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PERFECT ISOMETRIES BETWEEN BLOCKS OF COMPLEX REFLECTION GROUPS

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Abstract. In this paper, we prove that, given any integers d, e, r and r', and a prime p not dividing de, any two blocks of the complex reflection groups $G(de,e,r)$ and $G(de,e,r')$ with the same p-weight are perfectly isometric.

1. Introduction

In the last 30 years, a lot of research in modular representation theory of finite groups has been fuelled by Broué's Abelian Defect Conjecture. This predicts that any p-block B of a finite group G which has abelian defect group P should be derived equivalent to its Brauer correspondent b in $N_G(P)$ (see [1]). Several refinements of this conjecture have been formulated, which involve deep structural correspondences, such as *splendid equivalences* or *Rickard equivalences*. At the level of complex irreducible characters, all of these conjectures predict the existence of a *perfect isometry* between B and b.

The first step towards proving Broué's Abelian Defect Conjecture for the symmetric group was proved by Enguehard in [3]. He showed that, if B and B' are p-blocks of the symmetric groups $\mathfrak{S}_m$ and $\mathfrak{S}_n$ respectively, and B and B' have the same *p-weight*, then B and B' are perfectly isometric. In this paper, we generalize Enguehard's result to the infinite family of complex reflection groups. More precisely, we show that, given any integers d, e, r and r', and a prime p not dividing de, any two blocks of the complex reflection groups $G(de,e,r)$ and $G(de,e,r')$ with the same p-weight are perfectly isometric (see Theorem 4.12).

The paper is organised as follows. In Section 2, we introduce some combinatorial tools we will need throughout the paper. We then present the already existing parametrizations, due to James and Kerber (§2.3) and to Marin and Michel (§2.4), of the irreducible representations of the wreath products $G(d,1,r)$, as well as a new parametrization which is more convenient for our purposes (§2.5). In Section 3, we construct the irreducible $G(de,e,r)$-modules (§3.1), and obtain some useful formulæ for the values of certain characters of $G(de,e,r)$ (§3.2). The results of this part are of independent interest; in particular, the character table of $G(de,e,r)$ is completely determined (see Theorem 3.7, Theorem 3.13 and Equality (18)). Note that we do not follow the same approach as that of [10].

Section 4 is devoted to perfect isometries and our main result, Theorem 4.12. In §4.1, §4.2 and §4.3, we describe the irreducible characters and p-blocks of $G(de,e,r)$, as well as bijections between p-blocks with the same p-weight. Finally, in §4.4, we introduce perfect isometries and prove our main theorem.

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2. Irreducible representations of $G(d,1,r)$

Let d and r be positive integers. Let $\mathcal{U}_d$ be the group of complex dth roots of unity. Define $G = G(d,1,r) = \mathcal{U}_d \wr \mathfrak{S}_r = \mathcal{U}_d^r \rtimes \mathfrak{S}_r$. The elements of G are denoted by $(z;\sigma)$, or simply $z\sigma$, with $z \in \mathcal{U}_d^r$ and $\sigma \in \mathfrak{S}_r$. In particular, $\mathfrak{S}_r$ and $\mathcal{U}_d^r$ are viewed as subgroups of G using the injections $\sigma \in \mathfrak{S}_r \mapsto (1;\sigma)$ and $z \in \mathcal{U}_d^r \mapsto (z;1)$, respectively. For any $(z_1,\ldots,z_r) \in \mathcal{U}_d^r$ and $\sigma \in \mathfrak{S}_r$, recall that

$$\sigma^{-1}(z_1,\ldots,z_r)\sigma = (z_{\sigma(1)},\ldots,z_{\sigma(r)}). \tag{1}$$

Let ζ be a generator of $\mathcal{U}_d$. Write $t = (\zeta,1,\ldots,1) \in \mathcal{U}_d^r$ and $s_i = (i\ i+1) \in \mathfrak{S}_r$ for $1 \leq i \leq r-1$. In particular, $G = \langle t, s_1, \ldots, s_{r-1}\rangle$.

2.1. **Tableaux.** Let λ be a partition of r and let T be a tableau of shape λ whose entries are distinct integers. For $(u,v) \in \mathbb{Z}^2$, denote by $E(T,(u,v))$ the entry of T in the box in row u and column v. Furthermore, we write $E(T)$ for the set of integers occurring in T. Set $r = |\lambda|$ and assume that $E(T) = \{t_1, \ldots, t_r\}$ with $t_1 < \cdots < t_r$. Denote by $\mathrm{ST}(\lambda; t_1,\ldots,t_r)$ the set of standard tableaux of shape λ with respect to $\{t_1, \ldots, t_r\}$, that is, tableaux T of shape λ filled by the set of integers $\{t_1, \ldots, t_r\}$ in such a way that the entries in T are increasing across the rows and the columns of T.

Now, for each $T \in \mathrm{ST}(\lambda; t_1,\ldots,t_r)$, define the tableau $\theta(T)$ of shape λ to be such that

$$E(T,(u,v)) = t_j \iff E(\theta(T),(u,v)) = j.$$

Write $\mathrm{ST}(\lambda) := \mathrm{ST}(\lambda;1,\ldots,r)$ for the set of usual standard tableaux of shape λ. Then

Lemma 2.1. *The map θ induces a bijection between $\mathrm{ST}(\lambda;t_1,\ldots,t_r)$ and $\mathrm{ST}(\lambda)$.*

2.2. **Coset representatives for Young subgroups.** Let r be a positive integer. A *composition* of r of length d is a d-tuple $(c_0,\ldots,c_{d-1})$ of non-negative integers such that $\sum_{i=0}^{d-1} c_i = r$. Let $c = (c_0,\ldots,c_{d-1})$ be a composition of r. Write I_c for the set of integers $0 \leq i \leq d-1$ such that $c_i \neq 0$. We set $C_0 = 0$ and $C_i = c_0 + \cdots + c_{i-1}$ for any $i \in I_c$, and $E_i = \{C_i+1,\ldots,C_i+c_i\}$. Now, we can associate to c the *Young subgroup* $\mathfrak{S}_c = \mathfrak{S}_{E_{i_0}} \times \cdots \times \mathfrak{S}_{E_{i_s}}$ of $\mathfrak{S}_r$, where $I_c = \{i_0,\ldots,i_s\}$. Furthermore, for any $i \in I_c$, we denote by $p_i : \mathfrak{S}_{E_i} \longrightarrow \mathfrak{S}_{c_i}$ the group isomorphism induced by the bijection $E_i \to \{1,\ldots,c_i\}$, $C_i + j \mapsto j$.

Let $E = \{1,\ldots,r\}$. For any composition $c = (c_0,\ldots,c_{d-1})$ of r, define

$$\mathcal{X}_c = \left\{ (X_0,\ldots,X_{d-1}) \,|\, \bigsqcup_{i=0}^{d-1} X_i = E,\ |X_i| = c_i \right\}. \tag{2}$$

Lemma 2.2. *Let $c = (c_0,\ldots,c_{d-1})$ be a composition of r. For $X = (X_0,\ldots,X_{d-1}) \in \mathcal{X}_c$, define $t_X \in \mathfrak{S}_r$ by setting, for all $i \in I_c$ and $1 \leq j \leq c_i$,*

$$t_X(C_i + j) = x_{i,j}, \tag{3}$$

where $X_i = \{x_{i,1},\ldots,x_{i,c_i}\}$ with $x_{i,1} < \cdots < x_{i,c_i}$.

Let $\sigma \in \mathfrak{S}_r$. For $i \in I_c$, define $X_i(\sigma) = \{\sigma(x) \,|\, x \in E_i\} = \{x_{i,1},\ldots,x_{i,c_i}\}$ with $x_{i,1} < \cdots < x_{i,c_i}$, and write $X = (X_0(\sigma),\ldots,X_{d-1}(\sigma)) \in \mathcal{X}_c$. Then

$$\sigma = t_X \widetilde{\sigma}_0 \cdots \widetilde{\sigma}_{d-1},$$

where $\widetilde{\sigma}_i \in \mathfrak{S}_{E_i}$ is defined as follows. If $i \notin I_c$, then $\widetilde{\sigma}_i = 1$. Otherwise, for any $x \in E_i$, there is a unique $m_x \in \{1, \dots, c_i\}$ such that $\sigma(x) = x_{i,m_x}$, and we set $\widetilde{\sigma}_i(x) = C_i + m_x$. For $x \notin E_i$, we set $\widetilde{\sigma}_i(x) = x$. In particular, $\mathcal{T}_c = \{t_X \mid X \in \mathcal{X}_c\}$ is a complete set of representatives for $\mathfrak{S}_r/\mathfrak{S}_c$.

Proof. Let $1 \leq x \leq r$. Since $X \in \mathcal{X}_c$, there is $i \in I_c$ such that $x \in X_i(\sigma)$, and $\widetilde{\sigma}_{i+1} \cdots \widetilde{\sigma}_{d-1}(x) = x$. Furthermore, $\widetilde{\sigma}_i(x) = C_i + m_x \in X_i(\sigma)$, and it follows that $\widetilde{\sigma}_1 \cdots \widetilde{\sigma}_{i-1}(C_i + m_x) = C_i + m_x$. Finally, $t_X(C_i + m_x) = x_{i,m_x} = \sigma(x)$, as required. By construction, t_X and $\widetilde{\sigma}_1, \dots, \widetilde{\sigma}_{d-1}$ are uniquely determined from σ, hence $\mathcal{T}_c$ is a complete set of representatives for $\mathfrak{S}_r/\mathfrak{S}_c$. □

For any composition $c = (c_0, \dots, c_{d-1})$ of r, we set

$$G_c = \mathcal{U}_d^r \rtimes \mathfrak{S}_c.$$

Write $\pi_c : G \to \mathfrak{S}_r$ for the natural projection with kernel $\mathcal{U}_d^r$. Note that $G_c = \pi^{-1}(\mathfrak{S}_c)$, and that $\pi(G_c) = \mathfrak{S}_c$, hence G/G_c is in bijection with $\mathfrak{S}_r/\mathfrak{S}_c$. The bijection is given by $tG_c \mapsto \pi(t)\mathfrak{S}_c$. Identifying $\mathfrak{S}_r$ to a subgroup of G as above, we can take $\mathcal{T}_c$ for a set of representatives for G/G_c. Furthermore, using Lemma 2.2 and Relation (1), we deduce that, if $g = z\sigma \in G$ with $z = (z_1, \dots, z_r) \in \mathcal{U}_d^r$ and $\sigma \in \mathfrak{S}_r$, then

$$g = t_X \underbrace{(z_{t_X(1)}, \dots, z_{t_X(r)})\widetilde{\sigma}_0 \cdots \widetilde{\sigma}_{d-1}}_{\in G_c}, \tag{4}$$

where $X = (X_0(\sigma), \dots, X_{d-1}(\sigma))$ and $\widetilde{\sigma}_0, \dots, \widetilde{\sigma}_{d-1}$ are as in Lemma 2.2.

2.3. **The James-Kerber parametrization.** For any partition λ of r, there is a corresponding irreducible Specht module V_λ of $\mathfrak{S}_r$. Write $\psi_\lambda : \mathfrak{S}_r \to \mathrm{GL}(V_\lambda)$ for the corresponding irreducible representation of $\mathfrak{S}_r$. Recall that V_λ has a $\mathbb{C}$-basis $v_\lambda = \{v_{\lambda,T} \mid T \in \mathrm{ST}(\lambda)\}$ such that

$$\psi_\lambda(s_i)(v_{\lambda,T}) = \frac{1}{a(i,i+1)} v_{\lambda,T} + (1 + \frac{1}{a(i,i+1)}) v_{\lambda,T_{i\leftrightarrow i+1}}, \tag{5}$$

where $a(i, i+1)$ denotes the distance between the diagonals of T where i and $i+1$ occur, and $T_{i\leftrightarrow i+1}$ is the standard tableau of shape λ obtained by exchanging the integers i and $i+1$ in T.

Let α denote the identity of $\mathcal{U}_d$. We can write $\mathrm{Irr}(\mathcal{U}_d) = \{\alpha^i \mid 0 \leq i \leq d-1\}$. A d-multipartition $\underline{\lambda} = (\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(d-1)})$ of r is a d-tuple of partitions such that $\sum_{i=0}^{d-1} |\lambda^{(i)}| = r$. We write this as $\underline{\lambda} \Vdash_d r$, and denote by $\mathcal{MP}_{r,d}$ the set of d-multipartitions of r. Recall that, up to G-isomorphism, the irreducible representations of G are parametrized by $\mathcal{MP}_{r,d}$ as follows.

For any $\underline{\lambda} = (\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(d-1)})$, write $c_i = |\lambda^{(i)}|$ for $0 \leq i \leq d-1$, and $c = (c_0, \dots, c_{d-1})$. Now, consider the irreducible character of $\mathcal{U}_d^r$

$$\alpha_c = \bigotimes_{i=0}^{d-1} \underbrace{\alpha^i \otimes \cdots \otimes \alpha^i}_{c_i \text{ times}}, \tag{6}$$

whose inertial subgroup in G is G_c. Extend α_c to G_c by setting $\alpha_c(z\sigma) = \alpha_c(z)$ for all $z \in \mathcal{U}_d^r$ and $\sigma \in \mathfrak{S}_c$, and denote by $\mathbb{C}w_c$ the corresponding representation space. Now, for any $i \in I_c$, the space $V_{\lambda^{(i)}}$ has a structure of $\mathfrak{S}_{E_i}$-module given by the homomorphism $\psi_{\lambda^{(i)}} \circ p_i : \mathfrak{S}_{E_i} \to \mathrm{GL}(V_{\lambda^{(i)}})$. Hence, $V_{\underline{\lambda}} = V_{\lambda^{(0)}} \otimes \cdots \otimes V_{\lambda^{(d-1)}}$ is an irreducible $\mathfrak{S}_c$-module, which gives an irreducible representation of G_c through

π_c. Furthermore, to simplify the notation, we identify $\mathbb{C}w_c \otimes V_{\underline{\lambda}}$ with $V_{\underline{\lambda}}$, by setting $zv = \alpha_c(z)v$ for all $z \in \mathcal{U}_d^r$ and $v \in V_{\underline{\lambda}}$. Now, by Clifford theory, the G-module

$$W_{\underline{\lambda}} = \operatorname{Ind}_{G_c}^{G}(V_{\underline{\lambda}}) \tag{7}$$

is irreducible, and $\{W_{\underline{\lambda}} \,|\, \underline{\lambda} \in \mathcal{MP}_{r,d}\}$ is a complete set of non-isomorphic irreducible G-modules. For any $\underline{\lambda} \in \mathcal{MP}_{r,d}$, write $\vartheta_{\underline{\lambda}} : G_c \to \mathrm{GL}(\mathrm{V}_{\underline{\lambda}})$ and $\rho_{\underline{\lambda}} : G \to \mathrm{GL}(W_{\underline{\lambda}})$ for the corresponding representation of G_c and G, respectively.

By definition of the induction representation, the set

$$\mathfrak{b}_{\underline{\lambda}} = \left\{ t_X \otimes v_{\lambda^{(0)},T_0} \otimes \cdots \otimes v_{\lambda^{(d-1)},T_{d-1}} \,|\, X \in \mathcal{X}_c,\, T_i \in \mathrm{ST}(\lambda^{(i)}) \text{ for } 0 \leq i \leq d-1 \right\}$$

is a $\mathbb{C}$-basis of $W_{\underline{\lambda}}$. Furthermore, for $z = (z_1, \ldots, z_r) \in \mathcal{U}_d^r$, $\sigma \in \mathfrak{S}_r$, and $t_X \in \mathcal{T}_c$, there are $t_{X_\sigma} \in \mathcal{T}_c$ and $\widetilde{\sigma}_0 \in \mathfrak{S}_{c_0}, \ldots, \widetilde{\sigma}_{d-1} \in \mathfrak{S}_{d-1}$ such that

$$z\sigma t_X = t_{X_\sigma}(z_{t_{X_\sigma}(1)}, \ldots, z_{t_{X_\sigma}(r)})\widetilde{\sigma}_0 \cdots \widetilde{\sigma}_{d-1} \in \mathfrak{S}_c \tag{8}$$

(see Relation (4) applied to $g = z\sigma t_X \in G$). Therefore, if $v = t_X \otimes v_{\lambda^{(0)},T_0} \otimes \cdots \otimes v_{\lambda^{(d-1)},T_{d-1}}$, then

$$\rho_{\underline{\lambda}}(z\sigma)(v) = \alpha_c(z_{t_{X_\sigma}(1)}, \ldots, z_{t_{X_\sigma}(r)}) t_{X_\sigma} \otimes \widetilde{\sigma}_0(v_{\lambda^{(0)},T_0}) \otimes \cdots \otimes \widetilde{\sigma}_{d-1}(v_{\lambda^{(d-1)},T_{d-1}}), \tag{9}$$

where $\widetilde{\sigma}_i(v_{\lambda^{(i)},T_i}) = \psi_{\lambda^{(i)}} \circ p_i(\widetilde{\sigma}_i)(v_{\lambda^{(i)},T_i})$ for all $0 \leq i \leq d-1$.

2.4. **The Marin-Michel parametrization.** In [5, §2.3], Marin and Michel give the following model for $\mathrm{Irr}(G)$. Let $\underline{\lambda} = (\lambda^{(0)}, \ldots, \lambda^{(d-1)}) \in \mathcal{MP}_{r,d}$. Define $\mathcal{T}(\underline{\lambda})$ to be the set of standard multi-tableaux of shape $\underline{\lambda}$, that is, the set of tuples of tableaux $\underline{T} = (T_0, \ldots, T_{d-1})$ where

- For all $0 \leq i \leq d-1$, the tableau T_i is of shape $\lambda^{(i)}$.
- The tableaux $T_0, \ldots, T_{d-1}$ are filled by the set of integers $\{1, \ldots, r\}$ in such a way that each integer appears exactly once in one of the tableaux, and, for each i, the integers appearing in T_i are increasing across the rows and columns of T_i.

Now, the $\mathbb{C}$-vector space $W'_{\underline{\lambda}}$ with basis $\mathcal{T}(\underline{\lambda})$ can be given a G-module structure so that $\{W'_{\underline{\lambda}} \,|\, \underline{\lambda} \in \mathcal{MP}_{r,d}\}$ is a complete set of non-isomorphic irreducible G-modules. Write $\rho'_{\underline{\lambda}} : G \to \mathrm{GL}(W'_{\lambda})$ for the corresponding irreducible representation of G.

Denote by $\underline{T}(1)$ the index of the tableau of $\underline{T}$ containing the integer 1, and for $1 \leq i \leq r-1$, write $\underline{T}_{i\leftrightarrow i+1} \in \mathcal{T}(\underline{\lambda})$ for the multi-tableau obtained from $\underline{T}$ by exchanging the integers i and $i+1$ in $\underline{T}$.

With this notation, we have (see [5, §2.3]) $\rho'_{\underline{\lambda}}(t)(\underline{T}) = \zeta^{\underline{T}(1)}\underline{T}$. Furthermore, for $1 \leq i \leq r-1$, if i and $i+1$ do not belong to the same tableau of $\underline{T}$, then $\rho'_{\underline{\lambda}}(s_i)(\underline{T}) = \underline{T}_{i\leftrightarrow i+1}$. Otherwise,

$$\rho'_{\underline{\lambda}}(s_i)\underline{T} = \frac{1}{a(i,i+1)}\underline{T} + \left(1 + \frac{1}{a(i,i+1)}\right)\underline{T}_{i\leftrightarrow i+1}. \tag{10}$$

Proposition 2.3. *Let $\underline{\lambda} = (\lambda^{(0)}, \ldots, \lambda^{(d-1)}) \in \mathcal{MP}_{r,d}$. Then the linear map $f_{\underline{\lambda}} : W'_{\underline{\lambda}} \to W_{\underline{\lambda}}$ defined on the basis $\{\underline{T} \,|\, T \in \mathcal{T}(\underline{\lambda})\}$ of $W'_{\underline{\lambda}}$ by setting, for every $\underline{T} = (T_0, \ldots, T_{d-1}) \in \mathcal{T}(\underline{\lambda})$,*

$$f_{\underline{\lambda}}(T_0, \ldots, T_{d-1}) = t_X \otimes v_{\lambda^{(0)},\theta(T_0)} \otimes \cdots \otimes v_{\lambda^{(d-1)},\theta(T_{d-1})},$$

where $X = (E(T_0), \ldots, E(T_{d-1}))$ and θ is the map constructed before Lemma 2.1, is an isomorphism of G-modules.

