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<tr>
<td>Complete List of Authors:</td>
<td>Archbold, Robert J.; University of Aberdeen, Institute of Mathematics, King's College</td>
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<td></td>
<td>Gogic, Ilja; Sveuciliste u Zagrebu Prirodoslovno-matematicki fakultet, Department of Mathematics</td>
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The centre-quotient property and weak centrality for $C^*$-algebras

Robert J. Archbold$^1$ and Ilja Gogić$^2$

$^1$Institute of Mathematics, University of Aberdeen, King’s College, Aberdeen AB24 3UE, Scotland, United Kingdom

$^2$Department of Mathematics, Faculty of Science, University of Zagreb, Bijenička 30, 10000 Zagreb, Croatia

Correspondence to be sent to: ilja@math.hr

We give a number of equivalent conditions (including weak centrality) for a general $C^*$-algebra to have the centre-quotient property. We show that every $C^*$-algebra $A$ has a largest weakly central ideal $J_{wc}(A)$. For an ideal $I$ of a unital $C^*$-algebra $A$, we find a necessary and sufficient condition for a central element of $A/I$ to lift to a central element of $A$. This leads to a characterisation of the set $V_A$ of elements of an arbitrary $C^*$-algebra $A$ which prevent $A$ from having the centre-quotient property. The complement $\text{CQ}(A) := A \setminus V_A$ always contains $Z(A) + J_{wc}(A)$ (where $Z(A)$ is the centre of $A$), with equality if and only if $A/J_{wc}(A)$ is abelian. Otherwise, $\text{CQ}(A)$ fails spectacularly to be a $C^*$-subalgebra of $A$.

1 Introduction

Let $A$ be a $C^*$-algebra with centre $Z(A)$. If $I$ is a closed two-sided ideal of $A$, it is immediate that

$$
(Z(A) + I)/I = q_I(Z(A)) \subseteq Z(A/I), \tag{1.1}
$$

where $q_I : A \to A/I$ is the canonical map. A $C^*$-algebra $A$ is said to have the centre-quotient property ([44], [4, Section 2.2] and [8, p. 2671]) if for any closed two-sided ideal $I$ of $A$, equality holds in (1.1). For the sake of brevity we shall usually refer to the centre-quotient property as the $CQ$-property.

In 1971, Vesterstrøm [44] proved the following theorem.

Theorem 1.1 (Vesterstrøm). If $A$ is a unital $C^*$-algebra, then the following conditions are equivalent:

(i) $A$ has the $CQ$-property.

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(ii) $A$ is weakly central, that is for any pair of maximal ideals $M$ and $N$ of $A$, $M \cap Z(A) = N \cap Z(A)$ implies $M = N$.

Weakly central $C^*$-algebras were introduced by Misonou and Nakamura in [30, 37] in the unital context. The most prominent examples of weakly central $C^*$-algebras $A$ are those satisfying the Dixmier property, that is for each $x \in A$ the closure of the convex hull of the unitary orbit of $x$ intersects $Z(A)$ [21, p. 275]. In particular, von Neumann algebras are weakly central (see [20, Théorème 7] and [37, Theorem 3]). It was shown by Haagerup and Zsidó in [26] that a unital simple $C^*$-algebra satisfies the Dixmier property if and only if it admits at most one tracial state. In particular, weak centrality does not imply the Dixmier property. However, in [35] Magajna gave a characterisation of weak centrality in terms of averaging involving unital completely positive elementary operators. Recently, Robert, Tikuisis and the first-named author found the exact gap between weak centrality and the Dixmier property for unital $C^*$-algebras [8, Theorem 2.6] and showed that a postliminal $C^*$-algebra has the (singleton) Dixmier property if and only if it has the CQ-property [8, Theorem 2.12]. Also, in a recent paper [15], Brešar and the second-named author studied an analogue of the CQ-property in a wider algebraic setting (so called ‘centrally stable algebras’).

In this paper we study weak centrality, the CQ-property and several equivalent conditions for general $C^*$-algebras that are not necessarily unital. We then investigate the failure of weak centrality in two different ways. Firstly, we show that every $C^*$-algebra $A$ has a largest weakly central ideal $J_{wc}(A)$, which can be readily determined in several examples. Secondly, we study the set $V_A$ of individual elements of $A$ which prevent the weak centrality (or the CQ-property) of $A$. The set $V_A$ is contained in the complement of $J_{wc}(A)$ and, in certain cases, is somewhat smaller than one might expect. In the course of this, we address a fundamental lifting problem that is closely linked to the CQ-property: for a fixed ideal $I$ of a unital $C^*$-algebra $A$, we find a necessary and sufficient condition for a central element of $A/I$ to lift to a central element of $A$.

The paper is organised as follows. After some preliminaries in Section 2, the main results are obtained in Section 3 and Section 4. In Section 3, we study weak centrality and the CQ-property for arbitrary $C^*$-algebras. In the non-unital context, the appropriate maximal ideals for the definition of weak centrality are the modular maximal ideals (Definition 3.5). In Theorem 3.16, we give a number of conditions (including the CQ-property) that are equivalent to the weak centrality of a $C^*$-algebra $A$. In Theorem 3.22, we show that every $C^*$-algebra $A$ has a largest ideal $J_{wc}(A)$ that is weakly central. In doing so, we obtain a formula for $J_{wc}(A)$ in terms of the set $T_A$ of those modular maximal ideals of $A$ which witness the failure of the weak centrality of $A$. This formula leads easily to the explicit description of $J_{wc}(A)$ in a number of examples. For example, $J_{wc}(A) = \{0\}$ when either $A$ is the rotation algebra (the $C^*$-algebra of the discrete three-dimensional Heisenberg group, Example 3.24) or $A = C^*(\mathbb{F}_2)$ (the full $C^*$-algebra of the free group on two generators, Example 3.25), and $J_{wc}(A) = K(H)$ for Dixmier’s classic example of a $C^*$-algebra in which the Dixmier property fails (Example 3.28). We also obtain
the stability of weak centrality and the CQ-property in the context of arbitrary $C^*$-tensor products (Theorem 3.29).

In Section 4, we undertake the more difficult task of describing the individual elements which prevent a $C^*$-algebra $A$ from having the CQ-property. We say that an element $a \in A$ is a CQ-element if for every closed two-sided ideal $I$ of $A$, $a + I \in Z(A/I)$ implies $a \in Z(A) + I$ (Definition 4.1). We denote by $\text{CQ}(A)$ the set of all CQ-elements of $A$.

Clearly, $A$ has the CQ-property if and only if $\text{CQ}(A) = A$ and the complement $V_A := A \setminus \text{CQ}(A)$ is precisely the set of elements which prevent the CQ-property for $A$. For an ideal $I$ of a unital $C^*$-algebra $A$, we use the complete regularization map, the Tietze extension theorem and the Dauns-Hofmann theorem to obtain a necessary and sufficient condition for a central element of $A/I$ to lift to a central element of $A$ (Theorem 4.7). This then leads to a description of $V_A$ (and hence $\text{CQ}(A)$) for an arbitrary $C^*$-algebra $A$ in terms of the subset $T_A$ (Theorem 4.8).

We show that $\text{CQ}(A)$ contains $Z(A) + J_{wc}(A)$ (Corollary 4.4), all commutators $[a,b]$ ($a,b \in A$) and all products $ab$, $ba$ where $a \in A$ and $b$ is a quasi-nilpotent element of $A$ (Proposition 4.5). It follows from this, together with a result of Pop [40, Theorem 1], that if $A$ is not weakly central (so that $\text{CQ}(A) \neq A$), $\text{CQ}(A)$ contains the norm-closure $[A,A]$ of $[A,A]$ (the linear span of all commutators in $A$) if and only if all quotients $A/M$ ($M \in T_A$) admit tracial states (Theorem 4.10). In particular, if $A$ is postliminal or an AF-algebra, then $[A,A] \subseteq \text{CQ}(A)$ (Corollary 4.11). On the other hand, if the tracial condition is not satisfied, then $\text{CQ}(A)$ does not even contain $[A,A]$ (Theorem 4.10 (b)).

Further, we show that for any $C^*$-algebra $A$ the following conditions are equivalent:

(i) $A/J_{wc}(A)$ is abelian.
(ii) $\text{CQ}(A) = Z(A) + J_{wc}(A)$.
(iii) $\text{CQ}(A)$ is closed under addition.
(iv) $\text{CQ}(A)$ is closed under multiplication.
(v) $\text{CQ}(A)$ is norm-closed.

(Theorem 4.12). If $A$ is postliminal or an AF-algebra, then the conditions (i)-(v) are also equivalent to the condition

(vi) For any $x \in \text{CQ}(A)$, $x^n \in \text{CQ}(A)$ for all positive integers $n$.

(Corollary 4.16). We also show that (vi) does not have to imply (i)-(v) for general (separable nuclear) $C^*$-algebras (Example 4.21). The methods for these results involve the lifting of nilpotent elements, commutators and simple projectionless $C^*$-algebras.

We finish with an example of a separable continuous-trace $C^*$-algebra $A$ for which $J_{wc}(A) = Z(A) = \{0\}$ but $\text{CQ}(A)$ is norm-dense in $A$ (Example 4.25). In other words, although no non-zero ideal of $A$ has the CQ-property, the set $V_A$ of elements which prevent the CQ-property of $A$ has empty interior in $A$. 
2 Preliminaries

Throughout this paper $A$ will be a $C^*$-algebras with the centre $Z(A)$. By $S(A)$ we denote the set of all states on $A$. As usual, if $x, y \in A$ then $[x, y]$ stands for the commutator $xy - yx$. If $A$ is non-unital, we denote the (minimal) unitization of $A$ by $A^\sharp$. If $A$ is unital we assume that $A^\sharp = A$.

By an ideal of $A$ we shall always mean a closed two-sided ideal. If $X$ is a subset of $A$, then $\text{Id}_A(X)$ denotes the ideal of $A$ generated by $X$. An ideal $I$ of $A$ is said to be modular if the quotient $A/I$ is unital. If $I$ is an ideal of $A$ then it is well-known that $Z(I) = I \cap Z(A)$.

The set of all primitive ideals of $A$ is denoted by $\text{Prim}(A)$. As usual, we equip $\text{Prim}(A)$ with the Jacobson topology. It is well-known that any proper modular ideal of $A$ (if such exists) is contained in some modular maximal ideal of $A$ (see e.g. [30, Lemma 1.4.2]) and that all modular maximal ideals of $A$ are primitive. We denote the set of all modular maximal ideals of $A$ by $\text{Max}(A)$, so that $\text{Max}(A) \subseteq \text{Prim}(A)$. Note that $\text{Max}(A)$ can be empty (e.g. the algebra $A = K(\mathcal{H})$ of compact operators on a separable infinite-dimensional Hilbert space $\mathcal{H}$).

**Remark 2.1.** Every non-modular primitive ideal of a $C^*$-algebra $A$ contains $Z(A)$. Indeed, if $z \in Z(A)$ then for $P \in \text{Prim}(A)$, $z + P \in Z(A/P)$ and so is either zero or a multiple of the identity in $A/P$ if $A/P$ has one. Therefore, if $P$ is non-modular, we have $z \in P$ for all $z \in Z(A)$, so $Z(A) \subseteq P$. In particular, if the set of all non-modular primitive ideals of $A$ is dense in $\text{Prim}(A)$, then $Z(A) = \{0\}$.

For any subset $S \subseteq \text{Prim}(A)$ we define its kernel $\text{ker} S$ as the intersection of all elements of $S$. For the case $S = \emptyset$, we define $\text{ker} S = A$. Note that $S$ is closed in $\text{Prim}(A)$ if and only if for any $P \in \text{Prim}(A)$, $\text{ker} S \subseteq P$ implies $P \in S$.

For any ideal $I$ of $A$ we define the following two subsets of $\text{Prim}(A)$:

$$\text{Prim}_I(A) := \{P \in \text{Prim}(A) : I \not\subseteq P\} \quad \text{and} \quad \text{Prim}^I(A) := \{P \in \text{Prim}(A) : I \subseteq P\}.$$  

Then $\text{Prim}_I(A)$ is an open subset of $\text{Prim}(A)$ homeomorphic to $\text{Prim}(I)$ via the map $P \mapsto P \cap I$, while $\text{Prim}^I(A)$ is a closed subset of $\text{Prim}(A)$ homeomorphic to $\text{Prim}(A/I)$ via the map $P \mapsto P/I$ (see e.g. [41, Proposition A.27]).

Similarly, we introduce the following subsets of $\text{Max}(A)$:

$$\text{Max}_I(A) := \{M \in \text{Max}(A) : I \not\subseteq M\} \quad \text{and} \quad \text{Max}^I(A) := \{M \in \text{Max}(A) : I \subseteq M\}.$$  

We shall frequently use the next simple fact which is probably well-known but as we have been unable to find a reference we include a proof for completeness.

**Lemma 2.2.** Let $A$ be a $C^*$-algebra and let $I$ be an arbitrary ideal of $A$. Then the assignment $M \mapsto M \cap I$ defines a homeomorphism from the set $\text{Max}_I(A)$ onto the set $\text{Max}(I)$.

\[ \square \]
By the Dauns-Hofmann theorem [41, Theorem A.34], there exists an isomorphism 
\[ P, Q \] next. For all \( A \) if a well-defined continuous surjection. We shall continue to assume that 
\[ Z \] of 
\[ P, Q \in Z \] for all 
\[ A \approx / \] follows there is one-to-one correspondence between the quotient set \( \operatorname{Prim}(A) \approx / \) follows and a set of ideals of \( A \) given by

\[ [P]_\approx \leftrightarrow \ker[P]_\approx \quad (P \in \operatorname{Prim}(A)), \]

**Proof.** For \( M \in \operatorname{Max}_*(A) \) set \( \psi(M) := M \cap I. \)

Let \( M \in \operatorname{Max}_*(A) \). Since \( I \not\subseteq M \), by maximality and modularity of \( M \) we have \( I + M = A \) and \( A/M \) is a simple unital \( C^* \)-algebra. Using the canonical isomorphism \( I/(M \cap I) \cong (I + M)/M = A/M \), we conclude that 
\[ I/(M \cap I) \] is also a simple unital \( C^* \)-algebra, so \( \psi(M) = M \cap I \in \operatorname{Max}(I) \).

The injectivity of the map \( \psi \) follows directly from the injectivity of the assignment \( \operatorname{Prim}_*(A) \to \operatorname{Prim}(I) \), 
\[ P \mapsto P \cap I, \] and the fact that all modular maximal ideals of \( A \) are primitive.

To show the surjectivity of \( \psi \), choose an arbitrary \( N \in \operatorname{Max}(I) \). Then there is \( M \in \operatorname{Prim}_*(A) \) such that 
\[ N = M \cap I. \] Since \( I/N \cong (I + M)/M \), it follows that \( (I + M)/M \) is a unital ideal of the primitive \( C^* \)-algebra \( A/M \). This forces \( (I + M)/M = A/M \), so \( A/M \) is a unital simple \( C^* \)-algebra. Thus, \( M \in \operatorname{Max}_*(A) \).

