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MASSEY PRODUCTS IN THE HOMOLOGY OF THE LOOP SPACE OF A P-COMPLETED CLASSIFYING SPACE: FINITE GROUPS WITH CYCLIC SYLOW P-SUBGROUPS

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Abstract Let G be a finite group with cyclic Sylow p-subgroups, and let k be a field of characteristic p. Then $H^*(BG;k)$ and $H_*(\Omega BG^{\wedge}_p;k)$ are A_∞ algebras whose structure we determine up to quasi-isomorphism.

Keywords: cyclic Sylow subgroups; cohomology of groups; p-completed classifying space; loop spaces; Massey products; A_∞ algebras

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1. Introduction

The general context is that we have a finite group G, and a field k of characteristic p. We are interested in the differential graded cochain algebra $C^*(BG;k)$ and the differential graded algebra $C_*(\Omega(BG^{\wedge}_p);k)$ of chains on the loop space: these two are Koszul dual to each other, and the Eilenberg–Moore and Rothenberg–Steenrod spectral sequences relate the cohomology ring $H^*(BG;k)$ to the homology ring $H_*(\Omega(BG^{\wedge}_p);k)$, see § 5. Of course if G is a p-group, BG is p-complete so $\Omega(BG^{\wedge}_p)\simeq G$, but, in general, $H_*(\Omega(BG^{\wedge}_p);k)$ is infinite dimensional. Henceforth, we will omit the brackets from $\Omega(BG^{\wedge}_p)$.

We consider a simple case where the two rings are not formal, but we can identify the A_∞ structures precisely (see § 3 for a brief summary on A_∞-algebras). From now on, we suppose specifically that G is a finite group with cyclic Sylow p-subgroup P, and let BG be its classifying space. Then the inclusion of the Sylow p-normaliser $N_G(P)\to G$ and the quotient map $N_G(P)\to N_G(P)/O_{p'}N_G(P)$ induce mod p cohomology equivalences

$$B(N_G(P)/O_{p'}N_G(P)) \leftarrow BN_G(P) \to BG,$$

see Swan [15] or Theorem II.6.8 of Adem and Milgram [1]. Hence, after p-completion, we have homotopy equivalences

$$B(N_G(P)/O_{p'}N_G(P))^{\wedge}_p \xleftarrow{\sim} BN_G(P)^{\wedge}_p \xrightarrow{\sim} BG^{\wedge}_p,$$

see Lemma I.5.5 of Bousfield and Kan [3]. Here, $O_{p'}N_G(P)$ denotes the largest normal p'-subgroup of $N_G(P)$. Thus, $N_G(P)/O_{p'}N_G(P)$ is a semidirect product $\mathbb{Z}/p^n \rtimes \mathbb{Z}/q$, where q is a divisor of $p-1$, and $\mathbb{Z}/q$ acts faithfully as a group of automorphisms of $\mathbb{Z}/p^n$. In particular, the isomorphism type of $N_G(P)/O_{p'}N_G(P)$ only depends on $|P| = p^n$ and the inertial index $q = |N_G(P) : C_G(P)|$, and therefore so does the homotopy type of $BG^{\wedge}_p$. Our main theorem determines the multiplication maps m_i in the A_∞ structure on $H^*(BG;k)$ and $H_*(\Omega(BG^{\wedge}_p);k)$ arising from $C^*(BG;k)$ and $C_*(\Omega(BG^{\wedge}_p);k)$ respectively. We will suppose from now on that $p^n > 2$, $q > 1$ since the case of a p-group is well understood.

The starting point is the cohomology ring

$$H^*(BG;k) = H^*(B\mathbb{Z}/p^n;k)^{\mathbb{Z}/q} = k[x] \otimes \Lambda(t) \text{ with } |x| = -2q, \quad |t| = -2q+1.$$

There is a preferred generator $t_1 \in H^1(B\mathbb{Z}/p^n;k) = \mathrm{Hom}(\mathbb{Z}/p^n, k)$ and we take $x_1 \in H^2(B\mathbb{Z}/p^n;k)$ to be the nth Bockstein of t_1. Now take $x = x_1^q$, $t = x_1^{q-1}t_1$.

Before stating our result, we should be clear about grading and signs.

Remark 1.1. We will be discussing both homology and cohomology, so we should be explicit that everything is graded homologically, so that differentials always lower degree. Explicitly, the degree of an element of $H^i(G;k)$ is $-i$.