Proof. First, we remark that $f_{\underline{\lambda}}$ sends a basis to a basis, whence is a bijective linear map. To prove the result, it suffices to show that $\rho_{\underline{\lambda}}(g) \circ f_{\underline{\lambda}} = f_{\underline{\lambda}} \circ \rho'_{\underline{\lambda}}(g)$ for all $g \in \{t, s_1, \ldots, s_{r-1}\}$.

Let $\underline{T} = (T_0, \ldots, T_{d-1}) \in \mathcal{T}(\underline{\lambda})$. Write $c_i = |\lambda^{(i)}|$ for all $0 \leq i \leq d-1$, and set $c = (c_0, \ldots, c_{d-1})$. Define $t' = (1, \ldots, 1, \zeta, 1, \ldots, 1)$, where ζ lies in position $t_X^{-1}(1)$. Then Relation (4) gives $tt_X = t_X t'$. Furthermore, $1 \in E(T_{\underline{T}(1)})$, thus $t_X^{-1}(1) \in E_{\underline{T}(1)}$. It follows from the linearity of $f_{\underline{\lambda}}$, and from Relations (6) and (9), that

$$\begin{aligned}
\rho_{\underline{\lambda}}(t)(f_{\underline{\lambda}}(\underline{T})) &= \rho_{\underline{\lambda}}(t)(t_X \otimes v_{\lambda^{(0)},\theta(T_0)} \otimes \cdots \otimes v_{\lambda^{(d-1)},\theta(T_{d-1})}) \\
&= \alpha_c(t') t_X \otimes v_{\lambda^{(0)},\theta(T_0)} \otimes \cdots \otimes v_{\lambda^{(d-1)},\theta(T_{d-1})} \\
&= \alpha^{\underline{T}(1)}(\zeta) t_X \otimes v_{\lambda^{(0)},\theta(T_0)} \otimes \cdots \otimes v_{\lambda^{(d-1)},\theta(T_{d-1})} \\
&= \zeta^{\underline{T}(1)} f_{\underline{\lambda}}(\underline{T}) \\
&= f_{\underline{\lambda}}(\zeta^{\underline{T}(1)} \underline{T}) \\
&= f_{\underline{\lambda}}(\rho'_{\underline{\lambda}}(t)(\underline{T})).
\end{aligned}$$

Now, let $1 \leq i \leq r-1$. Assume i and $i+1$ do not lie in the same tableau of $\underline{T}$, say $i \in E(T_k)$ and $i+1 \in E(T_\ell)$. Then $s_i T_X = T_{X_{i\leftrightarrow i+1}}$, where $X_{i\leftrightarrow i+1} \in \mathcal{X}_c$ is obtained from X by exchanging i and $i+1$. It follows from Relation (9) that

$$\rho_{\underline{\lambda}}(s_i)(f_{\underline{\lambda}}(\underline{T})) = t_{X_{i\leftrightarrow i+1}} \otimes v_{\lambda^{(0)},\theta(T_0)} \otimes \cdots \otimes v_{\lambda^{(d-1)},\theta(T_{d-1})} = f_{\underline{\lambda}}(T_{i\leftrightarrow i+1}) = f_{\underline{\lambda}}(\rho'_{\underline{\lambda}}(s_i)(\underline{T})).$$

Assume now that i and $i+1$ lie in the same tableau of $\underline{T}$, say $i, i+1 \in E(T_k) = \{t_1, \ldots, t_m\}$ with $t_1 < \cdots < t_m$. Let $1 \leq i' \leq m$ be such that $i = t_{i'}$. Necessarily, we have $t_{i'+1} = i+1$, and $s_i t_X = t_X s_{C_k + i'}$. Thus, Relations (9) and (5) give

$$\begin{aligned}
\rho_{\underline{\lambda}}(s_i)(f_{\underline{\lambda}}(\underline{T})) &= t_X \otimes v_{\lambda^{(0)},\theta(T_0)} \otimes \cdots \otimes \psi_{\lambda^{(k)}}(s_{i'}) v_{\lambda^{(k)},\theta(T_k)} \otimes \cdots \otimes v_{\lambda^{(d-1)},\theta(T_{d-1})} \\
&= \frac{1}{a(i', i'+1)} f_{\underline{\lambda}}(\underline{T}) + \left(1 + \frac{1}{a(i', i'+1)}\right) f_{\underline{\lambda}}(\underline{T}_{i\leftrightarrow i+1}).
\end{aligned}$$

Let (u, v) and (u', v') be such that $E(T_k, (u, v)) = i$ and $E(T_k, (u', v')) = i+1$. Then by construction, we have $E(\theta(T_k), (u, v)) = i'$ and $E(\theta(T_k), (u', v')) = i'+1$. In particular, $a(i, i+1) = a(i', i'+1)$ and we deduce from the linearity of $f_{\underline{\lambda}}$ and Relation (10) that

$$\begin{aligned}
\rho_{\underline{\lambda}}(s_i)(f_{\underline{\lambda}}(\underline{T})) &= f_{\underline{\lambda}}\left(\frac{1}{a(i, i+1)} \underline{T} + \left(1 + \frac{1}{a(i, i+1)}\right) \underline{T}_{i\leftrightarrow i+1}\right) \\
&= f_{\underline{\lambda}}\left(\rho'_{\underline{\lambda}}(s_i)(\underline{T})\right),
\end{aligned}$$

as required. □

2.5. **Other descriptions in some special cases.** In this section, we assume that there are integers q, r' and d' such that $d = qd'$ and $r = qr'$, and we consider multi-partitions $\underline{\lambda} \in \mathcal{MP}_{r,d}$ of the form

$$(11) \quad \underline{\lambda} = (\lambda^{(0)}, \ldots, \lambda^{(d'-1)}, \lambda^{(0)}, \ldots, \lambda^{(d'-1)}, \ldots, \lambda^{(0)}, \ldots, \lambda^{(d'-1)}) = (\underbrace{\underline{\mu}, \ldots, \underline{\mu}}_{q \text{ times}}),$$

where $\underline{\mu} = (\lambda^{(0)}, \dots, \lambda^{(d'-1)}) \in \mathcal{MP}_{r',d'}$. Write $c = (c_0, \dots, c_{d-1})$ and $E_0, \dots, E_{d-1}$ as above, and $c' = (c_0, \dots, c_{d'-1})$.

Let $0 \leq i \leq q-1$. We set $L_i = \mathfrak{S}_{E_i} \times \cdots \times \mathfrak{S}_{E_{i+d'-1}}$, $K_i = \mathcal{U}_d^{r'} \rtimes L_i$,

$$E'_i = \bigsqcup_{k=0}^{d'-1} E_{i+k},$$

and $H_i = \mathcal{U}_d^{r'} \rtimes \mathfrak{S}_{E'_i}$. Note that the character $\alpha^i_{r'} = \alpha^i \otimes \cdots \otimes \alpha^i \in \mathrm{Irr}(\mathcal{U}_d^{r'})$ extends to H_i. Recall that $V_{\underline{\mu}}$ is an L_i-module. We endow $V_{\underline{\mu}}$ with a structure of K_i-module where the action of $\mathcal{U}_d^{r'} \leq K_i$ is given by $\alpha^{d'i}_{r'} \otimes \alpha_{c'}$, and we denote by $V_{\underline{\mu},i}$ the resulting K_i-module. Now set $W_{\underline{\mu},i} = \mathrm{Ind}_{K_i}^{H_i}(V_{\underline{\mu},i})$, and define $X_{\underline{\mu},i}$ to be the subset of elements $Y = (Y_0, \cdots, Y_{d-1}) \in \mathcal{X}_c$ such that for all $0 \leq j \leq q-1$ and $0 \leq k \leq d'-1$, if $j \neq i$, then $Y_{jd'+k} = \{jr' + C_k + 1, \dots, jr' + C_{k+1}\}$. In particular, $|Y_{id'+k}| = c_k$ and $\bigsqcup_k Y_{id'+k} = \{ir'+1, \dots, (i+1)r'\}$. Therefore, $\{t_{Y_i} \mid Y_i \in X_{\underline{\mu},i}\}$ is a system of coset representatives of H_i/K_i. We also consider the set $\mathcal{T}'$ of tuples $T' = (T_0, \dots, T_{d'-1})$ with $T_j \in \mathrm{ST}(\lambda^{(j)})$ for $0 \leq j \leq d'-1$. For $T' \in \mathcal{T}'$, write $v_{T'} = v_{\lambda^{(0)},T_0} \otimes \cdots \otimes v_{\lambda^{(d'-1)},T_{d'-1}}$. In particular, $\{v_{T'} \mid T' \in \mathcal{T}'\}$ is a basis of $V_{\underline{\mu},i}$. Hence, $\{t_{Y_i} \otimes v_{T'_i} \mid Y_i \in X_{\underline{\mu},i},\, T'_i \in \mathcal{T}'\}$ is a basis of $W_{\underline{\mu},i}$.

Now, set

$$H = H_0 \times \cdots \times H_{q-1}. \tag{12}$$

Consider the H-module

$$U_{\underline{\mu}} = W_{\underline{\mu}} \otimes W_{\underline{\mu},1} \otimes \cdots \otimes W_{\underline{\mu},q-1}, \tag{13}$$

and define

$$W''_{\underline{\lambda}} = \mathrm{Ind}_H^G(U_{\underline{\mu}}).$$

We write $\rho''_{\underline{\lambda}} : G \to \mathrm{GL}(W''_{\underline{\lambda}})$ for the corresponding representation of G.

Proposition 2.4. *The G-module $W''_{\underline{\lambda}}$ has basis*

$$\mathfrak{b}''_{\underline{\lambda}} = \{t_{X'} \otimes (t_{Y_0} \otimes v_{T'_0}) \otimes \cdots \otimes (t_{Y_{q-1}} \otimes v_{T'_{q-1}}) \mid X' \in \mathcal{X}_{(r',\dots,r')},\, Y_i \in X_{\underline{\mu},i},\, T'_i \in \mathcal{T}'\}.$$

For $X \in \mathcal{X}_c$, define $l(X) = (X'_0, \dots, X'_{q-1}) \in \mathcal{X}_{(r',\dots,r')}$, where

$$X'_i = \bigsqcup_{k=0}^{d'-1} X_{id'+k}, \quad \text{for all} \quad 0 \leq i \leq q-1.$$

Write $X'_i = \{x'_{i,1}, \dots, x'_{i,r'}\}$ with $x'_{i,1} < \cdots < x'_{i,r'}$, and, for all $0 \leq i \leq q-1$ and $0 \leq k \leq d'-1$, consider the element $Y_i(X) \in X_{\underline{\mu},i}$ such that $Y_{i,k}(X) = \{id' + j \mid x'_{i,j} \in X_{id'+k}\}$. Then the linear map $f'_{\underline{\lambda}}$ defined by

$$f'_{\underline{\lambda}}(t_X \otimes v_{T'_0} \otimes \cdots \otimes v_{T'_{q-1}}) = t_{l(X)} \otimes (t_{Y_0(X)} \otimes v_{T'_0}) \otimes \cdots \otimes (t_{Y_{q-1}(X)} \otimes v_{T'_{d-1}})$$

is an isomorphism of G-modules between $W_{\underline{\lambda}}$ and $W''_{\underline{\lambda}}$.

Proof. Note that

$$G_c = K_0 \times \cdots \times K_{q-1}, \tag{14}$$

and, viewed as a G_c-representation with respect to the direct product (14), we have

$$V_{\underline{\lambda}} = V_{\underline{\mu},0} \otimes \cdots \otimes V_{\underline{\mu},q-1}.$$

By Lemma 2.2, $\{t_{X'} \mid X' \in \mathcal{X}_{(r',\ldots,r')}\}$ is a system of coset representatives of G/H, and $\{t_{Y_0} \cdots t_{Y_{q-1}} \mid Y_i \in X_{\underline{\mu},i}\}$ is a system of coset representatives for H/G_c. Then there is an isomorphism of G-modules $\kappa_1 : \mathrm{Ind}_H^G(\mathrm{Ind}_{G_c}^H(V_{\underline{\lambda}})) \longrightarrow \mathrm{Ind}_{G_c}^G(V_{\underline{\lambda}})$ given on any basis $\{v\}$ of $V_{\underline{\lambda}}$ by

$$\kappa_1(t_{X'} \otimes t_{Y_0} \cdots t_{Y_{q-1}} \otimes v) = t_{X'} t_{Y_0} \cdots t_{Y_{q-1}} \otimes v,$$

where $X' \in \mathcal{X}_{(r',\ldots,r')}$ and $Y_i \in X_{\underline{\mu},i}$. Furthermore, we have

$$\begin{aligned} \mathrm{Ind}_{G_c}^H(V_{\underline{\lambda}}) &= \mathrm{Ind}_{K_0 \times \cdots \times K_{q-1}}^{H_0 \times \cdots \times H_{q-1}}(V_{\underline{\mu},0} \otimes \cdots \otimes V_{\underline{\mu},q-1}) \\ &\cong \mathrm{Ind}_{K_0}^{H_0}(V_{\underline{\mu},0}) \otimes \cdots \otimes \mathrm{Ind}_{K_{q-1}}^{H_{q-1}}(V_{\underline{\mu},q-1}). \end{aligned}$$

The last isomorphism of H-modules is for example given by

$$t_{Y_0} \cdots t_{Y_{q-1}} \otimes v_{T'_0} \otimes \cdots v_{T'_{q-1}} \mapsto (t_{Y_0} \otimes v_{T'_0}) \otimes \cdots \otimes (t_{Y_{q-1}} \otimes v_{T'_{q-1}})$$

for all $Y_i \in X_{\underline{\mu},i}$ and $T'_i \in \mathcal{T}'$. We thus obtain an isomorphism of G-modules $\kappa_2 : \mathrm{Ind}_H^G(\mathrm{Ind}_{G_c}^H(V_{\underline{\lambda}})) \longrightarrow \mathrm{Ind}_H^G(U_{\underline{\mu}})$ given by

$$\kappa_2(t_{X'} \otimes t_{Y_0} \cdots t_{Y_{q-1}} \otimes v_{T'_0} \otimes \cdots v_{T'_{q-1}}) = t_{X'} \otimes (t_{Y_0} \otimes v_{T'_0}) \otimes \cdots \otimes (t_{Y_{q-1}} \otimes v_{T'_{q-1}}).$$

Now, note that the map

$$\kappa : \mathcal{X}_c \to \mathcal{X}_{(r',\ldots,r')} \times X_{\underline{\mu},0} \times \cdots \times X_{\underline{\mu},q-1},\ X \mapsto (l(X), Y_0(X), \ldots, Y_{q-1}(X)) \tag{15}$$

is bijective and that

$$t_X = t_{l(X)} t_{Y_0(X)} \cdots t_{Y_{q-1}(X)}.$$

It follows that $f'_{\underline{\lambda}} = \kappa_2 \circ \kappa_1^{-1}$ has the required property. □

Remark 2.5. Note that $H_0 = G(d,1,r')$ and that $W_{\underline{\mu},0}$ is the irreducible representation of H_0 labeled by $(\underline{\mu}, \emptyset, \ldots, \emptyset)$. In the same way, for every $0 \leq i \leq q-1$, the group H_i can be viewed as a complex reflexion group $G(d,1,r')$ with support E_i. The irreducible representation $W_{\underline{\mu},i}$ of H_i is then labeled by $(\emptyset, \ldots, \emptyset, \underline{\mu}, \emptyset, \ldots, \emptyset)$, where $\underline{\mu}$ lies in ith-coordinate. In the following, we will identify H_i with H_0 as well as $W_{\underline{\mu},i}$ with $W_{\underline{\mu},0}$ as follows. Let $0 \leq i \leq q-1$. Write

$$\tau_i : \{ir'+1 \ldots (i+1)r'\} \to \{1, \ldots, r'\},\ ir'+j \mapsto j.$$

Then τ_i induces a group isomorphism between H_i and H_0. Furthermore, for $Y \in X_{\underline{\mu},i}$, define $Y^0 \in X_{c'}$ by setting $Y^0_k = \tau_i(Y_{id'+k})$ for all $0 \leq k \leq d'-1$. Then the H_i-module $W_{\underline{\mu},i}$ and the H_0-module $\alpha_{r'}^{d'i} \otimes W_{\underline{\mu}}$ are isomorphic. An isomorphism $\mathfrak{f}_i$ is given on the basis $\{t_Y \otimes v_{T'}\}$ by

$$t_Y \otimes v_{T'} \mapsto t_{\tau_i(Y)} \otimes v_{T'}, \tag{16}$$

for all $Y \in X_{\underline{\mu},i}$ and $T' \in \mathcal{T}'$.

3. Character formula for the irreducible representations of $G(de,e,r)$

Let e, d and r be positive integers, and write $G = G(de,1,r)$.
Let $\underline{\varepsilon} = (\emptyset, \cdots, \emptyset, (r), \emptyset, \cdots, \emptyset) \in \mathcal{MP}_{r,de}$, where the non-empty part of $\underline{\varepsilon}$ lies in position $de-1-d$. Then $\varepsilon = \rho_{\underline{\varepsilon}}$ is a linear character of G of order e, and we denote by $N = G(de,e,r)$ its kernel. In particular, if $z = (z_1, \ldots, z_r) \in \mathcal{U}_{de}^r$ and $\sigma \in \mathfrak{S}_r$, then $z\sigma$ lies in N if and only if $\varepsilon(z) = 1$, that is $z_1 \cdots z_r \in \mathcal{U}_d$.

3.1. **Representations of $G(de,e,r)$.** Let $\underline{\lambda}=(\lambda^{(0)},\ldots,\lambda^{(de-1)})\in\mathcal{MP}_{r,de}$. Note that, by construction, $\operatorname{Res}^G_{\mathcal{U}^r_{de}}(\varepsilon)=\alpha^d\otimes\cdots\otimes\alpha^d\in\operatorname{Irr}(\mathcal{U}^r_{de})$. It follows from Equation (7) that

$$\varepsilon\otimes\rho_{\underline{\lambda}}=\varepsilon\otimes\operatorname{Ind}^G_{G_c}(\vartheta_{\underline{\lambda}})\cong\operatorname{Ind}^G_{G_c}(\varepsilon\otimes\vartheta_{\underline{\lambda}})=\operatorname{Ind}^G_{G_c}(\vartheta_{\varepsilon(\underline{\lambda})})=\rho_{\varepsilon(\underline{\lambda})},$$

where $\varepsilon(\underline{\lambda})=(\lambda^{(d)},\lambda^{(d+1)},\ldots,\lambda^{(de+d)})$; note that, here, the indices are taken modulo de.

