# Measuring financial exclusion of firms

**Abstract**

Financial ratios such as leverage or indices based on firm characteristics (Kaplan-Zingales, Whited-Wu, Hadlock-Pierce) have been used to measure whether a firm has too much debt. Let's assume a firm does not have any debt. Does this 'choice' reflect financial strength or exclusion? To measure the latter, this paper develops a theory to estimate the value of financial constraints. Based on a strictly concave production function, firms that face financial constraints take longer to reach their steady-state. This added time diminishes firm value, which translates into a shadow price of relaxing financial constraints.

*JEL codes*: G30, G32

*Keywords:* Impulse control, financial constraints, financial exclusion

## 1. Introduction

Improving firms' access to finance has become a major policy goal, and a growing literature on measuring financial inclusion has emerged (Chakravarty and Pal, 2013; Mailou et al., 2017; Sarma and Pais, 2011). Many measures rely on surveys but they are limited in terms of coverage and responses (e.g. binary variables) (World Bank, 2014).[1] Firm-level secondary data sources such as Compustat of-

[1] Binary responses include whether a firm has a bank account or not.

fer better coverage (e.g. number of firms and time periods). These data sources provide financial information, which can be used to derive ratios such as leverage or indices based on firm characteristics (e.g. Kaplan-Zingales, Whited-Wu) (Kaplan and Zingales, 1997; Whited and Wu, 2006). However, Farre-Mensa and Ljungqvist (2016) outline the empirical challenges of measuring financial constraints. Financial ratios are difficult to interpret as low levels of debt could signal financial exclusion (Kling, 2018). Recently, alternative methods have be considered such as textual analysis from annual reports to identify firms facing financial constraints (Buehlmaier and Whited, 2018; Hoberg and Maksimovic, 2015). Yet, these methods rely on firms' willingness to disclose their financial difficulties. Another empirical approach uses the reduction in firms profitability due to binding budget constraints, which limit the acquisition of inputs, to quantify financial constraints (Cherchye et al., 2020). Their empirical approach considers a set of inputs, derives optimal production and assesses the deviation from an optimal allocation of inputs due to a lack of funding.

This paper focuses on finance as the ultimate input that allows the purchase of all required inputs for production. It develops a model to quantify the value of financial constraints. The idea is that firms without access to finance cannot grow fast enough to reach their steady-state. This *growth constraint* has a value in the form of the shadow price $\lambda$. Firms can raise capital through debt or equity finance, which shifts budget constraints instantaneously. This 'lumpy' nature of external financing is captured using a deterministic impulse control framework.

An alternative theoretical approach to quantify 'lost growth opportunities'

refers to real option theory. A recent study in this strand of literature by Bolton et al. (2019) considers different growth phases of the firm. A real option approach requires the assumption that cash flows or earnings follow a geometric Brownian motion. Empirically and theoretically, this assumption can be challenged as it implicitly assumes that cash flows or earnings are exogenous, i.e. independent from management decisions (Kling, 2018). The model setting is discrete in the sense that there exists a growth phase and a mature phase. Firms switch from the growth to the mature phase after exercising an investment option. Hence, this theoretical approach does not derive a steady-state or a time into steady-state endogenously.

The main contribution of this paper is threefold. First, the paper solves a deterministic impulse control problem with state dependent continuous control constraints. Impulses shift the state variable, altering subsequent continuous control constraints, making established methods difficult to apply.[2] The insight is to express the functional as the present value of a steady state perpetuity shown in Theorem 1. This theorem can be applied to other theoretical models involving deterministic impulse controls. Second, Corollary 1 determines the value of financial constraints analytically for the production function $f(K) = K^{\alpha}$ with $0 < \alpha < 1$. Hence, the value of financial constraints refers to a shadow price $\lambda$, which is the value of relaxing the *growth constraint* through debt or equity issues. Corollary 1 could be used as an input for further theoretical work, requiring endogenous

[2]There are two approaches to solve deterministic impulse control problems: the maximum principle Fleming and Rishel (1975), Rempala and Zabczyk (1988) and quasi-variational inequalities Bensoussan and Lions (1980). Yet, many applications violate sufficient conditions Chahim et al. (2011).

measures of financial constraints. Third, an empirical specification of Corollary 1 could derive a measure of financial constraints, defined as 'slow growth into a steady-state', by estimating a production function.

