

SIMULTANEOUS CORE PARTITIONS WITH NONTRIVIAL COMMON DIVISOR

JEAN-BAPTISTE GRAMAIN, RISHI NATH, AND JAMES A. SELLERS

ABSTRACT. A tremendous amount of research has been done in the last two decades on (s, t) -core partitions when s and t are relatively prime integers. Here we change perspective slightly and explore properties of (s, t) -core and (\bar{s}, \bar{t}) -core partitions for s and t with a nontrivial common divisor g .

We begin by recovering, using the g -core and g -quotient construction, the generating function for (s, t) -core partitions first obtained by D. Aukerman, D. Kane and L. Sze. Then, using a construction developed by the first two authors, we obtain a generating function for the number of (\bar{s}, \bar{t}) -core partitions of n . Our approach allows for new results on t -cores and self-conjugate t -cores that are *not* g -cores and \bar{t} -cores that are *not* \bar{g} -cores, thus strengthening positivity results of K. Ono and A. Granville, J. Baldwin et. al., and I. Kiming.

We then move to bijections between bar-core partitions and self-conjugate partitions. We give a new, short proof of a correspondence between self-conjugate t -core and \bar{t} -core partitions when t is odd and positive first due to J. Yang. Then, using certain lattice-path labelings due to B. Ford, H. Mai, and L. Sze on the one hand, and C. Bessenrodt and J. Olsson on the other, we give a bijection between self-conjugate (s, t) -core and (\bar{s}, \bar{t}) -core partitions when s and t are odd and $g = 1$. We end this section with a bijection between self-conjugate (s, t) -core and (\bar{s}, \bar{t}) -core partitions when s and t are odd and $g > 1$ which uses the results stated above.

We end the paper by noting (s, t) -core and (\bar{s}, \bar{t}) -core partitions inherit Ramanujan-type congruences from those of g -core and \bar{g} -core partitions.

2010 Mathematics Subject Classification: 05A17, 11P83

Keywords: partition, congruence, generating function

1. OUTLINE AND BACKGROUND

We begin, in this section, by reviewing definitions of core partitions, generating functions, and simultaneous core partitions. In Section 2.1, we reprove, in a succinct fashion, Theorems 1.6 and 1.8 and Corollary 1.7 of Aukerman, Kane, and Sze using known properties of the g -core and g -quotient. We extend these results to self-conjugate (s, t) -core partitions and (\bar{s}, \bar{t}) -core partitions in Section 2.2 and Section 2.3.

In Section 3.1 we again reprove Corollary 1.9 in a succinct fashion before strengthening Theorem 1.10 for $g \geq 4$. In Section 3.2 and Section 3.3 we move to analogous results for self-conjugate (s, t) -core partitions and (\bar{s}, \bar{t}) -core partitions. In each case, we also set $s = t$ to strengthen existing positivity.

We give a new, shorter proof of Theorem 1.5 in Section 4.1. In Section 4.2 we provide a bijection between self-conjugate (s, t) -core partitions and (\bar{s}, \bar{t}) -partitions when s and t are odd and $g = 1$ using the work of Ford, Mai and Sze, Bessenrodt

Date: May 8, 2020.

and Olsson. We combine those results with our work from Section 2.3 to show in Section 4.3 that there is a bijection between self-conjugate (s, t) -core partitions and (\bar{s}, \bar{t}) -core partitions when s, t, g are all odd and nontrivial.

We conclude, in Section 5 by showing how the generating functions mentioned above inherit Ramanujan-type congruences.

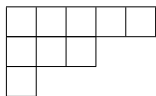
1.1. partitions, cores and bar-cores. Let n be a positive integer. A *partition* λ of a nonnegative integer n is a non-decreasing sequence of positive integers, called *parts*, that sum to n . We let $|\lambda| = n$ be the *size* of λ . A partition λ can be visualized in terms of its *Young diagram* $[\lambda]$, a graphic representation of λ in which left- and top-aligned rows of boxes correspond sequentially to the parts in the partition. [When the context is clear we will abuse notation and let λ refer to both the Young diagram and the partition.] A *hook* $\mathcal{H}_{\iota, \gamma}$ of $[\lambda]$ with *corner* (ι, γ) , using matrix notation, is the set of boxes in the same row and to the right of (ι, γ) , in the same column and below (ι, γ) , and (ι, γ) itself. Then $\mathcal{H} = \{\mathcal{H}_{\iota, \gamma} \mid \iota, \gamma \geq 1\}$ is the *hook set* of λ . The set $\{\mathcal{H}_{\iota 1} \mid \iota \geq 1\}$ consists of the *first column hooks* of $[\lambda]$. The *hook length* $h_{\iota, \gamma}$ (or $|\mathcal{H}_{\iota, \gamma}|$) of $\mathcal{H}_{\iota, \gamma}$ is the number of boxes in the hook. The set $\{h_{\iota, \gamma}\}$ will be called the *set of hook lengths* of λ . In particular, the set $\{h_{\gamma}\}$ of *first column hook lengths* of λ completely determine the partition. The Young diagram $[\lambda^\vee]$ of the *conjugate partition* λ^\vee of λ is obtained by exchanging rows and columns of the Young diagram of $[\lambda]$. We say λ is a *self-conjugate* partition if $\lambda = \lambda^\vee$. A self-conjugate partition is completely determined by its *set of diagonal hook lengths* $\Delta(\lambda) = \{h_{ii}\}$.

Let $\{h_{\iota, \gamma} \mid \iota, \gamma \geq 1\}$ be the *multiset of hook lengths* of λ (including repetitions). A *t-hook* is a hook of length t . For fixed t , the partitions whose set of hook lengths does not contain any multiple of t are known as *t-core partitions*. The *positivity* of *t-core partitions* (see Theorem 1.1 below) was proved in several steps (see [8], [13], [18], [28]).

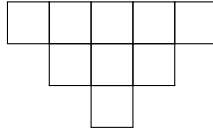
Theorem 1.1 (Positivity). *Every non-negative integer n has at least one t -core partition for $t \geq 4$.*

Bar partitions are partitions with distinct parts. One obtains the *shifted Young diagram* $S(\lambda)$ of a bar partition by shifting the i th row in the Young diagram $i - 1$ boxes to the right. The *shift-symmetric diagram* $SS(\lambda)$ is then obtained by adjoining the parts of λ as columns to the shifted diagram in the following way: a column consisting of λ_i boxes is attached one position to the left of the (i, i) position in the shift diagram. Then, seen on the shift-symmetric Young diagram, there is an analogous definition of a *bar* and *bar length*. The bar lengths of λ are the hook lengths of the boxes of $S(\lambda)$ embedded in the shift-symmetric Young diagram $SS(\lambda)$ [4].

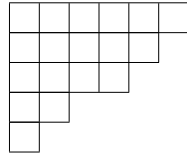
Example 1.2. *Consider the Young diagram of the bar partition $\lambda = (5, 3, 1)$.*



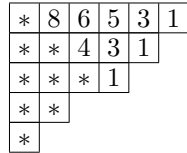
We have $S(\lambda)$



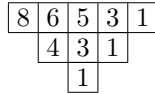
and $SS(\lambda)$



Now consider the hook lengths of the boxes in $S(\lambda)$ embedded in $SS(\lambda)$.



Returning to $S(\lambda)$ we obtain the bar lengths associated to λ .



We define a t -bar-core, or \bar{t} -core, to be a partition with no bars of length t . J. Baldwin, M. Depweg, B. Ford, A. Kunin, and L. Sze [5] proved a version of the positivity conjecture for self-conjugate core partitions; I. Kiming [19] proved a version of positivity for bar-core partitions. [Kiming’s theorem was originally proven for p prime (relevant for the so-called p -defect zero spin blocks), but the argument goes through without change for all odd $t \geq 7$ [10]. We will use the more general statement.]

Theorem 1.3. *Every non-negative integer n has at least one self-conjugate t -core partition for $t = 8$ or $t \geq 10$.*

Theorem 1.4. *Every non-negative integer n has a \bar{t} -core partition for $t \geq 7$ and odd.*

The next theorem is a recent result of J. Yang which connects self-conjugate t -core partitions and \bar{t} -core partitions.

Theorem 1.5. [31, Theorem 1.1] *Let t be odd. There is a correspondence between the self-conjugate t -cores and \bar{t} -core partitions.*

We note that for the remainder of the paper, $s' := \frac{s}{g}$ and $t' := \frac{t}{g}$.

1.2. Generating functions. Euler determined the generating function P for the number $p(n)$ of partitions of n :

$$P(x) := \sum_{n=0}^{\infty} p(n)x^n = \prod_{n=1}^{\infty} \frac{1}{1-x^n}$$

The generating function F_t for the number $f_t(n)$ of t -core partitions of n was first obtained by J. B. Olsson [22].

$$F_t(x) = \sum_{n=0}^{\infty} f_t(n)x^n = \prod_{n=1}^{\infty} \frac{(1-x^{tn})^t}{(1-x^n)}$$

F. Garvan, D. Kim, and D. Stanton (Eqs. 7.1(a) and (b) in [11]) first described the generating function F_t^* for the number $f_t^*(n)$ of self-conjugate t -core partitions of n . We use a formulation that appears in [5].

$$F_t^*(x) = \sum_{n=0}^{\infty} f_t^*(n)x^n = \begin{cases} \prod_{n=1}^{\infty} (1-x^{2tn})^{\frac{t}{2}} (1+x^{2n-1}) & \text{if } t \text{ is even,} \\ \prod_{n=1}^{\infty} (1-x^{2tn})^{\frac{t-1}{2}} \left(\frac{1+x^{2n-1}}{1+x^{t(2n-1)}} \right) & \text{if } t \text{ is odd.} \end{cases}$$

The generating function $F_{\bar{t}}$ for the number $f_{\bar{t}}(n)$ of \bar{t} -core partitions of n was also found by Olsson [25].

$$F_{\bar{t}}(x) = \sum_{n=0}^{\infty} f_{\bar{t}}(n)x^n = \prod_{n=1}^{\infty} \frac{(1-x^{2n})(1-x^{tn})^{\frac{t+1}{2}}}{(1-x^n)(1-x^{2tn})}$$

1.3. (s, t) -core partitions, $g = 1$. A *simultaneous core partition* is a partition whose set of hook lengths avoids a specified subset of positive integers. An (s, t) -core partition is both an s -core and a t -core partition, where $s, t > 1$. For the rest of the paper we let $g := \gcd(s, t)$.