Let a be a divisor of e such that $\varepsilon^a(\underline{\lambda})=\underline{\lambda}$. Then $\lambda^{(da+k)}=\lambda^{(k)}$ for any k, and

$$r=|\langle\varepsilon^a\rangle|\sum_{k=0}^{da-1}|\lambda^{(k)}|. \tag{17}$$

The set $C_{\underline{\lambda}}=\{\varepsilon^j\,|\,\varepsilon^j\otimes\rho_{\underline{\lambda}}\cong\rho_{\underline{\lambda}}\}$ is a subgroup of the cyclic group $\langle\varepsilon\rangle$, hence there is a divisor $b_{\underline{\lambda}}$ of e such that $C_{\underline{\lambda}}=\langle\varepsilon^{b_{\underline{\lambda}}}\rangle$. Furthermore, by Clifford theory, $\operatorname{Res}^G_N(\rho_{\underline{\lambda}})$ is the sum of $|C_{\underline{\lambda}}|$ non isomorphic irreducible N-modules. Following [5, § 2.4], they can be described as follows. By Schur's Lemma and the fact that $\mathbb{C}$ is algebraically closed, we can choose a bijective linear map $M_{\underline{\lambda}}\in\operatorname{Hom}_G(\rho_{\underline{\lambda}},\varepsilon^{b_{\underline{\lambda}}}\otimes\rho_{\underline{\lambda}})$ such that $M_{\underline{\lambda}}$ has order $|C_{\underline{\lambda}}|$. On the other hand, $M_{\underline{\lambda}}$ is diagonalisable and has exactly $|C_{\underline{\lambda}}|$ eigenspaces with eigenvalues in $\mathcal{U}_{|C_{\underline{\lambda}}|}$. Denote by $W_{\underline{\lambda},k}$ the eigenspace attached to the eigenvalue $\zeta^{b_{\underline{\lambda}}dk}$, where $\mathcal{U}_{de}=\langle\,\zeta\,\rangle$ (so that $\mathcal{U}_{|C_{\underline{\lambda}}|}=\langle\,\zeta^{b_{\underline{\lambda}}d}\,\rangle$). Then $\{W_{\underline{\lambda},k}\,|\,0\leq k\leq|C_{\underline{\lambda}}|-1\}$ is the set of irreducible N-modules appearing in the decomposition of $W_{\underline{\lambda}}$ into simple N-modules.

For $0\leq k\leq|C_{\underline{\lambda}}|-1$, denote by $\chi_{\underline{\lambda},k}$ the character of the N-module $W_{\underline{\lambda},k}$ and by

$$\Delta_{\underline{\lambda},k}(g)=\operatorname{Tr}(M^k_{\underline{\lambda}}\circ\rho_{\underline{\lambda}}(g))\quad\text{for all } g\in N.$$

Then we have (see [5])

$$\chi_{\underline{\lambda},k}=\frac{1}{|C_{\underline{\lambda}}|}\sum_{j=0}^{|C_{\underline{\lambda}}|-1}\zeta^{-db_{\underline{\lambda}}kj}\Delta_{\underline{\lambda},j}. \tag{18}$$

Now, using the first orthogonality relation, we deduce that, for $0\leq k\leq|C_{\underline{\lambda}}|-1$,

$$\Delta_{\underline{\lambda},k}=\sum_{j=0}^{|C_{\underline{\lambda}}|-1}\zeta^{db_{\underline{\lambda}}kj}\chi_{\underline{\lambda},j}. \tag{19}$$

Proposition 3.1. *Let $\underline{\lambda}=(\lambda^{(0)},\ldots,\lambda^{(de-1)})\in\mathcal{MP}_{r,de}$ and $c=(c_0,\ldots,c_{de-1})$ be such that $c_i=|\lambda^{(i)}|$ for all $0\leq i\leq de-1$. Let $b_{\underline{\lambda}}$ be a divisor of e such that $C_{\underline{\lambda}}=\langle\varepsilon^{b_{\underline{\lambda}}}\rangle$. Define $m_{\underline{\lambda}}:\mathcal{X}_c\to\mathcal{X}_c$ by setting, for any $X=(X_0,\ldots,X_{de-1})\in\mathcal{X}_c$,*

$$m_{\underline{\lambda}}(X)=(X_{db_{\underline{\lambda}}},\ldots,X_{db_{\underline{\lambda}}+de-1}),$$

where indices are taken modulo de. Then the linear map $M_{\underline{\lambda}}\in\operatorname{Hom}_G(\rho_{\underline{\lambda}},\varepsilon^{b_{\underline{\lambda}}}\otimes\rho_{\underline{\lambda}})$ as above can be described on the basis $\mathfrak{b}_{\underline{\lambda}}$ of $W_{\underline{\lambda}}$ as follows.

$$M_{\underline{\lambda}}(t_X\otimes v_{\lambda^{(0)},T_0}\otimes\cdots\otimes v_{\lambda^{(de-1)},T_{de-1}})=t_{m_{\underline{\lambda}}(X)}\otimes v_{\lambda^{(db_{\underline{\lambda}})},T_{db_{\underline{\lambda}}}}\otimes\cdots\otimes v_{\lambda^{(db_{\underline{\lambda}}+de-1)},T_{db_{\underline{\lambda}}+de-1}},$$

where $X\in\mathcal{X}_c$ and $T_i\in\operatorname{ST}(\lambda^{(i)})$.

Proof. In [5, § 2.4], a bijective linear map $M'_{\underline{\lambda}} \in \mathrm{Hom}_G(\rho'_{\underline{\lambda}}, \varepsilon^{b_{\underline{\lambda}}} \otimes \rho'_{\underline{\lambda}})$ of order $|C_{\underline{\lambda}}|$ is described on the basis $\mathcal{T}(\underline{\lambda})$ of $W'_{\underline{\lambda}}$ as follows. For every $T = (T_0, \ldots, T_{de-1}) \in \mathcal{T}(\underline{\lambda})$, we set

$$M'_{\underline{\lambda}}(T) = (T_{db_{\underline{\lambda}}}, \ldots, T_{db_{\underline{\lambda}}+de-1}). \tag{20}$$

Now, using Proposition 2.3, we check that

$$M_{\underline{\lambda}} \circ f_{\underline{\lambda}} = f_{\underline{\lambda}} \circ M'_{\underline{\lambda}}.$$

The result follows. □

Let b' be a divisor of $|C_{\underline{\lambda}}|$. Then $q = \frac{|C_{\underline{\lambda}}|}{b'} = \frac{e}{b_{\underline{\lambda}} b'}$ is the order of $M^{b'}$ and $\varepsilon^{b_{\underline{\lambda}} b'}(\underline{\lambda}) = \underline{\lambda}$. Hence, Relation (17) applied to $a = b_{\underline{\lambda}} b'$ gives that q divides r. Let $r' \in \mathbb{N}$ be such that $r = qr'$.

Proposition 3.2. *We keep the notation as above. Write* $\underline{\mu} = (\lambda^{(0)}, \ldots, \lambda^{(db_{\underline{\lambda}} b'-1)})$ *so that* $\underline{\lambda} = (\underline{\mu}, \ldots, \underline{\mu})$ *as in Relation (11). Define* $m''_{\underline{\lambda}} : \mathcal{X}_{(r',\ldots,r')} \to \mathcal{X}_{(r',\ldots,r')}$ *by*

$$m''_{\underline{\lambda}}(X'_0, \ldots, X'_{q-1}) = (X'_1, \ldots, X'_{q-1}, X'_0),$$

and, for all $\mathfrak{u} = t_{X'} \otimes (t_{Y_0} \otimes v_{T'_0}) \otimes \cdots \otimes (t_{Y_{q-1}} \otimes v_{T'_{q-1}})$ *with* $X' \in \mathcal{X}_{(r',\ldots,r')}$, $Y_i \in X_{\underline{\mu},i}$ *and* $T'_i \in \mathcal{T}'$*, we set*

$$M''_{\underline{\lambda}}(\mathfrak{u}) = t_{m''_{\underline{\lambda}}(X')} \otimes (t_{Y_1} \otimes v_{T'_0}) \otimes \cdots \otimes (t_{Y_{q-1}} \otimes v_{T'_{q-2}}) \otimes (t_{Y_0} \otimes v_{T'_{q-1}}).$$

Then

$$M''_{\underline{\lambda}} \circ f'_{\underline{\lambda}} = f'_{\underline{\lambda}} \circ M^{b'}_{\underline{\lambda}}.$$

Proof. We remark that for all $X \in \mathcal{X}_c$, one has $l(m^{b'}_{\underline{\lambda}}(X)) = m''_{\underline{\lambda}}(l(X))$, where $l : \mathcal{X}_c \to \mathcal{X}_{(r',\ldots,r')}$ is the map defined in Proposition 2.4. Let $\kappa : \mathcal{X}_c \to \mathcal{X}_{(r',\ldots,r')} \times X_{\underline{\mu},0} \times \cdots \times X_{\underline{\mu},q-1}$ be the bijection defined in Relation (15). Then for all $X \in \mathcal{X}_c$,

$$\kappa \circ m^{b'}_{\underline{\lambda}}(X) = (m''_{\underline{\lambda}}(l(X)), Y_1(X), \ldots, Y_{q-1}(X), Y_0(X)).$$

The result then follows from Proposition 2.4. □

3.2. Values of $\Delta_{\underline{\lambda},k}$.

Lemma 3.3. *Let* $\underline{\lambda} \in \mathcal{MP}_{r,de}$ *be such that* $C_{\underline{\lambda}} = \langle \varepsilon^{b_{\underline{\lambda}}} \rangle$ *for some divisor* $b_{\underline{\lambda}}$ *of* e*. Let* b' *be a divisor of* $|C_{\underline{\lambda}}|$, $q = |C_{\underline{\lambda}}|/b'$ *and* $r' \in \mathbb{N}$ *be such* $r = qr'$*. Write* $\mu = (\lambda^{(0)}, \ldots, \lambda^{(db_{\underline{\lambda}} b'-1)})$ *so that* $\underline{\lambda} = (\underline{\mu}, \ldots, \underline{\mu})$*. For* $g \in G$ *and* $X \in \mathcal{X}_{(r',\ldots,r')}$*, define* $X_g \in \mathcal{X}_{(r',\ldots,r')}$ *and* $g_i^X \in H_i$ *for* $0 \leq i \leq q-1$ *such that* $g t_X = t_{X_g} g_0^X \cdots g_{q-1}^X$*. Then*

$$\Delta_{\underline{\lambda},b'}(g) = \sum_{X \mid m''_{\underline{\lambda}}(X_g) = X} \left(\prod_{i=0}^{q-1} \alpha_{r'}^{db_{\underline{\lambda}} b' i}(g_i^X) \right) \chi_{\underline{\mu}}(\overline{g}_0^X \cdots \overline{g}_{q-1}^X),$$

where $\chi_{\underline{\mu}}$ *denotes the character of the irreducible representation of* $\mathrm{Irr}(H_0)$ *labeled by* $(\underline{\mu}, \emptyset, \ldots, \emptyset)$ *and* $\overline{g}_i^X \in H_0$ *is the image of* g_i^X *by the isomorphism* $H_i \to H_0$ *induced by the bijection* τ_i *given in Remark 2.5.*

Proof. By Propositions 2.4 and 3.2, we have $\rho_{\underline{\lambda}}(g) = f'^{-1}_{\underline{\lambda}} \circ \rho''_{\underline{\lambda}}(g) \circ f'_{\underline{\lambda}}$ and $M^{b'}_{\underline{\lambda}} = f'^{-1}_{\underline{\lambda}} \circ M''_{\underline{\lambda}} \circ f'_{\underline{\lambda}}$. Hence

$$\Delta_{\underline{\lambda},b'}(g) = \mathrm{Tr}(M^{b'}_{\underline{\lambda}} \circ \rho_{\underline{\lambda}}(g)) = \mathrm{Tr}(f'^{-1}_{\underline{\lambda}} \circ M''_{\underline{\lambda}} \circ \rho''_{\underline{\lambda}}(g) \circ f'_{\underline{\lambda}}) = \mathrm{Tr}(M''_{\underline{\lambda}} \circ \rho''_{\underline{\lambda}}(g)).$$

Let $\mathfrak{u} = t_X \otimes (t_{Y_0} \otimes v_{T'_0}) \otimes \cdots \otimes (t_{Y_{q-1}} \otimes v_{T'_{q-1}}) \in \mathfrak{b}''_{\underline{\lambda}}$. We have

$$M''_{\underline{\lambda}} \circ \rho''_{\underline{\lambda}}(g)(\mathfrak{u}) = \left(\prod_{i=0}^{q-1} \alpha_{r'}^{db_{\underline{\lambda}} b' i}(g_i^X)\right) t_{m''_{\underline{\lambda}}(X_g)} \left(\bigotimes_{i=1}^{q} g_i^X \cdot (t_{Y_i} \otimes v_{T'_{i-1}})\right),$$

where the indices are taken modulo q. To simplify the notation, we denote the basis $\{t_{Y_0} \otimes v_{T'_0} \,|\, Y_0 \in X_{\underline{\mu},0},\, T'_0 \in \mathcal{T}'\}$ of $W_{\underline{\mu}}$ by $\mathfrak{e} = \{e_1, \ldots, e_s\}$, and the basis of $W_{\underline{\mu},i}$ is then equal to $\mathfrak{f}_i^{-1}(\mathfrak{e})$ by (16). Then, by Remark 2.5, for all $0 \leq i \leq q-1$, the matrix of $g_i^X \cdot (t_{Y_i} \otimes v_{T'_{i-1}})$ (where the indices are taken modulo q) with respect to the basis $\mathfrak{f}_i^{-1}(\mathfrak{e})$ is the same as that of $\rho_{\underline{\mu}}(\overline{g}_i^X)$ with respect to the basis $\mathfrak{e}$. For $h \in H_0$, we denote by $A_{\underline{\mu}}(h) = (a_{ij}(h))_{ij}$ the matrix of $\rho_{\underline{\mu}}(h)$ with respect to the basis $\mathfrak{e}$. In particular, for $i_0, \ldots, i_{q-1}$, if we decompose $g_1^X \cdot e_{i_1} \otimes \cdots \otimes g_{q-1}^X \cdot e_{i_{q-1}} \otimes g_0^X \cdot e_{i_0}$ with respect to the basis $\{e_{i_0} \otimes \cdots \otimes e_{i_{q-1}}\}$, then its coefficient in the $e_{i_0} \otimes \cdots \otimes e_{i_{q-1}}$-coordinate is

$$a_{i_0 i_1}(\overline{g}_1^X) \cdots a_{i_{q-2} i_{q-1}}(\overline{g}_{q-1}^X)\, a_{i_{q-1} i_0}(\overline{g}_0^X).$$

Write $M''_{\underline{\lambda}} \circ \rho''_{\underline{\lambda}}(g)(\mathfrak{u}) = \sum_{v \in \mathfrak{b}''_{\underline{\lambda}}} a_v\, v$. Thus, if $a_{\mathfrak{u}} \neq 0$, then $m''_{\underline{\lambda}}(X_g) = X$ and, in this case, one has

$$a_{\mathfrak{u}} = \left(\prod_{i=0}^{q-1} \alpha_{r'}^{db_{\underline{\lambda}} b' i}(g_i^X)\right) a_{i_0 i_1}(\overline{g}_1^X) \cdots a_{i_{q-2} i_{q-1}}(\overline{g}_{q-1}^X)\, a_{i_{q-1} i_0}(\overline{g}_0^X).$$

Furthermore, note that $\sum_{i_1, \ldots i_{q-1}} a_{i_0 i_1}(\overline{g}_1^X) \cdots a_{i_{q-2} i_{q-1}}(\overline{g}_{q-1}^X)\, a_{i_{q-1} i_0}(\overline{g}_0^X)$ is the coefficient (i_0, i_0) of the matrix

$$A_{\underline{\mu}}(\overline{g}_1^X) \cdots A_{\underline{\mu}}(\overline{g}_{q-1}^X) A_{\underline{\mu}}(\overline{g}_0^X) = A_{\underline{\mu}}(\overline{g}_1^X \cdots \overline{g}_{q-1}^X \overline{g}_0^X),$$

because $\rho_{\underline{\mu}}$ is a representation of H_0. It follows that

$$\begin{aligned} \sum_{i_0, \ldots, i_{q-1}} a_{i_0 i_1}(\overline{g}_1^X) \cdots a_{i_{q-2} i_{q-1}}(\overline{g}_{q-1}^X)\, a_{i_{q-1} i_0}(\overline{g}_0^X) &= \sum_{i_0} a_{i_0 i_0}(\overline{g}_1^X \cdots \overline{g}_{q-1}^X \overline{g}_0^X) \\ &= \chi_{\underline{\mu}}(\overline{g}_1^X \cdots \overline{g}_{q-1}^X \overline{g}_0^X) \\ &= \chi_{\underline{\mu}}(\overline{g}_0^X \cdots \overline{g}_{q-1}^X). \end{aligned}$$

The result follows. □

Lemma 3.4. *We keep the notation of Lemma 3.3. Let $g = (z; \sigma)$ with $z \in \mathcal{U}_{de}^r$ and $\sigma \in \mathfrak{S}_r$. Write $\sigma = \sigma_1 \cdots \sigma_s$ the cycle decomposition with disjoint support of σ. Assume that σ_i has length ℓ_i, and that, for $1 \leq j \leq s$,*

$$\sigma_j = (L_j + 1 \;\; \cdots \;\; L_j + \ell_j), \tag{21}$$

where $L_1 = 0$ and $L_j = \ell_1 + \cdots + \ell_{j-1}$. Let $X = (X_0, \ldots, X_{q-1}) \in \mathcal{X}_{(r', \ldots, r')}$, and, for $0 \leq i \leq q-1$, write

$$X_i = \{x_{i,1}, \ldots, x_{i,r'}\} \quad \textit{with } x_{i,1} < \cdots < x_{i,r'}.$$

If $m''_{\underline{\lambda}}(X_g) = X$, then ℓ_j is divisible by q for all $1 \leq i \leq s$. Let $1 \leq j \leq s$. Write $l_j = q l'_j$ and $L_j = q L'_j$. Then there is $0 \leq i_0 \leq q-1$ such that $x_{i_0,1} = L_j + 1$, and, for all $1 \leq k \leq \ell'_j$ and $0 \leq l \leq q-1$, we have

$$x_{i_0 - l, L'_j + k} = L_j + (k-1)q + l + 1,$$

where $i_0 - l$ is taken modulo q.

Proof. Assume that $m''_{\underline{\lambda}}(X_g) = X$. We have $X_g = (\sigma(X_0), \ldots, \sigma(X_{q-1}))$. Thus, for all $i \geq 0$, one has

$$\sigma(X_{i+1}) = X_i,$$

where the indices are taken modulo q. Let $1 \leq j \leq s$. Assume $L_j + 1 \in X_{i_0}$ for some $0 \leq i_0 \leq q-1$. Now, we prove by induction on l that

$$L_j + l \in X_{i_0 - l + 1}. \tag{22}$$

Indeed, it is true for $l = 1$ and, if we assume it holds for some $l \geq 1$, then one has

$$L_j + l + 1 = \sigma(L_j + l) \in \sigma(X_{i_0 - l + 1}) = X_{i_0 - l},$$

as required. In particular, $L_j + \ell_j \in X_{i_0 - \ell_j + 1}$. However, $\sigma(L_j + \ell_j) = L_j + 1$, hence $X_{i_0 - \ell_j} = \sigma(X_{i_0 - \ell_j + 1}) = X_{i_0}$ and $i_0 - \ell_j \equiv i_0 \mod q$, that is q divides ℓ_j. The result now follows from Relation (22). □

Remark 3.5. In fact, for all $1 \leq j \leq s$, the position of $L_j + 1$ completely determines the integer $x_{i, L'_j + k}$ for $0 \leq i \leq q-1$ and $1 \leq k \leq \ell'_j$. Since there are q choices for the place of $L_j + 1$, we deduce that the number of $X \in \mathcal{X}_{(r', \ldots, r')}$ such that $m''_{\underline{\lambda}}(X_g) = X$ is q^s.

Recall that the conjugacy classes of G are labeled by $\mathcal{MP}_{de,r}$ as follows. Let $g = (z; \sigma) \in G$ be with $z = (z_1, \ldots, z_r) \in \mathcal{U}_{de}^r$ and $\sigma \in \mathfrak{S}_r$ with disjoint cycle decomposition $\sigma_1 \cdots \sigma_s$. For $1 \leq j \leq s$, write $\widetilde{\sigma}_j = (z_{(j)}; \sigma_j)$ where $z_{(j)\,k} = z_k$ if k lies in the support of σ_j, and $z_{(j)\,k} = 1$ otherwise. The cycle product $\mathfrak{c}(\widetilde{\sigma}_j)$ of $\widetilde{\sigma}_j$ is then defined to be $\prod_k z_{(j)\,k}$. Now, we associate to g the multi-partition $\underline{\eta} = (\eta_0, \ldots, \eta_{de-1}) \in \mathcal{MP}_{r,de}$, called the *cyclic structure* $\mathfrak{c}(g)$ of g, in such a way that, for all $1 \leq j \leq s$, η_u has a part of length $|\sigma_j|$ if and only if $\mathfrak{c}(\widetilde{\sigma}_j) = \zeta^u$, where ζ is a generator of $\mathcal{U}_{de}$. Then two elements g and g' of G are conjugate if and only if $\mathfrak{c}(g) = \mathfrak{c}(g')$.