## 2. The model

The model is set in a deterministic world without market frictions and continuous time. Equity holders maximize future discounted free cash flows paid to equity holders. Investment $I$ drives capital stock $K$, which determines gross cash flows through a strictly concave production function $f(K)$. Investment has to be financed through cash flows generated from existing capital stock, which determines control constraints captured in the feasible set $\Omega_I$. The optimal control problem is shown in (1). To simplify notation, we omit the argument time except if we refer to a specific point in time. Capital stock $K$ is the state variable, investment $I$ is the continuous control variable, $E(t_j)$ are impulse controls and $\delta(t_j - t)$ refer to Dirac delta functions, which take the value one if $t = t_j$ and zero otherwise. We follow Sethi and Thompson (2006) by treating debt and equity issues as impulse controls that affect the state variable, capital stock $K$, directly, and not via the control constraint of the continuous control variable. The equity holders can be regarded as entrepreneurs with limited funds financing the initial capital stock $K(0) = K_0$. They have monopolistic access to an idea or technology that allows them to translate capital stock $K$ into gross cash flows.

Impulses can occur at $n$ points. Costs associated with the magnitudes and timings of impulses affect the functional directly and also the feasible set $\Omega_I$ denoted

$c(E(t_1), \ldots, E(t_n), t_1, \ldots, t_n) \geq 0$, e.g. interest expenses. Capital stock cannot become negative as the minimum change is zero, the lower bound of the control constraint $\Omega_I$. Hence, we do not need an additional state inequality constraint.

$$\max_{\{I\}_0^\infty, \{E(t_j)\}_1^n} \left( V = \int_0^\infty (f(K) - I - c(E(t_1), \ldots, E(t_n), t_1, \ldots, t_n)) e^{-it} \mathrm{dt} \right)$$
$$\dot{K} = I + \sum_{j=1}^{n} \delta(t_j - t) E(t_j), K(0) = K_0$$
$$I \in \Omega_I = [0, f(K) - c(E(t_1), \ldots, E(t_n), t_1, \ldots, t_n)] \tag{1}$$

The idea is to rewrite problem (1) as a discounted stead-state cash flow. First, determine the optimal investment plan $\{I\}_0^\infty$ by setting $E(t_1) = \ldots = E(t_n) = 0$. Second, derive the steady state capital stock $\bar{K}$. Third, establish the optimal path of the state variable $K^*(t)$. Fourth, the time into steady state $\kappa$ follows from $K^*(\kappa) = \bar{K}$. Fifth, show that there is only one impulse at $t_1 = 0^+$. Finally, solve a standard optimization problem where the time into steady state $\kappa$ depends on the impulse $E_{0^+}$. Theorem 1 summarizes the main finding, and Appendix A provides the proof.

**Theorem 1.** *The optimization problem* (1) *can be rewritten as follows, where* $\kappa(E_{0^+})$ *is the time into steady state and* $\bar{K}$ *is the steady state capital stock.*

$$\max_{\{I\}_0^\infty, \{E(t_j)\}_1^n} V = \max_{E_{0^+}} \left( \frac{f(\bar{K} - c(E_{0^+}))}{i} e^{-i\kappa(E_{0^+})} \right), \; K(0) < \bar{K}$$

Theorem 1 generalizes the *Irrelevance Theorem* derived by Modigliani and

Miller (1958) by adding $e^{-i\kappa(E_{0^+})}$. Once the steady state is reached, i.e. $\kappa = 0$, the model converges to Modigliani and Miller (1958). If firms cannot jump into steady-state, i.e. $\kappa > 0$, access to finance has a value, which can be measured. Firms with capital stocks below their steady state have an incentive to invest their internal cash flow as the marginal return on capital exceeds the cost of equity. Hence, the *growth constraint* is binding, and relaxing this constraint through debt or equity issues adds value.

## 3. The value of financial constraints

Using standard methods (e.g. translog cost model), $f(K)$ can be estimated, which then permits an estimate of $\lambda$ based on (A.6) and (A.7). This shadow price gives the value of relaxing the *growth constraint*.This is a different measure compared to financial ratios as it focuses on the lack of opportunity to grow quickly. To illustrate this approach, we select $f(K) = K^{\alpha}$ with $0 < \alpha < 1$, which provides a closed from solution of the value of financial constraints denoted $\lambda(t)$. Appendix B provides the proof.

