \alpha \in \Delta_-\}$. The element n can be uniquely represented in the form $n = X_{\beta_1}(r_1) \cdots X_{\beta_D}(r_D)$. Assume that the root vectors X_α used in the definition of one-parametric subgroups $X_\alpha(t) = \exp(tX_\alpha)$ are chosen in such a way that $\omega(X_\alpha) = X_{-\alpha}$. Then according to [4, Props. 2.6 and 2.11] we have

$$r_p = d_p \frac{(w_{p-1}v_{\omega_p}, nw^{-1}hw_{p-1}v_{\omega_p})}{(w_{p-1}v_{\omega_p}, nw^{-1}hw_{p-1}v_{\omega_p})} = d_p \frac{(w_{p-1}v_{\omega_p}, gn_{w^{-1}}^{-1}w_{p-1}v_{\omega_p})}{(w_{p-1}v_{\omega_p}, gn_{w^{-1}}^{-1}w_{p-1}v_{\omega_p})}, \tag{2.19}$$

where d_p is a non-zero constant only depending on the choice of the representative $s_{\beta_p} \in G$ and on the choice of the root vector X_{β_p} , and it is assumed that $w_0 = 1$.

Finally, once we know the coefficients r_i we can find n and $h = wn^{-1}gn_{w^{-1}}^{-1}$. Thus we have completely described g in terms of some matrix elements of finite-dimensional irreducible representations. The functions q_i, r_i defined by (2.18) and (2.19) together with the fundamental weights evaluated at $h = wn^{-1}gn_{w^{-1}}^{-1}$ can be regarded as natural coordinates on the Bruhat cell $B_+w^{-1}B_+$.

Now we introduce functions required for the definition of the operator Π . Let $g \in NZs^{-1}N$. By Proposition 2.8 g can be uniquely represented in the form $g = n^{-1}n_szs^{-1}n, n \in N, n_s \in N_s$.

Proposition 2.11. *Let $g = n^{-1}n_szs^{-1}n \in NZs^{-1}N$. Let α_i be the simple roots of a system of positive roots Δ_+^s associated to s, s_i the corresponding simple reflections, $\beta_1, \dots, \beta_D, \beta_j = s_{i_1} \cdots s_{i_{j-1}} \alpha_{i_j}$ a normal ordering (2.11) of Δ_+^s . Denote $\Delta_0 \cap \Delta_+^s = \{\beta_{d+1}, \dots, \beta_D\}$. Let $\omega_i, i = 1, \dots, l$ be the fundamental weights corresponding to Δ_+^s, v_{ω_i} a non-zero highest weight vector in the irreducible highest weight representation V_{ω_i} of highest weight $\omega_i, (\cdot, \cdot)$ the contravariant non-degenerate bilinear form on V_{ω_i} such that $(v_{\omega_i}, v_{\omega_i}) = 1$. Then n can be uniquely factorized as $n = X_{\beta_d}(t_d) \cdots X_{\beta_1}(t_1)$ and the numbers t_i can be found inductively by the following formula*

$$t_p = c_p \frac{(w_{p-1}v_{\omega_{i_p}}, sg_p w_{p-1}v_{\omega_{i_p}})}{(w_{p-1}v_{\omega_{i_p}}, sg_p w_{p-1}v_{\omega_{i_p}})}, \tag{2.20}$$

where $w_p = s_{\beta_p} \cdots s_{\beta_1}, w_{p-1} = s_{\beta_{p-1}} \cdots s_{\beta_1}, c_p$ is a non-zero constant only depending on the choice of the representative $s_{\beta_p} \in G$ and on the choice of the root vector $X_{\beta_p} \in \mathfrak{g}, g_p = n_p gn_p^{-1}, n_p = X_{\beta_{p-1}}(t_{p-1}) \cdots X_{\beta_1}(t_1)$ and it is assumed that $n_1 = 1, w_0 = 1$.

Proof. The proof of this proposition is similar to that of the previous proposition. The numbers t_p can be found by induction starting with $p = 1$. We shall establish the induction step. The case $p = 1$ corresponding to the base of the induction can be considered in a similar way.