Convention 3.6. *Now, for any $\underline{\eta} = (\eta_0, \ldots, \eta_{de-1}) \in \mathcal{MP}_{r,de}$, we choose as representative for the class of G labeled by $\underline{\eta}$ the element $g_{\underline{\eta}} = (z; \sigma)$, where the cycles of σ are as in (21), and, if $\sigma_j = (L_j + 1 \cdots L_j + \ell_j)$ is a cycle of σ such that $\mathfrak{c}(\widetilde{\sigma}_j) = \zeta^u$, then $z_{(j)\,k} = 1$ if $k \neq L_j + 1$, and $z_{(j)\,L_j+1} = \zeta^u$.*

For $r \in \mathbb{N}$, we denote by $\mathcal{P}_r$ the set of partitions of r, and we let $\mathcal{P} = \bigcup_{r \in \mathbb{N}} \mathcal{P}_r$. For any $\pi = (\pi_1, \ldots, \pi_t) \in \mathcal{P}$ and any positive integer q, we let

$$q \star \pi = (q\pi_1, \ldots, q\pi_t) \in \mathcal{P}. \tag{23}$$

Furthermore, we write $\ell(\pi) = t$, and, for $\underline{\eta} = (\eta_0, \ldots, \eta_{de-1}) \in \mathcal{MP}_{r,de}$, we set $\ell(\underline{\eta}) = \sum \ell(\eta_u)$. Note that, if $g = (z; \sigma)$ with $\sigma = \sigma_1 \cdots \sigma_s$ has cyclic structure $\underline{\eta}$, then $s = \ell(\underline{\eta})$.

Theorem 3.7. *We keep the notation as in Lemma 3.3. Let $\underline{\eta} = (\eta_0, \ldots, \eta_{de-1}) \in \mathcal{MP}_{r,de}$ and $g_{\underline{\eta}} = (z; \sigma)$ with $z = (z_1, \ldots, z_r) \in \mathcal{U}_{de}^r$ and $\sigma = \sigma_1 \cdots \sigma_s \in \mathfrak{S}_r$ be as in Convention 3.6. For any $1 \leq j \leq s$, write $\xi_j = \mathfrak{c}(\widetilde{\sigma}_j)$.*

(i) If there is $1 \leq u \leq de - 1$ such that $\eta_u \notin q \star \mathcal{P}$, or if there is $1 \leq j \leq s$ such that $\xi_j \notin \mathcal{U}_{db_{\underline{\lambda}} b'}$ then, $\Delta_{\underline{\lambda}, b'}(g_{\underline{\eta}}) = 0$.

(ii) *Assume* $\eta_u = q \star \eta'_u$ *for all* $0 \leq u \leq de-1$ *and* $\xi_j \in \mathcal{U}_{db_{\underline{\lambda}}b'}$ *for all* $1 \leq j \leq s$. *Then*

$$\Delta_{\underline{\lambda},b'}(g_{\underline{\eta}}) = q^{\ell(\underline{\eta})}\chi_{\underline{\mu}}(g'_{\underline{\eta}}),$$

where $g'_{\underline{\eta}} \in H_0$ *has cyclic structure* $(\eta'_0, \ldots, \eta'_{de-1})$ *and* $\underline{\mu} \in \mathcal{MP}_{r',de/q}$ *is as in Lemma 3.3.*

Proof. If there is $1 \leq u \leq de-1$ such that q does not divide $|\eta_u|$, then σ has a cycle of length not divisible by q. By Lemma 3.4, there are no $X \in \mathcal{X}_{(r',\ldots,r')}$ such that $m''_{\underline{\lambda}}(X_{g_{\underline{\eta}}}) = X$, thus $\Delta_{\underline{\lambda},b'}(g_{\underline{\eta}}) = 0$ by Lemma 2.1, proving the first part of (i).

Denote by $\mathfrak{X}$ the set of $X \in \mathcal{X}_{(r',\ldots,r')}$ such that $m''_{\underline{\lambda}}(X_{g_{\underline{\eta}}}) = X$. Set $Q = \{0,\ldots,q-1\}$ and, for $\underline{i} = (i_1,\ldots,i_s) \in Q^s$, define $\mathfrak{X}_{\underline{i}}$ to be the set of $(X_0,\ldots,X_{q-1}) \in \mathfrak{X}$ such that, for all $1 \leq j \leq s$, the integer L_j+1 lies in X_{i_j}.

Let $X \in \mathfrak{X}$. Then there is a unique $\underline{i} \in Q^s$ such that $X \in \mathfrak{X}_{\underline{i}}$. For $1 \leq j \leq s$, write $q\ell'_j$ for the length of σ_j. Then $\sigma_j t_X = t_{X_{\sigma_j}}\sigma'_j$, where

$$\sigma'_j = (C_{i_j}+L'_j+\ell'_j \;\; C_{i_j}+L'_j+\ell'_j-1 \;\cdots\; C_{i_j}+L'_j+1). \tag{24}$$

Furthermore, Relation (8) gives $z_{(j)}t_{X_{\sigma_j}} = t_{X_{\sigma_j}}z'$, where $z' = (z'_1,\ldots,z'_k) \in \mathcal{U}_{de}^r$ is such that $z'_{C_{i_j}+L'_j+1} = z_{(j)\,L_j+1}$ and $z'_k = 1$ otherwise. So, if we set $\widetilde{\sigma}_k'^X = 1$ if $k \neq i_j$ and $\widetilde{\sigma}_{i_j}'^X = z''\sigma'_j$, where $z'' = (z''_1,\ldots,z''_{r'}) \in \mathcal{U}_{de}^{r'}$ is such that $z''_{L'_j+1} = z'_{C_{i_j}+L'_j+1}$ and $z''_k = 1$ otherwise, then

$$\widetilde{\sigma}_j t_X = t_{X_{\sigma_j}}\widetilde{\sigma}_0'^X \cdots \widetilde{\sigma}_{q-1}'^X, \tag{25}$$

where $\widetilde{\sigma}_k'^X \in H_k$. Note that $\mathfrak{c}(\widetilde{\sigma}_{i_j}'^X) = \mathfrak{c}(\widetilde{\sigma}_j)$. Now, Lemma 3.4 implies that $t_{X_g} = t_{X_{\sigma_1}}\cdots t_{X_{\sigma_s}}$, so that applying Relation (25) iteratively to the cycles of $g_{\underline{\eta}}$, we obtain

$$g_{\underline{\eta}}t_X = t_{X_g}g_0^X\cdots g_{q-1}^X,$$

where

$$g_k^X = \prod_{\{1\leq j\leq s \,|\, i_j=k\}} \widetilde{\sigma}_j'^X \in H_k \quad \text{for all } 0 \leq k \leq q-1.$$

By Relation (24), the cycles $\widetilde{\sigma}_j'^X$ have disjoint support, and

$$\overline{\sigma}'_j = (L'_j+\ell'_j \;\cdots L'_j+1).$$

Hence, the element $g'_{\underline{\eta}} = \overline{g}_0^X\cdots\overline{g}_{q-1}^X \in H_0$ has cyclic structure $\underline{\eta}' = (\eta'_0,\ldots,\eta'_{de-1})$ and does not depend on $X \in \mathfrak{X}$. Therefore, Lemma 3.3 implies

$$\Delta_{\underline{\lambda},b'}(g_{\underline{\eta}}) = \chi_{\underline{\mu}}(g'_{\underline{\eta}})\sum_{X\in\mathfrak{X}}\prod_{i=0}^{q-1}\alpha_{r'}^{db_{\underline{\lambda}}b'i}(g_i^X). \tag{26}$$

Let $1 \leq j \leq s$ and take $X \in \mathfrak{X}$. For $0 \leq k \leq q-1$, write $h_k^X \in H_k$ such that $\widetilde{\sigma}_j^{-1}gt_X = t_{X_{\widetilde{\sigma}_j^{-1}g}}h_0^X\cdots h_{q-1}^X$. We denote by $\mathfrak{X}_i$ the set of $X \in \mathfrak{X}_{(r',\ldots,r')}$ such that $L_j+1 \in X_i$. If $X \in \mathfrak{X}_i$, then $g_k^X = h_k^X$ for $k \neq i$ and $g_i^X = \widetilde{\sigma}_i'^X h_i^X$, by Lemma 3.4 and Relation (24), and it follows that

$$\prod_{k=0}^{q-1}\alpha_{r'}^{db_{\underline{\lambda}}b'k}(g_k^X) = \alpha_{r'}^{db_{\underline{\lambda}}b'i}(\widetilde{\sigma}_j'^X)\prod_{k=0}^{q-1}\alpha_{r'}^{db_{\underline{\lambda}}b'k}(h_k^X) = \xi_j^{db_{\underline{\lambda}}b'i}\prod_{k=0}^{q-1}\alpha_{r'}^{db_{\underline{\lambda}}b'k}(h_k^X).$$

Note that $J = \sum_{X \in \mathfrak{X}_i} \prod_{k=0}^{q-1} \alpha_{r'}^{db_{\underline{\lambda}} b' k}(h_k^X)$ does not depend on $i \in Q$. Hence, we obtain

$$\begin{aligned}
\sum_{X \in \mathfrak{X}} \prod_{k=0}^{q-1} \alpha_{r'}^{db_{\underline{\lambda}} b' k}(g_k^X) &= \sum_{i_j=0}^{q-1} \sum_{X \in \mathfrak{X}_{i_j}} \prod_{k=0}^{q-1} \alpha_{r'}^{db_{\underline{\lambda}} b' k}(g_k^X) \\
&= \sum_{i_j=0}^{q-1} \sum_{X \in \mathfrak{X}_{i_j}} \xi_j^{db_{\underline{\lambda}} b' i_j} \prod_{k=0}^{q-1} \alpha_{r'}^{db_{\underline{\lambda}} b' k}(h_k^X) \\
&= \sum_{i_j=0}^{q-1} \left(\xi_j^{db_{\underline{\lambda}} b' i_j} \sum_{X \in \mathfrak{X}_{i_j}} \prod_{k=0}^{q-1} \alpha_{r'}^{db_{\underline{\lambda}} b' i}(h_k^X) \right) \\
&= J \sum_{i_j=0}^{q-1} \xi_j^{db_{\underline{\lambda}} b' i_j}.
\end{aligned}$$

Now, if $\xi_j \notin \mathcal{U}_{db_{\underline{\lambda}} b'}$, then

$$\sum_{i_j=0}^{q-1} \xi_j^{db_{\underline{\lambda}} b' i_j} = \frac{\xi_j^{db_{\underline{\lambda}} b' q} - 1}{\xi_j^{db_{\underline{\lambda}} b'} - 1} = 0,$$

since $db_{\underline{\lambda}} b' q = de$ and $\xi_j \in \mathcal{U}_{de}$. This concludes the proof of (i). If, on the other hand, $\xi_j \in \mathcal{U}_{db_{\underline{\lambda}} b'}$ for all $1 \leq j \leq s$, then $\alpha_{r'}^{db_{\underline{\lambda}} b' i}(g_i^X) = 1$ for all i. Equation (26) now gives $\Delta_{\underline{\lambda}, b'}(g_{\underline{\eta}}) = \chi_{\underline{\mu}}(g'_{\underline{\eta}})|\mathfrak{X}|$, and (ii) follows from Remark 3.5 since $s = \ell(\underline{\eta})$. □

Remark 3.8. Let $\underline{\lambda} = (\underline{\mu}, \dots, \underline{\mu}) \in \mathcal{MP}_{r,de}$, where $\underline{\mu} \in \mathcal{MP}_{r',de/q}$ is as in Lemma 3.3. Let $\underline{\eta} = (\eta_0, \dots, \eta_{de-1}) \in \mathcal{MP}_{r,de}$ and $g_{\underline{\eta}} = (z; \sigma)$ be as in Convention 3.6. Set $\sigma = \sigma_1 \cdots \sigma_s$ and $\xi_j = \mathfrak{c}(\widetilde{\sigma}_j)$. Assume that q divides η_j or all $1 \leq j \leq de-1$ and that $\xi_j \in \mathcal{U}_{de/q}$ for all $1 \leq j \leq s$. Since $\xi_j \in \mathcal{U}_{de/q}$ for all $1 \leq j \leq s$, we deduce that $\eta_u \neq \emptyset$ only if q divides u. Let $g'_{\underline{\eta}}$ be the element of $G(de, 1, r')$ with cyclic structure $(\eta'_0, \dots, \eta'_{de-1})$ described in Convention 3.6, where $\eta_j = q\eta'_j$, and let $g_{\underline{\eta}}^{(q)} \in G(de/q, 1, r')$ be the element with cyclic structure $(\eta'_0, \eta'_q, \eta'_{2q}, \dots)$ described in Convention 3.6.

Denote by $\widetilde{\chi}_{\underline{\mu}}$ the irreducible character of $G(de/q, 1, r')$ labeled by $\underline{\mu}$. Following §2.3, note that the representation space of H_0 labeled by $(\underline{\mu}, \emptyset, \dots, \emptyset)$ and that of H labeled by $\underline{\mu}$ have the same basis $\mathfrak{b}$. Furthermore, using the fact that $\mathrm{Irr}(\mathcal{U}_{de/q}) = \{\alpha^i \downarrow_{\mathcal{U}_{de/q}} | \, 0 \leq i \leq r'\}$, we deduce from (8) and (9) that the actions of $g'_{\underline{\eta}}$ and $g_{\underline{\eta}}^{(q)}$ on $\mathfrak{b}$ are the same. In particular, we have $\chi_{\underline{\mu}}(g'_{\underline{\eta}}) = \widetilde{\chi}_{\underline{\mu}}(g_{\underline{\eta}}^{(q)})$ and

$$\Delta_{\underline{\lambda}, b'}(g_{\underline{\eta}}) = q^s \widetilde{\chi}_{\underline{\mu}}(g_{\underline{\eta}}^{(q)}). \tag{27}$$

Example 3.9. *Consider $G = G(6,1,6)$ and $N = G(6,3,6)$. Let $\underline{\eta} = (\emptyset, \emptyset, \emptyset, (6), \emptyset, \emptyset)$. Then the representative for the conjugacy class of G labeled by $\underline{\eta}$ described in Convention 3.6 is $g_{\underline{\eta}} = (\zeta^3, 1, 1, 1, 1, 1; \sigma)$, where $\sigma = (1\ 2\ 3\ 4\ 5\ 6)$. Let $(\lambda_0, \lambda_1) \in \mathcal{MP}_{2,2}$. Write $\underline{\lambda} = (\lambda_0, \lambda_1, \lambda_0, \lambda_1, \lambda_0, \lambda_1) \in \mathcal{MP}_{6,6}$. We will compute $\Delta_{\underline{\lambda},1}(g_{\underline{\eta}})$. To this end, consider the set $\mathfrak{X}$ as in the proof of Theorem 3.7. Then Lemma 3.4 gives*

$\mathfrak{X} = \{X_1, X_2, X_3\}$ where $X_1 = (\{1,4\}, \{3,6\}, \{2,5\})$, $X_2 = (\{2,5\}, \{1,4\}, \{3,6\})$ and $X_3 = (\{3,6\}, \{2,5\}, \{1,4\})$. Furthermore,

$$\begin{aligned} g_{\underline{\eta}} t_{X_1} &= t_{X_1} g_0^{X_1} g_1^{X_1} g_2^{X_1} = t_{X_1} \left((\zeta^3,1,1,1,1,1;(1\ 2)),1,1\right) \\ g_{\underline{\eta}} t_{X_2} &= t_{X_2} g_0^{X_2} g_1^{X_2} g_2^{X_2} = t_{X_2} \left(1,(1,1,\zeta^3,1,1,1;(3\ 4)),1\right) \\ g_{\underline{\eta}} t_{X_3} &= t_{X_3} g_0^{X_3} g_1^{X_3} g_2^{X_3} = t_{X_3} \left(1,1,(1,1,1,1,\zeta^3,1;(5\ 6))\right). \end{aligned}$$

Note that

$$\overline{g}_0^{X_1}\overline{g}_1^{X_1}\overline{g}_2^{X_1} = \overline{g}_0^{X_2}\overline{g}_1^{X_2}\overline{g}_2^{X_2} = \overline{g}_0^{X_3}\overline{g}_1^{X_3}\overline{g}_2^{X_3} = (\zeta^3,1,1,1,1,1;(1\ 2)) \in H_0,$$

and can be identifed with the element $(\zeta^3,1;(1\ 2)) \in G(2,1,2)$. We have

$$\Delta_{\underline{\lambda},1}(g_{\underline{\mu}}) = 3\widetilde{\chi}_{(\lambda_0,\lambda_1)}(\zeta^3,1;(1\ 2)),$$

where $\widetilde{\chi}_{(\lambda_0,\lambda_1)}$ is the irreducible character of $G(2,1,2)$ labeled by (λ_0,λ_1).

Theorem 3.10. *With the notation as above, if $0 \leq k \leq |C_{\underline{\lambda}}|-1$, then $\Delta_{\underline{\lambda},k}(g_{\underline{\eta}}) = \Delta_{\underline{\lambda},b'}(g_{\underline{\eta}})$, where $1 \leq b' \leq |C_{\underline{\lambda}}|$ is such that the order of $M_{\underline{\lambda}}^k$ is $|C_{\underline{\lambda}}|/b'$.*

Proof. Write $q = |C_{\underline{\lambda}}|/b'$. First, we remark that the matrix $M_{\underline{\lambda}}^k$ has order q if and only if $M_{\underline{\lambda}}^k$ is a generator of the cyclic group $\langle M_{\underline{\lambda}}^{b'}\rangle$. In particular, there is an integer $1 \leq t \leq q$ coprime to q such that $M_{\underline{\lambda}}^k = M_{\underline{\lambda}}^{b't}$. Now, using Proposition 3.2, we deduce that $M_{\underline{\lambda}}^{\prime\prime s} \circ f'_{\underline{\lambda}} = f'_{\underline{\lambda}} \circ M_{\underline{\lambda}}^k$. Let $g = (z;\sigma) \in G(de,1,r)$ be such that $\sigma = \sigma_1 \cdots \sigma_s$. Write $\mathcal{Y}$ for the set of $Y \in \mathfrak{X}_{r',\ldots,r'}$ such that $m_{\underline{\lambda}}^{\prime\prime t}(Y_g) = Y$. Now, if $gt_Y = t_{Y_g} g_0^Y \cdots g_{q-1}^Y$, then we derive from the proof of Lemma 3.3 that

$$\Delta_{\underline{\lambda},k}(g) = \sum_{Y\in\mathcal{Y}} \left(\prod_{j=0}^{q-1} \alpha_{r'}^{db_{\underline{\lambda}}b'j}(g_j^Y)\right) \chi_{\underline{\mu}}(\overline{g}_s^Y \cdots \overline{g}_{s+q-1}^Y), \tag{28}$$

where the indices are taken modulo q. Furthermore, using the fact that $\chi_{\underline{\mu}}$ is a trace, we obtain $\chi_{\underline{\mu}}(\overline{g}_s^Y \cdots \overline{g}_{s+q-1}^Y) = \chi_{\underline{\mu}}(\overline{g}_0^Y \cdots \overline{g}_{q-1}^Y)$.