Finally, since \( \psi \) is a restriction of a canonical homeomorphism \( \operatorname{Prim}_*(A) \to \operatorname{Prim}(I) \) on \( \operatorname{Max}_*(A) \) with the image \( \operatorname{Max}(I) \), it is itself a homeomorphism. \qed

**Remark 2.3.** It is also easy to see that for any ideal \( I \) of a \( C^* \)-algebra \( A \), the assignment \( M \mapsto M/I \) defines a homeomorphism from the set \( \operatorname{Max}_*(A) \) onto the set \( \operatorname{Max}(A/I) \), but we shall not use this fact in this paper. \( \square \)

If \( A \) is a \( C^* \)-algebra and \( P \) a primitive ideal of \( A \) such that \( Z(A) \not\subseteq P \), then \( P \cap Z(A) \) is a maximal ideal of \( Z(A) \). In particular, if \( A \) is unital, then the map

\[ \operatorname{Prim}(A) \to \operatorname{Max}(Z(A)) \quad \text{defined by} \quad P \mapsto P \cap Z(A) \]

if a well-defined continuous surjection. We shall continue to assume that \( A \) is unital in this paragraph and the next. For all \( P, Q \in \operatorname{Prim}(A) \) we define

\[ P \approx Q \quad \text{if} \quad P \cap Z(A) = Q \cap Z(A). \]

By the Dauns-Hofmann theorem [11, Theorem A.34], there exists an isomorphism

\[ \Psi_A : Z(A) \to C(\operatorname{Prim}(A)) \quad \text{such that} \quad z + P = \Psi_A(z)(P)1 + P \]

for all \( z \in Z(A) \) and \( P \in \operatorname{Prim}(A) \) (note that \( \operatorname{Prim}(A) \) is compact, as \( A \) is unital [14, II.6.5.7]). Hence, for all \( P, Q \in \operatorname{Prim}(A) \) we have

\[ P \approx Q \quad \iff \quad f(P) = f(Q) \quad \text{for all} \quad f \in C(\operatorname{Prim}(A)). \]

Note that \( \approx \) is an equivalence relation on \( \operatorname{Prim}(A) \) and the equivalence classes are closed subsets of \( \operatorname{Prim}(A) \). It follows there is one-to-one correspondence between the quotient set \( \operatorname{Prim}(A)/\approx \) and a set of ideals of \( A \) given by

\[ [P]_\approx \leftrightarrow \ker[P]_\approx \quad (P \in \operatorname{Prim}(A)), \]
where \([P]_\approx\) denotes the equivalence class of \(P\). The set of ideals obtained in this way is denoted by \(\text{Glimm}(A)\), and its elements are called \(\text{Glimm ideals}\) of \(A\). The quotient map

\[
\phi_A : \text{Prim}(A) \to \text{Glimm}(A), \quad \phi_A(P) := \ker[P]_\approx
\]

is known as the complete regularization map. We equip \(\text{Glimm}(A)\) with the quotient topology, which coincides with the complete regularization topology, since \(A\) is unital. In this way \(\text{Glimm}(A)\) becomes a compact Hausdorff space. In fact, \(\text{Glimm}(A)\) is homeomorphic to \(\text{Max}(Z(A))\) via the assignment \(G \mapsto G \cap Z(A)\) (see [9] for further details).

**Definition 2.4.** A \(C^*\)-algebra \(A\) is said to be quasi-central if no primitive ideal of \(A\) contains \(Z(A)\).

Quasi-central \(C^*\)-algebras were introduced by Delaroche in [19]. We have the following useful characterisation of quasi-central \(C^*\)-algebras.

**Proposition 2.5.** [5, Proposition 1] Let \(A\) be a \(C^*\)-algebra. The following conditions are equivalent:

(i) \(A\) is quasi-central.

(ii) \(A\) admits a central approximate unit, i.e. there exists an approximate unit \((e_\alpha)\) of \(A\) such that \(e_\alpha \in Z(A)\) for all \(\alpha\).

**Remark 2.6.** It is easily seen from Proposition 2.5 (ii) that quasi-centrality passes to quotients and tensor products.

We have the following prominent examples of quasi-central \(C^*\)-algebras.

**Example 2.7.** (a) Every unital \(C^*\)-algebra is obviously quasi-central.

(b) Every abelian \(C^*\)-algebra is quasi-central. More generally, a \(C^*\)-algebra \(A\) is said to be \(n\)-homogeneous if all irreducible representations of \(A\) have the same finite dimension \(n\) (note that the abelian \(C^*\)-algebras are precisely 1-homogeneous \(C^*\)-algebras). Then by [32, Theorem 4.2] \(\text{Prim}(A)\) is a (locally compact) Hausdorff space and by a well-known theorem of Fell [22, Theorem 3.2] and Tomiyama-Takesaki [33, Theorem 5] there is a locally trivial bundle \(E\) over \(\text{Prim}(A)\) with fibre \(M_n(\mathbb{C})\) and structure group \(\text{Aut}(M_n(\mathbb{C})) \cong PU(n)\) (the projective unitary group) such that \(A\) is isomorphic to the \(C^*\)-algebra \(\Gamma_0(E)\) of continuous sections of \(E\) that vanish at infinity. Using the local triviality of the underlying bundle \(E\) one can now easily check that \(A \cong \Gamma_0(E)\) is quasi-central (see also [32, p. 236]).

(c) For a locally compact group \(G\), the following conditions are equivalent:

(i) The full group \(C^*\)-algebra \(C^*(G)\) is quasi-central.

(ii) The reduced group \(C^*\)-algebra \(C^*_r(G)\) is quasi-central.
(iii) $G$ is an SIN-group (that is, the identity has a base of neighbourhoods that are invariant under conjugation by elements of $G$).

(See [34, Corollary 1.3] and the remark which follows it.)

Remark 2.8. Let $A$ be an arbitrary $C^*$-algebra.

(a) Using the Hewitt-Cohen factorization theorem (see e.g. [13, Theorem A.6.2]) we have

$$K_A := \text{Id}_A(Z(A)) = Z(A)A = \{za : z \in Z(A), a \in A\}$$

(finite sums are not needed). In particular, $A$ is quasi-central if and only if $A = Z(A)A$ ([23 Proposition 3.2]).

(b) The ideal $K_A$ is in fact the largest quasi-central ideal of $A$ [19]. Indeed, $K_A$ contains $Z(A)$, so $Z(K_A) = K_A \cap Z(A) = Z(A)$. Therefore, $Z(K_A)K_A = K_A$, so $K_A$ is quasi-central. On the other hand, if $K$ is an arbitrary quasi-central ideal of $A$, then (a) implies $K = Z(K)K$. Since $Z(K) = K \cap Z(A) \subseteq Z(A)$, $K \subseteq Z(A)A = K_A$.

(c) Since $P \in \text{Prim}(A)$ contains $Z(A)$ if and only if $P$ contains $K_A$, it follows

$$K_A = \ker\{P \in \text{Prim}(A) : Z(A) \subseteq P\}.$$

(d) If $A$ is quasi-central, then all primitive ideals of $A$ are modular. This follows directly from Remark 2.1.

The following well-known example shows that the converse of Remark 2.8(d) is not true in general. First recall that a $C^*$-algebra $A$ is called $n$-subhomogeneous ($n \in \mathbb{N}$) if all irreducible representations of $A$ have dimension at most $n$ and $A$ also admits an $n$-dimensional irreducible representation.

Example 2.9. Consider the $C^*$-algebra $A$ that consists of all continuous functions $f : [0, 1] \to M_2(\mathbb{C})$ such that $f(1) = \text{diag}(\lambda(f), 0)$, for some scalar $\lambda(f) \in \mathbb{C}$. Since $A$ is 2-subhomogeneous, all primitive ideals of $A$ are modular. On the other hand,

$$Z(A) = \{\text{diag}(f, f) : f \in C([0, 1]), f(1) = 0\}$$

is contained in the kernel of the one-dimensional (hence irreducible) representation $\lambda : A \to \mathbb{C}$, defined by the assignment $\lambda : f \mapsto \lambda(f)$. Hence, $A$ is not quasi-central. In fact the largest quasi-central ideal $K_A$ of $A$ consists of all $f \in A$ such that $f(1) = 0$, since the primitive ideals of this ideal have the form

$$\{f \in A : f(t) = 0 \text{ and } f(1) = 0\}$$
3 Characterisations of $C^*$-algebras with the CQ-property

We begin this section with the following $C^*$-algebraic version of [16, Proposition 2.1] in which for $a \in A$,

$$[a, A] := \{[a, x] : x \in A \}.$$  

Since the proof requires only obvious changes, we omit it.

**Proposition 3.1.** Let $A$ be a $C^*$-algebra. The following conditions are equivalent:

(i) $A$ has the CQ-property.

(ii) For every $*$-epimorphism $\phi : A \to B$, where $B$ is another $C^*$-algebra, $\phi(Z(A)) = Z(B)$.

(iii) For every $a \in A$, $a \in Z(A) + \text{Id}_A([a, A])$.

The next fact was obtained in [4, Lemma 2.2.3] but we include the details here for completeness.

**Proposition 3.2.** If a $C^*$-algebra $A$ has the CQ-property, so do all ideals and quotients of $A$.

**Proof.** Assume that $A$ has the CQ-property and let $I$ be an ideal of $A$.

If $J$ is an ideal of $I$, then $J$ is an ideal of $A$ and $I/J$ is an ideal of $A/J$. The CQ-property of $A$ implies

$$Z(I/J) = (I/J) \cap Z(A/J) = (I/J) \cap ((Z(A) + J)/J). \quad (3.1)$$

Let $a \in I$ such that $a + J \in (Z(A) + J)/J$. Then there is $z \in Z(A)$ such that $a - z \in J \subseteq I$, so $z \in I \cap Z(A) = Z(I)$. It follows that

$$(I/J) \cap ((Z(A) + J)/J) = (Z(I) + J)/J,$$

so by (3.1),

$$Z(I/J) = (Z(I) + J)/J.$$

Therefore, $I$ has the CQ-property.

We now show that $A/I$ has the CQ-property. Let $q_I : A \to A/I$ be the canonical map and $\phi : A/I \to B$ any $*$-epimorphism, where $B$ is another $C^*$-algebra. Then $\phi \circ q_I : A \to B$ is a $*$-epimorphism, so the CQ-property of $A$ implies

$$Z(B) = Z((\phi \circ q_I)(A)) = (\phi \circ q_I)(Z(A)) = \phi(q_I(Z(A))) = \phi(Z(A/I)).$$

Therefore, $A/I$ has the CQ-property.
Proposition 3.3. For a $C^*$-algebra $A$ the following conditions are equivalent:

(i) $Z(A) = \{0\}$ and $A$ has the CQ-property.

(ii) Every primitive ideal of $A$ is non-modular.

\[\square\]

Proof. (i) $\implies$ (ii). Assume that $Z(A) = \{0\}$ and that $A$ has the CQ-property. Then for any $P \in \text{Prim}(A)$ we have $Z(A/P) = (Z(A) + P)/P = \{0\}$, so $P$ is non-modular.

(ii) $\implies$ (i). Assume that all primitive ideals of $A$ are non-modular. By Remark 2.1 $Z(A) = \{0\}$. Also, for any ideal $I$ of $A$, all primitive ideals of $A/I$ are non-modular, so Remark 2.1 again implies $Z(A/I) = \{0\}$. Thus, $A$ has the CQ-property.

\[\square\]

The following result was obtained in [4, Proposition 2.2.4] but we give a shorter argument in one direction by using the method of [10, Proposition 2.15].

Proposition 3.4. For a non-unital $C^*$-algebra $A$ the following conditions are equivalent:

(i) $A$ has the CQ-property.

(ii) $A^\sharp$ has the CQ-property.

\[\square\]

Proof. (i) $\implies$ (ii). Suppose that $A$ has the CQ-property and let $\lambda 1 + a \in A^\sharp$, where $a \in A$ and $\lambda \in \mathbb{C}$. Then, by Proposition 3.1, we have $a \in Z(A) + \text{Id}_A([a, A])$. Since

$$\text{Id}_{A^\sharp}([\lambda 1 + a, A^\sharp]) = \text{Id}_A([a, A]),$$

it follows that $a \in Z(A) + \text{Id}_{A^\sharp}([\lambda 1 + a, A^\sharp])$. Since $Z(A^\sharp) = \mathbb{C}1 + Z(A)$ we conclude that

$$\lambda 1 + a \in Z(A^\sharp) + \text{Id}_{A^\sharp}([\lambda 1 + a, A^\sharp]).$$

Therefore, by Proposition 3.1, $A^\sharp$ has the CQ-property.

(ii) $\implies$ (i). Since $A$ is an ideal of $A^\sharp$, this follows directly from Proposition 3.2

\[\square\]

We now extend the notion of weak centrality to arbitrary $C^*$-algebras.

Definition 3.5. We say that a $C^*$-algebra $A$ is weakly central if the following two conditions are satisfied:

(a) No modular maximal ideal of $A$ contains $Z(A)$.

(b) For each pair of modular maximal ideals $M_1$ and $M_2$ of $A$, $M_1 \cap Z(A) = M_2 \cap Z(A)$ implies $M_1 = M_2$.

\[\square\]
Remark 3.6. Since all modular maximal ideals of a $C^*$-algebra $A$ are primitive and since each modular primitive ideal of $A$ is contained in a modular maximal ideal of $A$, the condition (a) in Definition 3.5 can be restated as:

(a') No modular primitive ideal of $A$ contains $Z(A)$.

The justification of Definition 3.5 will be given in the following series of results. First consider one example.

Example 3.7. Let $X$ be a compact Hausdorff space, $H$ a separable infinite-dimensional Hilbert space and $A := C(X, K(H))$. Then each primitive ideal of $A$ is of the form $P_t := \{ f \in A : f(t) = 0 \}$ for some $t \in X$. Since $A/P_t \cong K(H)$ for all $t \in X$, all primitive ideals of $A$ are maximal and non-modular. It follows from Proposition 3.3 that $Z(A) = \{0\}$ and $A$ has the CQ-property. Secondly, $\text{Max}(A) = \emptyset$ so that $A$ is trivially weakly central even though $P_t \cap Z(A) = \{0\}$ for all of the maximal ideals $P_t$. On the other hand, $A^Z$ can be identified with the $C^*$-subalgebra of $B := C(X, B(H))$ that consists of all functions $f \in B$ for which there exists a scalar $\lambda$ such that $f(t) - \lambda 1 \in K(H)$ for all $t \in X$. Then $A$ is the unique maximal ideal of $A^Z$ and hence $A^Z$ is weakly central.

Proposition 3.8. For a non-unital $C^*$-algebra $A$ the following conditions are equivalent:

(i) $A$ is weakly central.

(ii) $A^Z$ is weakly central.

Proof. (i) $\implies$ (ii). Assume that $A$ is weakly central and let $M_1, M_2 \in \text{Max}(A^Z)$ such that $M_1 \cap Z(A^Z) = M_2 \cap Z(A^Z)$.

If one of $M_1$ or $M_2$ is $A$, so is the other. Indeed, assume for example that $M_1 = A$. If $M_2 \neq A$ then, by Lemma 2.2, $M_2 \cap A$ is a modular maximal ideal of $A$. We have $Z(A) = A \cap Z(A^Z) = M_2 \cap Z(A^Z)$ and so $Z(A)$ is contained in $M_2 \cap A$, contradicting the weak centrality of $A$.

Therefore assume that both $M_1$ and $M_2$ are not $A$. Again, by Lemma 2.2, $N_1 := M_1 \cap A$ and $N_2 := M_2 \cap A$ are modular maximal ideals of $A$ such that $N_1 \cap Z(A) = N_2 \cap Z(A)$. The weak centrality of $A$ forces $N_1 = N_2$, so Lemma 2.2 implies $M_1 = M_2$. Hence, $A^Z$ is weakly central.

(ii) $\implies$ (i). Suppose that $A^Z$ is weakly central. Let $M \in \text{Max}(A)$. By Lemma 2.2 there exists $N \in \text{Max}(A^Z)$ such that $M = N \cap A$. Since $N \neq A$, it follows that

$$N \cap Z(A^Z) \neq A \cap Z(A^Z) = Z(A).$$

But $Z(A)$ is a maximal ideal in $Z(A^Z)$. Thus $Z(A)$ is not contained in $N$ and consequently neither in $M$. 