Assume that t_1, \dots, t_{p-1} have already been found. Then

$$sg_p = sn_p gn_p^{-1} = sX_{\beta_p}(-t_p) \cdots X_{\beta_d}(-t_d)n_szs^{-1}X_{\beta_d}(t_d) \cdots X_{\beta_p}(t_p),$$

where $n_p = X_{\beta_{p-1}}(t_{p-1}) \cdots X_{\beta_1}(t_1)$. By Lemma 2.10 we can write

$$X_{\beta_d}(t_d) \cdots X_{\beta_p}(t_p) = X_{\beta_p}(t_p)m_1, m_1 \in N_{[\beta_{p+1}, \beta_d]}.$$

Since $\Delta_{w_p^{-1}} = \{\beta_1, \dots, \beta_p\}$ we have $w_p^{-1}N_{[\beta_{p+1}, \beta_d]}w_p \subset N_+^s$, and hence

$$\begin{aligned} X_{\beta_d}(t_d) \cdots X_{\beta_p}(t_p)w_pv_{\omega_{i_p}} &= X_{\beta_p}(t_p)m_1w_pv_{\omega_{i_p}} \\ &= X_{\beta_p}(t_p)w_pw_p^{-1}m_1w_pv_{\omega_{i_p}} \\ &= X_{\beta_p}(t_p)w_pv_{\omega_{i_p}} \end{aligned}$$

as $v_{\omega_{i_p}}$ is a highest weight vector. We deduce that

$$(w_{p-1}v_{\omega_{i_p}}, sg_p w_p v_{\omega_{i_p}}) = (w_{p-1}v_{\omega_{i_p}}, sX_{\beta_p}(-t_p) \cdots X_{\beta_d}(-t_d)n_s z s^{-1} X_{\beta_p}(t_p)w_pv_{\omega_{i_p}}).$$

Now observe that $X_{\beta_p}(-t_p) \cdots X_{\beta_d}(-t_d)n_s \in N_{[\beta_p, \beta_d]}N_s$. Note that the normal ordering in Δ_+^s associated to s has the property that the set Δ_s is a segment of the form β_k, \dots, β_d . Therefore the union $[\beta_p, \beta_d] \cup [\beta_k, \beta_d]$ is also a minimal segment and the subgroup $N_{[\beta_p, \beta_d]}N_s$ is generated by one-parametric subgroups corresponding to the roots from that segment. Thus using Lemma 2.10 one can uniquely factorize the element $X_{\beta_p}(-t_p) \cdots X_{\beta_d}(-t_d)n_s$ as

$$X_{\beta_p}(-t_p) \cdots X_{\beta_d}(-t_d)n_s = m_2 m_3, m_2 \in N_{[\beta_k, \beta_d]}, \quad m_3 \in N_{[\beta_p, \beta_{k-1}]},$$

where it is assumed that $N_{[\beta_p, \beta_{k-1}]} = 1$ if $p > k - 1$. If $\alpha \in [\beta_p, \beta_{k-1}]$ then $s\alpha \in \Delta_+^s$ and by the properties of the normal ordering in Δ_+^s associated to s we have $s\alpha > \alpha$, and if $s\alpha + \alpha_0 \in \Delta_+^s$ for $\alpha_0 \in \Delta_0$ then $s\alpha + \alpha_0 > \alpha$. Observing also that Z is generated by one-parametric subgroups corresponding to roots from Δ_0 and by the centralizer of s in H which normalizes all one-parametric subgroups corresponding to roots, we deduce $sX_{\beta_p}(-t_p) \cdots X_{\beta_d}(-t_d)n_s z s^{-1} = sm_2 s^{-1} z_1 m_4$, $m_4 = z_1^{-1} s m_3 s^{-1} z_1 \in N_{[\beta_{p+1}, \beta_D]}$, $z_1 = s z s^{-1} \in Z$.

Observe now that using Lemma 2.10 we can uniquely factorize the element $m_4 X_{\beta_p}(t_p) \in N_{[\beta_p, \beta_D]}$ as $m_4 X_{\beta_p}(t_p) = X_{\beta_p}(t_p)m_5$, $m_5 \in N_{[\beta_{p+1}, \beta_D]}$. Remark that $w_p^{-1}N_{[\beta_{p+1}, \beta_D]}w_p \subset N_+^s$ as $\Delta_{w_p^{-1}} = \{\beta_1, \dots, \beta_p\}$ and hence

$$\begin{aligned} (w_{p-1}v_{\omega_{i_p}}, sg_p w_p v_{\omega_{i_p}}) &= (w_{p-1}v_{\omega_{i_p}}, sm_2 s^{-1} z_1 m_4 X_{\beta_p}(t_p)w_pv_{\omega_{i_p}}) \\ &= (w_{p-1}v_{\omega_{i_p}}, sm_2 s^{-1} z_1 X_{\beta_p}(t_p)m_5 w_pv_{\omega_{i_p}}) \\ &= (\omega(z_1)\omega(sm_2 s^{-1})w_{p-1}v_{\omega_{i_p}}, X_{\beta_p}(t_p)w_pv_{\omega_{i_p}}) \end{aligned}$$

as $v_{\omega_{i_p}}$ is a highest weight vector.