Define $f : \mathfrak{X} \to \mathcal{Y}$, $(X_0,\ldots,X_{q-1}) \mapsto (X_0, X_t, \ldots, X_{(q-1)t})$ where the indices are taken modulo q. The map f is well defined because t is coprime to q, whence $j \mapsto jt$ is a bijection of $\mathbb{Z}/q\mathbb{Z}$, and, if $X \in \mathfrak{X}$, then, for all $0 \leq j \leq q-1$,

$$m_{\underline{\lambda}}^{\prime\prime t}(\sigma(X_{jt})) = \sigma(X_{(j+1)t}) = \sigma(f(X_{j+1})) = f(\sigma(X_{j+1})) = f(X_j) = X_{jt}.$$

Furthermore, f is bijective since t is coprime to q. Let $X \in \mathfrak{X}$ and $1 \leq j \leq s$. Write $Y = f(X)$. Then $L_j + 1 \in X_{i_j}$ if and only if $L_j + 1 \in Y_{i_j t}$. In particular, $\{\widetilde{\sigma}_0^{\prime Y}, \ldots, \widetilde{\sigma}_{q-1}^{\prime Y}\}$ is a permutation of $\{\widetilde{\sigma}_0^{\prime X}, \ldots, \widetilde{\sigma}_{q-1}^{\prime X}\}$ (for the notation, we refer to the proof of Theorem 3.7). Hence, $\overline{g}_0^X \cdots \overline{g}_{q-1}^X$ and $\overline{g}_0^Y \cdots \overline{g}_{q-1}^Y$ have the same cyclic structure, that does not depend on X and Y. We can now conclude as in the end of the proof of Theorem 3.7. □

Example 3.11. *We continue with Example 3.9. We will now compute $\Delta_{\underline{\lambda},2}(g_{\underline{\eta}})$. We consider $\mathcal{Y}$ and $f : \mathfrak{X} \to \mathcal{Y}$ as in the proof of Theorem 3.10.*

Then $\mathcal{Y} = \{Y_1, Y_2, Y_3\}$ *where* $Y_1 = f(X_1) = (\{1,4\}, \{2,5\}, \{3,6\})$, $Y_2 = f(X_2) = (\{2,5\}, \{3,6\}, \{1,4\})$ *and* $Y_3 = f(X_3) = (\{3,6\}, \{1,4\}, \{2,5\})$. *We have*

$$g_{\underline{\eta}} t_{Y_1} = t_{Y_1} g_0^{Y_1} g_1^{Y_1} g_2^{Y_1} = t_{Y_1}\left((\zeta^3, 1, 1, 1, 1, 1; (1\ 2)), 1, 1\right)$$
$$g_{\underline{\eta}} t_{Y_2} = t_{Y_2} g_0^{Y_2} g_1^{Y_2} g_2^{Y_2} = t_{Y_2}\left(1, 1, (1, 1, 1, 1, \zeta^3, 1; (5\ 6))\right)$$
$$g_{\underline{\eta}} t_{Y_3} = t_{Y_3} g_0^{Y_3} g_1^{Y_3} g_2^{Y_3} = t_{Y_3}\left(1, (1, 1, \zeta^3, 1, 1, 1; (3\ 4)), 1\right).$$

We again have, for all $1 \le i,\ j \le 3$

$$\overline{g}_0^{Y_i}\overline{g}_1^{Y_i}\overline{g}_2^{Y_i} = \overline{g}_0^{X_j}\overline{g}_1^{X_j}\overline{g}_2^{X_j} = (\zeta^3, 1, 1, 1, 1, 1; (1\ 2)) \in H_0,$$

and

$$\Delta_{\underline{\lambda},1}(g_{\underline{\mu}}) = 3\widetilde{\chi}_{(\lambda_0,\lambda_1)}(\zeta^3, 1; (1\ 2)),$$

as required.

Proposition 3.12. *Let* $0 \le k \le |C_{\underline{\lambda}}| - 1$. *For any* $g \in G(de,1,r)$ *and* $x \in G(de,e,r)$, *we have*

$$\Delta_{\underline{\lambda},k}({}^g x) = \varepsilon^{kb_{\underline{\lambda}}}(g)\Delta_{\underline{\lambda},k}(x).$$

Proof. Recall that $M_{\underline{\lambda}}\rho_{\underline{\lambda}}M_{\underline{\lambda}}^{-1} = \varepsilon^{b_{\underline{\lambda}}} \otimes \rho_{\underline{\lambda}}$, so that $M_{\underline{\lambda}}^k\rho_{\underline{\lambda}}M_{\underline{\lambda}}^{-k} = \varepsilon^{b_{\underline{\lambda}}k} \otimes \rho_{\underline{\lambda}}$. Thus

$$\begin{aligned}
\Delta_{\underline{\lambda},k}({}^g x) &= \operatorname{Tr}(M_{\underline{\lambda}}^k \rho_{\underline{\lambda}}({}^g x)) \\
&= \operatorname{Tr}(M_{\underline{\lambda}}^k \rho_{\underline{\lambda}}(g)\rho_{\underline{\lambda}}(x)\rho_{\underline{\lambda}}(g)^{-1}) \\
&= \operatorname{Tr}(\varepsilon^{kb_{\underline{\lambda}}} \otimes \rho_{\underline{\lambda}}(g) M_{\underline{\lambda}}^k \rho_{\underline{\lambda}}(x)\rho_{\underline{\lambda}}(g)^{-1}) \\
&= \operatorname{Tr}(\varepsilon^{kb_{\underline{\lambda}}}(g) \rho_{\underline{\lambda}}(g) M_{\underline{\lambda}}^k \rho_{\underline{\lambda}}(x)\rho_{\underline{\lambda}}(g)^{-1}) \\
&= \varepsilon^{kb_{\underline{\lambda}}}(g)\operatorname{Tr}(\rho_{\underline{\lambda}}(g) M_{\underline{\lambda}}^k \rho_{\underline{\lambda}}(x)\rho_{\underline{\lambda}}(g)^{-1}) \\
&= \varepsilon^{kb_{\underline{\lambda}}}(g)\operatorname{Tr}(M_{\underline{\lambda}}^k \rho_{\underline{\lambda}}(g_{\underline{\eta}})),
\end{aligned}$$

whence

$$\Delta_{\underline{\lambda},i}(x) = \varepsilon^{kb_{\underline{\lambda}}}(g)\Delta_{\underline{\lambda},k}(x). \tag{29}$$

□

Let $\mathfrak{g} \in G(de,1,r)$ be an element of order e such that

$$G(de,1,r) = G(de,e,r) \rtimes \langle \mathfrak{g} \rangle.$$

Suppose that $\varepsilon(\mathfrak{g}) = \omega = \zeta^d$. For any divisor q of e and r, define

$$\mathcal{P}_{r,ed,q} = \{(\eta_0, \emptyset, \ldots, \emptyset, \eta_q, \emptyset, \ldots, \emptyset, \ldots, \eta_{de-q}, \emptyset, \ldots, \emptyset) \mid (\eta_0, \eta_q, \ldots, \eta_{de-q}) \in q\mathcal{P}_{r/q,de/q}\}.$$

Furthermore, for any $0 \le j \le q-1$ and $\underline{\eta} \in \mathcal{P}_{r,ed,q}$, write

$$g_{\underline{\eta},j} = {}^{\mathfrak{g}^j} g_{\underline{\eta}}.$$

Theorem 3.13. *The set*

$$\bigsqcup_{q|e} \{g_{\underline{\eta},j} \mid \underline{\eta} \in \mathcal{P}_{r,de,q},\ 0 \le j \le q-1\}$$

is a system of representatives for the conjugacy classes of $G(de,e,r)$.

Proof. Write $\mathcal{E}$ for a system of representatives of the $\langle\varepsilon\rangle$-orbits of $\mathcal{P}_{r,de}$. By Clifford theory from $G(de,1,r)$ to $G(de,e,r)$, the elements of

$$\mathfrak{P} = \{(\underline{\lambda},k) \mid \underline{\lambda}\in\mathcal{E},\, 0\leq k\leq |C_{\underline{\lambda}}|-1\}$$

label $\mathrm{Irr}(G(de,e,r))$. For any divisor q of e, write $\mathfrak{P}_q=\{(\underline{\lambda},k)\in\mathfrak{P}\mid |C_{\underline{\lambda}}|=q\}$. In particular,

$$\mathfrak{P}=\bigsqcup_{q\mid e}\mathfrak{P}_q.$$

Note that $(\underline{\lambda},k)\in\mathfrak{P}_q$ if and only if $\underline{\lambda}=(\underline{\mu},\ldots,\underline{\mu})$, where $\underline{\mu}\in\mathcal{P}_{r/q,de/q}$ is repeated q times. Let q be a divisor of e. For any $\underline{\mu}=(\mu_0,\ldots,\mu_{de/q-1})\in\mathcal{P}_{r/q,de/q}$, we define $f_1(\underline{\mu})=(\underline{\mu},\ldots,\underline{\mu})\in\mathcal{P}_{r,de}$ and $f_2(\underline{\mu})=(\lambda_0,\ldots,\lambda_{de-1})\in\mathcal{P}_{r,de,q}$ where $\lambda_{qj}=q\mu_j$ for $0\leq j\leq de/q-1$ and $\lambda_u=\emptyset$ otherwise. The maps f_1 and f_2 are bijective. Let $\underline{\eta}_1$ and $\underline{\eta}_2$ be two distinct elements of $\mathcal{P}_{r,de,q}$. Then for all $0\leq j_1,\, j_2\leq q-1$, the elements $g_{\underline{\eta}_1,j_1}$ and $g_{\underline{\eta}_2,j_2}$ are not conjugate in $G(de,e,r)$ since they are not conjugate in $G(de,1,r)$. Let $\underline{\eta}\in\mathcal{P}_{r,de,q}$. Write $\underline{\eta}'=f_2^{-1}(\eta)$. There exists a character $\widetilde{\chi}_{\underline{\mu}}$ of $G(de/q,1,r/q)$ such that $\widetilde{\chi}_{\underline{\mu}}(g_{\underline{\eta}'})\neq 0$ (we can take for example $\underline{\mu}$ such that $\widetilde{\chi}_{\underline{\mu}}$ is the trivial character of $G(de/q,1,r/q)$). Then by Proposition 3.12, for $0\leq j\leq q-1$, we have

$$\begin{aligned}\Delta_{f_1(\underline{\mu}),1}(g_{\underline{\eta},j}) &= \varepsilon^{b_{\underline{\lambda}}}(\mathfrak{g}^j)\Delta_{f_1(\underline{\mu}),1}(g_{\underline{\eta}})\\ &= \omega^{b_{\underline{\lambda}}j}\Delta_{f_1(\underline{\mu}),1}(g_{\underline{\eta}})\\ &= (\omega^{e/q})^j\Delta_{f_1(\underline{\mu}),1}(g_{\underline{\eta}}).\end{aligned}$$

Furthermore, $\Delta_{f_1(\underline{\mu}),1}(g_{\underline{\eta}})\neq 0$ by Remark 3.8 and Theorem 3.7, and $\Delta_{f_1(\underline{\mu}),1}(g_{\underline{\eta},j_1})\neq \Delta_{f_1(\underline{\mu}),1}(g_{\underline{\eta},j_2})$ for all $j_1\neq j_2$ since $\omega^{e/q}$ is a primitive qth-root of unity. Now, using that $\Delta_{f_1(\underline{\mu}),1}$ is a class function of $G(de,e,r)$, we conclude that the elements $g_{\underline{\eta},j}$ for $0\leq j\leq q-1$ are not conjugate in $G(de,e,r)$. Finally, the result follows from the fact that $f_1\circ f_2^{-1}$ induces a bijection between the sets $\{g_{\underline{\eta},j}\mid \underline{\eta}\in\mathcal{P}_{r,de,q},\, 0\leq j\leq q-1\}$ and $\{\chi_{\underline{\lambda},j}\mid (\underline{\lambda},j)\in\mathfrak{P}_q\}$. □

Example 3.14. *Let e be a prime number and r be a positive integer. By Theorem 3.13, the elements $g_{\underline{\eta},j}$ where $\underline{\eta}=(\eta,\emptyset,\ldots,\emptyset)$ with $\eta\in\mathcal{P}_r$ and $0\leq j\leq e-1$ form a system of representatives for the split classes of $G(e,e,er)$. For $\lambda\in\mathcal{P}_r$, set $\underline{\lambda}=(\lambda,\lambda,\ldots,\lambda)\in\mathcal{P}_{er,e}$. By Theorem 3.7, Remark 3.8, Theorem 3.10 and Proposition 3.12, for $1\leq k\leq e-1$, we have*

$$\Delta_{\underline{\lambda},k}(g_{\underline{\eta},j})=\zeta^{kj}e^{\ell(\eta)}\chi_\lambda(\eta),$$

where $\chi_\lambda(\eta)$ is the value of the irreducible character of $\mathfrak{S}_r$ labeled by λ on a element with cyclic structure η. Now, using Equality (18), we obtain

$$\chi_{\underline{\lambda},k}(g_{\underline{\eta},j})=\begin{cases}\dfrac{1}{e}\Big(\chi_{\underline{\lambda}}(g_{\underline{\eta}}))-e^{\ell(\eta)}\chi_\lambda(\eta)\Big) & \text{if } k\neq j,\\[2ex] \dfrac{1}{e}\Big(\chi_{\underline{\lambda}}(g_{\underline{\eta}}))+(e-1)e^{\ell(\eta)}\chi_\lambda(\eta)\Big) & \text{if } k=j.\end{cases}$$

In particular, for $e=2$, we recover with our method the result of [9, Thm. 5.1].

4. Perfect Isometries

Throughout this section, we consider $G = G(de, 1, r)$ and its normal subgroup $N = G(de, e, r)$, and we use the notation of Section 3.1.

4.1. **Characters of N.** In order to describe $\mathrm{Irr}(N)$, we will apply Clifford theory from G to N. We therefore consider the orbits of $\mathrm{Irr}(G)$ under the action of $G/N \cong \langle \varepsilon \rangle$, or, equivalently, the $\langle \varepsilon \rangle$-orbits of the parametrizing set $\mathcal{MP}_{r,de}$. For any $\underline{\lambda} \in \mathcal{MP}_{r,de}$, we denote by $[\underline{\lambda}]$ the $\langle \varepsilon \rangle$-orbit of $\underline{\lambda}$. Hence $\underline{\mu} \in [\underline{\lambda}]$ if and only if there exists $s \in \mathbb{N}$ such that $\underline{\mu} = \varepsilon^s(\underline{\lambda})$. In particular, for any $\underline{\mu} \in [\underline{\lambda}]$, we have $b_{\underline{\lambda}} = b_{\underline{\mu}}$ and $|C_{\underline{\lambda}}| = |C_{\underline{\mu}}|$. Furthermore, we see that $|[\underline{\lambda}]| = \dfrac{e}{|C_{\underline{\lambda}}|} = b_{\underline{\lambda}}$.

Lemma 4.1. *If $\underline{\lambda}, \underline{\mu} \in \mathcal{MP}_{r,de}$ are such that $[\underline{\lambda}] = [\underline{\mu}]$, then, with the notation of Section 3.1, $\chi_{\underline{\lambda},i} = \chi_{\underline{\mu},i}$ for all $0 \leq i < |C_{\underline{\lambda}}|$.*

Proof. We have $[\underline{\lambda}] = [\underline{\mu}]$, so there exists $s \in \mathbb{N}$ such that $\rho_{\underline{\lambda}} = \varepsilon^s \otimes \rho_{\underline{\mu}}$, and $W_{\underline{\lambda}} = W_{\underline{\mu}} = W$. In particular, there exist endomorphisms $M_{\underline{\lambda}}$ and $M_{\underline{\mu}}$ of W such that $M_{\underline{\lambda}}\rho_{\underline{\lambda}} = \varepsilon_{\underline{\lambda}}\rho_{\underline{\lambda}}M_{\underline{\lambda}}$ and $M_{\underline{\mu}}\rho_{\underline{\mu}} = \varepsilon_{\underline{\mu}}\rho_{\underline{\mu}}M_{\underline{\mu}}$, where, furthermore, $\varepsilon_{\underline{\lambda}} = \varepsilon^{b_{\underline{\lambda}}} = \varepsilon^{b_{\underline{\mu}}} = \varepsilon_{\underline{\mu}}$.

We therefore have

$$\varepsilon^s M_{\underline{\lambda}}\rho_{\underline{\mu}} = M_{\underline{\lambda}}\varepsilon^s\rho_{\underline{\mu}} = M_{\underline{\lambda}}\rho_{\underline{\lambda}} = \varepsilon_{\underline{\lambda}}\rho_{\underline{\lambda}}M_{\underline{\lambda}} = \varepsilon^s\varepsilon_{\underline{\lambda}}\rho_{\underline{\mu}}M_{\underline{\lambda}},$$

so that $M_{\underline{\lambda}}\rho_{\underline{\mu}} = \varepsilon_{\underline{\lambda}}\rho_{\underline{\mu}}M_{\underline{\lambda}} = \varepsilon_{\underline{\mu}}\rho_{\underline{\mu}}M_{\underline{\lambda}}$ (since $\varepsilon_{\underline{\lambda}} = \varepsilon_{\underline{\mu}}$).

Now, since $\varepsilon_{\underline{\mu}}\rho_{\underline{\mu}}M_{\underline{\mu}} = M_{\underline{\mu}}\rho_{\underline{\mu}}$, we also have $M_{\underline{\mu}}^{-1}\varepsilon_{\underline{\mu}}\rho_{\underline{\mu}} = \rho_{\underline{\mu}}M_{\underline{\mu}}^{-1}$, so that $M_{\underline{\lambda}}\rho_{\underline{\mu}} = \varepsilon_{\underline{\mu}}\rho_{\underline{\mu}}M_{\underline{\lambda}}$ yields $M_{\underline{\mu}}^{-1}M_{\underline{\lambda}}\rho_{\underline{\mu}} = M_{\underline{\mu}}^{-1}\varepsilon_{\underline{\mu}}\rho_{\underline{\mu}}M_{\underline{\lambda}} = \rho_{\underline{\mu}}M_{\underline{\mu}}^{-1}M_{\underline{\lambda}}$. Hence $M_{\underline{\mu}}^{-1}M_{\underline{\lambda}} \in \mathrm{End}_G(\rho_{\underline{\mu}})$, and Schur's Lemma shows that $M_{\underline{\lambda}} = \xi M_{\underline{\mu}}$ for some $\xi \in \mathbb{C}$.

We will show that $\xi = 1$. First note that, since $M_{\underline{\lambda}}$ and $M_{\underline{\mu}}$ both have order $|C_{\underline{\lambda}}| = |C_{\underline{\mu}}|$, we must have $\xi^{|C_{\underline{\lambda}}|} = 1$. Now fix any order on the elements of the bases $\mathfrak{b}_{\underline{\lambda}}$ and $\mathfrak{b}_{\underline{\mu}}$ of W. By Proposition 3.1, we have, for any $t_X \otimes v_{\underline{\lambda},\underline{T}} \in \mathfrak{b}_{\underline{\lambda}}$,

$$M_{\underline{\lambda}}(t_X \otimes v_{\underline{\lambda},\underline{T}}) = t_{\varepsilon^{b_{\underline{\lambda}}}(X)} \otimes v_{\varepsilon^{b_{\underline{\lambda}}}(\underline{\lambda}),\varepsilon^{b_{\underline{\lambda}}}(\underline{T})} = t_{\varepsilon^{b_{\underline{\lambda}}}(X)} \otimes v_{\underline{\lambda},\varepsilon^{b_{\underline{\lambda}}}(\underline{T})} \in \mathfrak{b}_{\underline{\lambda}}$$

(since, by definition, $\varepsilon^{b_{\underline{\lambda}}}(\underline{\lambda}) = \underline{\lambda}$). Hence $\mathrm{Mat}(M_{\underline{\lambda}}, \mathfrak{b}_{\underline{\lambda}})$ is a permutation matrix. Similarly, $\mathrm{Mat}(M_{\underline{\mu}}, \mathfrak{b}_{\underline{\mu}})$ is a permutation matrix.