J. Anderson (Section 3, [2]) describes, when $g = 1$, the possible first-column hook lengths of an (s, t) -core partition. Recall a lattice path is *monotonic* if it only moves one position up or one position to the right at every step. Then, in particular, Anderson's results imply a correspondence between (s, t) -cores and a family of monotonic paths that stay above the diagonal in a certain $s \times t$ lattice, constructed in the following way: The left-and-topmost box, with upper-left corner labeled $(0, s)$, contains the value $st - s - t$. One horizontal position right from $(0, s)$ is the box whose upper left corner is labeled $(1, s)$, and the value in that box is $st - 2s - t$ (a reduction by s). One vertical position down from $(0, s)$ is the box with label $(0, s - 1)$; the value inside is $st - s - 2t$. Complete the lattice coordinates with their values in this fashion.

Example 1.6. Let $s = 7$ and $t = 11$. Anderson's lattice is shown below. The monotonic path π appearing in the lattice corresponds to the $(7, 11)$ -core partition λ_π whose first-column hook lengths $\{h_{\gamma 1}\}_{1 \leq \gamma \leq 5} = \{13, 10, 9, 8, 6, 5, 3, 2, 1\}$ are the values trapped between the (solid) path and the (dashed) \pm border. This corresponds to the partition $\lambda_\pi = (5, 3^3, 2^2, 1^3)$.

59	52	45	38	31	24	17	10	3	-4	-11
48	41	34	27	20	13	6	-1	-15	-22	
37	30	23	16	9	2	-5	-12	-19	-26	-33
26	19	12	5	-2	-9	-16	-23	-30	-37	-44
15	8	1	-6	-13	-20	-27	-34	-41	-48	-55
4	-3	-10	-17	-24	-31	-38	-45	-52	-59	-66
-7	-14	-21	-28	-35	-42	-49	-56	-63	-70	-77

Anderson uses this to prove that the number of (s, t) -core partitions is $\frac{1}{(s+t)} \binom{s+t}{t}$. J. B. Olsson and D. Stanton [27] showed that the largest size of such an (s, t) -core partition is $\frac{(s^2-1)(t^2-1)}{24}$, verifying a conjecture of B. Kane [17].

1.4. **(s, t) -core and (\bar{s}, \bar{t}) -core partitions, $g = 1$.** Let $s, t > 1$ be odd. We use the notation (s, t) -core partition to indicate an (s, t) -core partition that is also self-conjugate. B. Ford, H. Mai and L. Sze [9] construct a bijection between (s, t) -core partitions and monotonic paths in a certain $\lfloor \frac{s}{2} \rfloor \times \lfloor \frac{t}{2} \rfloor$ lattice which we call the *diagonal hooks diagram*. As a consequence, they find that, when $g = 1$, the number of self-conjugate (s, t) -core partitions is

$$\binom{\lfloor \frac{s}{2} \rfloor + \lfloor \frac{t}{2} \rfloor}{\lfloor \frac{s}{2} \rfloor}.$$

A bar-partition is said to be an (\bar{s}, \bar{t}) -core if it is both an \bar{s} -core and a \bar{t} -core. In the case that $s, t > 1$ are odd and $g = 1$, C. Bessenrodt and J. B. Olsson [6] construct a bijection between (\bar{s}, \bar{t}) -core partitions and monotonic paths in a $\frac{s-1}{2} \times \frac{t-1}{2}$ array, called the *Yin-Yang diagram*. As a consequence they find the number of (\bar{s}, \bar{t}) -core partitions and the size of the (\bar{s}, \bar{t}) -core which “contains” all others.

We outline the Ford, Mai and Sze construction as well as that of Bessenrodt and Olsson in Section 4.2.

1.5. **(s, t) -core and (s, t) -core partitions, $g > 1$.** When $g > 1$, there are infinitely many (s, t) -cores, self-conjugate (s, t) -cores and (\bar{s}, \bar{t}) -cores, so there is no largest such partition. However, some properties of (s, t) -cores for $g = 1$ remain true when $g > 1$ (e.g. [12] and [20] generalize results in [26]). D. Aukerman, B. Kane and L. Sze first characterized an (s, t) -core partition λ in terms of its g -core $Cor_g(\lambda)$ and g -quotient $Quo_g(\lambda)$, which we define formally in Section 2.1.

Theorem 1.7. [3, Theorem 1.3] *Let λ be any integer partition, $s = s'g$ and $t = t'g$, and let $Quo_g(\lambda) = (\lambda_1, \dots, \lambda_g)$ be the g -quotient of λ . Then λ is an (s, t) -core if and only if $\lambda_1, \dots, \lambda_g$ are (s', t') -cores.*

They obtain, as a corollary, a generating function $\Psi_{s,t}$ for the number of (s, t) -core partitions of n .

Corollary 1.8. [3, Corollary 4.3] *With the above notation, we have*

$$\Psi_{s,t}(x) = \Psi_{s',t'}(x^g)^g F_g(x) = \Psi_{s',t'}(x^g)^g \prod_{n=1}^{\infty} \frac{(1-x^{gn})^g}{1-x^n},$$

where F_g is the generating function for the number of g -core partitions.

They also find the generating function for the number of self-conjugate (s, t) -cores when $g > 1$.

Theorem 1.9. [3, Theorem 7.1] *The generating function $\Psi_{s,t}^*$ for the number of self-conjugate (s, t) -cores with $g > 1$ is given by*

$$\Psi_{s,t}^*(x) = \begin{cases} F_g^*(x)(\Psi_{s',t'}(x^{2g}))^{\frac{g}{2}} & \text{if } g \text{ is even,} \\ F_g^*(x)(\Psi_{s',t'}(x^{2g}))^{\frac{g-1}{2}} \Psi_{s',t'}^*(x^g) & \text{if } g \text{ is odd.} \end{cases}$$

Aukerman, Kane, and Sze also obtain a result on the number of (s, t) -core partitions which are not g -cores, and the existence of a t -core partition of n that is not a g -core.

Theorem 1.10. [3, Corollary 4.4] *If $g > 1$, $s' > 1$ and $t' > 1$, then there are infinitely many $(s'g, t'g)$ -cores which are not g -cores.*

Theorem 1.11. [3, Corollary 1.2] *Let $g \geq 4$ be an integer. Then, for any integers $n \geq g$ and $t' > 1$, there exists a partition of n which is a $t'g$ -core but not a g -core.*

Aukerman, Kane and Sze do not consider (\bar{s}, \bar{t}) -core partitions with $g > 1$; they are considered here using a construction of the first two authors [12].

2. GENERATING FUNCTIONS

2.1. (s, t) -cores with $g > 1$. We start with some relevant background. For more details we refer the reader to [25, Section 3]. A t -core or t -core partition is any partition whose set of hook lengths avoids t , or has no t -hooks. More generally, the t -core of λ , denoted $Cor_t(\lambda)$, is obtained by removing from λ a sequence of t -hooks in succession until no further t -hooks can be removed. If a partition has w such t -hooks, then the t -quotient $Quo_t(\lambda)$ is an t -tuple $(\lambda_1, \dots, \lambda_t)$ of partitions whose sizes add up to w . The integer w is then called the t -weight of λ , and $Cor_t(\lambda)$ is a partition of $n - wt$. The fundamental property of $Quo_t(\lambda)$ is that it contains all the information about the hooks of length divisible by t in λ . The following lemma and proposition are well-known.

Lemma 2.1. *If λ is a partition of n , and if t is any positive integer, then λ is completely and uniquely determined by its t -core $Cor_t(\lambda)$ and t -quotient $Quo_t(\lambda)$.*

Proposition 2.2. *For any integer $k \geq 1$, there is a canonical bijection between the set of hooks of length kt in λ and the set of hooks of length k in $Quo_t(\lambda)$, where a hook in the t -quotient is simply a hook in any of $\lambda_1, \dots, \lambda_t$.*

Let $s = s'g$ and $t = t'g$, with $g > 1$. We now provide new, succinct proofs of Theorem 1.7 and Corollary 1.8. We note that the approach here provides the blueprint for many of the results that follow.

Proof of Theorem 1.7. Recall that λ is an s -core if and only if λ has no hook of length divisible by $s = s'g$. Now, by Proposition 2.2, there is a bijection between the set of hooks of length $ks = ks'g$ in λ and the set of hooks of length ks' in $Quo_g(\lambda)$ for any $k \geq 1$. Hence λ is an s -core if and only if $\lambda_1, \dots, \lambda_g$ are s' -cores. Similarly, λ is a t -core if and only if $\lambda_1, \dots, \lambda_g$ are t' -cores. The result follows. ■

We write $Q_{s',t'}(g, w)$ for the number of g -tuples of (s', t') -cores of weight w , and $\psi_{s,t}(n)$ for the number of (s, t) -cores of an integer n .