**Corollary 1.** *Setting $f(K) = K^{\alpha}$ with $0 < \alpha < 1$ yields an analytic solution of the value of financial constraints.*

$$\lambda(t) = \left(\frac{\bar{K}}{K^*(t)}\right)^{\alpha} e^{i(t-\kappa)} - 1, \quad \textit{for } 0 \leq t \leq \kappa \tag{2}$$

## 4. Conclusion

Traditionally, the finance literature has adopted two definitions of financial constraints. First, the inelasticity of the supply of capital suggests that constrained firms face high cost of capital for additional borrowing (Stiglitz and Weiss, 1981; Whited and Wu, 2006). Second, in line with Fazzari et al. (1987) financial constraints can be measured by the difference between firms' opportunity cost of internal capital and their cost of external capital. This paper follows an alternative perspective focusing on measuring the slowness of firms' growth due to their lack of access to external finance. This idea is related to recent empirical work by Cherchye et al. (2020); however, their approach considers a reduction in firms' profitability due to their inability to acquire inputs as a measure of financial constraints.

Using a deterministic impulse control setup, this paper derives a measure of firms' financial exclusion. Firms face a *growth constraint* if they do not have sufficient access to external finance; hence, they cannot reach their steady-state instantaneously. This delay results in a loss in firm value as indicated by Theorem 1. Hence, the model can estimate the value of financial constraints based on the shadow price $\lambda$. Empirically, theoretical predictions could be tested based on Corollary 1, which assumes a simplified production function $f(K) = K^{\alpha}$ with $0 < \alpha < 1$. Taking logs on both sides provides a suitable regression equation, which can identify $\alpha$. As derived in the proof of Corollary in Appendix B, the steady-state capital stock can be estimated using cost of capital denoted $i$ and the

estimated $\alpha$. The time into steady state $\kappa$ follows from equation B.2. Hence, the shadow price can be estimated using the current capital stock as a proxy for $K^*(t)$.

As the main aim is to derive a closed-form expression for financial constraints, assumptions have to be imposed. Obviously, the true production function does deviate from $f(K) = K^\alpha$, and market frictions (e.g. tax shields) have implications for the steady-state capital stock depending on whether firms have access to debt finance. However, Theorem 1 holds in general for any deterministic impulse control problem as defined in equation 1. A closed-form expression for financial constraints is only possible if assumptions are made as in Corollary 1. Future research could explore empirical tests of Corollary 1, which might offer an alternative measure of financial exclusion.

## Appendix A. Proof of Theorem 1

Setting $E(t_1) = \ldots = E(t_n) = 0$ implying $c(E(t_1), \ldots, E(t_n), t_1, \ldots, t_n) = 0$ in (1), the current value Hamiltonian $\mathcal{H}$ (A.1) is linear in the control variable $I$, the solution is bang-bang limited by control constraints and triggered by the shadow price $\Psi$.

$$\mathcal{H} = f(K) - I + \Psi I \tag{A.1}$$

Growth occurs if $\Psi(0) > 1$ as $d\mathcal{H}/dI > 0$ implying that the firm has not reached its steady-state yet. In the steady-state at time $t = \kappa$, growth stops, i.e.

$\Psi(\kappa) = 1$. Thus, the optimal investment policy refers to (A.2).

$$I^*(t) = \begin{cases} f(K) & \text{for } 0 \leq t < \kappa \\ \text{undefined} & \text{for } t \geq \kappa \end{cases} \tag{A.2}$$

The optimal time path of capital stock $K^*(t)$ follows from $\frac{dK}{dt} = I^*$.

$$\int \frac{1}{f(K)} \, \mathrm{d}K = t, \text{ for } 0 \leq t < \kappa \tag{A.3}$$

We obtain the steady state capital stock $\bar{K}$ from (A.4) setting $\Psi(\kappa) = 1$ and $\dot{\Psi}(\kappa) = 0$.

$$\left.\frac{\partial \mathcal{H}}{\partial K}\right|_{t=\kappa} = f'(\bar{K}) = -\dot{\Psi}(\kappa) + i\Psi(\kappa) = i \tag{A.4}$$