Now, since $\underline{\mu} = \varepsilon^s(\underline{\lambda})$, there is a bijection $\sigma\colon \mathfrak{b}_{\underline{\lambda}} \longrightarrow \mathfrak{b}_{\underline{\mu}}$, given by $\sigma(t_X \otimes v_{\underline{\lambda},\underline{T}}) = t_{\varepsilon^s(X)} \otimes v_{\underline{\mu},\varepsilon^s(\underline{T})})$ for all $t_X \otimes v_{\underline{\lambda},\underline{T}} \in \mathfrak{b}_{\underline{\lambda}}$. The corresponding change of basis matrix P_σ from $\mathfrak{b}_{\underline{\lambda}}$ to $\mathfrak{b}_{\underline{\mu}}$ is therefore also a permutation matrix.

By construction, we have

$$\mathrm{Mat}(M_{\underline{\lambda}}, \mathfrak{b}_{\underline{\lambda}}) = P_\sigma^{-1}\,\mathrm{Mat}(M_{\underline{\lambda}}, \mathfrak{b}_{\underline{\mu}})P_\sigma = \xi P_\sigma^{-1}\,\mathrm{Mat}(M_{\underline{\mu}}, \mathfrak{b}_{\underline{\mu}})P_\sigma$$

(since $M_{\underline{\lambda}} = \xi M_{\underline{\mu}}$). Since all of these matrices have entries in $\mathbb{N}$, we deduce that $\xi = 1$, and thus that $M_{\underline{\lambda}} = M_{\underline{\mu}}$.

In particular, with the notation of Section 3.1, the eigenspaces $W_{\underline{\lambda},i}$ and $W_{\underline{\mu},i}$ coincide for all $0 \leq i < |C_{\underline{\lambda}}|$, and $\chi_{\underline{\lambda},i} = \chi_{\underline{\mu},i}$ for all $0 \leq i < |C_{\underline{\lambda}}|$. □

Corollary 4.2. *If $\underline{\lambda}, \underline{\mu} \in \mathcal{MP}_{r,de}$ are such that $[\underline{\lambda}] = [\underline{\mu}]$, then $\Delta_{\underline{\lambda},i} = \Delta_{\underline{\mu},i}$ for all $0 \leq i < |C_{\underline{\lambda}}|$.*

Proof. This follows immediately from Lemma 4.1 and (19) (since $b_{\underline{\lambda}} = b_{\underline{\mu}}$ and $|C_{\underline{\lambda}}| = |C_{\underline{\mu}}|$).

□

Remark 4.3. Suppose $\underline{\mu} \in \mathcal{MP}_{r,de}$ is such that $|C_{\underline{\mu}}|$ is even, and take $\delta \in \{\pm 1\}$. Then $M_{\underline{\mu}}$ and $\delta M_{\underline{\mu}}$ have the same eigenspaces, and the same set $\mathcal{U}_{|C_{\underline{\mu}}|} = \langle \omega \rangle$ of eigenvalues, where $\omega = \zeta^{b_{\underline{\mu}} d}$. For any $\omega^j \in \mathcal{U}_{|C_{\underline{\mu}}|}$, we set

$$\chi_{\underline{\mu},j,\delta M_{\underline{\mu}}} = \mathrm{Tr}(\rho_{\underline{\mu}} \,|\, E_{\omega^j}),$$

where E_{ω^j} is the eigenspace of $\delta M_{\underline{\mu}}$ corresponding to the eigenvalue ω^j. We also set, for any $0 \leq i < |C_{\underline{\mu}}|$,

$$\Delta_{\underline{\mu},i,\delta M_{\underline{\mu}}} = \mathrm{Tr}((\delta M_{\underline{\mu}})^i \rho_{\underline{\mu}} \,|\, W_{\underline{\mu}}).$$

In particular, we have $\Delta_{\underline{\mu},i,\delta M_{\underline{\mu}}} = \delta^i \Delta_{\underline{\mu},i}$.

We also have, as in Section 3.1,

$$\Delta_{\underline{\mu},i,\delta M_{\underline{\mu}}} = \sum_{j=0}^{|C_{\underline{\mu}}|-1} \omega^{ij} \chi_{\underline{\mu},j,\delta M_{\underline{\mu}}} \quad \text{and} \quad \chi_{\underline{\mu},j,\delta M_{\underline{\mu}}} = \frac{1}{|C_{\underline{\mu}}|} \sum_{i=0}^{|C_{\underline{\mu}}|-1} \omega^{-ij} \Delta_{\underline{\mu},i,\delta M_{\underline{\mu}}}.$$

4.2. **Blocks of G and N.** We now take any prime p not dividing de. The p-blocks of G can be described as follows (see [8, Theorem 1]). Two irreducible characters $\widetilde{\chi}_{\underline{\mu}}$ and $\widetilde{\chi}_{\underline{\nu}}$ of G, corresponding to $\underline{\mu} = (\mu^{(0)}, \ldots, \mu^{(de-1)})$ and $\underline{\nu} = (\nu^{(0)}, \ldots, \nu^{(de-1)})$ in $\mathcal{MP}_{r,de}$ lie in the same p-block B of G if and only if, for every $0 \leq i \leq de-1$, the partitions $\mu^{(i)}$ and $\nu^{(i)}$ have the same p-core $(\mu^{(i)})_{(p)} = (\nu^{(i)})_{(p)} = \gamma^{(i)}$ and same p-weight $w_p(\mu^{(i)}) = w_p(\nu^{(i)}) = w_i$. The de-tuple $\underline{w} = (w_0, \ldots, w_{de-1})$ (respectively $\underline{\gamma} = (\gamma^{(0)}, \ldots, \gamma^{(de-1)})$) is called the p-weight of B (respectively the p-core of B). Note that B has p-defect 0 if and only if $w = (0, \ldots, 0)$. We denote by $\mathcal{E}_{\underline{\gamma},\underline{w}}$ the set of de-multipartitions $\underline{\mu} = (\mu^{(0)}, \ldots, \mu^{(de-1)})$ such that $(\mu^{(i)})_{(p)} = \gamma^{(i)}$ and $w_p(\mu^{(i)}) = w_i$.

We can now describe the p-blocks of N using Clifford theory. If B is a p-block of G of defect 0, then, since $(p,e) = 1$, B only covers p-blocks of defect 0 of N. Conversely, a p-block of N of defect 0 can only been covered by p-blocks of G of defect 0. Hence suppose B is a p-block of G of positive defect, and take k dividing e minimal such that B is ε^k-stable (i.e. $\varepsilon^k \otimes B = B$). Then B has p-core $\underline{\gamma} = (\gamma^{(0)}, \gamma^{(1)}, \ldots, \gamma^{(kd-1)}, \gamma^{(0)}, \ldots, \gamma^{(kd-1)}, \ldots, \gamma^{(0)}, \ldots, \gamma^{(kd-1)})$ and p-weight $\underline{w} = (w_0, w_1, \ldots, w_{kd-1}, w_0, \ldots, w_{kd-1}, \ldots, w_0, \ldots, w_{kd-1})$, where $w_0 + \cdots + w_{kd-1} \neq 0$. Without loss of generality, we can furthermore suppose that $w_0 \neq 0$. Now consider any $\underline{\lambda} \in \mathcal{MP}_{r,de}$ given by

$$\underline{\lambda} = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(kd-1)}, \mu^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(kd-1)}, \ldots, \mu^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(kd-1)}),$$

where

- for $1 \leq i \leq kd-1$, $(\lambda^{(i)})_{(p)} = \gamma^{(i)}$ and $w_p(\lambda^{(i)}) = w_i$,
- $(\lambda^{(0)})_{(p)} = (\mu^{(0)})_{(p)} = \gamma^{(0)}$, $\lambda^{(0)}$ and $\mu^{(0)}$ have p-quotients $Q_p(\lambda^{(0)}) = ((w_0), \emptyset, \ldots, \emptyset)$ and $Q_p(\mu^{(0)}) = (\emptyset, \ldots, \emptyset, (w_0))$ (so that $\lambda^{(0)} \neq \mu^{(0)}$),
- and, for $1 \leq j \leq k-1$,

$$\lambda^{(jd)} = \begin{cases} \mu^{(0)} & \text{if } w_{jd} = w_0 \text{ and } \gamma^{(jd)} = \gamma^{(0)} \\ \text{any } \mu \text{ with } \mu_{(p)} = \gamma^{(jd)} \text{ and } w_p(\mu) = w_{jd} & \text{if } w_{jd} \neq w_0 \text{ or } \gamma^{(jd)} \neq \gamma^{(0)} \end{cases}$$

(so that $\lambda^{(jd)} \neq \lambda^{(0)}$).

Then $\underline{\lambda}_{(p)} = \underline{\gamma}$ and $w_p(\underline{\lambda}) = \underline{w}$, so that $\widetilde{\chi}_{\underline{\lambda}} \in B$. And $\lambda^{(jd)} \neq \lambda^{(0)}$ for all $0 < j < e$, so that $\varepsilon^j(\underline{\lambda}) \neq \underline{\lambda}$, and $\widetilde{\chi}_{\underline{\lambda}}$ is not ε^j-stable for any $0 < j < e$.

This shows that any p-block B of G of positive defect contains an irreducible character which is not ε^j-stable for any $0 < j < e$. By Clifford theory, such a character must restrict irreducibly to N, and its restriction to N is therefore G-stable. By [7, Corollary (9.3)], this implies that B covers a unique p-block b of N.

4.3. **Bijections and isometries between blocks.** We now fix the positive integers d and e, a prime p not dividing de, and consider two positive integers r and r'. We let $G = G(de,1,r)$, $N = G(de,e,r)$, $G' = G(de,1,r')$ and $N' = G(de,e,r')$. Suppose b is a p-block of N, covered by the p-block B of G of p-core $\underline{\gamma} = (\gamma^{(0)},\ldots,\gamma^{(de-1)})$ and p-weight $\underline{w} = (w_0,\ldots,w_{de-1})$, and b' is a p-block of N', covered by the p-block B' of G' of p-core $\underline{\gamma}' = (\gamma'^{(0)},\ldots,\gamma'^{(de-1)})$ and p-weight $\underline{w}' = \underline{w}$. Suppose furthermore that $w_0 + \cdots + w_{de-1} \neq 0$. Then there is a bijection ψ between the subsets $\mathcal{E}_{\underline{\gamma},\underline{w}}$ and $\mathcal{E}_{\underline{\gamma}',\underline{w}}$ of $\mathcal{MP}_{r,de}$ and $\mathcal{MP}_{r',de}$ (which parametrize the irreducible characters in B and B' respectively) described as follows. For any $\underline{\lambda} = (\lambda^{(0)},\ldots,\lambda^{(de-1)}) \in \mathcal{E}_{\underline{\gamma},\underline{w}}$, we have $\psi(\underline{\lambda}) = (\Psi(\lambda^{(0)}),\ldots,\Psi(\lambda^{(de-1)}))$, where, for each $0 \leq i \leq de-1$, $\Psi(\lambda^{(i)})$ is the partition defined by $\Psi(\lambda^{(i)})_{(p)} = \gamma'^{(i)}$ and $Q_p(\Psi(\lambda^{(i)})) = Q_p(\lambda^{(i)})$.

With the notation of Section 3.1, we see that, for any $\underline{\lambda} \in \mathcal{E}_{\underline{\gamma},\underline{w}}$, we have $|C_{\psi(\underline{\lambda})}| = |C_{\underline{\lambda}}|$ and $b_{\psi(\underline{\lambda})} = b_{\underline{\lambda}}$. Furthermore, for any $\underline{\lambda},\underline{\mu} \in \mathcal{E}_{\underline{\gamma},\underline{w}}$, we have, with the notation of Section 4.1, $[\underline{\lambda}] = [\underline{\mu}]$ if and only if $[\psi(\underline{\lambda})] = [\psi(\underline{\mu})]$. In particular, ψ also induces a bijection between $\mathrm{Irr}(b)$ and $\mathrm{Irr}(b')$.

Before our next definition, we need a few more pieces of notation. If s and t are positive integers, and if $n \in \mathbb{N}$, then, for any $\alpha \in \mathcal{MP}_{s,n}$, we set $t\alpha = (\alpha,\ldots,\alpha) \in \mathcal{MP}_{ts,tn}$. If $\beta = t\alpha$, then we write $\alpha = \beta/t$.
Finally, for any $k > 0$ and any k-multipartition $\underline{\lambda} = (\lambda^{(0)},\lambda^{(1)},\ldots,\lambda^{(k-1)})$, we set $\delta_p(\underline{\lambda}) = \delta_p(\lambda^{(0)})\delta_p(\lambda^{(1)})\cdots\delta_p(\lambda^{(k-1)})$, where, for each $0 \leq i < k$, $\delta_p(\lambda^{(i)})$ is the p-sign of $\lambda^{(i)}$ (see [6, §2]).

Definition 4.4. *With the notation above, we define an isometry* $I\colon \mathbb{C}\,\mathrm{Irr}(b) \longrightarrow \mathbb{C}\,\mathrm{Irr}(b')$ *by letting, for any* $\underline{\lambda} \in \mathcal{E}_{\underline{\gamma},\underline{w}}$ *and any* $0 \leq i < |C_{\underline{\lambda}}|$,

$$I(\chi_{\underline{\lambda},i}) = \begin{cases} \delta_p(\underline{\lambda})\delta_p(\psi(\underline{\lambda}))\chi_{\psi(\underline{\lambda}),i} & \text{if } |C_{\underline{\lambda}}| \text{ is odd,} \\ \delta_p(\underline{\lambda})\delta_p(\psi(\underline{\lambda}))\chi_{\psi(\underline{\lambda}),i,\delta_{\underline{\lambda}}M_{\psi(\underline{\lambda})}} & \text{if } |C_{\underline{\lambda}}| \text{ is even,} \end{cases}$$

where $\delta_{\underline{\lambda}} = \delta_p(\underline{\lambda}/|C_{\underline{\lambda}}|)\delta_p(\psi(\underline{\lambda})/|C_{\psi(\underline{\lambda})}|)$.

Remark 4.5. Note that I is well-defined. Indeed, if $[\underline{\lambda}] = [\underline{\mu}]$, then $[\psi(\underline{\lambda})] = [\psi(\underline{\mu})]$, so that $\delta_p(\underline{\lambda}) = \delta_p(\underline{\mu})$ and $\delta_p(\psi(\underline{\lambda})) = \delta_p(\psi(\underline{\mu}))$. Also, by Lemma 4.1, $\chi_{\underline{\lambda},i} = \chi_{\underline{\mu},i}$ for all $0 \leq i < |C_{\underline{\lambda}}|$. Furthermore, by the proof of Lemma 4.1, $M_{\psi(\underline{\lambda})} = M_{\psi(\underline{\mu})}$ and thus $\delta_{\underline{\lambda}}M_{\psi(\underline{\lambda})} = \delta_{\underline{\mu}}M_{\psi(\underline{\mu})}$ (since $\delta_{\underline{\lambda}} = \delta_{\underline{\mu}}$). Finally, by Remark 4.3 and Lemma 4.1, $\chi_{\psi(\underline{\lambda}),i,\delta_{\underline{\lambda}}M_{\psi(\underline{\lambda})}} = \chi_{\psi(\underline{\mu}),i,\delta_{\underline{\mu}}M_{\psi(\underline{\mu})}}$ for all $0 \leq i < |C_{\underline{\lambda}}|$.

4.4. **Perfect isometries.** We keep the notation as in the previous section. Our aim is now to show that the isometry I described in Definition 4.4 is actually a perfect isometry between b and b', thereby generalizing to complex reflection groups the results known about the symmetric groups (see [3, Theorem 11]), wreath products (see [2, Theorem 5.4]) and Weyl groups of type B and D (see [2, Corollary 5.6 and Theorem 5.8]). We start by recalling the definition of perfect isometry.

Definition 4.6. *(See [1] and [2, §2.5]) Let H and H' be finite groups, p be a prime, and $(K, \mathcal{R}, k)$ a splitting p-modular system for H and H'. Let $\mathcal{B}$ and $\mathcal{B}'$ be unions of p-blocks of H and H' respectively, and $J\colon \mathbb{C}\operatorname{Irr}(\mathcal{B}) \longrightarrow \mathbb{C}\operatorname{Irr}(\mathcal{B}')$ an isometry such that $J(\mathbb{Z}\operatorname{Irr}(\mathcal{B})) = \mathbb{Z}\operatorname{Irr}(\mathcal{B}')$. Let $(e_1, \ldots, e_n)$ be any $\mathbb{C}$-basis for $\mathbb{C}\operatorname{Irr}(\mathcal{B})$ and $(e_1^\vee, \ldots, e_n^\vee)$ its dual with respect to the usual hermitian product $\langle\ ,\ \rangle_H$ on $\mathbb{C}\operatorname{Irr}(H)$, and let $\widehat{J} = \sum_{i=1}^{n} \overline{e_i^\vee} \otimes J(\varepsilon_i)$. Then J is a perfect isometry between $\mathcal{B}$ and $\mathcal{B}'$ if the following hold:*

(1) For every $(x, x') \in H \times H'$, $\widehat{J}(x, x') \in |C_H(x)|_p\mathcal{R} \cap |C_{H'}(x')|_p\mathcal{R}$.
(2) If $\widehat{J}(x, x') \neq 0$, then x and x' are both p-regular or both p-singular.

Remark 4.7. Note that, in Definition 4.6, $\widehat{J}$ does not in fact depend on the choice of basis for $\mathbb{C}\operatorname{Irr}(\mathcal{B})$ (see [2, §2.3]).

If we let $[\mathcal{E}_{\underline{\gamma},\underline{w}}]$ be a set of representatives for the $\langle\varepsilon\rangle$-orbits of $\mathcal{E}_{\underline{\gamma},\underline{w}}$, then $\{\chi_{\underline{\lambda},i},\ \underline{\lambda} \in [\mathcal{E}_{\underline{\gamma},\underline{w}}]$ and $0 \leq i < |C_{\underline{\lambda}}|\}$ is a (self-dual) $\mathbb{C}$-basis for $\mathbb{C}\operatorname{Irr}(b)$. By (19), $\{\Delta_{\underline{\lambda},i},\ \underline{\lambda} \in [\mathcal{E}_{\underline{\gamma},\underline{w}}]$ and $0 \leq i < |C_{\underline{\lambda}}|\}$ is also a $\mathbb{C}$-basis for $\mathbb{C}\operatorname{Irr}(b)$, and this is the basis we will use to prove that I is a perfect isometry between b and b'.