Proof of Corollary 1.8. The proof is similar to that of [25, Proposition (9.4)(iii)], and follows from a counting argument using Theorem 1.7. By Lemma 2.1, each (s, t) -core partition λ of n is completely and uniquely determined by its g -core $Cor_g(\lambda)$ and g -quotient $Quo_g(\lambda)$. And, by Theorem 1.7, if λ has g -weight w , then $Quo_g(\lambda)$ can be any g -tuple of (s', t') -cores of weight w . We have

$$(1) \quad \psi_{s,t}(n) = \sum_{w \geq 0} Q_{s',t'}(g, w) f_g(n - gw).$$

Recall that $F_g(x)$ is the generating function for the number of g -cores of n . Then, using an argument similar to Equation (1) can be transformed into the generating function

$$\Psi_{s,t}(x) = \Psi_{s',t'}(x^g)^g F_g(x),$$

as claimed. ■

2.2. $(s, t)^*$ -cores with $g > 1$. We consider self-conjugate (s, t) -core partitions, or $(s, t)^*$ -core partitions. Our description of the core and quotient of a self-conjugate (s, t) -core partition is a consequence of the following lemma which appears in [14].

Lemma 2.3. *Let t be any positive integer, λ be a partition, and $\text{Quo}_t(\lambda) = (\lambda_0, \dots, \lambda_{t-1})$. Then λ is self-conjugate if and only if $\text{Cor}_t(\lambda)$ is self-conjugate and $\lambda_i^\vee = \lambda_{t-i-1}$ for all $0 \leq i \leq t-1$.*

Lemma 2.4. *A partition λ is an self-conjugate (s, t) -core partition if and only if $\text{Cor}_g(\lambda)$ is self-conjugate and each part of $\text{Quo}_g(\lambda) = (\lambda_0, \dots, \lambda_{g-1})$ is an (s', t') -core such that $\lambda_i = \lambda_{g-i-1}^\vee$.*

Proof. This follows from Theorem 1.7 and Lemma 2.3. ■

Let $\psi_{s,t}^*(n)$ be the number of self-conjugate (s, t) -core partitions of n .

Proof of Theorem 1.9. If λ is any self-conjugate (s, t) -core of n of g -weight v , then $\text{Cor}_g(\lambda)$ can be any self-conjugate g -core of $n - vg$.

If g is even, then $\text{Quo}_g(\lambda)$ has the form $(\lambda_0, \dots, \lambda_{\frac{g}{2}-1}, \lambda_{\frac{g}{2}-1}^\vee, \dots, \lambda_0^\vee)$, and $(\lambda_0, \dots, \lambda_{\frac{g}{2}-1})$ can be any $\frac{g}{2}$ -tuple of (s', t') -cores whose sizes add up to w such that $v = 2w$. The number of such λ 's is thus

$$(2) \quad \hat{\psi}_{s,t}^*(n) = \sum_{w \geq 0}^{\frac{n}{2g}} Q_{s',t'}\left(\frac{g}{2}, w\right) f_g^*(n - 2wg)$$

in this case. [Note: Here $\frac{n}{g} \geq 2w_1 + w_2$.]

If, on the other hand, g is odd, then $\text{Quo}_g(\lambda) = (\lambda_0, \dots, \lambda_{\frac{g-3}{2}}, \lambda_{\frac{g-1}{2}}, \lambda_{\frac{g-3}{2}}^\vee, \dots, \lambda_0^\vee)$, where $(\lambda_0, \dots, \lambda_{\frac{g-3}{2}})$ can be any $\frac{g-1}{2}$ -tuple of (s', t') -cores whose sizes add up to w_1 say, and $\lambda_{\frac{g-1}{2}}$ can be any self-conjugate (s', t') -core of w_2 , with $v = 2w_1 + w_2$. The number of such λ (over all such g -weights ν) is thus

$$(3) \quad \psi_{s,t}^*(n) = \sum_{2w_1+w_2 \geq 0}^{\frac{n}{g}} Q_{s',t'}\left(\frac{g-1}{2}, w_1\right) f_{s',t'}^*(w_2) f_g^*(n - (2w_1 + w_2)g)$$

in this case.

These yield the desired generating functions. ■

2.3. (\bar{s}, \bar{t}) -cores with $g > 1$. In this section, we state and prove new bar-analogues of Theorem 1.7 and Corollary 1.8.

We start with combinatorial properties of bar partitions. For more details, we refer the reader to [25, Section 4]. A \bar{t} -core, or \bar{t} -core partition, is any bar partition which has no bar of length divisible by t . More generally, $Cor_{\bar{t}}(\lambda)$ the \bar{t} -core of λ , is obtained by removing successive bars of length divisible by t from λ until no such bars of length t remain. If there are w such bars of length t , then the \bar{t} -quotient $Quo_{\bar{t}}(\lambda)$ is an $\frac{t+1}{2}$ -tuple $(\lambda_0, \lambda_1, \dots, \lambda_{\frac{t-1}{2}})$ of partitions whose sizes add up to w , where λ_0 is itself a bar-partition. The integer w is then called the \bar{t} -weight of λ , and $Cor_{\bar{t}}(\lambda)$ is a bar-partition of $n - wt$. The \bar{t} -quotient $Quo_{\bar{t}}(\lambda)$ is said to have t -weight w .

Lemma 2.5. [25, Proposition (4.2)] *If λ is a bar-partition, and if t is any odd positive integer, then λ is completely and uniquely determined by $Cor_{\bar{t}}(\lambda)$ and $Quo_{\bar{t}}(\lambda)$.*

In analogy with Proposition 2.2, we have the following:

Proposition 2.6. [25, Theorem (4.3)] *If t is an odd positive integer and λ is a bar-partition, then, for any integer $k \geq 1$, there is a canonical bijection between the set of bars of length kt in λ and the set of bars of length k in $Quo_{\bar{t}}(\lambda)$, where a bar in $Quo_{\bar{t}}(\lambda)$ is defined to be a bar in λ_0 , or a hook in any of $\lambda_1, \dots, \lambda_{\frac{t-1}{2}}$.*

Now consider odd integers $s = s'g$ and $t = t'g$, where $g > 1$. We denote by $\Psi_{\bar{s}, \bar{t}}$ the generating function for the number of (\bar{s}, \bar{t}) -core partitions. We now have the following bar-analogues of Theorem 1.7 and Corollary 1.8.

Theorem 2.7. *Let λ be any bar-partition, and let $Quo_{\bar{g}}(\lambda) = (\lambda_0, \lambda_1, \dots, \lambda_{\frac{g-1}{2}})$ be the \bar{g} -quotient of λ . Then λ is an (\bar{s}, \bar{t}) -core if and only if*

- (1) λ_0 is an (\bar{s}', \bar{t}') -core and
- (2) $\lambda_1, \dots, \lambda_{\frac{g-1}{2}}$ are (s', t') -cores.

Proof. Note that λ is an \bar{s} -core if and only if λ has no bar of length divisible by $s = s'g$. Now, by Proposition 2.6, for any $k \geq 1$, there is a bijection between the set of bars of length $ks = kgs'$ in λ and the set of bars of length ks' in $Quo_{\bar{g}}(\lambda)$. Hence λ is an \bar{s} -core if and only if λ_0 is an \bar{s}' -core and $\lambda_1, \dots, \lambda_{\frac{g-1}{2}}$ are s' -cores. Similarly, λ is a \bar{t} -core if and only if λ_0 is a \bar{t}' -core and $\lambda_1, \dots, \lambda_{\frac{g-1}{2}}$ are t' -cores. The result follows. \blacksquare

Corollary 2.8. *With the above notation, we have*

$$\Psi_{\bar{s}, \bar{t}}(x) = \Psi_{\bar{s}', \bar{t}'}(x^g) \Psi_{s', t'}(x^g)^{\frac{g-1}{2}} F_{\bar{g}}(x),$$

where $F_{\bar{g}}(x)$ is the generating function for the number of \bar{g} -core partitions.

Proof. Each (\bar{s}, \bar{t}) -core partition λ of n is completely and uniquely determined by its \bar{g} -core $Cor_{\bar{g}}(\lambda)$ and \bar{g} -quotient $Quo_{\bar{g}}(\lambda)$. And, by Theorem 2.7, if λ has \bar{g} -weight w , then $Quo_{\bar{g}}(\lambda)$ can be any \bar{g} -quotient $(\lambda_0, \lambda_1, \dots, \lambda_{\frac{g-1}{2}})$ of weight w such that λ_0 is an (\bar{s}', \bar{t}') -core and $\lambda_1, \dots, \lambda_{\frac{g-1}{2}}$ are (s', t') -cores. Writing $Q_{\bar{s}', \bar{t}'}(g, w)$ for the number of such \bar{g} -quotients, this shows that the enumerating function $\psi_{\bar{s}, \bar{t}}(n)$ of (\bar{s}, \bar{t}) -cores of n is

$$(4) \quad \psi_{\bar{s}, \bar{t}}(n) = \sum_{w \geq 0} Q_{\bar{s}', \bar{t}'}(g, w) f_{\bar{g}}(n - gw).$$

Viewed as a generating function, this transforms into

$$\Psi_{\bar{s}, \bar{t}}(x) = \Psi_{\bar{s}', \bar{t}'}(x^g) \Psi_{s', t'}(x^g)^{\frac{g-1}{2}} F_{\bar{g}}(x),$$

as claimed. ■

We will return to these functions in Section 3 and Section 5.

3. t -CORES THAT ARE NOT g -CORES.

In this section we consider t -cores and self-conjugate t -cores that are *not* g -cores, and \bar{t} -cores that are *not* \bar{g} -cores. We call these $t \setminus g$ -cores, $t^* \setminus g$ -cores and $\bar{t} \setminus \bar{g}$ -cores respectively. We first study (s, t) -core partitions that are *not* g -cores and (\bar{s}, \bar{t}) -cores that are *not* \bar{g} -cores. Again our convention is to fix positive integers s and t where $s = s'g$ and $t = t'g$, and $g = \gcd(s, t)$, where $s' > 1$ and $t' > 1$.