By the definition of m_2 we have $sm_2 s^{-1} \in N_-^s$, and hence $\omega(sm_2 s^{-1}) \in N_+^s$. Therefore using arguments similar to those used above we can factorize $\omega(sm_2 s^{-1}) = m_6 m_7$, $m_6 \in N_{[\beta_1, \beta_{p-1}]}$, $m_7 \in N_{[\beta_p, \beta_D]}$ and obtain that

$$\begin{aligned} (w_{p-1}v_{\omega_{i_p}}, sg_p w_p v_{\omega_{i_p}}) &= (\omega(z_1)\omega(sm_2 s^{-1})w_{p-1}v_{\omega_{i_p}}, X_{\beta_p}(t_p)w_pv_{\omega_{i_p}}) \\ &= (z_2 m_6 m_7 w_{p-1}v_{\omega_{i_p}}, X_{\beta_p}(t_p)w_pv_{\omega_{i_p}}) \\ &= (z_2 m_6 w_{p-1}v_{\omega_{i_p}}, X_{\beta_p}(t_p)w_pv_{\omega_{i_p}}), \end{aligned}$$

where $z_2 = \omega(z_1) \in Z$.

As we already showed in the previous proof,

$$X_{\beta_p}(t_p)w_p v_{\omega_{i_p}} = s_{\beta_p} w_{p-1} v_{\omega_{i_p}} + \frac{t_p}{c_p} w_{p-1} v_{\omega_{i_p}} = cX_{-\beta_p} w_{p-1} v_{\omega_{i_p}} + \frac{t_p}{c_p} w_{p-1} v_{\omega_{i_p}}, \quad c \in \mathbb{C},$$

so the first term in the last sum has weight $-\beta_p + w_{p-1}\omega_{i_p}$, and the second one $w_{p-1}\omega_{i_p}$.

The vector $z_2 m_6 w_{p-1} v_{\omega_{i_p}}$ is a linear combination of vectors of weights of the form $w_{p-1}\omega_{i_p} + \sum_{q=1}^{p-1} c_q \beta_q + \omega_0$, where $\omega_0 \in \mathfrak{h}_0^*$, $c_q \in \{0, 1, 2, \dots\}$. Since weight spaces corresponding to different weights are orthogonal with respect to the contravariant bilinear non-degenerate form on $V_{\omega_{i_p}}$ the only nontrivial contributions to the product $(z_2 m_6 w_{p-1} v_{\omega_{i_p}}, X_{\beta_p}(t_p)w_p v_{\omega_{i_p}})$ come from the products of vectors of weights either $-\beta_p + w_{p-1}\omega_{i_p}$ or $w_{p-1}\omega_{i_p}$.

In the first case we must have $w_{p-1}\omega_{i_p} + \sum_{q=1}^{p-1} c_q \beta_q + \omega_0 = -\beta_p + w_{p-1}\omega_{i_p}$, and hence $\sum_{q=1}^{p-1} c_q \beta_q + \omega_0 = -\beta_p$. In particular, $\bar{h}_0(\sum_{q=1}^{p-1} c_q \beta_q + \omega_0) = \sum_{q=1}^{p-1} c_q \bar{h}_0(\beta_q) = -\bar{h}_0(\beta_p)$ which is impossible as $-\bar{h}_0(\beta_p) < 0$ and $c_q \bar{h}_0(\beta_q) \geq 0$.

In the second case we must have $w_{p-1}\omega_{i_p} + \sum_{q=1}^{p-1} c_q \beta_q + \omega_0 = w_{p-1}\omega_{i_p}$ or $\sum_{q=1}^{p-1} c_q \beta_q + \omega_0 = 0$. In particular, $\bar{h}_0(\sum_{q=1}^{p-1} c_q \beta_q + \omega_0) = \sum_{q=1}^{p-1} c_q \bar{h}_0(\beta_q) = 0$ which forces $c_q = 0$ for all q as $\bar{h}_0(\beta_q) > 0$ and $c_q \in \{0, 1, 2, \dots\}$, and hence $\omega_0 = 0$ as well.