Note that if $A$ is unital then the above definition agrees with the standard notion of weak centrality.
Now suppose that $M_1, M_2 \in \text{Max}(A)$ and $M_1 \cap Z(A) = M_2 \cap Z(A)$. By Lemma 2.2, there exist $N_1, N_2 \in \text{Max}(A^\sharp)$ such that $M_1 = N_1 \cap A$ and $M_2 = N_2 \cap A$. We have

$$(N_1 \cap Z(A^\sharp)) \cap Z(A) = M_1 \cap Z(A) = (N_2 \cap Z(A^\sharp)) \cap Z(A).$$

By the previous paragraph, $M_1$ and $M_2$ do not contain $Z(A)$. It follows that the maximal ideals $N_1 \cap Z(A^\sharp)$ and $N_2 \cap Z(A^\sharp)$ of $Z(A^\sharp)$ do not contain $Z(A)$ and hence must be equal by Lemma 2.2 (applied to the ideal $Z(A)$ of $Z(A^\sharp)$). By the weak centrality of $A^\sharp$, we have $N_1 = N_2$ and hence $M_1 = M_2$. Thus $A$ is weakly central. 

As a direct consequence of Vesterstrøm’s theorem (Theorem 1.1) and Propositions 3.8 and 3.4 we get the next characterisation.

**Corollary 3.9.** For a $C^*$-algebra $A$ the following conditions are equivalent:

(i) $A$ has the CQ-property.

(ii) $A$ is weakly central.

**Remark 3.10.** An immediate consequence of Proposition 3.2 and Corollary 3.9 is that the class of weakly central $C^*$-algebras is closed under forming ideals and quotients.

The next simple fact follows directly from Corollary 3.9, Remark 2.8 (d) and Remark 3.6.

**Proposition 3.11.** For a $C^*$-algebra $A$ the following conditions are equivalent:

(i) $A$ is quasi-central and weakly central.

(ii) $A$ has the CQ-property and every primitive ideal of $A$ is modular.

Part (b) of the next result overlaps with [8, Corollary 2.13], but the proof here avoids the use of a composition series.

**Corollary 3.12.** Let $A$ be a postliminal $C^*$-algebra.

(a) $A$ has the (singleton) Dixmier property if and only if $A$ is weakly central.

(b) If every irreducible representation of $A$ is infinite-dimensional, then $A$ has the CQ-property and the (singleton) Dixmier property and is weakly central.

Before proving Corollary 3.12, we record the next simple fact which will be also used in Section 4.

**Remark 3.13.** Let $A$ be a postliminal $C^*$-algebra. If $\text{Max}(A) \neq \emptyset$, then for each $M \in \text{Max}(A)$, $A/M$ is a unital simple postliminal $C^*$-algebra and thus $A/M \cong M_n(\mathbb{C})$ for some $n \in \mathbb{N}$.
Proof of Corollary 3.12. (a) By [8, Theorem 2.12] a postliminal $C^*$-algebra $A$ has the (singleton) Dixmier property if and only if it has the CQ-property. It remains to apply Corollary 3.9.

(b) Assume that $A$ contains a modular maximal ideal $M$. By Remark 3.13 $A/M \cong M_n(\mathbb{C})$ for some $n \in \mathbb{N}$. Therefore, $A$ has a finite-dimensional irreducible representation; a contradiction. Thus all primitive ideals of $A$ are non-modular (Remark 3.6), so by Proposition 3.3 $A$ has the CQ-property. The other properties follow from [8, Theorem 2.12] and Corollary 3.9.

For the main results of this section (Theorems 3.16 and 3.22), we shall need to consider the following subsets of $\text{Max}(A)$ for an arbitrary $C^*$-algebra $A$:

- $T^1_A$ as the set of all $M \in \text{Max}(A)$ such that $Z(A) \subseteq M$.
- $T^2_A$ as the set of all $M \in \text{Max}(A)$ for which exists $N \in \text{Max}(A)$ such that $M \neq N$, $Z(A) \subseteq M, N$ and $M \cap Z(A) = N \cap Z(A)$.
- $T_A := T^1_A \cup T^2_A$.

The set $T^1_A$ is obviously closed in $\text{Max}(A)$. The next example shows that this is not generally true for the set $T^2_A$ (and consequently $T_A$).

Example 3.14. Let $A$ be the $C^*$-algebra consisting of all functions $a \in C([0, 1], M_2(\mathbb{C}))$ which are diagonal at $1/n$ for all $n \in \mathbb{N}$ and scalar at zero. Then $A$ is a unital 2-subhomogeneous $C^*$-algebra, so $\text{Max}(A) = \text{Prim}(A)$ and

$$T_A = T^2_A = \{ P \in \text{Prim}(A) : \exists Q \in \text{Prim}(A), P \neq Q, P \cap Z(A) = Q \cap Z(A) \}.$$ 

Let $\lambda_n(a), \mu_n(a)$ and $\eta(a)$ be complex numbers such that

$$a(1/n) = \text{diag}(\lambda_n(a), \mu_n(a)) \quad (n \in \mathbb{N}) \quad \text{and} \quad a(0) = \text{diag}(\eta(a), \eta(a)).$$

If we denote by $\lambda_n, \mu_n$ and $\eta$ the 1-dimensional (irreducible) representations of $A$ defined respectively by the assignments $a \mapsto \lambda_n(a), a \mapsto \mu_n(a)$ and $a \mapsto \eta(a)$, it is easy to verify that

$$T_A = \{ \ker \lambda_n : n \in \mathbb{N} \} \cup \{ \ker \mu_n : n \in \mathbb{N} \}.$$ 

Then $\ker T_A$ consists of all functions in $A$ which vanish at $1/n (n \in \mathbb{N})$, and hence vanish at 0 too. Therefore, $\ker \eta \in T_A \setminus T_A$, so $T_A$ is not closed in $\text{Max}(A) = \text{Prim}(A)$. □

Lemma 3.15. If $A$ is a $C^*$-algebra, then $\ker T_A$ contains any weakly central ideal of $A$. □

Proof. Let $J$ be a weakly central ideal of $A$. Suppose that $M \in T^1_A$, so that $Z(A) \subseteq M$. Then $J \subseteq M$, for otherwise $M \cap J \in \text{Max}(J)$ (Lemma 2.2) and $M \cap J$ contains $Z(J)$, contradicting the weak centrality of
J. Secondly, suppose that $M \in T_A^2$. Then there exists $N \in \text{Max}(A)$ such that $N \neq M$, $Z(A) \not\subseteq M, N$ and $M \cap Z(A) = N \cap Z(A)$. We have

$$(M \cap J) \cap Z(J) = (M \cap Z(A)) \cap J = (N \cap J) \cap Z(J). \quad (3.2)$$

Suppose that $M$ does not contain $J$. Then $M \cap J \in \text{Max}(J)$ (Lemma 2.2) and $M \cap J$ does not contain $Z(J)$ by the weak centrality of $J$. By (3.2), $N \cap J$ does not contain $Z(J)$ and hence $N$ does not contain $J$. Since $J$ is weakly central, it follows from (3.2) that $M \cap J = N \cap J$ and hence (again by Lemma 2.2) $M = N$; a contradiction. Thus $J \subseteq M$ as required.

Given a $C^*$-algebra $A$ we also define

$$S_A := \{P \in \text{Prim}(A) : P \text{ is non-modular}\} \quad \text{and} \quad J_A := \ker S_A.$$

By Remarks 2.1 and 2.8 (c), $J_A$ contains the largest quasi-central ideal $K_A$ of $A$, so in particular $Z(J_A) = Z(A)$. Example 2.9 shows that $J_A$ can strictly contain $K_A$ (in this case $J_A = A$).

**Theorem 3.16.** For a $C^*$-algebra $A$ the following conditions are equivalent:

(i) $A$ has the CQ-property.

(ii) $A$ is weakly central.

(iii) $S_A$ is a closed subset of $\text{Prim}(A)$ and $J_A = K_A$ is a quasi-central weakly central $C^*$-algebra.

(iv) There is a weakly central ideal $J$ of $A$ such that all primitive ideals of $A$ that contain $J$ are non-modular.

(v) There is an ideal $J$ of $A$ such that both $J$ and $A/J$ have the CQ-property and $Z(A/J) = (Z(A) + J)/J$.

In the proof of implication $(v) \implies (i)$ of Theorem 3.16 we shall use the next simple fact.

**Lemma 3.17.** Let $A$ be a $C^*$-algebra and $J$ an ideal of $A$ such that $A/J$ has the CQ-property and $Z(A/J) = (Z(A) + J)/J$. Then $Z(A/I) = (Z(A) + I)/I$ for any ideal $I$ of $A$ that contains $J$.

**Proof.** Let $a \in A$ and suppose that $a + I \in Z(A/I)$. Let $\phi : A/I \to (A/J)/(I/J)$ be the canonical isomorphism. Then

$$\phi(a + I) \in Z((A/J)/(I/J)) = \frac{Z(A/J) + I/J}{I/J} = \frac{(Z(A) + J)/J + I/J}{I/J},$$

so there exists $z \in Z(A)$ such that $\phi(a + I) = \phi(z + I)$. Hence $a + I = z + I$ and so $a \in Z(A) + I$, as required.

**Proof of Theorem 3.16** (i) $\iff$ (ii). This is Corollary 3.9.
(ii) $\implies$ (iii). Assume that $A$ is weakly central. Let $P \in \text{Prim}(A)$ be in the closure of $S_A$ in $\text{Prim}(A)$, that is $J_A \subseteq P$. Then $Z(A) = Z(J_A) \subseteq P$. Since $A$ is weakly central, $P$ must be non-modular (Remark 3.6), so $P \in S_A$. Therefore $S_A$ is closed in $\text{Prim}(A)$.

Since $J_A$ is an ideal of $A$ and $A$ is weakly central, so is $J_A$ by Remark 3.10. It remains to show that $J_A = K_A$. Since $K_A \subseteq J_A$, it suffices to show that $J_A$ is quasi-central. Assume there exists $R \in \text{Prim}(J_A)$ that contains $Z(A) = Z(J_A)$ and let $P \in \text{Prim}_{J_A}(A)$ such that $R = P \cap J_A$. Obviously $Z(A) \subseteq P$. Since $S_A$ is closed in $\text{Prim}(A)$, the set $\text{Prim}_{J_A}(A)$ consists of all modular primitive ideals of $A$. In particular, $P$ is a modular primitive ideal of $A$ that contains $Z(A)$, which (together with Remark 3.6) contradicts the weak centrality of $A$.

(iii) $\implies$ (iv). Choose $J = J_A = K_A$.

(iv) $\implies$ (v). Let $J$ be a weakly central ideal of $A$ such that all primitive ideals in $\text{Prim}^J(A)$ are non-modular.

By Corollary 3.9 $J$ has the CQ-property. Also, all primitive ideals of $A/J$ are non-modular, so by Proposition 3.3 $Z(A/J) = \{0\}$ and $A/J$ has the CQ-property. Further, since $J = \ker \text{Prim}^J(A)$ and $Z(A)$ is contained in each $P \in \text{Prim}^J(A)$ (Remark 2.1), $Z(A) \subseteq J$. Thus

$$(Z(A) + J)/J = \{0\} = Z(A/J).$$

(v) $\implies$ (i). Assume that $A$ does not have the CQ-property. By Corollary 3.9 this is equivalent to say that $A$ is not weakly central. Since $J$ has the CQ-property, it is weakly central (Corollary 3.9), so by Lemma 3.15 $J$ is contained in $\ker T_A$. We have the following two possibilities.

Case 1. There is $M \in \text{Max}(A)$ such that $Z(A) \subseteq M$. Then $M \in T_A^1$ so $J \subseteq \ker T_A \subseteq M$. Thus, by Lemma 3.17

$$C \cong Z(A/M) = (Z(A) + M)/M = \{0\};$$

a contradiction.

Case 2. There are distinct $M, N \in \text{Max}(A)$ such that $Z(A) \nsubseteq M, N$ and $M \cap Z(A) = N \cap Z(A)$. Then

$$Z \left( \frac{A}{M \cap N} \right) \cong Z(A/M) \oplus Z(A/N) \cong C \oplus C.$$ 

On the other hand, $M, N \in T_A^2$, so $J \subseteq \ker T_A \subseteq M \cap N$. Since $Z(A) \nsubseteq M, N$, $M \cap Z(A)$ is a maximal ideal of $Z(A)$, so using Lemma 3.17 we get

$$Z \left( \frac{A}{M \cap N} \right) = \frac{Z(A) + (M \cap N)}{M \cap N} \cong \frac{Z(A)}{(M \cap N) \cap Z(A)} = \frac{Z(A)}{M \cap Z(A)} \cong C;$$

a contradiction. $\Box$

Recall that a $C^*$-algebra $A$ is said to be central if $A$ is quasi-central and for all $P_1, P_2 \in \text{Prim}(A)$, $P_1 \cap Z(A) = P_2 \cap Z(A)$ implies $P_1 = P_2$ (see [31, Section 9]).
Remark 3.18. It is well-known that a quasi-central \( C^* \)-algebra \( A \) is central if and only if \( \text{Prim}(A) \) is a Hausdorff space (see e.g. [18, Proposition 3]). In particular, by Example 2.7 (b), all homogeneous \( C^* \)-algebras are central. Further, all central \( C^* \)-algebras are obviously weakly central. \( \square \)

Corollary 3.19. A liminal \( C^* \)-algebra \( A \) has the CQ-property if and only if the set of all kernels of infinite-dimensional irreducible representations of \( A \) is closed in \( \text{Prim}(A) \) and the intersection of these kernels is a central \( C^* \)-algebra.

Proof. Since \( A \) is liminal, an irreducible representation of \( A \) is infinite-dimensional if and only if its kernel is a non-modular primitive ideal of \( A \). Thus

\[
S_A = \{ \ker \pi : [\pi] \in \hat{A}, \pi \text{ infinite-dimensional} \},
\]

where \( \hat{A} \) denotes the spectrum of \( A \). Hence, by Theorem 3.16 \( A \) has the CQ-property if and only if \( S_A \) is closed in \( \text{Prim}(A) \) and \( J_A = \ker S_A \) is a quasi-central weakly central \( C^* \)-algebra. Suppose that \( S_A \) is closed in \( \text{Prim}(A) \). Then all irreducible representations of \( J_A \) are finite-dimensional. In particular, all primitive ideals of \( J_A \) are modular and maximal, so weak centrality and quasi-centrality of \( J_A \) in this case is equivalent to centrality. \( \blacksquare \)

We also record the following special case of Corollary 3.19.

Corollary 3.20. If all irreducible representations of a \( C^* \)-algebra \( A \) are finite-dimensional, then \( A \) has the CQ-property if and only if \( A \) is central. \( \square \)

Remark 3.21. In contrast to Proposition 3.8 the multiplier algebras of weakly central \( C^* \)-algebras do not have to be weakly central. In fact, Somerset and the first-named author exhibited an example of a homogeneous (hence central) \( C^* \)-algebra \( A \) such that \( \text{Prim}(M(A)) \) is not Hausdorff [10, Theorem 1]. Specifically, the subhomogeneous \( C^* \)-algebra \( M(A) \) (see e.g. [14, Proposition IV.1.4.6]) is not (weakly) central. \( \square \)

It is possible to show that every \( C^* \)-algebra contains a largest ideal with the CQ-property by using Zorn’s lemma and the fact that the sum of two ideals with the CQ-property has the CQ-property. However, in view of Corollary 3.9 we are able to take a different approach that has the merit of obtaining a formula for this ideal in terms of the set \( T_A \) of those modular maximal ideals of \( A \) which witness the failure of the weak centrality of \( A \).

Theorem 3.22. Let \( A \) be a \( C^* \)-algebra. Then \( \ker T_A \) is the largest weakly central ideal of \( A \).