3.1. $t \setminus g$ -cores. We begin with a succinct proof of Theorem 1.10.

Proof of Theorem 1.10. Recall that there are infinitely many g -cores if $g > 1$. For any g -core γ , consider the (completely and uniquely defined) partition λ with $Cor_g(\lambda) = \gamma$ and $Quo_g(\lambda) = (\lambda_1, \dots, \lambda_g) = ((1), \emptyset, \dots, \emptyset)$. Then $\lambda_1, \dots, \lambda_g$ are certainly (s', t') -cores (since neither s' nor t' is 1), so that, by Theorem 1.7, λ is an $(s'g, t'g)$ -core. [Note that, in order to apply Theorem 1.7, we do not actually need the extra hypothesis that $g = \gcd(s'g, t'g)$.] Also, since $Quo_g(\lambda) \neq (\emptyset, \dots, \emptyset)$, the partition λ is certainly *not* a g -core. Allowing γ to vary produces an infinite number of (distinct) $(s'g, t'g)$ -cores which are not g -cores. ■

Theorem 1.11 shows that, if $g \geq 4$ and $t' > 1$, then, for any integer $n \geq g$, there exists a $t'g$ -core of n that is not a g -core. Our next theorem improves this result, particularly if $t' \geq 4$.

Theorem 3.1. *Fix integers $g \geq 4$ and $t' > 1$, and let $t = t'g$. Then, for every $n \geq g$, the number $\psi_{t \setminus g}(n)$ of $t \setminus g$ -cores of n is bounded below by $\sum_{w=1}^{\lfloor \frac{n}{g} \rfloor} Q_{t'}(g, w)$, where $Q_{t'}(g, w)$ is the number of g -quotients of weight w all of whose components are t' -cores.*

In particular, $\psi_{t \setminus g}(n) \geq g$. If, furthermore, $t' \geq 4$, then $\psi_{t \setminus g}(n) \geq g \lfloor \frac{n}{g} \rfloor$.

Proof. First note that a partition λ of n is not a g -core if and only if it has g -weight $w \geq 1$. We therefore have (as in the proof of Theorem 1.7)

$$\psi_{t \setminus g}(n) = \sum_{w \geq 1} Q_{t'}(g, w) f_g(n - wg) = \sum_{w=1}^{\lfloor \frac{n}{g} \rfloor} Q_{t'}(g, w) f_g(n - wg) \geq \sum_{w=1}^{\lfloor \frac{n}{g} \rfloor} Q_{t'}(g, w),$$

the last inequality holding by Theorem 1.1 since $g \geq 4$.

Note that, for any $w \geq 1$, we have $Q_{t'}(g, w) \geq g f_{t'}(w)$, as one can get g different g -tuples of t' -cores of weight w by placing a t' -core of w (if any exists) in any of the g positions available. We therefore get

$$\psi_{t \setminus g}(n) \geq \sum_{w=1}^{\lfloor \frac{n}{g} \rfloor} g f_{t'}(w).$$

Now, since $t' > 1$, the partition (1) is the only t' -core of 1, whence $f_{t'}(1) = 1$ (and, in fact, $Q_{t'}(g, 1) = g$), so that $\psi_{t \setminus g}(n) \geq g f_{t'}(1) = g$, as claimed.

If, furthermore, $t' \geq 4$, then, by Theorem 1.1, $f_{t'}(w) \geq 1$ for all w , and thus, in this case,

$$\psi_{t \setminus g}(n) \geq \sum_{w=1}^{\lfloor \frac{n}{g} \rfloor} g = g \left\lfloor \frac{n}{g} \right\rfloor.$$

■

3.2. self-conjugate $t \setminus g$ -cores. We have the following self-conjugate analogues to Theorem 1.10 and Theorem 3.1.

Theorem 3.2. *If $s, t, g > 1$, then there are infinitely many self-conjugate (s, t) -core partitions that are not g -core partitions.*

Proof. Recall that there are infinitely many self-conjugate g -cores for $g > 1$. For any self-conjugate g -core γ , consider the (completely and uniquely defined) partition λ with $Cor_g(\lambda) = \gamma$ and $Quo_g(\lambda) = ((1), \emptyset, \dots, \emptyset, (1))$. Then note that the partitions (1) is an (s', t') -core since $s' > 1$ and $t' > 1$. Thus λ is a self-conjugate $(s'g, t'g)$ -core that is not a g -core. Allowing γ to vary completes the proof. ■

Theorem 3.3. *Fix integers $g = 8$ or $g \geq 10$ and $t' > 1$, and let $t = t'g$. Then the number $\psi_{t \setminus g}^*(n)$ of self-conjugate t -cores of n which are not g -cores satisfies*

$$(1) \quad \psi_{t \setminus g}^*(n) \geq \sum_{w=1}^{\lfloor \frac{n}{2g} \rfloor} Q_{t'}\left(\frac{g}{2}, w\right) \text{ if } g \text{ is even, and}$$

$$(2) \quad \psi_{t \setminus g}(n) \geq \sum_{2w_1+w_2=1}^{\lfloor \frac{n}{g} \rfloor} Q_{t'}\left(\frac{g-1}{2}, w_1\right) f_{t'}^*(w_2) \text{ if } g \text{ is odd.}$$

In particular, if g is even and $n \geq 2g$, or if g is odd and $n \geq g$, then $\psi_{t \setminus g}^(n) > 0$ i.e there exists a self-conjugate t -core of n which is not a g -core. In addition, if $n \geq 2g$, then $\psi_{t \setminus g}^*(n) \geq \frac{g}{2}$.*

If, furthermore, g is even and $t' \geq 4$, then $\psi_{t \setminus g}^(n) \geq \frac{g}{2} \lfloor \frac{n}{2g} \rfloor$. If g is odd and $t' = 8$ or $t' \geq 10$, then $\psi_{t \setminus g}^*(n) \geq 1 + \frac{g-1}{2} \lfloor \frac{n}{2g} \rfloor$.*

Proof. Again, note that a partition λ of n is not a g -core if and only if it has g -weight $v \geq 1$. Following the proof of Theorem 1.9, we see that the number $\psi_{t \setminus g}^*(n)$ of self-conjugate t -cores of n which are not g cores is given by

$$\psi_{t \setminus g}^*(n) = \sum_{w \geq 1} Q_{t'}\left(\frac{g}{2}, w\right) f_g^*(n - 2wg) \text{ if } g \text{ is even,}$$

and

$$\psi_{t \setminus g}^*(n) = \sum_{2w_1+w_2 \geq 1} Q_{t'}\left(\frac{g-1}{2}, w_1\right) f_{t'}^*(w_2) f_g^*(n - (2w_1 + w_2)g) \text{ if } g \text{ is odd.}$$

Since $g = 8$ or $g \geq 10$, we get, by Theorem 1.3,

$$\psi_{t \setminus g}^*(n) = \sum_{w=1}^{\lfloor \frac{n}{2g} \rfloor} Q_{t'}\left(\frac{g}{2}, w\right) f_g^*(n - 2wg) \geq \sum_{w=1}^{\lfloor \frac{n}{2g} \rfloor} Q_{t'}\left(\frac{g}{2}, w\right) \text{ if } g \text{ is even,}$$

while

$$\psi_{t \setminus g}^*(n) \geq \sum_{2w_1+w_2=1}^{\lfloor \frac{n}{g} \rfloor} Q_{t'} \left(\frac{g-1}{2}, w_1 \right) f_{t'}^*(w_2) \text{ if } g \text{ is odd.}$$

Suppose first that g is even. If $\lfloor \frac{n}{2g} \rfloor = 0$, i.e. if $n < 2g$, then $\psi_{t \setminus g}^*(n) = 0$, since $n - 2wg \leq 0$ for each w in the sum above. If, on the other hand, $n \geq 2g$, then

$$\psi_{t \setminus g}^*(n) \geq \sum_{w=1}^{\lfloor \frac{n}{2g} \rfloor} Q_{t'} \left(\frac{g}{2}, w \right) \geq \sum_{w=1}^{\lfloor \frac{n}{2g} \rfloor} \frac{g}{2} f_{t'}(w).$$

Since $t' > 1$, we have $f_{t'}(1) = 1$ (and $Q_{t'}(\frac{g}{2}, 1) = \frac{g}{2}$), so that

$$\psi_{t \setminus g}^*(n) \geq \frac{g}{2} f_{t'}(1) = \frac{g}{2},$$

as claimed. In particular, there exists a self-conjugate t -core of n which is not a g -core.

If, furthermore, $t' \geq 4$, then, by Theorem 1.1, $f_{t'}(w) \geq 1$ for all w , and thus, in this case,

$$\psi_{t \setminus g}^*(n) \geq \sum_{w=1}^{\lfloor \frac{n}{2g} \rfloor} \frac{g}{2} = \frac{g}{2} \left\lfloor \frac{n}{2g} \right\rfloor,$$

as claimed.