We conclude that the only nontrivial contributions to the product

$$(z_2 m_6 w_{p-1} v_{\omega_{i_p}}, X_{\beta_p}(t_p)w_p v_{\omega_{i_p}})$$

come from the products of vectors of weights $w_{p-1}\omega_{i_p}$. By the above considerations only terms of the form $z_2 w_{p-1} v_{\omega_{i_p}}$ may give contributions of weight $w_{p-1}\omega_{i_p}$ in the weight decomposition of the element $z_2 m_6 w_{p-1} v_{\omega_{i_p}}$, and this yields

$$\begin{aligned} (w_{p-1} v_{\omega_{i_p}}, s g_p w_p v_{\omega_{i_p}}) &= (z_2 m_6 w_{p-1} v_{\omega_{i_p}}, X_{\beta_p}(t_p)w_p v_{\omega_{i_p}}) \\ &= \frac{t_p}{c_p} (z_2 w_{p-1} v_{\omega_{i_p}}, w_{p-1} v_{\omega_{i_p}}) = \frac{t_p}{c_p} (w_{p-1} v_{\omega_{i_p}}, s z s^{-1} w_{p-1} v_{\omega_{i_p}}). \end{aligned}$$

Therefore

$$t_p = c_p \frac{(w_{p-1} v_{\omega_{i_p}}, s g_p w_p v_{\omega_{i_p}})}{(w_{p-1} v_{\omega_{i_p}}, s z s^{-1} w_{p-1} v_{\omega_{i_p}})}.$$

Similar arguments show that

$$(w_{p-1} v_{\omega_{i_p}}, s z s^{-1} w_{p-1} v_{\omega_{i_p}}) = (w_{p-1} v_{\omega_{i_p}}, s g_p w_{p-1} v_{\omega_{i_p}}).$$

Combining the last two identities we obtain formula (2.20). \square

Remark 2.1. Note that in the proof of [15, Prop. 6.2] it is shown that $NZs^{-1}N$ is a closed subvariety in G . Therefore for each $p = 1, \dots, d$ the right-hand side of (2.20) is a regular function on $NZs^{-1}N$ as the composition of the regular function

$$n = X_{\beta_d}(t_d) \cdots X_{\beta_1}(t_1) \mapsto t_p$$

defined on N and of isomorphism (2.16) of varieties. Hence the denominator in (2.20) must be canceled.

Observing that in the notation of the previous proposition for $g = n^{-1}n_s z s^{-1}n$ we have $g_{d+1} = n_s z s^{-1} = n_{d+1} g n_{d+1}^{-1}$, $n = n_{d+1} = X_{\beta_d}(t_d) \cdots X_{\beta_1}(t_1)$ and that the map (2.16) is an isomorphism of varieties, we infer the following proposition from the previous statement.

Proposition 2.12. *Let A_p , $p = 1, \dots, d$ be the rational function on G defined by*

$$A_p(g) = c_p \frac{(w_{p-1} v_{\omega_{i_p}}, s g w_p v_{\omega_{i_p}})}{(w_{p-1} v_{\omega_{i_p}}, s g w_{p-1} v_{\omega_{i_p}})},$$

and Π_p the operator on the space of rational functions on G induced by conjugation by the element $\exp(A_p X_{\beta_p})$,

$$\Pi_p f(g) = f(\exp(A_p(g) X_{\beta_p}) g \exp(-A_p(g) X_{\beta_p})). \tag{2.21}$$

Then the composition

$$\Pi = \Pi_1 \circ \cdots \circ \Pi_d \tag{2.22}$$

gives rise to a well-defined operator

$$\Pi = \Pi_1 \circ \cdots \circ \Pi_d : \mathbb{C}[NZs^{-1}N] \rightarrow \mathbb{C}[NZs^{-1}N]^N,$$

which is a projection operator onto the subspace $\mathbb{C}[NZs^{-1}N]^N$ of N -invariant regular functions on $NZs^{-1}N$.

3. The Poisson structure of q-W algebras

In this section we obtain two descriptions of the Poisson structure of the q-W algebras. First following [11] we recall the definition of Poisson q-W algebras. We shall need some related facts on Poisson–Lie groups and on the definition of the Poisson structure of quasi-triangular Poisson–Lie groups and their dual groups which can be found in [8], [9], [10].

Let $s \in W$ be an element of the Weyl group W of the pair $(\mathfrak{g}, \mathfrak{h})$, Δ_+^s a system of positive roots associated to s , and \mathfrak{h}' the orthogonal complement in \mathfrak{h} , with respect to the Killing form, to the subspace of \mathfrak{h} fixed by the natural action of s on \mathfrak{h} . The restriction of the natural action of s on \mathfrak{h} to the subspace \mathfrak{h}' has no fixed points. Therefore one can define the Cayley transform $((1+s)/(1-s))P_{\mathfrak{h}'}$ of the restriction of s to \mathfrak{h}' , where $P_{\mathfrak{h}'}$ is the orthogonal projection operator onto \mathfrak{h}' in \mathfrak{h} , with respect to the Killing form.