Proof. Set \( J := \ker T_A \). By Lemma 3.15 it suffices to prove that \( J \) is weakly central. For this, we begin by assuming that \( A \) is unital (so that \( T_A = T_\hat{A} \)) and that \( J \) is not weakly central. We have two possibilities.

Case 1. There is \( M_0 \in \text{Max}(J) \) such that \( Z(J) \subseteq M_0 \). By Lemma 2.2 there exists \( N_0 \in \text{Max}_J(A) \) such that \( M_0 = N_0 \cap J \). Since

\[
\ker \{ N \cap Z(A) : N \in T_A \} = Z(J) \subseteq N_0 \cap Z(A) \in \text{Max}(Z(A)),
\]

we have that \( \ker T_A \) is the largest weakly central ideal of \( A \).
there is a net \((N_\alpha)\) in \(T_A\) such that
\[
\lim_{\alpha} N_\alpha \cap Z(A) = N_0 \cap Z(A). \tag{3.3}
\]
in \(\text{Max}(Z(A))\). Since \(A\) is unital, \(\text{Max}(A)\) is a compact subspace of \(\text{Prim}(A)\), so there is a subnet \((N_\alpha)\) of \((N_\alpha)\) that converges to some \(N'_0 \in \text{Max}(A)\). Then the continuity of the map \(\text{Max}(A) \to \text{Max}(Z(A))\), defined by
\[M \mapsto M \cap Z(A),\]
implies that
\[
\lim_{\beta} N_\alpha(\beta) \cap Z(A) = N'_0 \cap Z(A). \tag{3.4}
\]
Since \(\text{Max}(Z(A))\) is Hausdorff, \(\text{(3.3)}\) and \(\text{(3.4)}\) imply
\[
N_0 \cap Z(A) = N'_0 \cap Z(A). \tag{3.5}
\]
Obviously \(N'_0\) lies in the closure of \(T_A\), so \(J \subseteq N'_0\). Since \(N_0 \in \text{Max}_{J}(A)\), \(N_0 \neq N'_0\), so \(\text{(3.5)}\) implies \(N_0, N'_0 \in T_A\). In particular, \(J \subseteq N_0\); a contradiction.

Case 2. There are \(M_1, M_2 \in \text{Max}(J)\) such that \(M_1 \neq M_2\), \(Z(J) \nsubseteq M_1, M_2\) and \(M_1 \cap Z(J) = M_2 \cap Z(J)\).

By Lemma 2.2 there are \(N_1, N_2 \in \text{Max}(J)\) such that \(M_1 = N_1 \cap J\) and \(M_2 = N_2 \cap J\). Since \(Z(J) = J \cap Z(A)\) is an ideal of \(Z(A)\),
\[
N_1 \cap Z(A), N_2 \cap Z(A) \in \text{Max}_{Z(J)}(Z(A))
\]
and
\[
(N_1 \cap Z(A)) \cap Z(J) = M_1 \cap Z(J) = (N_2 \cap Z(A)) \cap Z(J),
\]
Lemma 2.2 (applied to \(Z(A)\) and its ideal \(Z(J)\)) implies that \(N_1 \cap Z(A) = N_2 \cap Z(A)\). Since \(N_1 \neq N_2\), we conclude that \(N_1, N_2 \in T_A\), so \(J \subseteq N_1 \cap N_2\); a contradiction.

We have now established that \(\ker T_A\) is weakly central in the case that \(A\) is unital. We suppose next that \(A\) is non-unital. Then, by the above arguments, \(\ker T_{A^2}\) is a weakly central ideal of \(A^2\). Since \(\ker T_{A^2} \cap A\) is an ideal of \(\ker T_{A^2}\), it is weakly central by Remark 3.10. Hence, it suffices to show that
\[
J := \ker T_A \subseteq \ker T_{A^2} \cap A.
\]
So let \(M \in T_{A^2}\). We only have to show that \(M\) contains \(J\). Since \(A^2\) is unital, \(T_{A^2} = T_{A^2}^2\), so there is \(M' \in \text{Max}(A^2)\) such that \(M' \neq M\) and \(M \cap Z(A^2) = M' \cap Z(A^2)\). We distinguish three possibilities.

- \(M = A\). Then clearly \(M\) contains \(J\).
- \(M' = A\). Then \(M\) does not contain \(A\), so by Lemma 2.2 \(M \cap A\) is a modular maximal ideal of \(A\) containing \(Z(A)\) (since \(M'\) does). Therefore, \(M \cap A\) is in \(T_A^1\) and hence contains \(J\).
- Both \(M\) and \(M'\) are not \(A\). Again, by Lemma 2.2 \(M \cap A\) and \(M' \cap A\) are distinct modular maximal ideals of \(A\) having the same intersection with \(Z(A)\). So either \(M \cap A\) is in \(T_A^1\) or it is in \(T_A^2\). In either case \(M\)
contains $J$. \hfill $\blacksquare$

In the sequel, for any $C^*$-algebra $A$ by $J_{wc}(A)$ we denote the largest weakly central ideal of $A$. By Corollary 3.9, $J_{wc}(A)$ is precisely the largest ideal of $A$ with the CQ-property.

**Corollary 3.23.** Let $A$ be a $C^*$-algebra.

(a) For any ideal $I$ of $A$ we have $J_{wc}(I) = I \cap J_{wc}(A)$.

(b) The sum of any two weakly central ideals of $A$ is a weakly central ideal of $A$.

□

**Proof.** (a) Since $J_{wc}(I)$ is a weakly central ideal of $I$, it is also a weakly central ideal of $A$, so $J_{wc}(I) \subseteq I \cap J_{wc}(A)$.

Conversely, since $I \cap J_{wc}(A)$ is an ideal of $J_{wc}(A)$, it is weakly central by Remark 3.10. Hence, $I \cap J_{wc}(A) \subseteq J_{wc}(I)$.

(b) If $I_1$ and $I_2$ are weakly central ideals of $A$, then by Theorem 3.22 both $I_1$ and $I_2$ are contained in $J_{wc}(A)$, so $I_1 + I_2 \subseteq J_{wc}(A)$. Thus, $I_1 + I_2$ is weakly central by Remark 3.10. \hfill $\blacksquare$

The next two examples demonstrate that there are non-trivial $C^*$-algebras whose largest weakly central ideal is zero.

**Example 3.24.** Let $A$ be the rotation algebra (the $C^*$-algebra of the discrete three-dimensional Heisenberg group, see [2] and the references therein). For each $t$ in the unit circle $\mathbb{T}$, there is an ideal $J_t$ of $A$ such that, with $A_t := A/J_t$, $A$ is $*$-isomorphic to a continuous field of $C^*$-algebras $(A_t)_{t \in \mathbb{T}}$ via the assignment $a \mapsto (a + J_t)_{t \in \mathbb{T}}$. This isomorphism maps $Z(A)$ onto $C(\mathbb{T})$. If $t$ is a root of unity then $A_t$ is a non-simple homogeneous $C^*$-algebra. If $P$ is a primitive ideal of $A$ that contains $J_t$ then $P \in \text{Max}(A)$, $P \cap Z(A) = J_t \cap Z(A) \neq Z(A)$ and hence $P \in T_A$. It follows that $J_{wc}(A) = \ker T_A \subseteq J_t$. Since the roots of unity form a dense subset of $\mathbb{T}$ and the field $(A_t)_{t \in \mathbb{T}}$ is continuous, $J_{wc}(A) = \{0\}$. Consequently, no non-zero ideal of $A$ has the CQ-property. □

**Example 3.25.** Consider the $C^*$-algebra $A = C^*(\mathbb{F}_2)$ (the full $C^*$-algebra of the free group $\mathbb{F}_2$ on two generators). Then by [17], $A$ is a unital primitive residually finite-dimensional $C^*$-algebra (that is, the intersection of the kernels of the finite-dimensional irreducible representations of $A$ is $\{0\}$). As $A$ is unital and primitive, $Z(A) = \mathbb{C}1$, so $T_A = \text{Max}(A)$. In particular, $J_{wc}(A) = \ker T_A$ is contained in the intersection of the kernels of the finite-dimensional irreducible representations of $A$ which is zero. Therefore $J_{wc}(A) = \{0\}$, as with the rotation algebra. □

**Remark 3.26.** Both $C^*$-algebras in Examples 3.24 and 3.25 are antiliminal. In Example 4.25 we shall also give an example of a (separable) continuous-trace $C^*$-algebra $A$ for which $J_{wc}(A) = \{0\}$.

On the other hand, if $A$ is a $C^*$-algebra for which all irreducible representations have finite dimension, then it follows from [24, Corollary 3.8] that the ideal $J_{wc}(A)$ is essential. □
We record next a slightly surprising result which can be used for a direct argument that the sum of two ideals with the CQ-property has the CQ-property.

**Proposition 3.27.** Let $A$ be a $C^*$-algebra and let $J$ and $K$ be ideals of $A$. If one of $J$ or $K$ has the CQ-property, then $Z(J + K) = Z(J) + Z(K)$. \qed

**Proof.** Assume that $J$ has the CQ-property and let $z \in Z(J + K)$. Then $z + K \in Z((J + K)/K)$. Let $\phi: (J + K)/K \to J/(J \cap K)$ be the canonical isomorphism. Then, since $J$ has the CQ-property and $J \cap K$ is an ideal of $J$,

$$\phi(z + K) \in Z \left( \frac{J}{J \cap K} \right) = \frac{Z(J) + (J \cap K)}{J \cap K}.$$ 

So there exists $y \in Z(J)$ such that

$$\phi(z + K) = y + (J \cap K) = \phi(y + K).$$

Hence $z + K = y + K$ and so $z - y \in K \cap Z(A) = Z(K)$. It follows that $Z(J + K) \subseteq Z(J) + Z(K)$. For the reverse inclusion, observe that

$$(J \cap Z(A)) + (K \cap Z(A)) \subseteq (J + K) \cap Z(A).$$

The next example shows that if both ideals $J$ and $K$ of a $C^*$-algebra $A$ fail to satisfy the CQ-property, then $Z(J + K)$ can strictly contain $Z(J) + Z(K)$.

**Example 3.28.** Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space and let $p \in B(\mathcal{H})$ be any projection with infinite-dimensional kernel and image. Set

$$A := K(\mathcal{H}) + \mathbb{C}p + \mathbb{C}(1 - p) \subset B(\mathcal{H}).$$

[20] NOTE 1, p.257]. Then $A$ has precisely two maximal ideals, namely

$$J := K(\mathcal{H}) + \mathbb{C}p \quad \text{and} \quad K := K(\mathcal{H}) + \mathbb{C}(1 - p).$$

Obviously $Z(J) = Z(K) = \{0\}$, but $Z(J + K) = Z(A) = \mathbb{C}1$.

For later use, we also note that

$$J_{wc}(A) = \ker T_A = J \cap K = K(\mathcal{H})$$

and hence $A/J_{wc}(A)$ is abelian. \qed
We finish this section with a generalization of \cite{[3]} Theorem 3.1 for arbitrary \( C^* \)-algebras. For \( C^* \)-algebras \( A_1 \) and \( A_2 \), we denote their algebraic tensor product by \( A_1 \odot A_2 \). If \( \beta \) is any \( C^* \)-norm on \( A_1 \odot A_2 \), we denote the completion of \( A_1 \odot A_2 \) with respect to \( \beta \) by \( A_1 \otimes_\beta A_2 \).

**Theorem 3.29.** Let \( A_1 \) and \( A_2 \) be \( C^* \)-algebras. The following conditions are equivalent:

(i) Both \( A_1 \) and \( A_2 \) have the CQ-property.

(ii) \( A_1 \otimes_\beta A_2 \) has the CQ-property for every \( C^* \)-norm \( \beta \).

(iii) \( A_1 \otimes_\beta A_2 \) has the CQ-property for some \( C^* \)-norm \( \beta \).

\( \Box \)

**Proof.** (i) \( \implies \) (ii). Suppose that \( A_1 \) and \( A_2 \) have the CQ-property and that \( \beta \) is a \( C^* \)-norm on \( A_1 \odot A_2 \). Since \( A_1^i \subseteq A_i^* \) \((i = 1, 2)\), it follows from \cite{[3]} Theorem 2 that there is a \( C^* \)-norm \( \beta' \) on \( A_1^i \odot A_2^i \) extending \( \beta \) (recall that by our convention \( A_i^i = A_i \) if \( A_i \) is unital). Since \( A_i \) \((i = 1, 2)\) has the CQ-property if and only if \( A_i^i \) is weakly central, by \cite{[3]} Theorem 3.1 \( A_1^i \otimes_{\beta'} A_2^i \) is weakly central. Hence, \( A_1^i \otimes_{\beta'} A_2^i \) has the CQ-property and so does its ideal \( A_1 \otimes_\beta A_2 \) (Proposition 3.2).

(ii) \( \implies \) (iii). This is trivial.

(iii) \( \implies \) (i). Assume that \( A_1 \otimes_\beta A_2 \) has the CQ-property for some \( C^* \)-norm \( \beta \). Since the minimal tensor product \( A_1 \otimes_{\text{min}} A_2 \) is \(*\)-isomorphic to a quotient of \( A_1 \otimes_\beta A_2 \), it follows from Proposition 3.2 that \( A_1 \otimes_{\text{min}} A_2 \) has the CQ-property. We show that \( A_1 \) has the CQ-property (a similar argument applies to \( A_2 \)). So let \( I \) be an ideal of \( A_1 \) and let \( q_I : A_1 \to A_1/I \) be the canonical map. Using the canonical \(*\)-epimorphism

\[
q_I \otimes \text{id}_{A_2} : A_1 \otimes_{\text{min}} A_2 \to (A/I) \otimes_{\text{min}} A_2
\]

and two applications of \cite{[27]} Corollary 1, we have

\[
Z(A_1/I) \otimes Z(A_2) = Z((A_1/I) \otimes_{\text{min}} A_2) = (q_I \otimes \text{id}_{A_2})(Z(A_1) \otimes Z(A_2)) = q_I(Z(A_1)) \otimes Z(A_2).
\]

(3.6)

Assume that \( q_I(Z(A_1)) \) is strictly contained in \( Z(A_1/I) \). Then there is a non-zero functional \( \varphi \in Z(A_1/I)^* \) that annihilates \( q_I(Z(A_1)) \). If \( \psi \) is any non-zero functional on \( Z(A_2) \), then \( \varphi \otimes \psi \) is a non-zero functional on \( Z(A_1/I) \otimes Z(A_2) \) that annihilates \( q_I(Z(A_1)) \otimes Z(A_2) \), contradicting (3.6). Thus

\[
Z(A_1/I) = q_I(Z(A_1)) = (Z(A_1) + I)/I
\]

as desired.

\( \blacksquare \)
4 CQ-elements in $C^*$-algebras

Motivated by [16], and with a view to identifying the individual elements which prevent the CQ-property, we now introduce a local version of the CQ-property.

**Definition 4.1.** Let $A$ be a $C^*$-algebra. We say that an element $a \in A$ is a CQ-element of $A$ if for every ideal $I$ of $A$, $a + I \in Z(A/I)$ implies $a \in Z(A) + I$.

By $CQ(A)$ we denote the set of all CQ-elements of $A$. Obviously $A$ has the CQ-property if and only if $CQ(A) = A$. We also define $V_A := A \setminus CQ(A)$, which is the set of elements which prevent $A$ from having the CQ-property.

We state the following $C^*$-algebraic version of [16, Proposition 2.2].

**Proposition 4.2.** Let $A$ be a $C^*$-algebra and let $a \in A$. The following conditions are equivalent:

(i) $a \in CQ(A)$.

(ii) For every $*$-epimorphism $\phi : A \to B$, where $B$ is another $C^*$-algebra, $\phi(a) \in Z(B)$ implies $a \in Z(A) + \ker \phi$.