Suppose now that g is odd. For any $w_1 \geq 1$, we have $Q_{t'}(\frac{g-1}{2}, w_1) \geq \frac{g-1}{2} f_{t'}(w_1)$ while, for $w_1 = 0$, we have $Q_{t'}(\frac{g-1}{2}, w_1) = 1$. Separating the sum according to $w_1 = 0$ and $w_1 \geq 1$, we thus obtain

$$\begin{aligned} \psi_{t \setminus g}^*(n) &\geq \sum_{w_2=1}^{\lfloor \frac{n}{g} \rfloor} Q_{t'} \left(\frac{g-1}{2}, 0 \right) f_{t'}^*(w_2) + \sum_{\substack{w_1 \geq 1 \\ 2w_1+w_2=1}}^{\lfloor \frac{n}{g} \rfloor} \frac{g-1}{2} f_{t'}(w_1) f_{t'}^*(w_2) \\ &= \sum_{w_2=1}^{\lfloor \frac{n}{g} \rfloor} f_{t'}^*(w_2) + \sum_{\substack{w_1 \geq 1 \\ 2w_1+w_2=1}}^{\lfloor \frac{n}{g} \rfloor} \frac{g-1}{2} f_{t'}(w_1) f_{t'}^*(w_2). \end{aligned}$$

Since $t' > 1$, we have $f_{t'}^*(1) = 1$ (corresponding to the self-conjugate t' -core (1)). In particular, if $n \geq g$, then $\psi_{t \setminus g}^*(n) \geq 1$, and there exists a self-conjugate t -core of n which is not a g -core.

If $n < 2g$, then this is all we can say, as the second sum is empty. If, on the other hand, $n \geq 2g$, then, for $w_1 = 1$ and $w_2 = 0$ (which does contribute to the second sum), we get $f_{t'}(w_1) = f_{t'}(1) = 1$ and $f_{t'}^*(w_2) = f_{t'}^*(0) = 1$ (corresponding to the empty partition). Hence, in this case, we have $\psi_{t \setminus g}^*(n) \geq 1 + \frac{g-1}{2} = \frac{g+1}{2} \geq \frac{g}{2}$, as claimed.

If, furthermore, $t' = 8$ or $t' \geq 10$, then, by Theorem 1.1 and Theorem 1.3, $f_{t'}(w_1) \geq 1$ and $f_{t'}^*(w_2) \geq 1$ for all w_1 and w_2 , and thus, in this case

$$\psi_{t \setminus g}^*(n) \geq 1 + \sum_{\substack{w_1 \geq 1 \\ 2w_1+w_2=1}}^{\lfloor \frac{n}{g} \rfloor} \frac{g-1}{2} \geq 1 + \sum_{w_1=1}^{\lfloor \frac{n}{2g} \rfloor} \frac{g-1}{2} = 1 + \frac{g-1}{2} \left\lfloor \frac{n}{2g} \right\rfloor,$$

as claimed. ■

3.3. $\bar{t}\backslash\bar{g}$ -cores. We have the following bar-analogues of Theorem 1.10 and Theorem 3.1.

Theorem 3.4. *Let s', t', g be odd, nontrivial integers. Then there are infinitely many $(\overline{s'g}, \overline{t'g})$ -cores which are not \bar{g} -cores.*

Proof. There are infinitely many \bar{g} -cores for $g > 1$. For any \bar{g} -core γ , consider the (completely and uniquely defined) bar-partition λ with $\mathcal{C}or_g(\lambda) = \gamma$ and $\mathcal{Q}uo_g(\lambda) = (\lambda_0, \lambda_1, \dots, \lambda_{\frac{g-1}{2}}) = ((1), \emptyset, \dots, \emptyset)$. Then λ_0 is an $(\overline{s'}, \overline{t'})$ -core (since $s', t' \neq 1$), and $\lambda_1, \dots, \lambda_{\frac{g-1}{2}}$ are certainly (s', t') -cores. Thus, by Theorem 2.7, λ is a $(\overline{s'g}, \overline{t'g})$ -core (note here that Theorem 2.7 does not require $g = \gcd(s'g, t'g)$). Also, since $\mathcal{Q}uo_{\bar{g}}(\lambda) \neq (\emptyset, \dots, \emptyset)$, the bar-partition λ is certainly not a \bar{g} -core. Allowing γ to vary produces an infinite number of (distinct) $(\overline{s'g}, \overline{t'g})$ -cores which are not \bar{g} -cores. ■

For odd integers g and t' , let $Q_{\bar{t}'}(g, w)$ be the number of \bar{g} -quotients $(\lambda_0, \lambda_1, \dots, \lambda_{\frac{g-1}{2}})$ of weight w such that λ_0 is a \bar{t}' -core and $\lambda_1, \dots, \lambda_{\frac{g-1}{2}}$ are t' -cores.

Theorem 3.5. *Let $g \geq 7$ and $t' > 1$ be odd integers, and let $t = t'g$. Then, for any $n \geq g$, the number $\psi_{\bar{t}\backslash\bar{g}}(n)$ of \bar{t} -core partitions of n which are not \bar{g} -cores satisfies*

$$\psi_{\bar{t}\backslash\bar{g}}(n) \geq \sum_{w=1}^{\lfloor \frac{n}{g} \rfloor} Q_{\bar{t}'}(g, w).$$

In particular, $\psi_{\bar{t}\backslash\bar{g}}(n) \geq \frac{g+1}{2}$ and there exists a \bar{t} -core partition of n which is not a \bar{g} -core.

Proof. Note that a bar-partition λ is a \bar{g} -core if and only if it has \bar{g} -weight 0. We thus have

$$\psi_{\bar{t}\backslash\bar{g}}(n) = \sum_{w \geq 1} Q_{\bar{t}'}(g, w) f_{\bar{g}}(n - gw) = \sum_{w=1}^{\lfloor \frac{n}{g} \rfloor} Q_{\bar{t}'}(g, w) f_{\bar{g}}(n - gw) \geq \sum_{w=1}^{\lfloor \frac{n}{g} \rfloor} Q_{\bar{t}'}(g, w)$$

(since, by Theorem 1.4, $f_{\bar{g}}(n - gw) \geq 1$ for all $1 \leq w \leq \lfloor \frac{n}{g} \rfloor$).

In particular, $\psi_{\bar{t}\backslash\bar{g}}(n) \geq Q_{\bar{t}'}(g, 1)$, and $Q_{\bar{t}'}(g, 1) = \frac{g+1}{2}$ since $t' > 1$, so that (1) is both the only \bar{t}' -core of 1 and the only t' -core of 1. The result follows immediately. ■

4. BIJECTIONS

Our main result in this section is Theorem 4.14: a bijection between (\bar{s}, \bar{t}) -core partitions and $(s, t)^*$ -core partitions when $g \geq 1$ and $s, t > 1$ are odd. In Section 4.1, we reprove a recent bijection of J. Yang between \bar{t} -core partitions and self-conjugate t -core partitions for odd t (Theorem 1.5). In Section 4.2 we show that when $s, t > 1$ are odd and $g = 1$, then self-conjugate (s, t) -core partitions and (\bar{s}, \bar{t}) -core partitions are in bijection. These results, combined with results from Section 2.2 and Section 2.3, give, are the tools we use to construct our bijection in Section 4.3.

[We note that J. Wang and J. Yang [30] have recently extended the Yin-Yang diagram of Bessenrodt and Olsson to the case when s is even and t is odd. We do not consider this case here, although the calculations will be similar.]

4.1. A bijection between t^* -cores and \bar{t} -cores. The following result is Bijection 2 in Garvan, Kim and Stanton [11].

Lemma 4.1. *For any integer $t \geq 1$, there is a correspondence φ between the set of t -core partitions and*

$$\left\{ (a_0, \dots, a_{t-1}) \mid a_i \in \mathbb{Z} \ (0 \leq i \leq t-1) \text{ and } \sum_{i=0}^{t-1} a_i = 0 \right\}.$$

Furthermore, if $\varphi(\lambda) = (a_0, \dots, a_{t-1})$ for some t -core λ , then the conjugate partition λ^\vee (which is also a t -core) satisfies $\varphi(\lambda^\vee) = (-a_{t-1}, \dots, -a_0)$.

[Note: Garvan, Kim and Stanton prove the existence of φ using t -residue diagrams. Its existence can also be derived from the study of so-called t -abacus configurations for t -cores (see [15, Chapter 2]).] The following follows immediately from the description of $\varphi(\lambda^\vee)$ in Lemma 4.1.

Corollary 4.2. *Let $t > 1$ be an odd integer. Then there is a correspondence between the set of self-conjugate t -cores and*

$$\left\{ (a_0, \dots, a_{\frac{t-3}{2}}, 0, -a_{\frac{t-3}{2}}, \dots, -a_0 \mid a_i \in \mathbb{Z} \ (0 \leq i \leq \frac{t-3}{2})) \right\}.$$

Recall $\Delta(\lambda) = \{h_{ii}\}$ is the set of diagonal hook lengths of a self-conjugate partition λ . Ford, Mai, and Sze (Proposition 3, [9]) have another characterization of self-conjugate t -core partitions, in terms of $\Delta(\lambda)$, which we rewrite in the following way.

Lemma 4.3. *Let λ be a t -core partition labeled by $\{(a_0, \dots, a_{t-1})\}$. Then λ is self-conjugate if and only if for every $a_\gamma > 0$:*

- (1) $2(\gamma + \ell t) + 1 \in \Delta(\lambda)$ for all $0 \leq \ell < a_\gamma$, and
- (2) If $h \in \Delta(\lambda)$ and $h + h' \equiv 0 \pmod{2t}$ then $h' \notin \Delta(\lambda)$.

In particular Lemma 4.3 (1) tells us how to recover $\Delta(\lambda)$, the set of diagonal hooks of a self-conjugate t -core partition, from its labeling t -tuple. Olsson has given the following $(\frac{t-1}{2})$ -tuple characterization of \bar{t} -core partitions (Proposition (4.1) and Proposition (4.2), [25]).