Using the steady state capital stock $\bar{K}$ from (A.4) and the initial capital stock $K(0) = K_0$, we solve (A.3) and determine the time into steady state $\kappa$. After establishing the optimal time paths of investment and capital stock and determining the time to steady state $\kappa$, we use the full Lagrangian $L$ to derive the time paths of shadow prices by converting the control constraints $\Omega_I$ into inequality constraints. The inequality constraint of the lower bound is not binding as $K_0 < \bar{K}$; hence, it is dropped. The Lagrangian (A.5), optimality condition (A.6) and costate equation

(A.7) are as follows.

$$L = f(K) - I + \Psi I + \lambda(f(K) - I) \tag{A.5}$$

$$\frac{\partial L}{\partial I} = -1 + \Psi - \lambda = 0 \tag{A.6}$$

$$\frac{\partial L}{\partial K} = f'(K) + \lambda f'(K) = -\dot{\Psi} + i\Psi \tag{A.7}$$

From the optimality condition (A.6), we derive the shadow price $\lambda(t)$ expressed in terms of $\Psi(t)$. The shadow price $\lambda(t)$ measures the marginal benefit of relaxing the *growth constraint*.

Relaxing the condition $E(t_1) = \ldots = E(t_n) = 0$, only one impulse at $t_1 = 0^+$ occurs as $\frac{\partial \lambda}{\partial t} < 0$, which implies that the best time to relax the *growth constraint* is at $t_1 = 0^+$. From (A.6) $\lambda = \Psi - 1$ into (A.7) gives $\dot{\Psi} = \Psi(i - f'(K)) < 0$ for $t < \kappa \Rightarrow \frac{\partial \lambda}{\partial t} < 0$. Hence, there is one impulse $E(0^+)$ at $t = 0^+$, which shifts the initial condition $K(0^+) = K(0) + E(0^+)$ reducing the time into steady state $\kappa(E(0^+))$. Hence, deriving optimality conditions under the assumption $E(t_1) = \ldots = E(t_n) = 0$ is valid as only one impulse occurs, and this impulse can be regarded as a change in the initial condition $K(0^+)$. Optimal investment $I^*$ in (A.2) refers to a step function; hence, the functional $V = 0$ for $t < \kappa$ and at $t = \kappa$ the steady state is reached resulting in a perpetuity discounted back to $t = 0$. □

## Appendix B. Proof of Corollary 1

First, we solve (A.3) with initial condition $K^*(0) = K_0$.

$$\int \frac{1}{K^\alpha}\,\mathrm{d}K = t$$
$$K^*(t) = \left[(1-\alpha)t + K_0^{1-\alpha}\right]^{\frac{1}{1-\alpha}} \tag{B.1}$$

Second set $\Psi = 1$ and $\dot{\Psi} = 0$ in (A.4) to derive the steady-state $\bar{K}$.

$$\alpha \bar{K}^{\alpha-1} = i$$
$$\bar{K} = \left(\frac{i}{\alpha}\right)^{\frac{1}{\alpha-1}}$$

Third, assess the time into steady state $\kappa$ by setting $K^*(\kappa) = \bar{K}$.

$$\left(\frac{i}{\alpha}\right)^{\frac{1}{\alpha-1}} = \left[(1-\alpha)\kappa + K_0^{1-\alpha}\right]^{\frac{1}{1-\alpha}}$$
$$(1-\alpha)\kappa + K_0^{1-\alpha} = \frac{\alpha}{i}$$
$$\kappa = \frac{\alpha - iK_0^{1-\alpha}}{i(1-\alpha)} \tag{B.2}$$

Fourth, derive $\Psi(t)$ using (A.7) and (A.6), i.e. $\lambda = \Psi - 1$, where $C$ is an arbitrary

constant.

$$\Psi f_K = -\dot{\Psi} + i\Psi$$
$$\frac{\dot{\Psi}}{\Psi} = i - \alpha K^*(t)^{\alpha-1}$$
$$ln\Psi = it - \int \alpha K^*(t)^{\alpha-1}\mathrm{dt}$$

Using (B.1), we can write $K^*(t)^{\alpha-1}$ as a function of time.

$$ln\Psi = it - \alpha \int \frac{1}{(1-\alpha)t + K_0^{1-\alpha}}\mathrm{dt}$$
$$ln\Psi = it - \alpha \frac{\ln\left((1-\alpha)t + K_0^{1-\alpha}\right)}{(1-\alpha)} + C$$
$$ln\Psi = it - \frac{\alpha}{1-\alpha}\ln\left((1-\alpha)t + K_0^{1-\alpha}\right) + C$$