(iii) $a \in Z(A) + \text{Id}_A([a, A])$.

**Proposition 4.3.** Let $A$ be a $C^*$-algebra.

(a) $CQ(A)$ is a self-adjoint subset of $A$ that is closed under scalar multiplication.

(b) $Z(A) + CQ(A) \subseteq CQ(A)$.

(c) If $I$ is an ideal of $A$ then $CQ(I) = I \cap CQ(A)$. In particular, $I$ has the CQ-property if and only if $I \subseteq CQ(A)$.

(d) If $A$ is unital, then for any $a \in CQ(A)$ and invertible $x \in A$ we have $xa_x^{-1} \in CQ(A)$.

**Proof.** (a) This is trivial.

(b) Let $a \in CQ(A)$ and $z \in Z(A)$. By Proposition 4.2, $a \in Z(A) + \text{Id}_A([a, A])$, so

$$z + a \in Z(A) + \text{Id}_A([a, A]) = Z(A) + \text{Id}_A([z + a, A]).$$

Using again Proposition 4.2 it follows that $z + a \in CQ(A)$.

(c) Let $a \in CQ(I)$. By Proposition 4.2 $a \in Z(I) + \text{Id}_I([a, I])$. Since $Z(I) = I \cap Z(A) \subseteq Z(A)$ and $\text{Id}_I([a, I]) \subseteq \text{Id}_A([a, A])$, we get $a \in Z(A) + \text{Id}_A([a, A])$. Therefore $a \in CQ(A)$, so $CQ(I) \subseteq I \cap CQ(A)$. 


Conversely, let \( a \in I \cap \text{CQ}(A) \) and \( \varepsilon > 0 \). By Proposition 4.2, there is a finite number of elements \( u_i, v_i, x_i \in A \) and \( z \in Z(A) \) such that
\[
\left\| a - z - \sum_i u_i[a, x_i]v_i \right\| < \frac{\varepsilon}{3}.
\] (4.1)

In particular, \( \|z + I\| < \varepsilon/3 \), so using the canonical isomorphism \((Z(A) + I)/I \cong Z(A)/(I \cap Z(A))\) we can find an element \( z' \in I \cap Z(A) = Z(I) \) such that
\[
\|z - z'\| < \frac{\varepsilon}{3}.
\] (4.2)

Let \((e_\alpha)\) be an approximate identity for \( I \). Then for all indices \( i \)
\[
\lim_{\alpha} u_i e_\alpha[a, e_\alpha x_i] e_\alpha v_i = u_i[a, x_i] v_i.
\] (4.3)

Indeed, since the multiplication on \( A \) is continuous, it suffices to show that for any \( x \in A \),
\[
\lim_{\alpha} e_\alpha[a, e_\alpha x] e_\alpha = [a, x].
\]

But this follows directly from the estimate
\[
\| e_\alpha[a, e_\alpha x] e_\alpha - [a, x] \| \leq \| e_\alpha ([a, e_\alpha x] - [a, x]) e_\alpha \| + \| e_\alpha ([a, x] e_\alpha - [a, x]) \| + \| e_\alpha [a, x] - [a, x] \|
\]
\[
\leq \| [a, e_\alpha x] - [a, x] \| + \| [a, x] e_\alpha - [a, x] \| + \| e_\alpha [a, x] - [a, x] \|.
\]

Hence, by (4.3) there are \( u'_i, v'_i, x'_i \in I \) such that
\[
\left\| \sum_i u_i[a, x_i] v_i - \sum_i u'_i[a, x'_i] v'_i \right\| < \frac{\varepsilon}{3}.
\] (4.4)

Then by (4.1), (4.2) and (4.4)
\[
\left\| a - z' - \sum_i u'_i[a, x'_i] v'_i \right\| < \varepsilon.
\]

Invoking again Proposition 4.2, we conclude that \( a \in \text{CQ}(I) \), so \( I \cap \text{CQ}(A) \subseteq \text{CQ}(I) \).

(d) Assume that \( A \) is unital, \( a \in \text{CQ}(A) \) and \( x \in A \) invertible. If \( I \) is an arbitrary ideal of \( A \) such that \( xax^{-1} + I \subseteq Z(A/I) \), then \( a + I \in Z(A/I) \). Since \( a \in \text{CQ}(A) \), this implies \( a \in Z(A) + I \). Then also \( xax^{-1} \in Z(A) + I \), so \( xax^{-1} \in \text{CQ}(A) \).

\[ \]

**Corollary 4.4.** If \( A \) is a \( C^* \)-algebra, then \( Z(A) + J_{wc}(A) \subseteq \text{CQ}(A) \).

**Proof.** By Theorem 3.22 and Corollary 3.9, \( J_{wc}(A) = \ker T_A \) has the CQ-property, so by Proposition 4.3 (c), \( J_{wc}(A) \subseteq \text{CQ}(A) \). It remains to apply Proposition 4.3 (b).

**Proposition 4.5.** Let \( A \) be a \( C^* \)-algebra.
(a) All commutators \([a, b] \ (a, b \in A)\) belong to \(\text{CQ}(A)\). In particular, \(\text{CQ}(A) = Z(A)\) if and only if \(A\) is abelian.

(b) All quasi-nilpotent elements of \(A\) belong to \(\text{CQ}(A)\). Moreover, if \(a \in A\) is quasi-nilpotent, then \(ab, ba \in \text{CQ}(A)\) for any \(b \in A\).

\(\Box\)

**Proof.** If \(x \in A\) is a commutator, quasi-nilpotent, or a product by a quasi-nilpotent element, we claim that for any primitive ideal \(P\) of \(A\), \(x + P \in Z(A/P)\) implies \(x \in P\). It then follows that \(x \in \text{CQ}(A)\). Indeed, assume that \(I\) is an ideal of \(A\) such that \(x + I \in Z(A/I)\). Then \(x + P \in Z(A/P)\) for any \(P \in \text{Prim}(A)\), so \(x \in P\). As \(\ker \text{Prim}(A) = I\), it follows that \(x \in I\) and thus \(x \in \text{CQ}(A)\) as claimed.

So assume that \(P\) is a primitive ideal of \(A\) such that \(x + P \in Z(A/P)\). If \(P\) is non-modular, then \(Z(A/P) = \{0\}\), so trivially \(x \in P\). Hence, assume that \(P\) is modular, so that \(Z(A/P) \cong \mathbb{C}\). Then there is a scalar \(\lambda\) such that

\[ x + P = \lambda I_{A/P}. \] (4.5)

(a) Assume that \(x\) is a commutator, so that \(x = [a, b]\) for some \(a, b \in A\). Then by (4.5),

\[ [a + P, b + P] = x + P = \lambda I_{A/P}. \]

As \(A/P\) is a unital \(\mathcal{C}^*\)-algebra, it is well-known that this is only possible if \(\lambda = 0\). Thus \(x \in P\) as claimed.

If \(A\) is non-abelian, then there are \(a, b \in A\) such that \(x := [a, b] \neq 0\). Then there is \(P \in \text{Prim}(A)\) such that \(x \notin P\), so by the above arguments \(x + P \notin Z(A/P)\). In particular, \(x \in \text{CQ}(A) \setminus Z(A)\).

(b) If \(x\) is quasi-nilpotent, so is \(x + P\), since by the spectral radius formula

\[ \nu(x + P) = \lim_n \|x^n + P\|^\frac{1}{n} = \lim_n \|x^n\|^\frac{1}{n} = \nu(x) = 0. \]

This together with (4.5) forces \(\lambda = 0\), so \(x \in P\).

Now assume that \(x = ab\), where \(a, b \in A\) and \(a\) is quasi-nilpotent. As \(a + P\) is quasi-nilpotent, it is a topological divisor of zero (see e.g. [25, Section XXIX.4]). Hence, there is a sequence of elements \((x_n)\) in \(A\) such that

\[ \|x_n + P\| = 1 \quad \forall n \in \mathbb{N} \quad \text{and} \quad \lim_n \|x_na + P\| = 0. \]

Then, by (4.5), for all \(n \in \mathbb{N}\) we have

\[ \lambda x_n + P = (x_n + P)(x + P) = (x_na + P)(b + P), \]

so

\[ |\lambda| \leq \|x_na + P\|b + P|. \] (4.6)
Since the right side in (4.6) tends to zero as \( n \) tends to infinity, we conclude that \( \lambda = 0 \). Therefore \( x \in P \) as claimed.

Finally, using the facts that \( a \in A \) is quasi-nilpotent if and only if \( a^* \) is quasi-nilpotent and that \( \text{CQ}(A) \) is a self-adjoint subset of \( A \) (Proposition 4.3 (a)), we also conclude that \( ba \in \text{CQ}(A) \) for any \( b \in A \) and quasi-nilpotent \( a \in A \).

**Remark 4.6.** By Proposition 4.5 (a), non-abelian \( C^* \)-algebras always contain non-central CQ-elements. On the other hand, in a purely algebraic setting, there are examples of non-abelian algebras in which all centrally stable elements are central \([16, \text{Examples } 2.5 \text{ and } 2.6]\) (where central stability is the algebraic counterpart of the CQ-property).

If a \( C^* \)-algebra \( A \) is unital, the following fundamental result gives a necessary and sufficient condition for a central element of \( A/I \) to lift to a central element of \( A \). Recall that by \( \Psi_A : Z(A) \to C(\text{Prim}(A)) \) we denote the Dauns-Hofmann isomorphism.

**Theorem 4.7.** Let \( A \) be a unital \( C^* \)-algebra and let \( I \) be an ideal of \( A \). Assume that an element \( a \in A \) satisfies \( a + I \in Z(A/I) \). Then \( a \in Z(A) + I \) if and only if

\[
\Psi_{A/I}(a + I)(P_1/I) = \Psi_{A/I}(a + I)(P_2/I)
\]

(4.7)

for all \( P_1, P_2 \in \text{Prim}^I(A) \) such that \( P_1 \cap Z(A) = P_2 \cap Z(A) \).

**Proof.** First assume that \( a \in Z(A) + I \), so that \( a - z \in I \) for some \( z \in Z(A) \). Suppose that \( P_1, P_2 \in \text{Prim}^I(A) \) and that \( P_1 \cap Z(A) = P_2 \cap Z(A) \). For \( i = 1, 2 \), there exists \( \lambda_i \in \mathbb{C} \) such that

\[
a + P_i = z + P_i = \lambda_i 1 + P_i \quad (\text{in } A/P_i).
\]

Then \( z - \lambda_1 1, z - \lambda_2 1 \in P_1 \cap Z(A) \) and so \( (\lambda_1 - \lambda_2) 1 \in P_1 \). It follows that \( \lambda_1 = \lambda_2 =: \lambda \), say. Hence

\[
(a + I) + P_1/I = \lambda(1 + I) + P_1/I \quad (\text{in } (A/I)/(P_1/I))
\]

and therefore

\[
\Psi_{A/I}(a + I)(P_1/I) = \lambda = \Psi_{A/I}(a + I)(P_2/I).
\]

Conversely, assume that the equality (4.7) holds for all \( P_1, P_2 \in \text{Prim}^I(A) \) such that \( P_1 \cap Z(A) = P_2 \cap Z(A) \). Since \( A \) is unital, \( \text{Prim}(A) \) is compact and so the closed subset \( \text{Prim}^I(A) \) is also compact. Define a function

\[
f \in C(\text{Prim}^I(A)) \quad \text{by the formula } \quad f(P) := \Psi_{A/I}(a + I)(P/I).
\]
Let $\phi_A : \text{Prim}(A) \to \text{Glimm}(A)$ be the complete regularization map (see Section 2). Since Prim$^I(A)$ is a compact subspace of Prim$(A)$, $\phi_A$ continuous and Glimm$(A)$ a compact Hausdorff space, $K := \phi_A(\text{Prim}$$^I(A))$ is a compact (hence closed) subset of Glimm$(A)$. Define a function

$$g : K \to \mathbb{C} \quad \text{by} \quad g(G) := f(P),$$

where $P$ is any primitive ideal in Prim$^I(A)$ such that $\phi_A(P) = G$. Since $f(P_1) = f(P_2)$ for any two $P_1, P_2 \in$ Prim$^I(A)$ such that $P_1 \cap Z(A) = P_2 \cap Z(A)$ (which is equivalent to $\phi_A(P_1) = \phi_A(P_2)$), $g$ is well-defined. We claim that $g$ is continuous on $K$. Indeed, let $(G_\alpha)$ be an arbitrary net in $K$ that converges to some $G_0 \in K$. By general topology, it suffices to show that for any subnet $(G_\alpha(\beta))$ of $(G_\alpha)$ there is a further subnet $(G_\alpha(\beta(\gamma)))$ such that $(g(G_\alpha(\beta(\gamma))))$ converges to $g(G_0)$. For each index $\beta$ choose $P_\alpha(\beta) \in$ Prim$^I(A)$ such that $\phi_A(P_\alpha(\beta)) = G_\alpha(\beta)$. Then $(P_\alpha(\beta))$ is a net in the compact space Prim$^I(A)$, so it has a subnet $(P_\alpha(\beta(\gamma)))$ convergent to some $P_0 \in$ Prim$^I(A)$. Since $\phi_A$ is continuous and Glimm$(A)$ Hausdorff, $G_0 = \phi_A(P_0)$. Further, since $f$ is continuous on Prim$^I(A)$, $(f(P_\alpha(\beta(\gamma))))$ converges to $f(P_0)$. Therefore

$$\lim_{\gamma} g(G_\alpha(\beta(\gamma))) = \lim_{\gamma} f(P_\alpha(\beta(\gamma))) = f(P_0) = g(G_0).$$

By the Tietze extension theorem, there exists a continuous function $\tilde{g} \in C(\text{Glimm}(A))$ that extends $g$. Then a function

$$\tilde{f} : \text{Prim}(A) \to \mathbb{C} \quad \text{defined by} \quad \tilde{f} := \tilde{g} \circ \phi_A$$

is continuous, so by the Dauns-Hofmann theorem there is $z \in Z(A)$ such that $\Psi_A(z) = \tilde{f}$. Since for any $P \in$ Prim$^I(A)$ we have $\tilde{f}(P) = f(P)$, we conclude $a - z \in P$. Thus $a - z \in I$, so $a \in Z(A) + I$ as desired.

We now describe the set CQ$(A)$ for an arbitrary C$^*$-algebra $A$. It is somewhat easier to describe its complement $V_A$. In order to do this, we introduce the following sets:

- $V_A^1$ as the set of all $a \in A$ for which there exists $M \in \text{Max}(A)$ such that $Z(A) \subseteq M$ and $a + M$ is a non-zero scalar in $A/M$,
- $V_A^2$ as the set of all $a \in A$ for which there exist $M_1, M_2 \in \text{Max}(A)$ and scalars $\lambda_1 \neq \lambda_2$ such that $Z(A) \nsubseteq M_i$, $M_1 \cap Z(A) = M_2 \cap Z(A)$ and $a + M_i = \lambda_i 1_{A/M_i}$ ($i = 1, 2$).

**Theorem 4.8.** If $A$ is a C$^*$-algebra then $V_A = V_A^1 \cup V_A^2$.

**Proof.** Assume there exists $a \in V_A^1 \setminus V_A$. Then $a \in$ CQ$(A)$ and there is $M \in \text{Max}(A)$ such that $Z(A) \subseteq M$ and $a + M$ is a non-zero scalar in $A/M$. In particular, $a + M \in Z(A/M)$, so the CQ-condition implies $a \in Z(A) + M = M$; a contradiction. This shows $V_A^1 \subseteq V_A$.