Lemma 4.4. *Let t be an odd integer. Then the \bar{t} -core partitions can be encoded as $(\frac{t-1}{2})$ -tuples: $(b'_1, b'_2, \dots, b'_{\frac{t-1}{2}})$, where $b'_i \in \mathbb{Z}$. In particular, the set of parts of λ can be recovered in the following way:*

- (1) If $b'_i > 0$ then $\gamma + \ell t$ is a part of λ for $0 \leq \ell < b'_i$.
- (2) If $b'_j < 0$ then $(t - \gamma) + \ell t$ is a part of λ for $0 \leq \ell < |b'_j|$.

We can now give a succinct proof of Theorem 1.5.

Proof of Theorem 1.5. By Corollary 4.2, if $t \geq 1$ is odd, then there is a correspondence between self-conjugate t -cores and t -tuples $(a_0, a_1, \dots, 0, \dots, -a_1, -a_0)$. By Lemma 4.4, \bar{t} -cores are similarly labeled by $(\frac{t-1}{2})$ -tuples of integers $(b'_1, \dots, b'_{\frac{t-1}{2}})$. We consider the map that sends a_i to b'_{i+1} for $0 \leq i \leq \frac{t-3}{2}$. ■

Example 4.5. *Let $t = 3$. Then the self-conjugate 3-core partition labeled by $(a_0) = (2, 0, -2)$ is in bijection with the 3-core partition labeled by $(b_1) = (2)$. In particular*

the self-conjugate 3-core $\lambda^* = (4, 2, 1, 1)$ such that $\Delta(\lambda^*) = \{7, 1\}$ is mapped to the $\bar{3}$ -core $\bar{\lambda} = (4, 1)$.

We note that the map in our proof of Theorem 1.5 does not work when t is even, as the tuples that label each self-conjugate t -core partition are associated with an infinite family of \bar{t} -core partitions.

4.2. A bijection between $(s, t)^*$ -core and (\bar{s}, \bar{t}) -cores when $g = 1$. In this section we show that, when $s, t > 1$ are odd and $g = 1$, there is a bijection between self-conjugate (s, t) -core partitions and (\bar{s}, \bar{t}) -core partitions via their corresponding lattice paths. First we detail the lattice constructions mentioned in Section 1.4.

Ford, Mai, and Sze, in (Section 4, [9]), construct an $\lfloor \frac{s}{2} \rfloor \times \lfloor \frac{t}{2} \rfloor$ lattice of odd numbers, which we call $\mathcal{DH}_{s,t}$ the *diagonal hooks diagram*, as follows. First place $st - s - t$ in the upper left-hand corner position, labelled $(1,1)$, in matrix notation. Then, applying Lemma 4.3 (1), we can deduct $2t$ for every move downwards, and $2s$ for every move rightward. The position (i, j) will be filled with $st - s(2j-1) - t(2i-1)$ for $1 \leq i \leq \lfloor \frac{s}{2} \rfloor$ and $1 \leq j \leq \lfloor \frac{t}{2} \rfloor$. Some values in the diagram of diagonals will be negative. We separate the positive values in the upper left quadrant (the *positive side*) from negative values in the bottom right portion (the *negative side*) of the $\lfloor \frac{s}{2} \rfloor \times \lfloor \frac{t}{2} \rfloor$ lattice by a monotonic path which starts at the bottom left-most corner of the diagonal hooks diagram and ends at its right-most top corner. We will call this *the \pm border*. Then we let $|\mathcal{DH}_{s,t}|$ be the set of absolute values of all entries in the diagonal hooks diagram, and we have the following.

Lemma 4.6. *Let $s > 1$ and $t > 1$ be such that $g = \gcd(s, t) = 1$. Then the diagonal hook lengths of any self-conjugate (s, t) -core partition is a subset of $|\mathcal{DH}_{s,t}|$.*

We can now use $\mathcal{DH}_{s,t}$ and the \pm border to describe the following bijection, which is Lemma 7 in [9].

Lemma 4.7. *Let $s, t > 1$ be such that $g = 1$. The self-conjugate (s, t) -core partitions are in correspondence with monotonic paths in $\mathcal{DH}_{s,t}$. In particular, if π is such a monotonic path and $\{d_1, \dots, d_k\}$ is the set of values trapped between π and the \pm border, then $\Delta(\lambda_\pi) = \{|d_1|, \dots, |d_k|\}$ for the corresponding self-conjugate (s, t) -core partition λ_π .*

In other words, firstly, for a given self-conjugate (s, t) -core partition λ , a value on the positive side of row or column of the diagonal hooks diagram can occur as a diagonal hook length of λ if and only if the absolute values in the same row or column but on the negative side *do not*. Secondly, if $|m|$ is a diagonal hook length of λ , and m appears in a column or row of the positive (negative) side of the \pm border, then the absolute value of all entries in the same column or row as m leading up to the \pm border also appear. [This corresponds to condition (2) in Lemma 4.3 when applied to both s and t .]

Example 4.8. *Let $s = 7$ and $t = 11$. Then the 3×5 lattice below is $\mathcal{DH}_{7,11}$. The \pm border is indicated by the dashed path. The self-conjugate $(7, 11)$ -core λ^* corresponding to the lattice path π (indicated by the solid black line) has diagonal hook set*

$$\Delta(\lambda^*) = \{5, 3, 1\}.$$

This is the partition $\lambda^ = (3^3)$.*

59	45	31	17	3
37	23	9	-5	-19
15	1	-13	-27	-41

Let $s, t > 1$ be odd and $g = 1$. Bessenrodt and Olsson [6] consider all possible values for parts of a simultaneous (\bar{s}, \bar{t}) -core partition. In an analogous argument to Anderson's, such values will be of the form $st - s - t - (\alpha s - \beta t)$ for $\alpha \geq 0$ and $\beta \geq 0$. We can visualize this in a way similar to Anderson, with a partially ordered set beginning with $st - s - t$ in the upper rightmost corner. Each move down reduces the value by t , each move to the right reduces the value by s . [We note that our poset orientation here is downward as opposed to Bessenrodt-Olsson's upward orientation; the reason for this will be made clear in the proof of Theorem 4.12.]

However, not all part-values contained in the poset are acceptable parts for a (\bar{s}, \bar{t}) -core partition. Consider $1 \leq j \leq t$. By Olsson's characterization of \bar{t} -core partitions (Equation 5(ii), Chapter 4, [25]) either parts of residue j modulo t or $t - j$ modulo t can appear, but not both. However, $st - s - t - (\frac{t-3}{2}s - \frac{s-3}{2}t) = \frac{t+s}{2}$, and $\frac{t+s}{2} - s = \frac{t-s}{2}$, and the inclusion of both values would violate this condition modulo s . Hence a rectangular section of $\frac{s-1}{2} \times \frac{t-1}{2}$ values in the poset, those *above* (in the sense of the partial order) and *including* $\frac{t+2}{2}$ must be eliminated. What remains are two sets of numbers: the *Yin* portion of the diagram, below the eliminated rectangular section, and the *Yang* portion of the diagram, to the right of the rectangular section. Then Bessenrodt and Olsson (Remark 4.1, [6]) show that by rotating the Yang portion of the diagram 180° , one can attach the Yang portion to the Yin portion to produce a $\frac{s-1}{2} \times \frac{t-1}{2}$ lattice with value $t(\frac{s-1}{2}) - s$ in the top leftmost corner, $(t - s)$ in the bottom leftmost corner, $s(\frac{t-1}{2}) - t$ in the bottom rightmost corner, and $(\frac{t-s}{2})$ in the top rightmost corner.

The following result appears as [6, Remark 3.1].

Lemma 4.9. *Let $s, t > 1$ be odd and $g = 1$. Then the Yin-Yang diagram contains all possible values of parts for an (\bar{s}, \bar{t}) -core partition.*

In particular, values from a column or row of the Yin diagram can be included as parts of an (\bar{s}, \bar{t}) -core partition if and only if no values from the same column or row but in the Yang diagram are included. Let the \mathcal{Y} -border be the path separating the Yin part from the Yang part. The following is a result of Bessenrodt and Olsson (Theorem 3.2, [6]).

Lemma 4.10. *Let $s, t > 1$ be such that $g = 1$. Then (\bar{s}, \bar{t}) -core partitions are labeled by monotonic paths π in the Yin-Yang diagram. In particular, $(\lambda_1, \dots, \lambda_k)$ are the parts of λ where λ_i are the values trapped between π and the \mathcal{Y} -border.*

[We note that recent results by C. Deng [7] describe precisely which monotonic paths correspond to the so-called "even" (\bar{s}, \bar{t}) -core partitions.]

Example 4.11. *Let $s = 7$ and $t = 11$. Then the Yin-Yang diagram is below, and the \mathcal{Y} -border between the two portions is indicated by a dashed line. The $(\bar{7}, \bar{11})$ -core partition $\bar{\lambda}$ that corresponds to the path below is $\bar{\lambda} = (6)$. [Here the elements of the Yang portion of the diagram are labeled with a negative sign (for clarity), so the absolute value of those values will need to taken.]*

26	19	12	5	-2
15	8	1	-6	-13
4	-3	-10	-17	-24

We are now in a position to prove the main result of this subsection.