Let $r \in \text{End } \mathfrak{g}$ be the endomorphism defined by

$$r = P_+ - P_- \frac{1+s}{1-s} P_{\mathfrak{h}'}, \tag{3.1}$$

where P_+ , P_- and $P_{\mathfrak{h}'}$ are the projection operators onto \mathfrak{n}_+^s , \mathfrak{n}_-^s and \mathfrak{h}' in the direct sum

$$\mathfrak{g} = \mathfrak{n}_+^s + \mathfrak{h}' + \mathfrak{h}'^\perp + \mathfrak{n}_-^s,$$

and \mathfrak{h}'^\perp is the orthogonal complement to \mathfrak{h}' in \mathfrak{h} with respect to the Killing form. The endomorphism r satisfies the modified classical Yang-Baxter equation

$$[rX, rY] - r([rX, Y] + [X, rY]) = -[X, Y], \quad X, Y \in \mathfrak{g} \tag{3.2}$$

and this ensures that

$$[X, Y]_* = \frac{1}{2}([rX, Y] + [X, rY]), \quad X, Y \in \mathfrak{g} \tag{3.3}$$

is a Lie bracket.

Identifying the dual space \mathfrak{g}^* with \mathfrak{g} using the Killing form one can check that this commutator is dual to a cocycle on \mathfrak{g} which makes $(\mathfrak{g}, \mathfrak{g}^*)$ a Lie bialgebra. Note also that r is skew-symmetric with respect to the Killing form.

Let G be a connected semi-simple Poisson-Lie group with the tangent Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$, G^* the dual connected simply-connected Poisson-Lie group.

Define operators $r_\pm \in \text{End } \mathfrak{g}$ by

$$r_\pm = \frac{1}{2}(r \pm \text{id}).$$

The classical Yang-Baxter equation implies that r_\pm , regarded as a mapping from \mathfrak{g}^* into \mathfrak{g} , is a Lie algebra homomorphism. Moreover, $r_+^* = -r_-$, and $r_+ - r_- = \text{id}$.

One can describe the dual group G^* in terms of G as follows. Put $\mathfrak{d} = \mathfrak{g} + \mathfrak{g}$ (direct sum of two copies). The mapping

$$\mathfrak{g}^* \rightarrow \mathfrak{d} : X \mapsto (X_+, X_-), \quad X_\pm = r_\pm X \tag{3.4}$$

is a Lie algebra embedding. Thus we may identify \mathfrak{g}^* with a Lie subalgebra in \mathfrak{d} .

Naturally, embedding (3.4) gives rise to a Lie group embedding

$$G^* \rightarrow G \times G, \quad L \mapsto (L_+, L_-).$$

We shall identify G^* with the image of this embedding in $G \times G$.

Now we explicitly describe the Poisson structures on the Poisson-Lie group G and on its dual group G^* .

For every group A with Lie algebra \mathfrak{a} and any function $\varphi \in C^\infty(A)$ we define left and right gradients $\nabla\varphi, \nabla'\varphi$, which are C^∞ -functions on A with values in \mathfrak{a}^* , by the formulae

$$\begin{aligned} \xi(\nabla\varphi(x)) &= \left(\frac{d}{dt}\right)_{t=0} \varphi(e^{t\xi}x), \\ \xi(\nabla'\varphi(x)) &= \left(\frac{d}{dt}\right)_{t=0} \varphi(xe^{t\xi}), \quad x \in A, \xi \in \mathfrak{a}. \end{aligned} \tag{3.5}$$

Denote by $\langle \cdot, \cdot \rangle$ the Killing form on \mathfrak{g} . Then the left and the right gradients of functions on G can be regarded as C^∞ -functions on G with values in \mathfrak{g} . They satisfy the following relations

$$\begin{aligned} \langle X, \nabla\varphi(g) \rangle &= \left(\frac{d}{dt}\right)_{t=0} \varphi(e^{tX}g), \\ \langle X, \nabla'\varphi(g) \rangle &= \left(\frac{d}{dt}\right)_{t=0} \varphi(ge^{tX}), \quad g \in G, X \in \mathfrak{g}. \end{aligned} \tag{3.6}$$

If $K \subset G$ is a Lie subgroup, $\mathfrak{k} \subset \mathfrak{g}$ its Lie algebra and $\bar{\mathfrak{k}} \subset \mathfrak{g}$ the image of \mathfrak{k} in $\mathfrak{g}^* \simeq \mathfrak{g}$ under the identification $\mathfrak{g} \simeq \mathfrak{g}^*$ induced by the Killing form then the left and the right gradients of functions on K can be regarded as C^∞ -functions on K with values in $\bar{\mathfrak{k}} \subset \mathfrak{g}$.