From (19), we see that, for any $\underline{\lambda}, \underline{\mu} \in [\mathcal{E}_{\underline{\gamma},\underline{w}}]$, $0 \leq i < |C_{\underline{\lambda}}|$ and $0 \leq j < |C_{\underline{\mu}}|$, we have

$$\begin{aligned}
\langle \Delta_{\underline{\lambda},i}, \Delta_{\underline{\mu},j} \rangle_N &= \langle \sum_{k=0}^{|C_{\underline{\lambda}}|-1} \zeta^{db_{\underline{\lambda}}ik} \chi_{\underline{\lambda},k}, \sum_{\ell=0}^{|C_{\underline{\mu}}|-1} \zeta^{db_{\underline{\mu}}j\ell} \chi_{\underline{\mu},\ell} \rangle_N \\
&= \sum_{k=0}^{|C_{\underline{\lambda}}|-1} \sum_{\ell=0}^{|C_{\underline{\mu}}|-1} \zeta^{db_{\underline{\lambda}}ik} \overline{\zeta^{db_{\underline{\mu}}j\ell}} \langle \chi_{\underline{\lambda},k}, \chi_{\underline{\mu},\ell} \rangle_N \\
&= \sum_{k=0}^{|C_{\underline{\lambda}}|-1} \sum_{\ell=0}^{|C_{\underline{\mu}}|-1} \zeta^{db_{\underline{\lambda}}ik - db_{\underline{\mu}}j\ell} \delta_{\underline{\lambda},\underline{\mu}} \delta_{k,\ell} \\
&= \delta_{\underline{\lambda},\underline{\mu}} \sum_{k=0}^{|C_{\underline{\lambda}}|-1} \zeta^{db_{\underline{\lambda}}(i-j)k} \\
&= \delta_{\underline{\lambda},\underline{\mu}} \delta_{i,j} |C_{\underline{\lambda}}|.
\end{aligned}$$

This shows that, for any $\underline{\lambda} \in [\mathcal{E}_{\underline{\gamma},\underline{w}}]$ and $0 \leq i < |C_{\underline{\lambda}}|$, we have

$$\Delta_{\underline{\lambda},i}^\vee = \frac{1}{|C_{\underline{\lambda}}|} \Delta_{\underline{\lambda},i}. \tag{30}$$

Furthermore, from (19) and Definition 4.4, we see that, for any $\underline{\lambda} \in \mathcal{E}_{\underline{\gamma},\underline{w}}$ and $0 \leq i < |C_{\underline{\lambda}}|$, we have

$$\begin{aligned} I(\Delta_{\underline{\lambda},i}) &= \sum_{j=0}^{|C_{\underline{\lambda}}|-1} \zeta^{db_{\underline{\lambda}}ij} I(\chi_{\underline{\lambda},j}) \\ &= \begin{cases} \delta_p(\underline{\lambda})\delta_p(\psi(\underline{\lambda})) \displaystyle\sum_{j=0}^{|C_{\underline{\lambda}}|-1} \zeta^{db_{\underline{\lambda}}ij} \chi_{\psi(\underline{\lambda}),j} & \text{if } |C_{\underline{\lambda}}| \text{ is odd,} \\ \delta_p(\underline{\lambda})\delta_p(\psi(\underline{\lambda})) \displaystyle\sum_{j=0}^{|C_{\underline{\lambda}}|-1} \zeta^{db_{\underline{\lambda}}ij} \chi_{\psi(\underline{\lambda}),j,\delta_{\underline{\lambda}}M_{\psi(\underline{\lambda})}} & \text{if } |C_{\underline{\lambda}}| \text{ is even.} \end{cases} \end{aligned}$$

Now, since $|C_{\underline{\lambda}}| = |C_{\psi(\underline{\lambda})}|$ and $b_{\underline{\lambda}} = b_{\psi(\underline{\lambda})}$, we have that, if $|C_{\underline{\lambda}}|$ is odd, then

$$\sum_{j=0}^{|C_{\underline{\lambda}}|-1} \zeta^{db_{\underline{\lambda}}ij} \chi_{\psi(\underline{\lambda}),j} = \sum_{j=0}^{|C_{\psi(\underline{\lambda})}|-1} \zeta^{db_{\psi(\underline{\lambda})}ij} \chi_{\psi(\underline{\lambda}),j} = \Delta_{\psi(\underline{\lambda}),i}$$

(by (19)), and, if $|C_{\underline{\lambda}}|$ is even, then

$$\sum_{j=0}^{|C_{\underline{\lambda}}|-1} \zeta^{db_{\underline{\lambda}}ij} \chi_{\psi(\underline{\lambda}),j,\delta_{\underline{\lambda}}M_{\psi(\underline{\lambda})}} = \sum_{j=0}^{|C_{\psi(\underline{\lambda})}|-1} \zeta^{db_{\psi(\underline{\lambda})}ij} \chi_{\psi(\underline{\lambda}),j,\delta_{\underline{\lambda}}M_{\psi(\underline{\lambda})}} = \Delta_{\psi(\underline{\lambda}),i,\delta_{\underline{\lambda}}M_{\psi(\underline{\lambda})}}$$

(by Remark 4.3). And, also by Remark 4.3, we have $\Delta_{\psi(\underline{\lambda}),i,\delta_{\underline{\lambda}}M_{\psi(\underline{\lambda})}} = \delta_{\underline{\lambda}}^i \Delta_{\psi(\underline{\lambda}),i}$.

This shows that, for any $\underline{\lambda} \in \mathcal{E}_{\underline{\gamma},\underline{w}}$ and $0 \leq i < |C_{\underline{\lambda}}|$, we have

$$I(\Delta_{\underline{\lambda},i}) = \begin{cases} \delta_p(\underline{\lambda})\delta_p(\psi(\underline{\lambda}))\Delta_{\psi(\underline{\lambda}),i} & \text{if } |C_{\underline{\lambda}}| \text{ is odd,} \\ \delta_p(\underline{\lambda})\delta_p(\psi(\underline{\lambda}))\delta_{\underline{\lambda}}^i\Delta_{\psi(\underline{\lambda}),i} & \text{if } |C_{\underline{\lambda}}| \text{ is even.} \end{cases} \tag{31}$$

When we compute $\widehat{I}$, we will regroup characters $\Delta_{\underline{\lambda},i}$ in "slices" according to the order modulo e of the integer $b_{\underline{\lambda}}i$. First note that, as an additive group, we have

$$\mathbb{Z}/e\mathbb{Z} = \coprod_{q|e} \{\bar{k} \in \mathbb{Z}/e\mathbb{Z} \mid \operatorname{ord}(\bar{k}) = q\} = \coprod_{q|e} \left\{ \overline{\left(\frac{e}{q}s\right)} \mid 0 \leq s < q \text{ and } (s,q) = 1 \right\}.$$

Since, whenever $0 \leq s < q$, we have $0 \leq \frac{e}{q}s < e$, we actually obtain

$$\{0, \ldots, e-1\} = \coprod_{q|e} \left\{ \frac{e}{q}s \mid 0 \leq s < q \text{ and } (s,q) = 1 \right\}.$$

Our "slices" are described by the following.

Proposition 4.8. *For any $0 \leq k \leq e-1$, we let*

$$\mathcal{P}_{\underline{\gamma},\underline{w},k} = \{(\underline{\lambda}, i) \mid \underline{\lambda} \in \mathcal{E}_{\underline{\gamma},\underline{w}}\ ,\ 0 \leq i < |C_{\underline{\lambda}}| \text{ and } b_{\underline{\lambda}}i = k\}.$$

Let q be the order of k modulo e. Then the maps $\alpha\colon \mathcal{P}_{\underline{\gamma},\underline{w},k} \longrightarrow \mathcal{E}_{\underline{\gamma}/q,\underline{w}/q}$ and $\beta\colon \mathcal{E}_{\underline{\gamma}/q,\underline{w}/q} \longrightarrow \mathcal{P}_{\underline{\gamma},\underline{w},k}$ given by $\alpha((\underline{\lambda}, i)) = \underline{\lambda}/q$ and $\beta(\underline{\mu}) = (q\underline{\mu}, k/b_{\underline{\mu}})$ are mutually inverse bijections.

Remark 4.9. Recall that $\mathcal{E}_{\underline{\gamma}/q,\underline{w}/q}$ is exactly the set of multipartitions labelling the irreducible characters which belong to the p-block of $G(de/q, 1, r/q)$ with p-core $\underline{\gamma}/q$ and p-weight $\underline{w}/q$.

Proof. We start by showing that α and β are indeed defined.

If $(\underline{\lambda}, i) \in \mathcal{P}_{\underline{\gamma},\underline{w},k}$, then $b_{\underline{\lambda}} i = k$. Since $k = \frac{e}{q} s$ for some $0 \leq s < q$ with $(s,q) = 1$, we have $b_{\underline{\lambda}} i = \frac{e}{q} s$. Hence $qi = s\frac{e}{b_{\underline{\lambda}}} = s|C_{\underline{\lambda}}|$. Since $(s,q) = 1$, this shows that q divides $|C_{\underline{\lambda}}|$. Hence $\underline{\lambda}/q$ is indeed defined, and so are $\underline{\gamma}/q$ and $\underline{w}/q$, and $\underline{\lambda}/q$ certainly has p-core $\underline{\gamma}/q$ and p-weight $\underline{w}/q$. Thus the map $\alpha \colon \mathcal{P}_{\underline{\gamma},\underline{w},k} \longrightarrow \mathcal{E}_{\underline{\gamma}/q,\underline{w}/q}$ is defined.

If, on the other hand, $\underline{\mu} \in \mathcal{E}_{\underline{\gamma}/q,\underline{w}/q}$, then $q\underline{\mu}$ certainly is defined, and $q\underline{\mu} \in \mathcal{E}_{\underline{\gamma},\underline{w}}$. Furthermore, by definition, $b_{\underline{\mu}} = \frac{e/q}{|C_{\underline{\mu}}|}$, so that $b_{\underline{\mu}}$ divides e/q, and also $(e/q)s = k$, whence $k/b_{\underline{\mu}}$ is an integer. Moreover, we have $|C_{q\underline{\mu}}| = q.|C_{\underline{\mu}}|$, and, since $0 \leq k < e$, we have

$$0 \leq \frac{k}{b_{\underline{\mu}}} < \frac{e}{b_{\underline{\mu}}} = q\frac{e/q}{b_{\underline{\mu}}} = q.|C_{\underline{\mu}}| = |C_{q\underline{\mu}}|.$$

Finally, since $b_{\underline{\mu}} = \frac{e/q}{|C_{\underline{\mu}}|} = \frac{e}{q.|C_{\underline{\mu}}|} = \frac{e}{|C_{q\underline{\mu}}|} = b_{q\underline{\mu}}$, we have $b_{q\underline{\mu}}\frac{k}{b_{\underline{\mu}}} = k$, whence $(q\underline{\mu}, k/b_{\underline{\mu}})$ is defined, and $(q\underline{\mu}, k/b_{\underline{\mu}}) \in \mathcal{P}_{\underline{\gamma},\underline{w},k}$.

It only remains to show that α and β are mutual inverses. For any $\underline{\mu} \in \mathcal{E}_{\underline{\gamma}/q,\underline{w}/q}$, we have $(\alpha \circ \beta)(\underline{\mu}) = \alpha((q\underline{\mu}, k/b_{\underline{\mu}})) = q\underline{\mu}/q = \underline{\mu}$. And, for any $(\underline{\lambda}, i) \in \mathcal{P}_{\underline{\gamma},\underline{w},k}$, we have

$$(\beta \circ \alpha)((\underline{\lambda}, i)) = \beta(\underline{\lambda}/q) = (q\underline{\lambda}/q, \frac{k}{b_{\underline{\lambda}/q}}) = (\underline{\lambda}, \frac{k}{b_{\underline{\lambda}/q}}) = (\underline{\lambda}, \frac{k}{b_{\underline{\lambda}}})$$

(since, as we've seen above, $b_{\underline{\lambda}/q} = b_{\underline{\lambda}}$). Finally, since $(\underline{\lambda}, i) \in \mathcal{P}_{\underline{\gamma},\underline{w},k}$, we have $\frac{k}{b_{\underline{\lambda}}} = i$, whence $(\beta \circ \alpha)((\underline{\lambda}, i)) = (\underline{\lambda}, i)$. This concludes the proof.

□

Remark 4.10. Note that, with the notation of Proposition 4.8, if $(\underline{\lambda}, i) \in \mathcal{P}_{\underline{\gamma},\underline{w},k}$ and $k = b_{\underline{\lambda}} i$ has order q modulo e (written $\text{ord}_e(k) = q$), then i has order q modulo $e/b_{\underline{\lambda}} = |C_{\underline{\lambda}}|$ (written $\text{ord}_{|C_{\underline{\lambda}}|}(i) = q$). Hence there exists s such that $(s,q) = 1$ and $i = s|C_{\underline{\lambda}}|/q$. Suppose furthermore that $|C_{\underline{\lambda}}|$ is even. If q is even, then $(s,q) = 1$ implies that s is odd, so that $|C_{\underline{\lambda}}|/q$ and $i = s|C_{\underline{\lambda}}|/q$ have the same parity. If, on the other hand, q is odd, then, since $|C_{\underline{\lambda}}|$ is even, $|C_{\underline{\lambda}}|/q$ is even, and so is $i = s|C_{\underline{\lambda}}|/q$. This shows that, whenever $|C_{\underline{\lambda}}|$ is even, $|C_{\underline{\lambda}}|/q$ and i have the same parity. Now we have $\underline{\lambda}/q = (|C_{\underline{\lambda}}|/q) \cdot \underline{\lambda}/|C_{\underline{\lambda}}|$. Taking p-signs, we have $\delta_p(\underline{\lambda}/q) = \delta_p((|C_{\underline{\lambda}}|/q) \cdot \underline{\lambda}/|C_{\underline{\lambda}}|) = \delta_p(\underline{\lambda}/|C_{\underline{\lambda}}|)^{|C_{\underline{\lambda}}|/q}$. And, since $|C_{\underline{\lambda}}|/q$ and i have the same parity, we obtain

$$\delta_p(\underline{\lambda}/|C_{\underline{\lambda}}|)^i = \delta_p(\underline{\lambda}/q) \text{ whenever } (\underline{\lambda}, i) \in \mathcal{P}_{\underline{\gamma},\underline{w},k} \text{ and } \text{ord}_e(k) = q. \tag{32}$$

From this, we easily deduce the following.

Lemma 4.11. *If I is the map described in Definition 4.4 and $(\underline{\lambda}, i) \in \mathcal{P}_{\underline{\gamma},\underline{w},k}$, where $k = b_{\underline{\lambda}} i$ has order q modulo e, then $I(\Delta_{\underline{\lambda},i}) = \delta_p(\underline{\lambda}/q)\delta_p(\psi(\underline{\lambda})/q)\Delta_{\psi(\underline{\lambda}),i}$.*

Proof. By (31), we know that

$$I(\Delta_{\underline{\lambda},i}) = \begin{cases} \delta_p(\underline{\lambda})\delta_p(\psi(\underline{\lambda}))\Delta_{\psi(\underline{\lambda}),i} & \text{if } |C_{\underline{\lambda}}| \text{ is odd,} \\ \delta_p(\underline{\lambda})\delta_p(\psi(\underline{\lambda}))\delta_{\underline{\lambda}}^i\Delta_{\psi(\underline{\lambda}),i} & \text{if } |C_{\underline{\lambda}}| \text{ is even.} \end{cases}$$

If $|C_{\underline{\lambda}}| = |C_{\psi(\underline{\lambda})}|$ is odd, then q must be odd (since q divides $|C_{\underline{\lambda}}|$), so that $\delta_p(\underline{\lambda}) = \delta_p(q \cdot \underline{\lambda}/q) = \delta_p(\underline{\lambda}/q)^q = \delta_p(\underline{\lambda}/q)$ and $\delta_p(\psi(\underline{\lambda})) = \delta_p(q \cdot \psi(\underline{\lambda})/q) = \delta_p(\psi(\underline{\lambda})/q)^q = \delta_p(\psi(\underline{\lambda})/q)$. Hence, in this case, $\delta_p(\underline{\lambda})\delta_p(\psi(\underline{\lambda})) = \delta_p(\underline{\lambda}/q)\delta_p(\psi(\underline{\lambda})/q)$.

If, on the other hand, $|C_{\underline{\lambda}}| = |C_{\psi(\underline{\lambda})}|$ is even, then $\delta_p(\underline{\lambda}) = \delta_p(|C_{\underline{\lambda}}| \cdot \underline{\lambda}/|C_{\underline{\lambda}}|) = \delta_p(\underline{\lambda}/|C_{\underline{\lambda}}|)^{|C_{\underline{\lambda}}|} = 1$ and $\delta_p(\psi(\underline{\lambda})) = \delta_p(|C_{\underline{\lambda}}| \cdot \psi(\underline{\lambda})/|C_{\underline{\lambda}}|) = \delta_p(\psi(\underline{\lambda})/|C_{\underline{\lambda}}|)^{|C_{\underline{\lambda}}|} = 1$. And, by (32) (and since $(\psi(\underline{\lambda}), i) \in \mathcal{P}_{\underline{\gamma}', \underline{w}, k}$),

$$\delta^i_{\underline{\lambda}} = \delta_p(\underline{\lambda}/|C_{\underline{\lambda}}|)^i \delta_p(\psi(\underline{\lambda})/|C_{\psi(\underline{\lambda})}|)^i = \delta_p(\underline{\lambda}/q)\delta_p(\psi(\underline{\lambda})/q).$$

Hence, in this case, $\delta_p(\underline{\lambda})\delta_p(\psi(\underline{\lambda}))\delta^i_{\underline{\lambda}} = \delta_p(\underline{\lambda}/q)\delta_p(\psi(\underline{\lambda})/q)$. □

We can now state and prove our main result.

Theorem 4.12. *Take any positive integers d, e, r and r', and a prime p not dividing de. Let $G = G(de, 1, r)$, $N = G(de, e, r)$, $G' = G(de, 1, r')$ and $N' = G(de, e, r')$. Suppose b is a p-block of N, covered by the p-block B of G of p-core $\underline{\gamma}$ and p-weight $\underline{w}$, and b' is a p-block of N', covered by the p-block B' of G' of p-core $\underline{\gamma}'$ and p-weight $\underline{w}' = \underline{w}$. Then there is a perfect isometry between b and b'.*

Proof. First note that, if $\underline{w} = (0, \dots 0)$, then both b and b' are p-blocks of defect 0, so that $b = \{\chi\}$ and $b' = \{\chi'\}$ for some irreducible characters χ and χ' (of N and N' respectively) which vanish on p-singular elements. If we define $I\colon \mathbb{C}\,\mathrm{Irr}(b) \longrightarrow \mathbb{C}\,\mathrm{Irr}(b')$ by $I(\chi) = \chi'$, then, with the notation of Definition 4.6, we have $\widehat{I} = \overline{\chi} \otimes \chi'$. Since χ and χ' vanish on p-singular elements, we have $\widehat{I}(x, x') = \overline{\chi(x)}\chi'(x') \neq 0$ only if x and x' are both p-regular, so that property (2) of Definition 4.6 holds. Furthermore, since $b = \{\chi\}$ and $b' = \{\chi'\}$, χ and χ' are actually projective indecomposable characters (of N and N' respectively). Hence, by [7, Lemma (2.21)], for all $(x, x') \in N \times N'$, $\dfrac{\chi(x)}{|C_N(x)|_p} \in \mathcal{R}$ and $\dfrac{\chi'(x')}{|C_{N'}(x')|_p} \in \mathcal{R}$. Property (1) of Definition 4.6 immediately follows. This shows that, if $\underline{w} = (0, \dots 0)$, then b and b' are perfectly isometric.

We therefore now suppose that $\underline{w} \neq (0, \dots 0)$. Let $I\colon \mathbb{C}\,\mathrm{Irr}(b) \longrightarrow \mathbb{C}\,\mathrm{Irr}(b')$ be the map described in Definition 4.4. We will decompose $\widehat{I}$ using the $\mathbb{C}$-basis $\{\Delta_{\underline{\lambda},i},\ \underline{\lambda} \in [\mathcal{E}_{\underline{\gamma},\underline{w}}]$ and $0 \leq i < |C_{\underline{\lambda}}|\}$ for $\mathbb{C}\,\mathrm{Irr}(b)$. We have, by Definition 4.6, $e\widehat{I} = e \displaystyle\sum_{\underline{\lambda} \in [\mathcal{E}_{\underline{\gamma},\underline{w}}]} \sum_{i=0}^{|C_{\underline{\lambda}}|-1} \overline{\Delta^{\vee}_{\underline{\lambda},i}} \otimes I(\Delta_{\underline{\lambda},i})$, so that, by (30),

$$e\widehat{I} = e \sum_{\underline{\lambda} \in [\mathcal{E}_{\underline{\gamma},\underline{w}}]} \sum_{i=0}^{|C_{\underline{\lambda}}|-1} \frac{1}{|C_{\underline{\lambda}}|} \overline{\Delta_{\underline{\lambda},i}} \otimes I(\Delta_{\underline{\lambda},i}).$$

Since $|[\underline{\lambda}]| = b_{\underline{\lambda}}$ and $b_{\underline{\lambda}}|C_{\underline{\lambda}}| = e$, Corollary 4.2 gives

$$e\widehat{I} = \sum_{\underline{\lambda} \in \mathcal{E}_{\underline{\gamma},\underline{w}}} \frac{1}{b_{\underline{\lambda}}} \sum_{i=0}^{|C_{\underline{\lambda}}|-1} \frac{1}{|C_{\underline{\lambda}}|} \overline{\Delta_{\underline{\lambda},i}} \otimes I(\Delta_{\underline{\lambda},i}) = \sum_{\underline{\lambda} \in \mathcal{E}_{\underline{\gamma},\underline{w}}} \sum_{i=0}^{|C_{\underline{\lambda}}|-1} \overline{\Delta_{\underline{\lambda},i}} \otimes I(\Delta_{\underline{\lambda},i}).$$

Using our "slices", we obtain

$$\begin{aligned} e\widehat{I} &= \sum_{k=0}^{e-1} \sum_{(\underline{\lambda},i)\in\mathcal{P}_{\underline{\gamma},\underline{w},k}} \overline{\Delta_{\underline{\lambda},i}} \otimes I(\Delta_{\underline{\lambda},i}) \\ &= \sum_{q|e} \sum_{\substack{0\le k<e \\ \mathrm{ord}_e(k)=q}} \sum_{(\underline{\lambda},i)\in\mathcal{P}_{\underline{\gamma},\underline{w},k}} \overline{\Delta_{\underline{\lambda},i}} \otimes I(\Delta_{\underline{\lambda},i}) \\ &= \sum_{q|e} \sum_{\substack{0\le k<e \\ \mathrm{ord}_e(k)=q}} \sum_{(\underline{\lambda},i)\in\mathcal{P}_{\underline{\gamma},\underline{w},k}} \overline{\Delta_{\underline{\lambda},i}} \otimes \delta_p(\underline{\lambda}/q)\delta_p(\psi(\underline{\lambda})/q)\Delta_{\psi(\underline{\lambda}),i} \end{aligned}$$

(by Lemma 4.11).