Now assume there exists $a \in V_A^2 \setminus V_A$ and let $M_1, M_2 \in \text{Max}(A)$ and $\lambda_1, \lambda_2 \in \mathbb{C}$, $\lambda_1 \neq \lambda_2$, such that $Z(A) \nsubseteq M_i$, $M_1 \cap Z(A) = M_2 \cap Z(A)$ and $a + M_i = \lambda_i 1_{A/M_i}$ ($i = 1, 2$). Then by maximality of $M_1$ and $M_2$...
we have $M_1 + M_2 = A$, so $A/(M_1 \cap M_2) \cong (A/M_1) \oplus (A/M_2)$. Hence $a + (M_1 \cap M_2) \in Z(A/(M_1 \cap M_2))$. Since $a \in \text{CQ}(A)$, this forces $a \in Z(A) + (M_1 \cap M_2)$. Choose a central element $z \in Z(A)$ such that $a - z \in M_1 \cap M_2$.

Obviously, $z + M_i = \lambda_i 1_{A/M_i}$, $(i = 1, 2)$. On the other hand, under the canonical isomorphisms

$$\frac{Z(A) + M_1}{M_1} \cong \frac{Z(A)}{M_1 \cap Z(A)} = \frac{Z(A)}{M_2 \cap Z(A)} \cong \frac{Z(A) + M_2}{M_2},$$

$z + M_1$ is mapped to $z + M_2$. This implies $\lambda_1 = \lambda_2$; a contradiction. Therefore $V_A^2 \subseteq V_A$.

Conversely, let $a \in V_A$.

**Case 1.** $A$ is unital.

In this case $V_A^1 = \emptyset$. Since $a \notin \text{CQ}(A)$, there exists an ideal $I$ of $A$ such that $a + I \in Z(A/I)$ but $a \notin Z(A) + I$. For each $\lambda \in \text{Prim}^I(A)$ set

$$\lambda_P := \Psi_{A/I}(a + I)(P/I),$$

where $\Psi_{A/I} : Z(A/I) \to C(\text{Prim}(A/I))$ is the Dauns-Hofmann isomorphism. By Theorem 4.7, there are $P_1, P_2 \in \text{Prim}^I(A)$ such that $P_1 \cap Z(A) = P_2 \cap Z(A)$ and $\lambda_{P_1} \neq \lambda_{P_2}$. Then $a - \lambda_{P_1} I \in P_i$ $(i = 1, 2)$. Choose maximal ideals $M_1, M_2$ of $A$ such that $P_1 \subseteq M_1$ and $P_2 \subseteq M_2$. Since $A/M_i$ is a quotient of $A/P_i$, it follows that $\lambda_{M_i} = \lambda_{P_i}$, so $a - \lambda_{P_i} I \in M_i$ $(i = 1, 2)$. Therefore, $a + M_1$ and $a + M_2$ are distinct scalars in $A/M_1$ and $A/M_2$, which implies $a \in V_A^2$.

**Case 2.** $A$ is non-unital.

In this case we work inside the unitization $A^\sharp$. By Proposition 4.3(c) $a \in V_A$. Then, using Case 1, there are maximal ideals $M_1$ and $M_2$ of $A^\sharp$ and scalars $\lambda_1 \neq \lambda_2$ such that $M_1 \cap Z(A^\sharp) = M_2 \cap Z(A^\sharp)$ and $a - \lambda_i I \in M_i$ $(i = 1, 2)$. We have two possibilities.

**Case 2.1.** One of $M_1, M_2$ coincides with $A$. Say $M_1 = A$. Then $\lambda_1 = 0$ (since $a$ belongs to $A$), so $\lambda_2 \neq 0$.

In this case $M_2 \neq A$, since otherwise $\lambda_1 = \lambda_2 = 0$; a contradiction. Then, by Lemma 2.2, $N_2 := M_2 \cap A$ is a modular maximal ideal of $A$. Since $Z(A) = A \cap Z(A^\sharp) = M_2 \cap Z(A^\sharp)$, $N_2$ contains $Z(A)$. Under the canonical isomorphism

$$\frac{A}{N_2} \cong \frac{A + M_2}{M_2} = \frac{A^\sharp}{M_2},$$

$a + N_2$ is mapped to $a + M_2 = \lambda_2 1 + M_2$, so $a + N_2 = \lambda_2 1_{A/N_2}$. Since $\lambda_2 \neq 0$, we conclude that $a$ belongs to $V_A^1$.

**Case 2.2.** $M_1 \neq A$ and $M_2 \neq A$. Then, by Lemma 2.2, $N_i := M_i \cap A$ $(i = 1, 2)$ are modular maximal ideals of $A$ that have the same intersection with $Z(A)$. Similarly as in Case 2.1, using the canonical isomorphisms $A/N_i \cong A^\sharp/M_i$, we conclude that $a + N_i = \lambda_i 1_{A/N_i}$ $(i = 1, 2)$, which implies that $a$ belongs to $V_A^2$.

Therefore $a \in V_A^1 \cup V_A^2$, so $V_A = V_A^1 \cup V_A^2$ as claimed.
Remark 4.9. Theorem 4.8 enables us to recapture Corollary 3.9 without using Vesterstrøm’s theorem for the unital case (Theorem 1.1). Indeed, if $A$ is weakly central then clearly $V_A$ is empty and so $A$ has the CQ-property. Conversely, suppose that $A$ is not weakly central. If there exists $M \in \text{Max}(A)$ such that $Z(A) \subseteq M$ then, taking any $a \in A$ such that $a + M = 1_{A/M}$, we obtain $a \in V_A^1$. Otherwise, there exist distinct $M_1, M_2 \in \text{Max}(A)$ such that $M_1 \cap Z(A) = M_2 \cap Z(A) \neq Z(A)$. Since $M_1 + M_2 = A$, there exists $a \in M_1$ such that $a + M_2 = 1_{A/M_2}$ and hence $a \in V_A^2$. Thus $V_A$ is non-empty and so $A$ does not have the CQ-property.

Also, the methods of this section enable us to give a short alternative proof of the fact that $\ker T_A$ is weakly central (Theorem 3.22). By the preceding paragraph $\ker T_A$ is weakly central if and only if $V_{\ker T_A} = 0$.

By Theorem 4.8 and Proposition 4.3 (c) it suffices to show that $a \in \ker T_A$ implies $a \notin V_A^1 \cup V_A^2 = V_A$. But this is trivial, since for any $M \in \text{Max}(A)$ that contains $Z(A)$ we have $M \in T_A^1$, so $a \in M$ and therefore $a + M$ is zero in $A/M$. Similarly, for all $M_1, M_2 \in \text{Max}(A)$ such that $M_1 \neq M_2$ and $M_1 \cap Z(A) = M_2 \cap Z(A) \neq Z(A)$ we have $M_1, M_2 \in T_A^2$, so $a \in M_1 \cap M_2$ and hence $a + M_i$ is zero in $A/M_i$ ($i = 1, 2$).

If $A$ is a $C^*$-algebra then by Proposition 4.5 (a) all commutators $[a, b]$ ($a, b \in A$) belong to $\text{CQ}(A)$. Let us denote by $[A, A]$ the linear span of all commutators of $A$ and by $[A, A]$ its norm-closure. We now characterise when $\text{CQ}(A)$ contains $[A, A]$.

Theorem 4.10. Let $A$ be a $C^*$-algebra that is not weakly central.

(a) If for all $M \in T_A$, $A/M$ admits a tracial state then $[A, A] \subseteq \text{CQ}(A)$.

(b) If there is $M \in T_A$ such that $A/M$ does not admit a tracial state, then $[A, A] \nsubseteq \text{CQ}(A)$.

Proof. (a) Let $x \in [A, A]$. In order to show that $x \in \text{CQ}(A)$, it suffices by Theorem 4.8 to prove that for each $M \in T_A$, $x + M \in Z(A/M)$ implies $x \in M$. Therefore, fix some $M \in T_A$ and assume that $x + M \in Z(A/M)$, so that $x + M = \lambda 1_{A/M}$ for some scalar $\lambda$. By assumption $A/M$ admits a tracial state $\tau$. As $x \in [A, A]$, clearly $x + M \in [A/M, A/M]$. Since $\tau([A/M, A/M]) = \{0\}$, we get

$$\lambda = \tau(\lambda 1_{A/M}) = \tau(x + M) = 0.$$  

Thus $x \in M$, as claimed.

(b) Assume that $A/M$ does not admit a tracial state for some $M \in T_A$. As $T_A = T_A^1 \cup T_A^2$, we have two possibilities.

Case 1. $M \in T_A^1$, so that $Z(A) \subseteq M$. By [10] Theorem 1] there is an integer $n > 1$, that depends only on $A/M$, such that any element of $A/M$ can be expressed as a sum of $n$ commutators. In particular, there are...
$a_1, \ldots, a_n, b_1, \ldots, b_n \in A$ such that

$$\sum_{i=1}^{n} [a_i, b_i] + M = \frac{1}{A/M},$$

so by Theorem 4.8

$$\sum_{i=1}^{n} [a_i, b_i] \in V_A^1 \subseteq V_A.$$

**Case 2.** $M \in T^2_A$. Then $Z(A) \not\subseteq M$ and there exists $N \in \text{Max}(A)$ such that $N \neq M$ and $M \cap Z(A) = N \cap Z(A)$. By Lemma 2.2, $M \cap N$ is a modular maximal ideal of $N$. As $N/(M \cap N) \cong A/M$, $N/(M \cap N)$ also does not admit a tracial state, so by [40, Theorem 1] there is an integer $n > 1$ and elements $a_1, \ldots, a_n, b_1, \ldots, b_n \in N$ such that

$$\sum_{i=1}^{n} [a_i, b_i] + M \cap N = 1_{N/(M \cap N)}.$$

Using the canonical isomorphism $N/(M \cap N) \cong A/M$, we get

$$\sum_{i=1}^{n} [a_i, b_i] + M = 1_{A/M}$$

and thus by Theorem 4.8

$$\sum_{i=1}^{n} [a_i, b_i] \in V_A^2 \subseteq V_A.$$

**Corollary 4.11.** If $A$ is a postliminal $C^*$-algebra or an AF-algebra, then $[A, A] \subseteq \text{CQ}(A)$. 

**Proof.** If $A$ is weakly central, then $\text{CQ}(A) = A$ so we have nothing to prove. Hence assume that $A$ is not weakly central, so that there is $M \in T_A$. If $A$ is postliminal then by Remark 3.13 $A/M \cong M_n(\mathbb{C})$ for some $n \in \mathbb{N}$, so $A/M$ has a (unique) tracial state. If, on the other hand, $A$ is an AF-algebra, then $A/M$ is a unital simple AF-algebra, so it also admits a tracial state (see e.g. [33, Proposition 3.4.11]). Therefore, the assertion follows directly from Theorem 4.10 (a). 

By Corollary 4.3 for any $C^*$-algebra $A$, $\text{CQ}(A)$ always contains $Z(A) + J_{wc}(A)$. The next result in particular demonstrates that $\text{CQ}(A)$ is a $C^*$-subalgebra of $A$ if and only if $\text{CQ}(A) = Z(A) + J_{wc}(A)$. In fact, when this does not hold, $\text{CQ}(A)$ fails dramatically to be a $C^*$-algebra.

**Theorem 4.12.** Let $A$ be a $C^*$-algebra. The following conditions are equivalent:

(i) $A/J_{wc}(A)$ is abelian.
(ii) $\text{CQ}(A) = Z(A) + J_{wc}(A)$.
(iii) $\text{CQ}(A)$ is closed under addition.
(iv) $\text{CQ}(A)$ is closed under multiplication.
(v) $\text{CQ}(A)$ is norm-closed.

\[\square\]

**Remark 4.13.** Since $T_A$ is dense in $\text{Prim}^{I_{wc}(A)}(A)$, it follows from [21 Proposition 3.6.3] that $A/J_{wc}(A)$ is non-abelian if and only if there is $M \in T_A$ such that $\dim(A/M) > 1$.

\[\square\]

**Proof of Theorem 4.12.** (i) $\implies$ (ii). Assume that $A/J_{wc}(A)$ is abelian. By Corollary 4.4 we already know that $Z(A) + J_{wc}(A) \subseteq \text{CQ}(A)$, so it suffices to show the reverse inclusion. For any $a \in A$ we have $a + J_{wc}(A) \in A/J_{wc}(A) = Z(A/J_{wc}(A))$, so if $a \in \text{CQ}(A)$, this forces $a \in Z(A) + J_{wc}(A)$. Therefore $\text{CQ}(A) = Z(A) + J_{wc}(A)$, as claimed.

(ii) $\implies$ (iii), (iv), (v) is trivial, since $Z(A) + J_{wc}(A)$ is a $C^*$-subalgebra of $A$.

(iii), (iv) or (v) $\implies$ (i). Assume that $A/J_{wc}(A)$ is non-abelian. By Remark 4.13 there is $M \in T_A$ such that $\dim(A/M) > 1$. We show that $\text{CQ}(A)$ is not norm-closed and is neither closed under addition nor closed under multiplication. As $A/M$ is non-abelian, by [29 Exercise 4.6.30] $A/M$ contains a nilpotent element $\dot{q}$ of nilpotency index 2. By [1 Proposition 2.8] (see also [38 Theorem 6.7]), we may lift $\dot{q}$ to a nilpotent element $q \in A$ of the same nilpotency index 2. As the norm function $\text{Prim}(A) \ni P \mapsto \|q + P\|$ is lower semi-continuous on $\text{Prim}(A)$ (see e.g. [14 Proposition II.6.5.6 (iii)]) and $q \notin M$, the set

$$U := \{P \in \text{Prim}(A) : \|q + P\| > 0\}$$

(4.8)

is an open neighbourhood of $M$ in $\text{Prim}(A)$. As $T_A = T_A^1 \cup T_A^2$ we have two possibilities.

**Case 1.** $M \in T_A^1$, so that $Z(A) \subseteq M$. Let $I$ be the ideal of $A$ that corresponds to $U$, so that $U = \text{Prim}_I(A)$. As $\text{CQ}(I) = I \cap \text{CQ}(A)$ (Proposition 4.3 (c)), it suffices to show that $\text{CQ}(I)$ is not norm-closed and is neither closed under addition nor closed under multiplication.

By Lemma 2.2 $M \cap I$ is a modular maximal ideal of $I$ that contains $Z(I) = I \cap Z(A)$, so that $M \cap I \in T_I^1$. Choose a self-adjoint element $a \in I$ such that

$$a + (M \cap I) = 1_{I/(M \cap I)}.$$  

(4.9)

For each non-zero scalar $\mu \in \mathbb{C}$ consider the element

$$x_\mu := a (a + \mu q) \in I.$$  

We claim that for any $N' \in \text{Max}(I)$, $x_\mu + N' \in Z(I/N')$ implies $x_\mu \in N'$, so that $x_\mu \in \text{CQ}(I)$ (Theorem 4.8). Indeed, assume there is $N' \in \text{Max}(I)$ such that $x_\mu + N' \in Z(I/N')$. By Lemma 2.2 there exists $N \in \text{Max}_I(A)$ such that $N' = N \cap I$. Then

$$x_\mu + (N \cap I) = \lambda 1_{I/(N \cap I)}$$
for some scalar $\lambda$, so using the canonical isomorphism $I/(N \cap I) \cong A/N$ we get

$$(a + N)((a + N) + \mu(q + N)) = x_\mu + N = \lambda 1_{A/N}. \tag{4.10}$$

Suppose that $\lambda \neq 0$. Then, by (4.10), the element $a + N$ is right invertible in $A/N$. Since $a + N$ is self-adjoint, it must be invertible in $A/N$. As $\mu \neq 0$, (4.10) implies

$$q + N = \frac{1}{\mu} \left( \lambda(a + N)^{-1} - (a + N) \right). \tag{4.11}$$

The right side in (4.11) defines a normal element of $A/N$, as a linear combination of two commuting self-adjoint elements of $A/N$. Hence, $q + N$ is a normal nilpotent element of $A/N$ which implies $q \in N$. But as $N \in \text{Max}_1(A)$, $N$ belongs to $U$, which contradicts (4.8). Thus $\lambda = 0$ and so $x_\mu \in \text{CQ}(I)$ as claimed.