The canonical Poisson bracket on the Poisson–Lie group G with the tangent bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ has the form:

$$\{ \varphi, \psi \} = \frac{1}{2} \langle r \nabla \varphi, \nabla \psi \rangle - \frac{1}{2} \langle r \nabla' \varphi, \nabla' \psi \rangle. \tag{3.7}$$

The canonical Poisson bracket on the dual Poisson–Lie group G^* can be described in terms of the original group G and the classical r-matrix r .

We shall use gradients of a function $\varphi \in C^\infty(G^*)$ with respect to the G^* group structure,

$$\begin{aligned} \langle X, \nabla \varphi(L_+, L_-) \rangle &= \left(\frac{d}{ds} \right)_{s=0} \varphi(e^{sX_+} L_+, e^{sX_-} L_-), \\ \langle X, \nabla' \varphi(L_+, L_-) \rangle &= \left(\frac{d}{ds} \right)_{s=0} \varphi(L_+ e^{sX_+}, L_- e^{sX_-}), \quad X \in \mathfrak{g}. \end{aligned}$$

Proposition 3.1 (cf. [9, §2], [10, §3]). *Let $(L_+, L_-) \in G^*$. Then for $f, g \in C^\infty(G^*)$ one has*

$$\begin{aligned} \{ f, g \}_{G^*}(L_+, L_-) &= \langle (\text{Ad } L_+ - \text{Ad } L_-) \nabla' f, \nabla g \rangle \\ &\quad - \langle \nabla f, (\text{Ad } L_+ - \text{Ad } L_-) \nabla' g \rangle. \end{aligned} \tag{3.8}$$

Denote by G_* the group G equipped with the following Poisson bracket

$$\{ \varphi, \psi \}_* = - \langle r \nabla \varphi, \nabla \psi \rangle - \langle r \nabla' \varphi, \nabla' \psi \rangle + 2 \langle r_- \nabla' \varphi, \nabla \psi \rangle + 2 \langle r_+ \nabla \varphi, \nabla' \psi \rangle, \tag{3.9}$$

where all the gradients are taken with respect to the original group structure on G .

Then the map $q : G^* \rightarrow G_*$ defined by

$$q(L_+, L_-) = L_- L_+^{-1} \tag{3.10}$$

is a Poisson mapping and the image of q is a dense open subset in G_* .

There exists a unique left Poisson group action

$$G \times G^* \rightarrow G^*, \quad (g, (L_+, L_-)) \mapsto g \circ (L_+, L_-),$$

i.e., the action map is Poisson assuming that $G \times G^*$ is equipped with the product Poisson structure, and if $q : G^* \rightarrow G_*$ is the map defined by formula (3.10) then

$$q(g \circ (L_+, L_-)) = g L_- L_+^{-1} g^{-1}.$$

Now we describe the Poisson structure of Poisson q-W-algebras. First restrict the action of G on G_* by conjugations to the subgroup $N \subset G$. Let $s \in G$ be a representative of the element $s \in W$. By Proposition 2.8 the quotient $N Z s^{-1} N / N \simeq s^{-1} Z N_s^{-1}$ is a subspace of the quotient G_*/N .

The following proposition summarizes the results of [11, Sect. 5].

Proposition 3.2. *Let G be a complex connected semi-simple algebraic group, \mathfrak{g} its Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra in \mathfrak{g} . Let $s \in W$ be an element of the Weyl group of the pair $(\mathfrak{g}, \mathfrak{h})$, \mathfrak{p} the parabolic subalgebra associated to s , \mathfrak{n} the nilradical of \mathfrak{p} , $N \subset G$ the subgroup corresponding to \mathfrak{n} , r the classical r -matrix (3.1) on \mathfrak{g} . Equip G with Poisson bracket (3.9) and denote the corresponding Poisson manifold by G_* . Restrict the conjugation action of G on G_* to the subgroup N . Then $C^\infty(G_*)^N$ is a Poisson subalgebra in $C^\infty(G_*)$ so that the Poisson structure on G_* induces a reduced Poisson structure on G_*/N , Hamiltonian vector fields of N -invariant functions on G_* are tangent to $NZs^{-1}N$ and $N_sZs^{-1} \simeq NZs^{-1}N/N$ is a smooth Poisson submanifold of G_*/N .*