Now take any $(x,x')\in N\times N'$. Write $x={}^g g_{\underline{\eta}}$ and $x'={}^{g'} g_{\underline{\eta}'}$, where $\underline{\eta}\in\mathcal{MP}_{r,de}$, $\underline{\eta}'\in\mathcal{MP}_{r',de}$, $g_{\underline{\eta}}\in N$ and $g_{\underline{\eta}'}\in N'$ are as in Convention 3.6, $g\in G$ and $g'\in G'$. Take any $0\le k\le e-1$, and $(\underline{\lambda},i)\in\mathcal{P}_{\underline{\gamma},\underline{w},k}$. We have $\varepsilon^k=\varepsilon^{b_{\underline{\lambda}}i}$ (since $(\underline{\lambda},i)\in\mathcal{P}_{\underline{\gamma},\underline{w},k}$). Hence, Proposition 3.12 gives

$$\Delta_{\underline{\lambda},i}(x)=\varepsilon^k(g)\Delta_{\underline{\lambda},i}(g_{\underline{\eta}}). \tag{33}$$

Similarly, since $(\psi(\underline{\lambda}),i)\in\mathcal{P}_{\underline{\gamma}',\underline{w},k}$, we have

$$\Delta_{\psi(\underline{\lambda}),i}(x')=\varepsilon^k(g')\Delta_{\psi(\underline{\lambda}),i}(g_{\underline{\eta}'}). \tag{34}$$

Now, if $\mathrm{ord}_e(k)=q$, then, for all $(\underline{\lambda},i)\in\mathcal{P}_{\underline{\gamma},\underline{w},k}$, we have, by Theorem 3.10,

$$\Delta_{\underline{\lambda},i}(g_{\underline{\eta}})=\Delta_{\underline{\lambda},|C_{\underline{\lambda}}|/q}(g_{\underline{\eta}}). \tag{35}$$

Furthermore, still supposing $\mathrm{ord}_e(k)=q$, Proposition 4.8 (applied twice) shows that

$$Q\colon \left\{\begin{array}{ccc} \mathcal{P}_{\underline{\gamma},\underline{w},k} & \longrightarrow & \mathcal{P}_{\underline{\gamma},\underline{w},e/q} \\ (\underline{\lambda},i) & \longmapsto & (\underline{\lambda},|C_{\underline{\lambda}}|/q) \end{array}\right. \quad \text{is a bijection.} \tag{36}$$

Similarly, for all $(\underline{\lambda},i)\in\mathcal{P}_{\underline{\gamma},\underline{w},k}$, we have, by Theorem 3.10,

$$\Delta_{\psi(\underline{\lambda}),i}(g_{\underline{\eta}'})=\Delta_{\psi(\underline{\lambda}),|C_{\underline{\lambda}}|/q}(g_{\underline{\eta}'}) \tag{37}$$

and, by Proposition 4.8,

$$Q'\colon \left\{\begin{array}{ccc} \mathcal{P}_{\underline{\gamma}',\underline{w},k} & \longrightarrow & \mathcal{P}_{\underline{\gamma}',\underline{w},e/q} \\ (\psi(\underline{\lambda}),i) & \longmapsto & (\psi(\underline{\lambda}),|C_{\underline{\lambda}}|/q) \end{array}\right. \quad \text{is a bijection.} \tag{38}$$

Note also that, by Proposition 4.8,

$$S\colon \left\{\begin{array}{ccc} \mathcal{P}_{\underline{\gamma},\underline{w},e/q} & \longrightarrow & \mathcal{E}_{\underline{\gamma}/q,\underline{w}/q} \\ (\underline{\lambda},|C_{\underline{\lambda}}|/q) & \longmapsto & \underline{\lambda}/q \end{array}\right. \quad \text{and} \tag{39}$$

$$S'\colon \left\{\begin{array}{ccc} \mathcal{P}_{\underline{\gamma}',\underline{w},e/q} & \longrightarrow & \mathcal{E}_{\underline{\gamma}'/q,\underline{w}/q} \\ (\psi(\underline{\lambda}),|C_{\underline{\lambda}}|/q) & \longmapsto & \psi(\underline{\lambda})/q=\psi(\underline{\lambda}/q) \end{array}\right. \quad \text{are bijections.}$$

Write $g_{\underline{\eta}}=(z;\sigma)$ with $\sigma=\sigma_1\cdots\sigma_s\in\mathfrak{S}_r$ and $g_{\underline{\eta}'}=(z';\sigma')$ with $\sigma'=\sigma_1'\cdots\sigma_{s'}'\in\mathfrak{S}_{r'}$ as in Convention 3.6. By Theorem 3.7, we see that, if there is $1\le u\le de-1$ such that q does not divide $|\eta_u|$, or if there is $1\le j\le s$ such that $\xi_j\notin\mathcal{U}_{de/q}$ (in which case we say that $g_{\underline{\eta}}$ is q-bad), then, for all $\underline{\lambda}\in\mathcal{E}_{\underline{\gamma},\underline{w}}$, we have $\Delta_{\underline{\lambda},|C_{\underline{\lambda}}|/q}(g_{\underline{\eta}})=0$. If, on the other hand, q divides $|\eta_u|$ for all $1\le u\le de-1$ and $\xi_j\in\mathcal{U}_{de/q}$ for all

$1 \leq j \leq s$ (in which case we say that $g_{\underline{\eta}}$ is q-good), then, with the notation of Remark 3.8, for all $\underline{\lambda} \in \mathcal{E}_{\underline{\gamma},\underline{w}}$, we have (by (27))

$$\Delta_{\underline{\lambda},|C_{\underline{\lambda}}|/q}(g_{\underline{\eta}}) = q^s \widetilde{\chi}_{\underline{\lambda}/q}(g_{\underline{\eta}}^{(q)}). \tag{40}$$

Similarly, if $g_{\underline{\eta}'}$ is q-bad, then, for all $\underline{\lambda} \in \mathcal{E}_{\underline{\gamma},\underline{w}}$, we have $\Delta_{\psi(\underline{\lambda}),|C_{\underline{\lambda}}|/q}(g_{\underline{\eta}'}) = 0$, while, if $g_{\underline{\eta}'}$ is q-good, then, with the notation of Remark 3.8, for all $\underline{\lambda} \in \mathcal{E}_{\underline{\gamma},\underline{w}}$, we have (by (27))

$$\Delta_{\psi(\underline{\lambda}),|C_{\underline{\lambda}}|/q}(g_{\underline{\eta}'}) = q^{s'} \widetilde{\chi}_{\psi(\underline{\lambda})/q}(g_{\underline{\eta}'}^{(q)}). \tag{41}$$

Recall that $e\widehat{I} = \sum_{q|e} \sum_{\substack{0\leq k<e \\ \mathrm{ord}_e(k)=q}} \sum_{(\underline{\lambda},i)\in\mathcal{P}_{\underline{\gamma},\underline{w},k}} \overline{\Delta_{\underline{\lambda},i}} \otimes \delta_p(\underline{\lambda}/q)\delta_p(\psi(\underline{\lambda})/q)\Delta_{\psi(\underline{\lambda}),i}$.

Writing $\delta_p^{\underline{\lambda},q}$ for $\delta_p(\underline{\lambda}/q)\delta_p(\psi(\underline{\lambda})/q)$ whenever $q|e$, we therefore have

$$e\widehat{I}(x,x') = \sum_{q|e} \sum_{\substack{0\leq k<e \\ \mathrm{ord}_e(k)=q}} \sum_{(\underline{\lambda},i)\in\mathcal{P}_{\underline{\gamma},\underline{w},k}} \delta_p^{\underline{\lambda},q}\overline{\Delta_{\underline{\lambda},i}(x)}\Delta_{\psi(\underline{\lambda}),i}(x').$$

By (33) and (34), this gives

$$e\widehat{I}(x,x') = \sum_{q|e} \sum_{\substack{0\leq k<e \\ \mathrm{ord}_e(k)=q}} \overline{\varepsilon^k(g)}\varepsilon^{k'}(g') \sum_{(\underline{\lambda},i)\in\mathcal{P}_{\underline{\gamma},\underline{w},k}} \delta_p^{\underline{\lambda},q}\overline{\Delta_{\underline{\lambda},i}(g_{\underline{\eta}})}\Delta_{\psi(\underline{\lambda}),i}(g'_{\underline{\eta}}).$$

By (35), (36), (37) and (38)), we obtain

$$e\widehat{I}(x,x') = \sum_{q|e} \sum_{\substack{0\leq k<e \\ \mathrm{ord}_e(k)=q}} \overline{\varepsilon^k(g)}\varepsilon^{k'}(g') \sum_{(\underline{\lambda},|C_{\underline{\lambda}}|/q)\in\mathcal{P}_{\underline{\gamma},\underline{w},e/q}} \delta_p^{\underline{\lambda},q}\overline{\Delta_{\underline{\lambda},|C_{\underline{\lambda}}|/q}(g_{\underline{\eta}})}\Delta_{\psi(\underline{\lambda}),|C_{\underline{\lambda}}|/q}(g'_{\underline{\eta}}).$$

Using (40) and (41)), we get

$$e\widehat{I}(x,x') = \sum_{\substack{q|e \\ g_{\underline{\eta}},g_{\underline{\eta}'} \\ q\text{-good}}} \sum_{\substack{0\leq k<e \\ \mathrm{ord}_e(k)=q}} \overline{\varepsilon^k(g)}\varepsilon^{k'}(g') \sum_{(\underline{\lambda},|C_{\underline{\lambda}}|/q)\in\mathcal{P}_{\underline{\gamma},\underline{w},e/q}} \delta_p^{\underline{\lambda},q} q^s\overline{\widetilde{\chi}_{\underline{\lambda}/q}(g_{\underline{\eta}}^{(q)})}q^{s'}\widetilde{\chi}_{\psi(\underline{\lambda})/q}(g_{\underline{\eta}'}^{(q)})$$

and, by (39), this finally gives

$$e\widehat{I}(x,x') = \sum_{\substack{q|e \\ g_{\underline{\eta}},g_{\underline{\eta}'} \\ q\text{-good}}} q^s q^{s'} \sum_{\substack{0\leq k<e \\ \mathrm{ord}_e(k)=q}} \overline{\varepsilon^k(g)}\varepsilon^{k'}(g') \sum_{\underline{\mu}\in\mathcal{E}_{\underline{\gamma}/q,\underline{w}/q}} \delta_p(\underline{\mu})\delta_p(\psi(\underline{\mu}))\overline{\widetilde{\chi}_{\underline{\mu}}(g_{\underline{\eta}}^{(q)})}\widetilde{\chi}_{\psi(\underline{\mu})}(g_{\underline{\eta}'}^{(q)}).$$

We can rewrite this as

$$e\widehat{I}(x,x') = \sum_{\substack{q|e \\ g_{\underline{\eta}},g_{\underline{\eta}'} \\ q\text{-good}}} q^s q^{s'} \sum_{\substack{0\leq k<e \\ \mathrm{ord}_e(k)=q}} \overline{\varepsilon^k(g)}\varepsilon^{k'}(g')\widehat{J}_{q,\underline{\gamma},\underline{\gamma}',\underline{w}}(g_{\underline{\eta}}^{(q)}, g_{\underline{\eta}'}^{(q)}), \tag{42}$$

where, for any q dividing e (and such that $\underline{\gamma}/q$, $\underline{\gamma}'/q$ and $\underline{w}/q$ are defined), $J_{q,\underline{\gamma},\underline{\gamma}',\underline{w}}$ is the perfect isometry described in [2, Theorem 5.4] between the p-block β of $G(de/q,1,r/q)$ with p-core $\underline{\gamma}/q$ and p-weight $\underline{w}/q$ and the p-block β' of $G(de/q,1,r'/q)$ with p-core $\underline{\gamma}'/q$ and (same) p-weight $\underline{w}/q$.

We now turn to Properties (1) and (2) of Definition 4.6.
Take any $q|e$ such that $g_{\underline{\eta}}$ and $g_{\underline{\eta}'}$ are q-good. Then $\underline{\eta} = (\eta_0,\ldots,\eta_{de-1}) \in \mathcal{MP}_{r,de}$, and $\eta_u \neq \emptyset$ only if q divides u. Furthermore, if $g_{\underline{\eta}} = (z;\sigma) \in G(de,1,r)$, then $g_{\underline{\eta}}^{(q)} = (z^{(q)};\sigma/q) \in G(de/q,1,r/q)$ has cycle type $(\eta_0/q, \eta_q/q, \ldots, \eta_{(de/q-1)q}/q) =$

$(\theta_0, \theta_1, \ldots, \theta_{de/q-1})$ (so that $\theta_i = \eta_{qi}/q$). Note that, since $q|e$ and $(p,e)=1$, p does not divide q.

Since $(p, de) = 1$, we have that $g_{\underline{\eta}}$ is p-singular if and only if $\sigma \in \mathfrak{S}_r$ is p-singular, i.e. if and only if σ has at least one cycle of length divisible by p. Since p does not divide q, this is equivalent to σ/q having at least one cycle of length divisible by p. Hence we obtain that $g_{\underline{\eta}}$ is p-singular if and only $g_{\underline{\eta}}^{(q)}$ is p-singular. Similarly, $g_{\underline{\eta'}}$ is p-singular if and only $g_{\underline{\eta'}}^{(q)}$ is p-singular.

By (42), if $\widehat{I}(x, x') \neq 0$, then there exists q dividing e such that $g_{\underline{\eta}}$ and $g_{\underline{\eta'}}$ are q-good, and $\widehat{J}_{q,\underline{\gamma},\underline{\gamma'},\underline{w}}(g_{\underline{\eta}}^{(q)}, g_{\underline{\eta'}}^{(q)}) \neq 0$. Since $J_{q,\underline{\gamma},\underline{\gamma'},\underline{w}}$ is a perfect isometry (by [2, Theorem 5.4]), this implies that $g_{\underline{\eta}}^{(q)}$ and $g_{\underline{\eta'}}^{(q)}$ are both p-regular or both p-singular, which, by the above, shows that $g_{\underline{\eta}}$ and $g_{\underline{\eta'}}$, and thus x and x', are both p-regular or both p-singular. Hence Property (2) of Definition 4.6 holds.

It remains to show that Property (1) holds. First note that, since $|G|/|N| = e$ is coprime to p, we have $|C_N(x)|_p = |C_G(x)|_p = |C_G(g_{\underline{\eta}})|_p$. Similarly, we have $|C_{N'}(x')|_p = |C_{G'}(g_{\underline{\eta'}})|_p$.

Now $|C_{G(de,1,r)}(g_{\underline{\eta}})| = \prod_{i,k} \eta_i^{\sharp k}!\,(kde)^{\eta_i^{\sharp k}}$, where $\eta_i^{\sharp k}$ is the number of k-cycles in η_i (see [4, Lemma 4.2.10]). Since all the cycles in any η_i have length divisible by q, and since $\eta_u \neq \emptyset$ only if q divides u, this can be rewritten as

$$|C_{G(de,1,r)}(g_{\underline{\eta}})| = \prod_{i,k} \eta_{qi}^{\sharp qk}!\,(qkde)^{\eta_{qi}^{\sharp qk}}.$$

However, by definition of the cycle type $(\theta_0, \theta_1, \ldots, \theta_{de/q-1})$ of $g_{\underline{\eta}}^{(q)}$, we have $\eta_{qi}^{\sharp qk} = (\eta_{qi}/q)^{\sharp k} = \theta_i^{\sharp k}$ for all i and k. Thus we obtain

$$\begin{aligned} |C_{G(de,1,r)}(g_{\underline{\eta}})| &= \prod_{i,k} \theta_i^{\sharp k}!\,(qkde)^{\theta_i^{\sharp k}} \\ &= \prod_{i,k} \theta_i^{\sharp k}!\,(q^2kde/q)^{\theta_i^{\sharp k}} \\ &= q^{2\sum_{i,k}\theta_i^{\sharp k}} \prod_{i,k} \theta_i^{\sharp k}!\,(kde/q)^{\theta_i^{\sharp k}} \\ &= q^{2s}|C_{G(de/q,1,r/q)}(g_{\underline{\eta}}^{(q)})| \qquad (\text{where } \sigma = \sigma_1\cdots\sigma_s) \end{aligned}$$

In particular, since $(p,q)=1$, we obtain

$$|C_N(x)|_p = |C_{G(de,1,r)}(g_{\underline{\eta}})|_p = |C_{G(de/q,1,r/q)}(g_{\underline{\eta}}^{(q)})|_p.$$

Similarly, we have $|C_{N'}(x')|_p = |C_{G(de,1,r')}(g_{\underline{\eta'}})|_p = |C_{G(de/q,1,r'/q)}(g_{\underline{\eta'}}^{(q)})|_p$.
From these, and from (42), we obtain

$$e\frac{\widehat{I}(x,x')}{|C_N(x)|_p} = \sum_{\substack{q|e \\ g_{\underline{\eta}},g_{\underline{\eta'}} \\ q\text{-good}}} q^s q^{s'} \sum_{\substack{0\le k<e \\ \mathrm{ord}_e(k)=q}} \overline{\varepsilon^k(g)}\varepsilon^{k'}(g')\frac{\widehat{J}_{q,\underline{\gamma},\underline{\gamma'},\underline{w}}(g_{\underline{\eta}}^{(q)}, g_{\underline{\eta'}}^{(q)})}{|C_{G(de/q,1,r/q)}(g_{\underline{\eta}}^{(q)})|_p}.$$

Since, for all q dividing e, $J_{q,\underline{\gamma},\underline{\gamma'},\underline{w}}$ is a perfect isometry (by [2, Theorem 5.4]), we have $\widehat{J}_{q,\underline{\gamma},\underline{\gamma'},\underline{w}}(g_{\underline{\eta}}^{(q)}, g_{\underline{\eta'}}^{(q)}) \in |C_{G(de/q,1,r/q)}(g_{\underline{\eta}}^{(q)})|_p\mathcal{R}$. Furthermore, the ring $\mathcal{R}$

contains the integers $q^s q^{s'}$, and the roots of unity $\overline{\varepsilon^k(g)}\varepsilon^{k'}(g')$. Hence $e\widehat{I}(x,x') \in |C_N(x)|_p\mathcal{R}$. Finally, since $(p,e)=1$, we obtain $\widehat{I}(x,x') \in |C_N(x)|_p\mathcal{R}$, as claimed. A similar argument shows that $\widehat{I}(x,x') \in |C_{N'}(x')|_p\mathcal{R}$, whence Property (1) of Definition 4.6 holds. This concludes the proof.

□

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