Fifth, express $\Psi$ in terms of $K^*(t)$.

$$ln\Psi = it - \frac{\alpha}{1-\alpha}\ln\left(K^*(t)\right)^{1-\alpha} + C$$
$$ln\Psi = it - \alpha \ln K^*(t) + C$$

To determine the constant $C$, we use the conditions in the steady state $\Psi = 1$, $t = \kappa$ and $K^*(\kappa) = \bar{K}$.

$$C = \alpha \ln \bar{K} - i\kappa$$

Finally, we obtain a closed-form solution for the time path of the shadow price $\Psi(t)$ and $\lambda(t)$.

$$
\begin{aligned}
ln\Psi &= i(t-\kappa) - \alpha\left(\ln K^*(t) - \ln \bar{K}\right) \\
ln\Psi &= i(t-\kappa) - \alpha \ln\left(\frac{K^*(t)}{\bar{K}}\right) \\
\Psi^*(t) &= \lambda(t) + 1 = e^{i(t-\kappa)}\left(\frac{\bar{K}}{K^*(t)}\right)^{\alpha} \qquad \square
\end{aligned}
\tag{B.3}
$$

## References


Bensoussan, A., Lions, J., 1980. Controle impulsionnel et inequations quasi- variationneltes. Dunod: Paris, France.

Bolton, P., Wang, N., Yang, J., 2019. Investment under uncertainty with financial constraints. Journal of Economic Theory 184, 104912.

Buehlmaier, M. M., Whited, T. M., 2018. Are financial constraints priced? evidence from textual analysis. The Review of Financial Studies 31 (7), 2693–2728.

Chahim, M., Hartl, R., Kort, P., 2011. The deterministic impulse control maximum principle in operations research: Necessary and sufficient optimality conditions.

Chakravarty, S. R., Pal, R., 2013. Financial inclusion in india: An axiomatic approach. Journal of Policy Modeling 35 (5), 813–837.

Cherchye, L., De Rock, B., Ferrando, A., Mulier, K., Verschelde, M., et al., 2020. Identifying financial constraints. Tech. rep., ULB–Universite Libre de Bruxelles.

Farre-Mensa, J., Ljungqvist, A., 2016. Do measures of financial constraints measure financial constraints? The Review of Financial Studies 29 (2), 271–308.

Fazzari, S., Hubbard, R. G., Petersen, B. C., 1987. Financing constraints and corporate investment. Tech. rep., National Bureau of Economic Research.

Fleming, W., Rishel, R., 1975. Deterministic and stochastic optimal control. Springer Verlag: Berlin, Germany.

Hoberg, G., Maksimovic, V., 2015. Redefining financial constraints: A text-based analysis. The Review of Financial Studies 28 (5), 1312–1352.

Kaplan, S. N., Zingales, L., 1997. Do investment-cash flow sensitivities provide useful measures of financing constraints? The Quarterly Journal of Economics 112 (1), 169–215.

Kling, G., 2018. A theory of operational cash-holding, endogenous financial constraints, and credit rationing. European Journal of Finance 24, 59–75.

Mailou, A., Amidzic, G., Massara, A., 2017. Assessing countries' financial inclusion standing: A new composite index. Journal of Banking and Financial Economics 2, 105–126.

Modigliani, F., Miller, M., 1958. The cost of capital, corporation finance, and the theory of investment. American Economic Review 48, 261–297.

Rempala, R., Zabczyk, J., 1988. On the maximum principle for deterministic impulse control problems. Journal of Optimization Theory and Applications 59, 281–288.

Sarma, M., Pais, J., 2011. Financial inclusion and development. Journal of International Development 23 (5), 613–628.

Sethi, S., Thompson, G., 2006. Optimal control theory: Applications to management science and economics. Berlin: Springer.

Stiglitz, J. E., Weiss, A., 1981. Credit rationing in markets with imperfect information. The American Economic Review 71 (3), 393–410.

Whited, T. M., Wu, G., 2006. Financial constraints risk. The Review of Financial Studies 19 (2), 531–559.

World Bank, 2014. Global financial development report 2014: Financial inclusion. Washington: World Bank.