Let $\pi : NZs^{-1}N \rightarrow N_sZs^{-1} \simeq NZs^{-1}N/N$ be the canonical projection. Then for any (locally defined) smooth functions φ, ψ on N_sZs^{-1} , and any (locally defined) smooth extensions $\overline{\varphi}, \overline{\psi}$ of $\varphi \circ \pi, \psi \circ \pi$ to G_* , we have

$$\{\varphi, \psi\} \circ \pi = \{\overline{\varphi}, \overline{\psi}\}_{G_*} \circ i = (P_{G_*}, d\overline{\varphi} \wedge d\overline{\psi}) \circ i, \tag{3.11}$$

where P_{G_*} is the Poisson tensor of G_* , (\cdot, \cdot) is the canonical pairing and $i : NZs^{-1}N \rightarrow G_*$ is the inclusion.

Note that for any $X \in \mathfrak{g}$ and any regular function φ on G the functions $\langle \nabla\varphi, X \rangle$ and $\langle \nabla'\varphi, X \rangle$ are regular. Therefore the space of regular functions on G is closed with respect to Poisson bracket (3.9). Since by Proposition 2.8 the projection $NZs^{-1}N \rightarrow N_sZs^{-1}$ induced by the map $G_* \rightarrow G_*/N$ is a morphism of varieties, Proposition 3.2 implies that the algebra of regular functions on N_sZs^{-1} is closed under the reduced Poisson bracket defined on N_sZs^{-1} in the previous theorem. This Poisson algebra is called the Poisson q-W-algebra associated to the Weyl group element $s \in W$, or, more precisely, to the conjugacy class of $s \in W$.

The reduced Poisson structure on the slice N_sZs^{-1} is explicitly described in the following proposition.

Proposition 3.3. *Identify N_sZs^{-1} with the subgroup $N_sZ \subset G$ using the right translation by s . Let \mathfrak{n}_s be the Lie algebra of N_s , \mathfrak{z} the Lie algebra of Z so that as a linear space the Lie algebra of N_sZ is $\mathfrak{n}_s + \mathfrak{z} \subset \mathfrak{g}$. Then the reduced Poisson bracket on $N_sZs^{-1} \simeq N_sZ$ is given by*

$$\begin{aligned} \{\varphi, \psi\}(m) = & -\langle r\nabla\varphi, \nabla\psi \rangle - \langle r\text{Ad}(sm^{-1})\nabla\varphi, \text{Ad}(sm^{-1})\nabla\psi \rangle \\ & + 2\langle r_-\text{Ad}(sm^{-1})\nabla\varphi, \nabla\psi \rangle + 2\langle r_+\nabla\varphi, \text{Ad}(sm^{-1})\nabla\psi \rangle, \end{aligned} \tag{3.12}$$

where $\varphi, \psi \in C^\infty(N_sZ)$, $m \in N_sZ$, and the left gradients $\nabla\varphi, \nabla\psi$ are regarded as C^∞ -functions on N_sZ with values in $\overline{\mathfrak{n}_s + \mathfrak{z}} \subset \mathfrak{g}$.

Proof. We shall use formula (3.11) to prove this proposition. Firstly we use the right trivialization of the tangent and of the cotangent bundle to G and identify them using the Killing form so that differentials of functions on G become their left gradients which can be regarded as functions on G with values in \mathfrak{g} . Observing that the right gradient of a function φ on G is related to its left gradient by

$$\nabla'\varphi(g) = \text{Ad } g^{-1}(\nabla\varphi(g))$$

we can rewrite formula (3.9) for the Poisson bracket on G_* in the form

$$\begin{aligned} \{\varphi, \psi\}(g) = & -\langle r\nabla\varphi, \nabla\psi \rangle - \langle r\text{Ad}(g^{-1})\nabla\varphi, \text{Ad}(g^{-1})\nabla\psi \rangle \\ & + 2\langle r_-\text{Ad}(g^{-1})\nabla\varphi, \nabla\psi \rangle + 2\langle r_+\nabla\varphi, \text{Ad}(g^{-1})\nabla\psi \rangle. \end{aligned} \quad (3.13)$$

Now we calculate the reduced Poisson bracket of two functions $\varphi, \psi \in C^\infty(N_s Z)$ using formulas (3.11) and (3.13). We evaluate the left-hand side and the right-hand side of (3.11) at point ms^{-1} , $m \in N_s Z$. Observe that the reduced space $N_s Z \simeq N_s Z s^{-1}$ can be regarded as a submanifold of $NZs^{-1}N$; therefore the right-hand side of (3.11) evaluated at point ms^{-1} is equal to the reduced Poisson bracket of φ and ψ at point m .