Remark 1.2. Sign conventions for Massey products and A_∞ algebras mean that a specific sign will enter repeatedly in our statements, so for brevity, we write

$$\epsilon(s) = (-1)^{s(s-1)/2} = \begin{cases} +1 & s \equiv 0,1 \mod 4 \\ -1 & s \equiv 2,3 \mod 4 \end{cases}.$$

Theorem 1.3. *Let G be a finite group with cyclic Sylow p-subgroup P of order p^n and inertial index q so that*

$$H^*(BG;k) = k[x] \otimes \Lambda(t) \text{ with } |x| = -2q, \quad |t| = -2q+1 \quad \text{and} \quad \beta_n t = x$$

Up to quasi-isomorphism, the A_∞ structure on $H^*(BG;k)$ is determined by

$$m_{p^n}(t,\ldots,t) = \epsilon(p^n)x^h$$

where $h = p^n - (p^n-1)/q$. This implies

$$m_{p^n}(x^{j_1}t,\ldots,x^{j_{p^n}}t) = \epsilon(p^n)x^{h+j_1+\cdots+j_{p^n}}$$

for all $j_1, \ldots, j_{p^n} \geq 0$. All m_i for $i > 2$ on all other i-tuples of monomials give zero.

If $q > 1$ and $p^n \neq 3$ then

$$H_*(\Omega BG^\wedge_p; k) = k[\tau] \otimes \Lambda(\xi) \quad \text{where } |\tau| = 2q - 2, \quad |\xi| = 2q - 1.$$

Up to quasi-isomorphism, the A_∞ structure is determined by

$$m_h(\xi, \dots, \xi) = \epsilon(h)\tau^{p^n}.$$

This implies

$$m_h(\tau^{j_1}\xi, \dots, \tau^{j_h}\xi) = \epsilon(h)\tau^{p^n + j_1 + \cdots + j_h}$$

for all $j_1, \dots, j_{p^n} \geq 0$. All m_i for $i > 2$ on all other i-tuples of monomials give zero.

If $q > 1$ and $p^n = 3$ then $q = 2$ and

$$H_*(\Omega BG^\wedge_p; k) = k[\tau, \xi]/(\xi^2 + \tau^3),$$

and all m_i are zero for $i > 2$.

2. The group algebra and its cohomology

We assume from now on, without loss of generality, that G has a normal cyclic Sylow p-subgroup $P = C_G(P)$, with inertial index $q = |G : P|$. We shall assume that $q > 1$, which then forces p to be odd. For notation, let

$$G = \langle g, s \mid g^{p^n} = 1, s^q = 1, sgs^{-1} = g^\gamma \rangle \cong \mathbb{Z}/p^n \rtimes \mathbb{Z}/q,$$

where γ is a primitive qth root of unity modulo p^n. Let $P = \langle g \rangle$ and $H = \langle s \rangle$ as subgroups of G.

In this section, we introduce a grading on kG. This comes from the fact that the radical filtration of kG is isomorphic to its associated graded in this case. This is a somewhat rare phenomenon, but when it happens, it induces an extra grading on mod p cohomology of BG and homology of $\Omega BG^\wedge_p$, that we can exploit to good effect.

Let k be a field of characteristic p. The action of H on kP by conjugation preserves the radical series, and since $|H|$ is not divisible by p, there are invariant complements. Thus, we may choose an element $U \in J(kP)$ such that U spans an H-invariant complement of $J^2(kP)$ in $J(kP)$. It can be checked that

$$U = \sum_{\substack{1 \leqslant j \leqslant p^n - 1, \\ j^p \equiv j \pmod{p}}} g^j/j$$

is such an element, and that $sUs^{-1} = \gamma U$. This gives us the following presentation for kG:

$$kG = k\langle s, U \mid U^{p^n} = 0, s^q = 1, sU = \gamma U s \rangle.$$

We shall regard kG as a $\mathbb{Z}[1/q]$-graded algebra with $|s| = 0$ and $|U| = 1/q$. Then the bar resolution is doubly graded, and taking homomorphisms into k, the cochains $C^*(BG; k)$

inherit a double grading. The differential decreases the homological grading and preserves the internal grading. Thus, the cohomology $H^*(G, k) = H^*(BG; k)$ is doubly graded:

$$H^*(BG; k) = k[x] \otimes \Lambda(t)$$

where $|x| = (-2q, -p^n)$, $|t| = (-2q+1, -h)$, and $h = p^n - (p^n - 1)/q$. Here, the first degree is homological, the second internal. The Massey product $\langle t, t, \ldots, t\rangle$ (p^n repetitions) is equal to $-x^h$. This may easily be determined by restriction to P, where it is well known that the p^n-fold Massey product of the degree one exterior generator is a non-zero degree two element. The usual convention is to make the constant -1, because this Massey product is minus the nth Bockstein of t [10, Theorem 19].