Theorem 4.12. *Let $s, t > 1$ be odd such that $g = \gcd(s, t) = 1$. Then self-conjugate (s, t) -core partitions are in bijection with (\bar{s}, \bar{t}) -core partitions.*

Proof. By Lemma 4.7, we know that the self-conjugate (s, t) -core partitions are labeled by monotonic paths in an $\lfloor \frac{s}{2} \rfloor \times \lfloor \frac{t}{2} \rfloor$ lattice (the diagonal hook diagram). Consider the self-conjugate (s, t) -core λ_π^* determined by the diagonal hook values trapped between a monotonic path π and the \pm boundary. By Lemma 4.10 we know that the (\bar{s}, \bar{t}) -core partitions are labeled by monotonic (as a consequence of our orientation) paths in a $\frac{s-1}{2} \times \frac{t-1}{2}$ lattice. We delete the values of the diagonals hook diagram and replace them with the values of the Yin-Yang diagram, leaving the path π unchanged. Then the corresponding (\bar{s}, \bar{t}) -core partition $\bar{\lambda}_\pi$ is the one described by the part-values trapped between π and the \mathcal{Y} -border. The map works the same in the other direction. ■

Example 4.13. *The self-conjugate $(7, 11)$ -core partition $\lambda^* = (3^3)$ in Example 4.8 corresponds to the $(\bar{7}, \bar{11})$ -core partition $\bar{\lambda} = (6)$ in Example 4.11.*

4.3. A bijection between $(s, t)^*$ -cores and (\bar{s}, \bar{t}) -cores when $g > 1$ and odd.

Let $s, t > 1$ be odd and $g > 1$ be odd. Using bijections from Section 4.1 and Section 4.2, we construct a correspondence between self-conjugate (s, t) -core partitions and (\bar{s}, \bar{t}) -core partitions. In particular, we apply results from Section 2.2 and Section 2.3 and move from the \bar{g} -core and \bar{g} -quotient of an (\bar{s}, \bar{t}) -core to the g -core and g -quotient of a self-conjugate (s, t) -core.

Theorem 4.14. *Let s, t be odd and such that $g > 1$. Then self-conjugate (s, t) -core partitions are in correspondence with (\bar{s}, \bar{t}) -core partitions.*

Proof. By Theorem 1.7, each self-conjugate (s, t) -core partition λ is labeled by a self-conjugate g -core $Cor_g(\lambda)$ and g -quotient $Quo_g(\lambda) = (\lambda_0, \dots, \lambda_{g-1})$ where each λ_i is an (s', t') -core for $0 \leq i \leq \frac{g-3}{2}$ and $\lambda_i^\vee = \lambda_{g-i-1}$. In particular, since g is odd, $\lambda_{\frac{g-1}{2}}$ is a $(s', t')^*$ -core. Consider the following map Γ .

- (1) Γ sends the self-conjugate g -core $Cor_g(\lambda)$ to its corresponding \bar{g} -core $\gamma^{(0)}$, using Theorem 1.5.
- (2) Γ sends the (s', t') -core λ_i , for $0 \leq i \leq \frac{g-3}{2}$, to itself, renaming it γ_{i+1} .
- (3) Γ sends the self-conjugate (s', t') -core $\lambda_{\frac{g-1}{2}}$ to its corresponding (\bar{s}', \bar{t}') -core γ_0 , using Theorem 4.12.

Then set $\gamma^{(0)} = Cor_{\bar{g}}(\gamma)$ and $(\gamma_0, \gamma_1, \dots, \gamma_{\frac{g-3}{2}}) = Quo_g(\gamma)$ for some partition γ . This, by Theorem 2.7, uniquely determines an (\bar{s}, \bar{t}) -core γ . Since the map goes in both directions, we are done. ■

Example 4.15. *Under Γ , the self-conjugate $(21, 33)$ -core partition*

$$\lambda = (21, 20, 12^4, 11^2, 10, 9, 8, 6, 2^8, 1)$$

where $Cor_3(\lambda) = (4, 2, 1, 1)$ and $Quo_3(\lambda) = ((5, 3^3, 2^2, 1^3), (3^3), (9, 6, 4, 1, 1))$ corresponds to a $(\bar{21}, \bar{33})$ -core partition γ where $Cor_{\bar{3}}(\gamma) = (4, 1)$ and $Quo_{\bar{3}}(\gamma) =$

$((6), (5, 3^3, 2^2, 1))$, since the self-conjugate $(7, 11)$ -core (3^3) corresponds to $(\bar{7}, \bar{11})$ -core (6) by Example 4.13. In particular $\gamma = (20, 19, 18, 10, 8, 7, 4)$.

The following corollary is immediate.

Corollary 4.16. *Let $t', g > 1$ be odd. The self-conjugate $t'g$ -core partitions that are not g -core partitions are in bijection with the $t'g$ -core partitions that are not \bar{g} -core partitions.*

Proof. This follows from Theorem 4.14 when $s = t$ and Theorem 1.5. ■

5. RAMANUJAN-TYPE CONGRUENCES

Srinivasa Ramanujan was the first to notice several remarkable arithmetic properties of the partition function $p(n)$.

In particular, he noted that, for all $k \geq 0$,

$$\begin{aligned} p(5k + 4) &= 0 \pmod{5}, \\ p(7k + 5) &= 0 \pmod{7}, \\ p(11k + 6) &= 0 \pmod{11}. \end{aligned}$$

In 1990, Garvan, Kim, and Stanton [11] proved the above congruences using, among other things, t -core partitions. They also showed that t -core partitions and self-conjugate t -cores of n satisfy Ramanujan-type congruences.

Theorem 5.1. *Let $f_t(n)$ be the number of t -core partitions of n . Then, for all $k \geq 0$,*

$$\begin{aligned} f_5(5k + 4) &\equiv 0 \pmod{5}, \\ f_7(7k + 5) &\equiv 0 \pmod{7}, \text{ and} \\ f_{11}(11k + 6) &\equiv 0 \pmod{11}. \end{aligned}$$

[Note: Theorem 5.1 is actually a special case of a much general result. In particular, Garvan, Kim and Stanton show (see [11, Corollary 1])

$$\begin{aligned} f_5(5^\alpha k - 1) &\equiv 0 \pmod{5^\alpha}, \\ f_7(7^\alpha k - 2) &\equiv 0 \pmod{7^\alpha}, \text{ and} \\ f_{11}(11^\alpha k - 5) &\equiv 0 \pmod{11^\alpha} \end{aligned}$$

for all $\alpha \geq 1$ and $k \geq 1$.]

Since then, congruences and generating functions of t -core partitions have become a rich topic of study. Recently, the second and third author made the following observation (Theorem 4.1, [21]) about properties of \bar{t} -core partitions.

Theorem 5.2. *Suppose $f_{\bar{p}}(n)$ is the number of \bar{p} -core partitions of n , where $p \geq 5$ is prime and r is such that $1 \leq r \leq p - 1$ and $24r + 1$ is a quadratic nonresidue modulo p . Then, for all $k \geq 0$,*

$$f_{\bar{p}}(pk + r) \equiv 0 \pmod{2}.$$

To see how Theorem 5.1 and Theorem 5.2 relate to congruences for the numbers of (s, t) -core partitions and (\bar{s}, \bar{t}) -core partitions for $g > 1$, we will need the following results.

Theorem 5.3. *Let*

$$A(x) = \sum_{n \geq 0} a(n)x^n \quad \text{and} \quad B(x) = \sum_{n \geq 0} b(n)x^n$$

be two generating functions and let $C(x) = A(x^g)B(x)$ for some positive integer g . Let r be an integer satisfying $1 \leq r \leq g - 1$. Then, for any $n \geq 0$,

$$c(gk + r) = \sum_{m \geq 0} a(k - m)b(gm + r).$$

where $C(x) = \sum_{n \leq 0} c(n)x^n$.

Proof. The proof of this result follows directly from the Cauchy product of $A(x^g)$ and $B(x)$. \blacksquare

Corollary 5.4. *Consider the generating functions $A(x), B(x)$, and $C(x)$ as defined in Theorem 5.3. If, for all $k \geq 0$, $b(gk + r) \equiv 0 \pmod{M}$ for some integer M , then for all $k \geq 0$, $c(gk + r) \equiv 0 \pmod{M}$.*

We use Corollary 5.4 to identify congruences in arithmetic progressions satisfied by $\psi_{s,t}(n)$ (which counts the number of (s, t) -core partitions of n) and $\psi_{\bar{s}, \bar{t}}(n)$ (which counts the (\bar{s}, \bar{t}) -core partitions) for specific families of values of s and t . We begin with $\psi_{s,t}(n)$.

Theorem 5.5. *Let $s > 1$ and $t > 1$. Then, for all $k \geq 0$,*

$$\begin{aligned} \psi_{s,t}(5k + 4) &\equiv 0 \pmod{5}, \\ \psi_{s,t}(7k + 5) &\equiv 0 \pmod{7}, \quad \text{and} \\ \psi_{s,t}(11k + 6) &\equiv 0 \pmod{11}. \end{aligned}$$

Proof. Using the notation from Corollary 1.8, let $F_g(n)$ be the number of g -core partitions of n . The result follows directly from Theorem 5.1, Corollary 1.8 and Corollary 5.4. \blacksquare

Let $\psi_{t \setminus g}(n)$ be the number of t -cores of n that are not g -cores. Then we have the following.

Corollary 5.6. *Let $t > 1$ and $g > 1$. Then, for all $k \geq 0$,*

$$\begin{aligned} \psi_{t \setminus g}(5k + 4) &\equiv 0 \pmod{5}, \\ \psi_{t \setminus g}(7k + 5) &\equiv 0 \pmod{7}, \quad \text{and} \\ \psi_{t \setminus g}(11k + 6) &\equiv 0 \pmod{11}. \end{aligned}$$

Proof. The number of t -cores of n that are not g -cores is given by

$$\psi_{t \setminus g}(n) = \sum_{w \geq 1} Q_{t'}(g, w) f_g(n - wg)$$

by Theorem 3.1. Since, when $n = gk + r$, each $f_g(n - wg)$ is of the form $f_g(gk + r - wg) = f_g(g(k - w) + r)$, the result follows from Theorem 5.1. \blacksquare

From Corollary 2.8, we see that

$$\Psi_{\bar{s}, \bar{t}}(x) = \Psi_{\bar{s}', \bar{t}'}(x^g) \Psi_{s', t'}(x^g)^{\frac{g-1}{2}} F_g^2(x),$$

where $F_{\bar{g}}(x) = \prod_{n=1}^{\infty} \frac{(1-x^{2n})(1-x^{gn})^{\frac{g+1}{2}}}{(1-x^n)(1-x^{2gn})}$ is the generating function for the number of \bar{g} -core partitions. Using the notation of Theorem 5.3, we see that $\Psi_{\bar{s},\bar{t}}(x) = A(x^g)B(x)$ where $A(x) = \Psi_{\bar{s}',\bar{t}'}(x^g)\Psi_{s',t'}(x^g)^{\frac{g-1}{2}}$ and $B(x) = F_{\bar{g}}(x)$. In this context, it is easy to prove the following parity results satisfied by $\psi_{\bar{s},\bar{t}}(n)$ for specific values of s and t .