We claim that

$$x_{-1} + x_1 \not\in \text{CQ}(I) \quad \text{and} \quad x_{-1}x_1 \not\in \text{CQ}(I).$$

Indeed, by (4.9) we have

$$x_{-1} + x_1 + (M \cap I) = (a(a - q) + (M \cap I)) + (a(a + q) + (M \cap I)) = 2a^2 + (M \cap I)$$

$$= 21_{I/(M \cap I)}.$$

Further, since $q^2 = 0$, we have

$$x_{-1}x_1 + (M \cap I) = (a(a - q) + (M \cap I))(a(a + q) + (M \cap I))$$

$$= (1_{I/(M \cap I)} - (q + (M \cap I)))(1_{I/(M \cap I)} + (q + (M \cap I)))$$

$$= 1_{I/(M \cap I)}.$$

Therefore, both $x_{-1} + x_1$ and $x_{-1}x_1$ belong to $V_I^1 \subseteq V_I = I \setminus \text{CQ}(I)$ (Theorem 4.8), which shows that $\text{CQ}(I)$ is neither closed under addition nor closed under multiplication.

It remains to show that $\text{CQ}(I)$ is not norm-closed. In order to do this, consider the sequence

$$y_k := x_k^\frac{1}{k} = a \left( a + \frac{1}{k}q \right) \quad (k \in \mathbb{N}).$$

Then $(y_k)$ is a sequence in $\text{CQ}(I)$ that converges to $a^2$. As $a^2 + (M \cap I) = 1_{I/(M \cap I)}$ (by (4.9)), we conclude that $a^2 \in V_I^1 \subseteq V_I$ (Theorem 4.8), so the proof for this case is finished.

Case 2. $M \in T_A^2$. Then $Z(A) \not\subseteq M$ and there exists $N \in \text{Max}(A)$ such that $N \neq M$ and $M \cap Z(A) = N \cap Z(A)$. As singleton subsets of $\text{Max}(A)$ are closed in $\text{Prim}(A)$, $U' := U \setminus \{N\}$ is also an open neighbourhood
of $M$ in Prim($A$). Let $J$ be the ideal of $A$ that corresponds to $U'$. Then, by Lemma 2.2, $M \cap J \in \text{Max}(J)$ and $\dim(J/(M \cap J)) = \dim(A/M) > 1$. Further, since $N \not\subseteq \langle U' \rangle$, $J \subseteq N$, so

$$Z(J) = (N \cap Z(A)) \cap J = (M \cap Z(A)) \cap J.$$  

This implies $Z(J) \subseteq M \cap J$ and therefore $M \cap J \in T_1^J$. Also, by (4.8), we have trivially $\|q + P\| > 0$ for all $P \in U'$. By applying the method of Case 1 to $J$ in place of $I$, we conclude that CQ($J$) is not norm-closed and is neither closed under addition nor closed under multiplication. As CQ($J$) = $J \cap$ CQ($A$) (Proposition 4.3 (c)), the same is true for CQ($A$).

**Remark 4.14.** Observe that Theorem 4.12 applies to Example 3.28, giving CQ($A$) = $C_1 + K(H)$.

**Corollary 4.15.** If $A$ is a 2-subhomogeneous $C^*$-algebra, then CQ($A$) = Z($A$) + $J_{wc}(A)$.

**Proof.** Since

$$\{P \in \text{Prim}(A) : \dim(A/P) = 1\}$$

is a closed subset of Prim($A$) (see e.g. [21 Proposition 3.6.3]), $A$ has a 2-homogeneous ideal $I$ such that $A/I$ is abelian. Then $I$ is a central $C^*$-algebra (Remark 3.18) and so $I \subseteq J_{wc}(A)$. Hence $A/J_{wc}(A)$ is abelian and so the result follows from Theorem 4.12.

In the case when a $C^*$-algebra $A$ is postliminal or an AF-algebra, we also show that the conditions (i)-(v) of Theorem 4.12 are equivalent to one additional condition.

**Corollary 4.16.** If $A$ is a postliminal $C^*$-algebra or an AF-algebra, then the conditions (i)-(v) of Theorem 4.12 are also equivalent to:

(vi) For any $x \in \text{CQ}(A)$, $x^n \in \text{CQ}(A)$ for all $n \in \mathbb{N}$.

In the proof of Corollary 4.16 we shall use the next two facts. In the sequel we say that a $C^*$-subalgebra $A$ of a unital $C^*$-algebra $B$ is *co-unital* if $A$ contains the identity of $B$.

**Lemma 4.17.** Let $B$ be a unital simple non-abelian AF-algebra. Then $B$ contains a co-unital finite-dimensional $C^*$-subalgebra with no abelian summand.

**Proof.** Let $(B_k)_{k \in \mathbb{N}}$ be an increasing sequence of finite-dimensional $C^*$-subalgebras of $B$ such that $1_B \in B_k$ for all $k \in \mathbb{N}$ and

$$B = \bigcup_{k \in \mathbb{N}} B_k.$$  

We claim that there exists $k \in \mathbb{N}$ such that $B_k$ has no direct summand $*$-isomorphic to $C$. On a contrary, suppose that for every $k \in \mathbb{N}$, $B_k$ has a direct summand $*$-isomorphic to $C$ and hence a multiplicative state $\omega_k$. For each
Thus, \( a, b \in S(B) \) be an extension of \( \omega_k \). By the weak*-compactness of \( S(B) \) there exists \( \psi \in S(B) \) and a subnet \((\psi_{k(\alpha)})\) of \((\psi_k)\) such that

\[
\psi = \psi^* - \lim_{\alpha} \psi_{k(\alpha)}.
\]

Let \( a \in B_{j_0} \) and \( b \in B_{k_0} \) for some \( j_0, k_0 \in \mathbb{N} \). There exists an index \( \alpha_0 \) such that \( k(\alpha) \geq \max\{j_0, k_0\} \) for all \( \alpha \geq \alpha_0 \). Thus

\[
\psi_{k(\alpha)}(ab) = \psi_{k(\alpha)}(a)\psi_{k(\alpha)}(b)
\]

for all \( \alpha \geq \alpha_0 \) and so

\[
\psi(ab) = \psi(a)\psi(b).
\]  (4.12)

Now suppose that \( a \in B_{j_0} \) for some \( j_0 \in \mathbb{N} \), that \( b \in B \) and that \( \varepsilon > 0 \). Then there exists \( k_0 \in \mathbb{N} \) and \( b_0 \in B_{k_0} \) such that

\[
\|b - b_0\| < \frac{\varepsilon}{2(1 + \|a\|)}.
\]

Then

\[
|\psi(ab) - \psi(a)\psi(b)| \leq |\psi(ab) - \psi(ab_0)| + |\psi(a)\psi(b_0) - \psi(a)\psi(b)|
\]

\[
\quad \leq 2\|a\|\|b - b_0\| < \varepsilon.
\]

Thus, again (4.12) holds. A similar approximation in the first variable shows that (4.12) holds for all \( a, b \in B \). Thus, \( B \) has a multiplicative state, contradicting the fact that \( B \) is simple but not \(*\)-isomorphic to \( \mathbb{C} \). \( \square \)

**Lemma 4.18.** Let \( A \) be a \( C^*\)-algebra such that for some \( M \in \text{Max}(A) \), \( A/M \) contains a co-unital finite-dimensional \( C^*\)-subalgebra with no abelian summand. Then there are \( a, b \in A \) and an integer \( n > 1 \) such that

\[
[a, b]^n + M = 1_{A/M}.
\]  (4.13)

**Proof.** Assume that \( B \) is a co-unital finite-dimensional \( C^*\)-subalgebra of \( A/M \) with no abelian summand. Then there are integers \( n_1, \ldots, n_k > 1 \) and a \(*\)-isomorphism \( \phi : B \to M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C}) \). For each \( i = 1, \ldots, k \) let \( \{\alpha_1^{(i)}, \ldots, \alpha_{n_i}^{(i)}\} \) be the set of all \( n_i\)-th roots of unity. It is well-known that the set of all commutators of \( M_{n_i}(\mathbb{C}) \) consists precisely of all matrices of trace zero. Hence, as \( \alpha_1^{(i)} + \cdots + \alpha_{n_i}^{(i)} = 0 \) for all \( i = 1, \ldots, k \), there are elements \( a, b \in A \) such that \( a + M, b + M \in B \) and

\[
\phi([a, b] + M) = \text{diag}(\alpha_1^{(1)}, \ldots, \alpha_{n_1}^{(1)}) \oplus \cdots \oplus \text{diag}(\alpha_1^{(k)}, \ldots, \alpha_{n_k}^{(k)}).
\]
Let $n$ be the least common multiple of $n_1, \ldots, n_k$. Then

$$\phi([a, b] + M)^n = 1_{M_{n_1}(\mathbb{C})} \oplus \cdots \oplus 1_{M_{n_k}(\mathbb{C})}.$$  

Since $B$ is co-unital in $A/M$, this is equivalent to (4.13). □

**Proof of Corollary 4.16.** By Theorem 4.12 we only have to prove the implication (vi) $\implies$ (i). Assume that (i) does not hold. By Remark 4.13 there is $M \in TA$ such that $\dim(A/M) > 1$. If $A$ is postliminal or AF (respectively), then $A/M$ is a unital simple $C^*$-algebra that is postliminal or AF (respectively). Hence, by Remark 4.13 and Lemma 4.17, $A/M$ certainly contains a co-unital finite-dimensional $C^*$-subalgebra with no abelian summand. As $TA = T_A^1 \cup T_A^2$ we have two possibilities.

**Case 1.** $M \in T_A^1$, so that $Z(A) \subseteq M$. By Lemma 4.18 there are $a, b \in A$ and an integer $n > 1$ such that for $x := [a, b]$ we have $x^n + M = 1_{A/M}$. In particular $x^n \in V_A^1$, so, by Theorem 4.8, $x^n \notin \text{CQ}(A)$. On the other hand, by Proposition 4.5 (a), $x \in \text{CQ}(A)$.

**Case 2.** $M \in T_A^2$. Then $Z(A) \nsubseteq M$ and there exists $N \in \text{Max}(A)$ such that $N \neq M$ and $M \cap Z(A) = N \cap Z(A)$. By Lemma 2.2, $M \cap N$ is a modular maximal ideal of $N$ and $N/(M \cap N) \cong A/M$.

By Lemma 4.18 (applied to $N$) there are $a, b \in N$ and an integer $n > 1$ such that for $x := [a, b]$ we have $x^n + M \cap N = 1_{N/(M \cap N)}$. Then, using the canonical isomorphism $N/(M \cap N) \cong A/M$, we get $x^n + M = 1_{A/M}$. As $x^n \in N$, we have $x^n \in V_A^2$, so $x^n \notin \text{CQ}(A)$ by Theorem 4.8. On the other hand, by Propositions 4.5 (a) and 4.3 (c), $x \in \text{CQ}(N) \subseteq \text{CQ}(A)$. □

The next example shows that 2-subhomogeneity in Corollary 4.15 cannot be replaced by $n$-subhomogeneity, where $n > 2$. It also provides an example of a liminal $C^*$-algebra for which the six equivalent conditions of Corollary 4.16 fail to hold.

**Example 4.19.** Let $A$ be the $C^*$-algebra consisting of all functions $a \in C([0, 1], M_3(\mathbb{C}))$ such that

$$a(1) = \begin{pmatrix} \lambda_{11}(a) & \lambda_{12}(a) & 0 \\ \lambda_{21}(a) & \lambda_{22}(a) & 0 \\ 0 & 0 & \mu(a) \end{pmatrix},$$

for some complex numbers $\lambda_{ij}(a), \mu(a)$ ($i, j = 1, 2$). Then $A$ is a unital 3-subhomogeneous $C^*$-algebra such that

$$Z(A) = \{\text{diag}(f, f, f) : f \in C([0, 1])\}$$

and

$$TA = T_A^2 = \{\ker \pi, \ker \mu\},$$
where \( \pi : A \to M_2(\mathbb{C}) \) and \( \mu : A \to \mathbb{C} \) are irreducible representations of \( A \) defined by the assignments \( \pi : a \mapsto (\lambda_{ij}(a)) \) and \( \mu : a \mapsto \mu(a) \). Hence, by Theorem 3.22,

\[
J_{\text{we}}(A) = \ker T_A = \{ a \in A : a(1) = 0 \}
\]

and so

\[
Z(A) + J_{\text{we}}(A) = \{ a \in A : a(1) \text{ is a scalar matrix} \}.
\]

As \( A/\ker \pi \cong M_2(\mathbb{C}) \), it follows from Theorem 4.12 and the proofs of Lemma 4.18 and Corollary 4.16 that \( \text{CQ}(A) \) is not closed under addition and is not norm-closed, and there is \( x \in \text{CQ}(A) \) such that \( x^2 \notin \text{CQ}(A) \). To show this explicitly, first by Theorem 4.8 we have

\[
V_A = A \setminus \text{CQ}(A) = \{ a \in A : \exists \lambda, \mu \in \mathbb{C}, \lambda \neq \mu, \text{ such that } a(1) = \text{diag}(\lambda, \lambda, \mu) \}.
\]

In particular, \( \text{CQ}(A) \) strictly contains \( Z(A) + J_{\text{we}}(A) \). Let \( b := \text{diag}(1,0,0) \) and \( c := \text{diag}(0,1,0) \) be elements of \( A \), considered as constant functions. Then, \( b, c \in \text{CQ}(A) \), but \( b + c = \text{diag}(1,1,0) \notin \text{CQ}(A) \). Similarly, the constant function \( x := \text{diag}(-1,1,0) \) belongs to \( \text{CQ}(A) \) but \( x^2 = \text{diag}(1,1,0) \) does not.

We now show that \( \text{CQ}(A) \) is not norm-closed in \( A \). In fact, we shall show that \( \text{CQ}(A) \) is norm-dense in \( A \), so as \( A \) is not weakly central, \( \text{CQ}(A) \) cannot be norm-closed (for a more general argument see Proposition 4.24). Choose any \( a \in A \setminus \text{CQ}(A) \). Then \( a(1) = \text{diag}(\lambda, \lambda, \mu) \) for some distinct scalars \( \lambda \) and \( \mu \). For any \( \varepsilon > 0 \), let \( b_\varepsilon := \text{diag}(\varepsilon,0,0) \) (as a constant function in \( A \)). Then \( a + b_\varepsilon \in \text{CQ}(A) \) and \( \| (a + b_\varepsilon) - a \| = \| b_\varepsilon \| = \varepsilon \). \( \square \)

We now demonstrate that Corollary 4.16 can fail when \( A \) is not assumed to be postliminal or an AF-algebra.

In order to do this, first recall that a \( C^* \)-algebra \( B \) is said to be projectionless if \( B \) does not contain non-trivial projections. The first example of a simple projectionless \( C^* \)-algebra was given by Blackadar [11] (the non-unital example) and [12] (the unital example). Also, the prominent examples of simple projectionless \( C^* \)-algebras include the reduced \( C^* \)-algebra \( C^*_r(F_n) \) for the free group \( F_n \) on \( n < \infty \) generators [39] and the Jiang-Su algebra \( Z \) [28], which also has the important property that it is KK-equivalent to \( \mathbb{C} \).

**Lemma 4.20.** Let \( B \) be a unital projectionless \( C^* \)-algebra and let \( p \in \mathbb{C}[z] \) be a separable polynomial. An element \( b \in B \) satisfies \( p(b) = 0 \) if and only if \( b = \mu 1 \), where \( \mu \) is a root of \( p \). \( \square \)

**Proof.** First note that since \( B \) is projectionless, all elements of \( B \) have connected spectrum. Indeed, otherwise by [29] Corollary 3.3.7 \( B \) would contain a non-trivial idempotent \( e \) and then by [13] Proposition 4.6.2, \( e \) would be similar to a (necessarily non-trivial) projection.