3. A_∞-algebras

An A_∞-algebra over a field is a $\mathbb{Z}$-graded vector space A with graded maps $m_n : A^{\otimes n} \to A$ of degree $n-2$ for $n \geq 1$ satisfying

$$\sum_{r+s+t=n} (-1)^{r+st} m_{r+1+t}(id^{\otimes r} \otimes m_s \otimes id^{\otimes t}) = 0$$

for $n \geq 1$. The map m_1 is therefore a differential, and the map m_2 induces a product on $H_*(A)$.

A theorem of Kadeishvili [7] (see also Keller [8, 9] or Merkulov [13]) may be stated as follows. Suppose that we are given a differential graded algebra A, over a field k. Let $Z^*(A)$ be the cocycles, $B^*(A)$ be the coboundaries, and $H^*(A) = Z^*(A)/B^*(A)$. Choose a vector space splitting $f_1 \colon H^*(A) \to Z^*(A) \subseteq A$ of the quotient. Then this gives by an inductive procedure an A_∞ structure on $H^*(A)$ so that the map f_1 is the degree one part of a quasi-isomorphism of A_∞-algebras.

If A happens to carry auxiliary gradings respected by the product structure and preserved by the differential, then it is easy to check from the inductive procedure that the maps in the construction may be chosen so that they also respect these gradings. It then follows that the structure maps m_i of the A_∞ structure on $H^*(A)$ also respect these gradings.

Let us apply this to $H^*(BG; k)$. We examine the elements $m_i(t, \ldots, t)$. By definition, we have $m_1(t) = 0$ and $m_2(t, t) = 0$. The degree of $m_i(t, \ldots, t)$ is i times the degree of t, increased in the homological direction by $i - 2$. This gives

$$|m_i(t, \ldots, t)| = i(-2q+1, h) + (i-2, 0) = (-2iq + 2i - 2, ih).$$

The homological degree is even, so if $m_i(t, \ldots, t)$ is non-zero then it is a multiple of a power of x. Comparing degrees, if $m_i(t, \ldots, t)$ is a non-zero multiple of x^α then we have

$$2iq - 2i + 2 = 2\alpha q, \quad ih = \alpha p^n.$$

Eliminating α, we obtain $(iq - i + 1)p^n = ihq$. Substituting $h = p^n - (p^n - 1)/q$, this gives $i = p^n$. Finally, since the Massey product of p^n copies of t is equal to $-x^h$, it follows that $m_{p^n}(t, \ldots, t) = \epsilon(p^n)x^h$, where the sign is as defined in Remark 1.2 ([11, Theorem 3.1], corrected in [4, Theorem 3.2]). Thus, we have

$$m_i(t, \ldots, t) = \begin{cases} \epsilon(p^n)x^h & i = p^n \\ 0 & \text{otherwise.} \end{cases}$$

We shall elaborate on this argument in a more general context in the next section, where we shall see that the rest of the A_∞ structure is also determined in a similar way.

4. A_∞ structures on a polynomial tensor exterior algebra

In this section, we shall examine the following general situation. Our goal is to establish that there are only two possible A_∞ structures satisfying Hypothesis 4.1, and that the Koszul dual also satisfies the same hypothesis with the roles of a and b, and of h and ℓ reversed.

Hypothesis 4.1. A is a $\mathbb{Z} \times \mathbb{Z}$-graded A_∞-algebra over a field k, where the operators m_i have degree $(i - 2, 0)$, satisfying

(1) $m_1 = 0$, so that m_2 is strictly associative,

(2) ignoring the m_i with $i > 2$, the algebra A is $k[x] \otimes \Lambda(t)$ where $|x| = (-2a, -\ell)$ and $|t| = (-2b - 1, -h)$, and

(3) $ha - \ell b = 1$.

Remarks 4.2. (i) The A_∞-algebra $H^*(BG; k)$ of the last section satisfies this hypothesis, with $a = q$, $b = q - 1$, $h = p^n - (p^n - 1)/q$, $\ell = p^n$.