Now we justify that the right-hand side of (3.11) is equal to the right-hand side of (3.12). Denote $\varphi^* = \varphi \circ \pi, \psi^* = \psi \circ \pi$ and let $P_{G_*}(d\bar{\varphi})$ be the Hamiltonian vector field corresponding to the function $\bar{\varphi}$. Without loss of generality we can assume that the extension $\bar{\varphi}$ is a (locally defined) N -invariant function on G_* . Since by Proposition 3.2 Hamiltonian vector fields of N -invariant functions on G_* are tangent to $NZs^{-1}N$ the right-hand side of (3.11) can be rewritten as

$$(P_{G_*}, d\bar{\varphi} \wedge d\bar{\psi}) \circ i(ms^{-1}) = (P_{G_*}(d\bar{\varphi}), d\bar{\psi})(ms^{-1}) = (P_{G_*}(d\bar{\varphi}), d\psi^*)(ms^{-1}).$$

Recall that we use the right trivialization of the tangent bundle to G and to $N_s Z$ and identify them with the corresponding cotangent bundles. The first identification also induces identifications of the tangent and of the cotangent bundles to $NZs^{-1}N$ with a subbundle of the tangent bundle to G , and of the tangent bundle to $N_s Z s^{-1}$ with a subbundle of the tangent bundle to $NZs^{-1}N$. Since the function ψ^* is the N -invariant function on $NZs^{-1}N$ the restriction of which to $N_s Z s^{-1} \simeq N_s Z$ is equal to ψ , under these identifications $d\psi^*$ at the point $ms^{-1} \in NZs^{-1}N$ is equal to the differential of ψ , so $d\psi^*(ms^{-1}) = d\psi(m) = \nabla\psi(m) \in \overline{\mathfrak{n}_s + \mathfrak{z}} \subset \mathfrak{g}$ and $P_{G_*}(d\bar{\varphi})(ms^{-1}) \in \mathfrak{g}$. This yields

$$(P_{G_*}, d\bar{\varphi} \wedge d\bar{\psi}) \circ i(ms^{-1}) = (P_{G_*}(d\bar{\varphi})(ms^{-1}), d\psi(m)),$$

where the pairing in the right-hand side of the last formula is induced by the Killing form.

Finally, by the skew-symmetry of the Poisson bracket

$$(P_{G_*}, d\bar{\varphi} \wedge d\bar{\psi}) \circ i(ms^{-1}) = (P_{G_*}(ms^{-1}), d\varphi(m) \wedge d\psi(m)), \quad (3.14)$$

where $d\varphi(m), d\psi(m) \in \overline{\mathfrak{n}_s + \mathfrak{z}} \subset \mathfrak{g}$, $P_{G_*}(ms^{-1}) \in \mathfrak{g} \wedge \mathfrak{g}$, and the pairing in the right-hand side of the last formula is induced by the Killing form. Now recalling that $d\varphi(m) = \nabla\varphi(m), d\psi(m) = \nabla\psi(m)$ and using (3.13) in the right-hand side of formula (3.14), we arrive at (3.12). \square

From Propositions 2.12 and 3.2 we can immediately deduce another description of the reduced Poisson bracket on $N_s Z s^{-1}$.

Proposition 3.4. *Assume that G is simply connected. Let $\bar{\varphi}, \bar{\psi}$ be two regular functions on G_* the restrictions of which to $N_s Z s^{-1} \subset G_*$ are equal to φ, ψ , respectively. Then the reduced Poisson bracket $\{\varphi, \psi\}$ of φ and ψ satisfies the following identity*

$$\{\varphi, \psi\}(ms^{-1}) = \{\Pi\bar{\varphi}, \Pi\bar{\psi}\}_{G_*(ms^{-1})}, m \in N_s Z, \quad (3.15)$$

where the action of Π on $\bar{\varphi}$ and $\bar{\psi}$ is defined by formulas (2.21) and (2.22).

Proof. Formula (3.15) is just formula (3.11) rewritten in terms of the operator Π . Indeed, by definition the operator $\Pi : \mathbb{C}[NZs^{-1}N] \rightarrow \mathbb{C}[NZs^{-1}N]^N$ is induced by the projection $\pi : NZs^{-1}N \rightarrow N_s Z s^{-1}$. Therefore the functions $\Pi\bar{\varphi}, \Pi\bar{\psi}$ are locally smooth extensions of $\varphi \circ \pi, \psi \circ \pi$ to G_* . It remains to apply formula (3.11) to φ and ψ and to these extensions. Note that the right-hand side of (3.11) only depends on φ and ψ but not on the extensions. \square

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