If \( p \in \mathbb{C}[z] \) is a separable polynomial of degree \( n \), we can factorize

\[
p(z) = \alpha(z - \mu_1) \cdots (z - \mu_n),
\]
where $\alpha \in \mathbb{C} \setminus \{0\}$ and $\mu_1, \ldots, \mu_n \in \mathbb{C}$ are distinct roots of $p$. If $b \in B$ satisfies $p(b) = 0$, then the spectral mapping theorem implies $\sigma(b) \subseteq \{\mu_1, \ldots, \mu_n\}$. As $\sigma(b)$ is connected, this forces $\sigma(b) = \{\mu_k\}$ for some $1 \leq k \leq n$. Then for all $i \in \{1, \ldots, n\} \setminus \{k\}$, the element $b - \mu_i 1$ is invertible so

$$0 = p(b) = \alpha(b - \mu_1 1) \cdots (b - \mu_n 1)$$

implies $b = \mu_k 1$ as claimed. The converse is trivial. \hfill \blacksquare

**Example 4.21.** Let $B$ be any unital simple projectionless non-abelian $C^*$-algebra (e.g. $B = \mathbb{Z}$, the Jiang-Su algebra).

Consider the $C^*$-algebra $C$ of all continuous functions $x : [0, 1] \to M_2(B)$ such that $x(1) = \text{diag}(b(x), 0)$ for some $b(x) \in B$ (note that $C$ can be identified with the tensor product $A \otimes B$, where $A$ is the $C^*$-algebra from Example 2.9 which is nuclear). As $B$ is unital and simple, $Z(B) = \mathbb{C}1_B$, so

$$Z(C) = \{\text{diag}(f1_B, f1_B) : f \in C([0, 1]), f(1) = 0\},$$

where $(f1_B)(t) = f(t)1_B$, for all $t \in [0, 1]$. Consider the ideal $M$ of $C$ defined by

$$M := \{x \in C : x(1) = 0\} = C_0([0, 1], M_2(B)).$$

As $C/M \cong B$, $M$ is a modular maximal ideal of $C$ that contains $Z(C)$, so that $M \subset T_C^1$. Since $Z(M) \cong C_0([0, 1])$ and Prim$(M)$ is canonically homeomorphic to $[0, 1)$, it is easy to check directly that $M$ is a central $C^*$-algebra (alternatively, $M \cong C_0([0, 1]) \otimes M_2(B)$ is weakly central by Theorem 3.29). Therefore,

$$J_{wc}(C) = M \quad \text{and} \quad T_C = T_C^1 = \{M\}.$$ 

By Theorem 4.8 we have

$$\text{CQ}(C) = \{x \in C : b(x) \text{ is not a non-zero scalar}\}.$$ 

As $C/J_{wc}(C) \cong B$ is non-abelian, by Theorem 4.12 $\text{CQ}(C)$ is not norm-closed and is neither closed under addition nor closed under multiplication.

On the other hand we claim that for any $x \in \text{CQ}(C)$, $x^n \in \text{CQ}(C)$ for all $n \in \mathbb{N}$. On a contrary, assume that there exists $x \in \text{CQ}(C)$ such that $x^n \notin \text{CQ}(C)$ for some $n > 1$. Then, by Theorem 4.8 there is a non-zero $\lambda \in \mathbb{C}$ such that $b(x)^n = \lambda 1_B$. Consider the polynomial $p(z) := z^n - \lambda$. As $\lambda \neq 0$, $p$ is separable. Since $B$ is projectionless and $p(b(x)) = 0$, Lemma 4.20 implies that $b(x) = \mu 1_B$, where $\mu$ is some $n$-th root of $\lambda$. But this contradicts the fact that $x \in \text{CQ}(C)$. \hfill \blacksquare

If a unital $C^*$-algebra $A$ is not weakly central then, even though $\text{CQ}(A)$ might be a $C^*$-subalgebra of $A$
(and hence equal to $Z(A) + J_{wc}(A)$ by Theorem 4.12, one may use matrix units to show that $CQ(M_2(A))$ is neither closed under addition nor closed under multiplication (for the algebraic counterpart, see the comment following [16, Remark 3.6]). In fact, this is a special case of the following more general result.

**Proposition 4.22.** Let $A$ be a unital $C^*$-algebra and let $B$ be a unital simple exact $C^*$-algebra.

(a) $J_{wc}(A \otimes \min B) = J_{wc}(A) \otimes \min B$.

(b) Suppose that $A$ is not weakly central and that $B$ is not abelian (that is, $B$ is not $^*$-isomorphic to $\mathbb{C}$). Then $CQ(A \otimes \min B)$ is not norm-closed and is neither closed under addition nor closed under multiplication. In particular, $CQ(M_n(A))$ is not a $C^*$-subalgebra of $M_n(A)$ for any $n > 1$.

\[ \square \]

**Proof.** (a) If $A$ is weakly central then, since $B$ is weakly central, we have that $A \otimes \min B$ is weakly central (see [3, Theorem 3.1] and Theorem 3.29). So we now assume that $A$ is not weakly central, so that $T_A \neq \emptyset$. Let $M \in T_A$. Then there is $N \in \text{Max}(A)$ such that $N \neq M$ and $M \cap Z(A) = N \cap Z(A)$. Since $B$ is exact,

\[
\frac{A \otimes \min B}{M \otimes \min B} \cong A \otimes \frac{\min B}{M}, \]

which is a simple $C^*$-algebra (see [12, Corollary]). Thus $M \otimes \min B \in \text{Max}(A \otimes \min B)$ and similarly $N \otimes \min B \in \text{Max}(A \otimes \min B)$. Let $x \in (M \otimes \min B) \cap (Z(A) \otimes C1_B)$. For a state $\omega \in S(B)$ let $L_\omega : A \otimes \min B \to A$ be the corresponding left slice map (i.e. $L_\omega(a \otimes b) = \omega(b)a$, see [15]). There exists $z \in Z(A)$ such that $x = z \otimes 1_B$ and hence $z = L_\omega(x) \in M$. Thus

\[
(M \otimes \min B) \cap (Z(A) \otimes C1_B) = (M \cap Z(A)) \otimes C1_B = (N \cap Z(A)) \otimes C1_B = (N \otimes \min B) \cap (Z(A) \otimes C1_B).
\]

Note that also $M \otimes \min B \neq N \otimes \min B$ (for otherwise, by using $L_\omega$, we would obtain $M \subseteq N$ and $N \subseteq M$). Since by [27, Corollary 1], $Z(A) \otimes C1_B = Z(A \otimes \min B)$, we have shown that $M \otimes \min B \in T_A \otimes \min B$. By Theorem 3.22

\[ J_{wc}(A \otimes \min B) \subseteq \bigcap_{M \in T_A} (M \otimes \min B) = J_{wc}(A) \otimes \min B. \tag{4.14} \]

For the equality in (4.14), let $y \in \bigcap_{M \in T_A} (M \otimes \min B)$ and $\psi \in B^*$. Then

\[ L_\psi(y) \in \bigcap_{M \in T_A} M = J_{wc}(A). \]

Hence

\[ 0 = q(L_\psi(y)) = L_\psi((q \otimes \text{id}_B)(y)), \]
where \( q : A \rightarrow A/J_{wc}(A) \) is the canonical map and \( L_\psi : (A/J_{wc}(A)) \otimes_{\min} B \rightarrow A/J_{wc}(A) \) is the left slice map.

It follows that
\[
y \in \ker(q \otimes \text{id}_B) = J_{wc}(A) \otimes_{\min} B,
\]
since \( B \) is exact.

On the other hand, it follows from Theorem 3.29 and Corollary 3.9 that \( J_{wc}(A) \otimes_{\min} B \) is weakly central. Thus \( J_{wc}(A \otimes_{\min} B) = J_{wc}(A) \otimes_{\min} B \), as claimed.

(b) Since \( B \) is exact, by (a)
\[
\frac{A \otimes_{\min} B}{J_{wc}(A \otimes_{\min} B)} = \frac{A \otimes_{\min} B}{J_{wc}(A) \otimes_{\min} B} \cong \frac{A}{J_{wc}(A)} \otimes_{\min} B,
\]
which is non-abelian. The result now follows from Theorem 4.12.

In contrast to the second paragraph of Remark 3.20, we now demonstrate there are even separable continuous-trace \( C^* \)-algebras \( A \) such that \( Z(A) = J_{wc}(A) = \{0\} \), while \( \text{CQ}(A) \) is norm-dense in \( A \). In order to do this, we shall use the following facts.

**Lemma 4.23.** Let \( A \) be a \( C^* \)-algebra such that all primitive ideals of \( A \) are maximal and both sets of all modular and non-modular primitive ideals are dense in \( \text{Prim}(A) \). Then \( Z(A) = J_{wc}(A) = \{0\} \).

**Proof.** That \( Z(A) = \{0\} \) follows from Remark 2.1. Let \( I \) be a non-zero ideal of \( A \). Then \( Z(I) = I \cap Z(A) = \{0\} \). On the other hand, the dense set of modular primitive ideals of \( A \) meets the open set \( \text{Prim}_A(I) \). If \( P \) is any modular primitive ideal of \( A \) that does not contain \( I \) then, by assumption, \( P \) is maximal, so by Lemma 2.2 \( P \cap I \) is a modular primitive ideal of \( I \) such that \( \{0\} = Z(I) \subseteq P \cap I \). Therefore, \( I \) is not weakly central. □

**Proposition 4.24.** Let \( A \) be a \( C^* \)-algebra.

(a) If either there is \( M \in \text{Max}(A) \) of codimension 1 such that \( Z(A) \subseteq M \) or there are distinct \( M_1, M_2 \in \text{Max}(A) \) of codimension 1 that satisfy \( M_1 \cap Z(A) = M_2 \cap Z(A) \neq Z(A) \), then \( \text{CQ}(A) \) is not norm-dense in \( A \).

(b) The converse of (a) is true if \( T_A \) is countable.

**Proof.** (a) Assume there is \( M \in \text{Max}(A) \) of codimension 1 that contains \( Z(A) \). Since \( A/M \cong \mathbb{C} \), by Theorem 4.8 for any \( a \in \text{CQ}(A) \), \( a + M \) is zero in \( A/M \), so \( a \in M \). Thus, \( \text{CQ}(A) \subseteq M \), so \( \text{CQ}(A) \) is clearly not norm-dense in \( A \).

Alternatively, assume there are distinct \( M_1, M_2 \in \text{Max}(A) \) of codimension 1 such that \( M_1 \cap Z(A) = M_2 \cap Z(A) \neq Z(A) \). Since \( A/(M_1 \cap M_2) \cong (A/M_1) \oplus (A/M_2) \cong \mathbb{C} \oplus \mathbb{C} \), Theorem 4.8 implies \( \text{CQ}(A) \subseteq \mathbb{C}1 + (M_1 \cap M_2) \), so \( \text{CQ}(A) \) is not norm-dense in \( A \).
(b) Now assume that all \( M \in \text{Max}(A) \) that contain \( Z(A) \) have codimension greater than 1 and for all distinct 
\( M_1, M_2 \in \text{Max}(A) \) that satisfy \( M_1 \cap Z(A) = M_2 \cap Z(A) \neq Z(A) \), at least one \( M_i \) has codimension greater than 1. We may assume that \( T_A \neq \emptyset \), for otherwise \( \text{CQ}(A) = A \), which is certainly dense in \( A \).

For each \( M \in T_A \) such that \( \dim(A/M) > 1 \), set

\[
U_M := \{ a \in A : a + M \text{ is not a scalar in } A/M \}.
\]

Evidently, \( U_M \) is an open subset of \( A \). We claim that \( U_M \) is norm-dense in \( A \). Let \( a \in A \setminus U_M \), so that \( a + M \) is a scalar in \( A/M \). Let \( \varepsilon > 0 \). Since \( A/M \) is non-abelian, there is a non-central element \( b \) of norm one in \( A/M \). Then by [46, Lemma 17.3.3], there is a norm one element \( b \in A \) such that \( b + M = b \). Then the element \( a + (\varepsilon/2)b \) lies in \( U_M \) and its distance from \( a \) is \( \varepsilon/2 \).

If \( T_A \) is countable, then the Baire category theorem implies that

\[
U := \bigcap \{ U_M : M \in T_A, \dim(A/M) > 1 \}
\]

is a dense subset of \( A \). Let \( a \in U \). If \( M \in T^1_A \) then \( a \in U_M \) and so \( a + M \) is not a scalar in \( A/M \). Also if \( M_1, M_2 \in \text{Max}(A) \), such that \( M_1 \neq M_2 \) and \( M_1 \cap Z(A) = M_2 \cap Z(A) \neq Z(A) \), then for some \( i \in \{1, 2\} \) we have \( \dim(A/M_i) > 1 \) so that \( a \in U_M \), and hence \( a + M_i \) is not a scalar in \( A/M_i \). Thus, by Theorem 4.8 \( U \subseteq \text{CQ}(A) \), so \( \text{CQ}(A) \) is norm-dense in \( A \).

The next example is a slight variant of [4] Example 4.4 where we have changed the quotient \( A(1) \) in order to avoid an abelian quotient.

**Example 4.25.** Let \( \mathcal{H} \) be a separable infinite-dimensional Hilbert space with orthonormal basis \( \{e_n : n \geq 0\} \). For each \( n \) let \( E_n \) be the projection from \( \mathcal{H} \) onto the linear span of the set \( \{e_0,e_1,\ldots,e_n\} \). We define \( A \) to be the subset of \( C([0,1],[K(\mathcal{H})]) \) consisting of all elements \( a \in C([0,1],[K(\mathcal{H})]) \) which satisfy the following requirement:

For any dyadic rational \( t = p/2^q \in [0,1) \), where \( p,q \) are positive integers such that \( 2^q/p \), then

\[
a(t) = E_q a(t) = a(t) E_q.
\]

Then \( A \) is a closed self-adjoint subalgebra of \( C([0,1],[K(\mathcal{H})]) \) and so is itself a \( C^* \)-algebra.

As in [3], standard arguments show that \( A \) is a continuous-trace \( C^* \)-algebra whose primitive ideal space can be identified with \([0,1], \) via the homeomorphism

\[
[0,1] \ni t \mapsto P_t := \text{ker } \pi_t \in \text{Prim}(A),
\]
where for each \( t \in [0, 1] \) and \( a \in A \), \( \pi_t(a) \) := \( a(t) \). Moreover, if for each \( t \in [0, 1] \) we denote the fibre of \( A \) at \( t \) by \( A(t) \) (i.e. \( A(t) = \{ a(t) : a \in A \} \)), then

\[
A(t) = \begin{cases} 
\{ K \in K(H) : E_{q}K = KE_{q} = K \} \cong M_{q+1}(\mathbb{C}) , & \text{if } t = p/2^{q} \text{ as above} \\
K(H), & \text{otherwise.}
\end{cases}
\]

In particular, all primitive ideals of \( A \) are maximal. Further, the sets of modular and non-modular primitive ideals of \( A \) are both dense in \( \text{Prim}(A) \), and so Lemma 4.23 implies \( Z(A) = J_{bc}(A) = \{0\} \). On the other hand, since

\[
T_{A} = T_{A}^{1} = \{ P_{t} : t \in [0, 1] \text{ is a dyadic rational} \}
\]

is countable and the codimension of each \( P_{t} \in T_{A} \) is larger than 1, Proposition 4.24 implies that \( \text{CQ}(A) \) is norm-dense in \( A \).

\[ \square \]

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**References**


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