(ii) By comparing degrees, if we have $m_\ell(t, \ldots, t) = \epsilon(\ell)x^h$ then $(2b + 1)\ell + 2 - \ell = 2ah$ and so $ha - \ell b = 1$. This explains the role of part (3) of the hypothesis. The consequence is, of course, that a and b are coprime, and so are h and ℓ.

Lemma 4.3. *If $m_i(t, \ldots, t)$ is non-zero, then $i = \ell > 2$ and $m_\ell(t, \ldots, t)$ is a multiple of x^h.*

Proof. The argument is the same as in the last section. The degree of $m_i(t, \ldots, t)$ is $i|t| + (i - 2, 0) = (-2ib - 2, -ih)$. Since the homological degree is even, if $m_i(t, \ldots, t)$ is non-zero then it is a multiple of some power of x, say x^α. Then we have

$$2ib + 2 = 2\alpha a, \qquad ih = \alpha\ell.$$

Eliminating α gives $(ib + 1)\ell = iha$, and so using $ha - \ell b = 1$ we have $i = \ell$. Substituting back gives $\alpha = h$. □

Elaborating on this argument gives the entire A_∞ structure. If $m_\ell(t, \ldots, t)$ is non-zero, then by rescaling the variables t and x if necessary we can assume that $m_\ell(t, \ldots, t) = \epsilon(\ell)x^h$ (note that we can even do this without extending the field, since ℓ and h are coprime).

Proposition 4.4. *If $m_\ell(t, \ldots, t) = 0$ then all m_i are zero for $i > 2$. If $m_\ell(t, \ldots, t) = \epsilon(\ell)x^h$ then $m_\ell(x^{j_1}t, \ldots, x^{j_\ell}t) = \epsilon(\ell)x^{h+j_1+\cdots+j_\ell}$, and all m_i for $i > 2$ on all other i-tuples of monomials give zero.*

Proof. All monomials live in different degrees, so we do not need to consider linear combinations of monomials. Suppose that $m_i(x^{j_1}t^{\varepsilon_1}, \ldots, x^{j_i}t^{\varepsilon_i})$ is some constant multiple of $x^j t^\varepsilon$, where each of $\varepsilon_1, \ldots, \varepsilon_i, \varepsilon$ is either zero or one. Then comparing degrees, we have

$$(j_1 + \cdots + j_i)|x| + (\varepsilon_1 + \cdots + \varepsilon_i)|t| + (i-2, 0) = j|x| + \varepsilon|t|.$$

Setting

$$\alpha = j_1 + \cdots + j_i - j, \quad \beta = \varepsilon_1 + \cdots + \varepsilon_i - \varepsilon$$

we have $\beta \leqslant i$, and

$$\alpha(-2a, -\ell) + \beta(-2b-1, -h) + (i-2, 0) = 0.$$

Thus

$$2\alpha a + 2\beta b + \beta + 2 - i = 0, \quad \alpha\ell + \beta h = 0.$$

Eliminating α, we obtain

$$-2\beta ha + 2\beta\ell b + \beta\ell + 2\ell - i\ell = 0.$$

Since $ha - \ell b = 1$, this gives $\beta = \ell(i-2)/(\ell-2)$. Combining this with $\beta \leqslant i$ gives $i \leqslant \ell$. If $i < \ell$ then β is not divisible by ℓ, and so $\alpha\ell + \beta h = 0$ cannot hold. So we have $\beta = i = \ell$, $\varepsilon_1 = \cdots = \varepsilon_\ell = 1$, $\varepsilon = 0$, $\alpha = -h$, and $j = h + j_1 + \cdots + j_\ell$. Finally, the identities satisfied by the m_i for an A_∞ structure show that all the constant multiples have to be the same, hence all equal to zero or after rescaling, all equal to $\epsilon(\ell)$. □

Theorem 4.5. *Under Hypothesis 4.1, if $\ell > 2$ then there are two possible A_∞ structures on A. There is the formal one, where m_i is equal to zero for $i > 2$, and the non-formal one, where after replacing x and t by suitable multiples, the only non-zero m_i with $i > 2$ is m_ℓ, and the only non-zero values on monomials are given by*

$$m_\ell(x^{j_1}t, \ldots, x^{j_\ell}t) = \epsilon(\ell)x^{h+j_1+\cdots+j_\ell}.$$