Theorem 5.7. *Let $s > 1$ and $t > 1$ be such that $g \geq 5$ is a prime. Let r be an integer, $1 \leq r \leq p-1$, such that $24r+1$ is a quadratic nonresidue modulo p . Then, for all $k \geq 0$,*

$$\psi_{\bar{s},\bar{t}}(gk+r) \equiv 0 \pmod{2}.$$

Proof. Theorem 5.2 gives us the parity of the values $F_{\bar{g}}(gn+r)$. An application of Corollary 5.4 completes the proof of this theorem. ■

Corollary 5.8. *Let $t = t'g$, where $g \geq 5$ is a prime. Let r be an integer, $1 \leq r \leq p-1$, such that $24r+1$ is a quadratic nonresidue modulo p . Then, for all $k \geq 0$, $\psi_{\bar{t}\setminus\bar{g}}(gk+r) \equiv 0 \pmod{2}$.*

Proof. Note that the result is obvious if $t' = 1$, in which case $\psi_{\bar{t}\setminus\bar{g}}(gk+r) = 0$ (as each \bar{t} -core is a \bar{g} -core). If, on the other hand, $t' > 1$, then the number of partitions of $n \geq g$ which are \bar{t} -cores but not \bar{g} -cores is $\psi_{\bar{t}\setminus\bar{g}}(n) = \sum_{w \geq 1} Q_{\bar{t}'}(g,w)F_{\bar{g}}(n-gw)$, by the proof of Theorem 3.5. ■

6. INDEX OF NOTATION FOR FUNCTIONS

Here we list the notation for enumerating and generating functions that appear throughout the paper.

enumerating function counts the number of

$p(n)$	integer partitions of n
$f_t(n)$	t -core partitions of n
$f_t^*(n)$	self-conjugate t -core partitions of n
$f_{\bar{t}}(n)$	\bar{t} -core partitions of n
$Q_t(n)$	t -quotients of n
$\psi_{s,t}(n)$	(s, t) -core partitions of n
$\psi_{s,t}^*(n)$	self-conjugate (s, t) -core partitions of n
$\psi_{\bar{s},\bar{t}}(n)$	(\bar{s}, \bar{t}) -core partitions of n
$\psi_{t \setminus g}(n)$	t -core partitions of n that are not g -core
$\psi_{t \setminus g}^*(n)$	self-conjugate t -core partitions of n that are not g -core
$\psi_{\bar{t} \setminus \bar{g}}(n)$	\bar{t} -core partitions of n that are not \bar{g} -core

generating function coefficients are

$P(x)$	$p(n)$
$F_t(x)$	$f_t(n)$
$F_t^*(x)$	$f_t^*(n)$
$F_{\bar{t}}(x)$	$f_{\bar{t}}(n)$
$\Psi_{s,t}(x)$	$\psi(n)$
$\Psi_{s,t}^*(x)$	$\psi_{s,t}^*(n)$
$\Psi_{\bar{s},\bar{t}}(x)$	$\psi_{\bar{s},\bar{t}}^*(n)$

Acknowledgements. Part of this work was done at the Centre Interfacultaire Bernoulli (CIB), in the École Polytechnique Fédérale de Lausanne (Switzerland), during the Semester *Local Representation Theory and Simple Groups*. The first two authors are grateful to the CIB for their financial and logistical support. The first author also acknowledges financial support from the Engineering and Physical Sciences Research Council grant *Combinatorial Representation Theory* EP/M019292/1. The second author was supported by PSC-TRADA-46-493 and thanks George Andrews who supported a visit to Penn State where this research began. The second author also thanks Christopher R. H. Hanusa for helpful conversations on diagrams and references, and notes that some diagrams were made using the ytab package. All of the authors thank the anonymous referee for the careful reading and detailed and helpful suggestions.

REFERENCES

- [1] D. Armstrong, C.H.R. Hanusa, B. C. Jones, Results and conjectures on simultaneous core partitions, *European J. Combin.* **41** (2014), 205–220.
- [2] J. Anderson, Partitions which are simultaneously t_1 - and t_2 -core, *Discrete Math.*, **248** (2002), 237–243.
- [3] D. Aukerman, B. Kane, L. Sze, On simultaneous s -cores/ t -cores, *Discrete Mathematics* 309 (2009) no.9, 2712–2720
- [4] C. Bessenrodt, Representations of the covering groups of the symmetric groups and their combinatorics, *Electronic J. Sm. Loth. Combinatoire* B33a (1995), 29 pp.
- [5] J. Baldwin, M. Depweg, B. Ford, A. Kunin, L. Sze, Self-conjugate t -core partitions, sums of squares, and p -blocks of A_n , *J. Algebra* **297** (2006) 438–452.
- [6] C. Bessenrodt and J. B. Olsson, Spin block inclusions, *J. Algebra* **306** (2006), 3–16

- [7] C. Deng, Even self-associate partitions and spin characters of \tilde{S}_n , *Discrete Mathematics*, 342 (2019) 540-545.
- [8] K. Erdmann and G. Michler, Blocks for Symmetric Groups and their Covering Groups and Quadratic Forms, *Contributions to Algebra and Geometry* Volume 37 (1972), No. 1 103-118
- [9] B. Ford, H. Mai, L. Sze, Self-conjugate p - and q -core partitions and blocks of A_n , *J. Num Th.* (2009) 858–865
- [10] Private communication between the second author and Joe Gallian. This was checked by Calvin Deng at the 2015 Duluth Summer REU.
- [11] F. Garvan, D. Kim, and D. Stanton, Cranks and t -cores. *Invent. Math.*, Vol. 101, no. 1 (1990), 1-17
- [12] J.-B. Gramain and R. Nath On core and bar-core partitions. *The Ramanujan Journal*, vol 27, no. 2, pp. 229-233.
- [13] A. Granville and K. Ono, Defect zero p -blocks for finite simple groups, *Trans. Amer. Soc.* **348** (1996) 221-228
- [14] C.R.H. Hanusa and R. Nath, The number of self-conjugate partitions, *J. Num. Th.* **77** (2013)
- [15] G. James and A. Kerber, Representation Theory of the Symmetric Group, Encyclopedia of Mathematics, 1981
- [16] P. Johnson Lattice points and simultaneous core partitions, *Electronic J. Combinatorics* Vol. 25 (2018)
- [17] B. Kane, Master's Thesis, Carnegie Mellon University.
- [18] I. Kiming, A note on a theorem of A. Granville and K. Ono, *J. Number Theory* 60 (1996), 97–102
- [19] I. Kiming, On the existence of \bar{p} -core partitions of natural numbers. *Quart. J. Math. Oxford* (2), Vol. 48, (1997) 59-65.
- [20] R. Nath, On the t -core of an s -core partition, *Integers*, 8 (2008), A28
- [21] R. Nath and J. A. Sellers, Congruences for the number of spin characters of the double covers of the symmetric and alternating groups. *Adv. in Appl. Math.*, Vol. 80 (2016) 114-130.
- [22] J.B. Olsson, McKay numbers and heights of characters, *Math. Scand.* 38 (1976), 25-42
- [23] J. B. Olsson, Frobenius symbols for partitions and degrees of spin characters, *Math. Scand.* 61 (1987), 223-247.
- [24] J. B. Olsson, On the p -blocks of the Symmetric and Alternating Groups and Their Covering Groups *Journal of Algebra* **128** s3-12:1 (1990) 188-213
- [25] J. B. Olsson, *Combinatorics of the Representation Theory of the Symmetric Groups* Vorlesungen aus dem FB Mathematik der Univ. Essen, Heft 20, 1993.
- [26] J.B. Olsson, The s -core of a t -core, *J. Comb. Theory, A* Volume 113, Issue 2 (2009)
- [27] J.B. Olsson, D. Stanton, Block inclusions and cores of partitions, *Aequationes mathematicae* 74 (2007), no. 1-2, 97–110
- [28] K. Ono, A note on the number of t -core partitions, *Rocky Mountain Journal of Mathematics* **25**, (1995) 1165–1169.
- [29] V. Wang, Simultaneous core partitions: parametrization and sums, *Electronic J. Comb.* **23**, (2016) P1.4
- [30] J.L.P Wang, J.Y.X. Yang, On the average size of an (\bar{s}, \bar{t}) -core partition, *Taiwanese Journal of Mathematics*, (2019)
- [31] J.X.Y. Yang, Bijections between bar-core and self-conjugate core partitions, *Ramanujan J.* (2019)

INSTITUTE OF MATHEMATICS, UNIVERSITY OF ABERDEEN, KINGS COLLEGE

ABERDEEN, AB24 3UE, UK

JBGRAMAIN@ABDN.AC.UK

DEPARTMENT OF MATHEMATICS, CUNY-YORK COLLEGE, JAMAICA, NY 11451, USA,

RNATH@YORK.CUNY.EDU

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF MINNESOTA DULUTH,

DULUTH, MN 55812, USA,