Proof. This follows from Lemma 4.3 and Proposition 4.4. □

Theorem 4.6. *Let $G = \mathbb{Z}/p^n \rtimes \mathbb{Z}/q$ as above, and k a field of characteristic p. Then the A_∞ structure on $H^*(G, k)$ given by Kadeishvili's theorem may be taken to be the non-formal possibility named in the above theorem, with $a = q$, $b = q - 1$, $h = p^n - (p^n - 1)/q$, $\ell = p^n$.*

Proof. Since we have $m_{p^n}(t, \ldots, t) = \epsilon(p^n)x^h$, the formal possibility does not hold. □

Remark 4.7. Dag Madsen's thesis [12] has an appendix in which the A_∞ structure is computed for the cohomology of a truncated polynomial ring, reaching similar conclusions by more direct methods. A similar computation appears in Examples 7.1.5 and 7.2.4 of the book by Witherspoon [16].

5. Loops on $BG^{\wedge}_p$

In general, for a finite group G, the classifying space BG is p-good, see Proposition VII.5.1 of Bousfield and Kan [3]. So its p-completion $BG^{\wedge}_p$ is p-complete. This space and its loop space have been the subject of considerable study, beginning with the work of Cohen and Levi [5]. We have $H^*(BG^{\wedge}_p; k) \cong H^*(BG; k) = H^*(G, k)$ and $\pi_1(BG^{\wedge}_p) = G/O^p(G)$, the largest p-quotient of G. In our case, $G = \mathbb{Z}/p^n \rtimes \mathbb{Z}/q$ with $q > 1$, we have $G = O^p(G)$ and so $BG^{\wedge}_p$ is simply connected. So the Eilenberg–Moore spectral sequence converges to the homology of its loop space:

$$\mathsf{Ext}^{*,*}_{H^*(BG;k)}(k, k) \cong \mathsf{Cotor}^{*,*}_{H_*(BG;k)}(k, k) \Rightarrow H_*(\Omega BG^{\wedge}_p; k)$$

(Eilenberg–Moore [6], Smith [14]). The internal grading on $C^*(BG; k)$ gives this spectral sequence a third grading that is preserved by the differentials, and $H_*(\Omega BG^{\wedge}_p; k)$ is again doubly graded. Since $H^*(G, k) = k[x] \otimes \Lambda(t)$ with $|x| = (-2q, -p^n)$ and $|t| = (-2q + 1, -h)$, it follows that the E^2 page of this spectral sequence is $k[\tau] \otimes \Lambda(\xi)$ where $|\xi| = (-1, 2q, p^n)$ and $|\tau| = (-1, 2q - 1, h)$ (recall $h = p^n - (p^n - 1)/q$). Provided that we are not in the case $h = 2$, which only happens if $p^n = 3$, ungrading E^∞ gives

$$H_*(\Omega BG^{\wedge}_p; k) = k[\tau] \otimes \Lambda(\xi)$$

with $|\tau| = (2q - 2, h)$ and $|\xi| = (2q - 1, p^n)$.

In the exceptional case $h = 2$, $p^n = 3$, we have $q = 2$, and the group G is the symmetric group Σ_3 of degree three. An explicit computation (for example by squeezed resolutions [2]) gives

$$H_*(\Omega(B\Sigma_3)^{\wedge}_3; k) = k[\tau, \xi]/(\xi^2 + \tau^3)$$

with $|\tau| = (2, 2)$ and $|\xi| = (3, 3)$, and the two gradings collapse to a single grading.

Applying Theorem 4.5, and using the fact that either formal case is Koszul dual to the other, we have the following.

Theorem 5.1. *Suppose that $p^n \neq 3$. Then the A_∞ structure on $H_*(\Omega BG^{\wedge}_p; k) = k[\tau] \otimes \Lambda(\xi)$ is given by*

$$m_h(\tau^{j_1}\xi, \ldots, \tau^{j_h}\xi) = \epsilon(h)\tau^{p^n + j_1 + \cdots + j_h},$$

and for $i > 2$, all m_i on all other i-tuples of monomials give zero.

Again using [11] corrected in [4], we may reinterpret this in terms of Massey products.

Corollary 5.2. *In $H_*(\Omega BG^{\wedge}_p; k)$, the Massey products $\langle \xi, \ldots, \xi \rangle$ (i times) vanish for $0 < i < h$, and give $-\tau^{p^n}$ for $i = h$.*

Note that the exceptional case $p^n = 3$ also fits the corollary, if we interpret a 2-fold Massey product as an ordinary product.

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