ON THE DE RHAM-WITT COMPLEX OVER PERFECTOID RINGS

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Abstract

Fix an odd prime p. The results in this paper are modeled after work of Hesselholt and Hesselholt-Madsen on the p-typical absolute de Rham-Witt complex in mixed characteristic. We have two primary results. The first is an exact sequence which describes the kernel of the restriction map on the de Rham-Witt complex over A, where A is the ring of integers in an algebraic extension of $\mathbf{Q}_p$, or where A is a p-torsion-free perfectoid ring. The second result is a description of the p-power torsion (and related objects) in the de Rham-Witt complex over A, where A is a p-torsion-free perfectoid ring containing a compatible system of p-power roots of unity. Both of these results are analogous to results of Hesselholt and Madsen. Our main contribution is the extension of their results to certain perfectoid rings. We also provide algebraic proofs of these results, whereas the proofs of Hesselholt and Madsen used techniques from topology.

1. Introduction

Let p denote an odd prime and let A denote a $\mathbf{Z}_{(p)}$-algebra. The (absolute, p-typical) de Rham-Witt complex over A is defined by Hesselholt and Madsen in (Hesselholt and Madsen, 2004, Introduction) as the initial object in the category of Witt complexes over A. It is closely related to trace invariants such as topological Hochschild and cyclic homologies which can be used to compute algebraic K-theory. The main goal of the present paper is to prove certain algebraic properties of the de Rham-Witt complex. These results extend earlier work of Hesselholt and Madsen to a certain class of perfectoid rings. (See Definition 2.19 for the definition of a perfectoid ring.) We also note that our proofs are purely algebraic, whereas the original proofs by Hesselholt and Madsen used topology.

Every element in the de Rham-Witt complex over A can be expressed using Witt vectors and the differential map $d : W_n\Omega^i_A \to W_n\Omega^{i+1}_A$. Given two such expressions, however, it is difficult to determine if they are equal. This phenomenon is already present in the module of Kähler differentials: every element in $\Omega^i_{A/R}$ can be expressed using A and the differential map, but in general it is difficult to determine if two such expressions are equal. The most challenging steps in our proofs involve showing that certain elements in $W_n\Omega^1_A$ are non-zero. (This includes the case $n = 1$, i.e., a difficult step involves showing that certain elements in $\Omega^1_A = \Omega^1_{A/\mathbf{Z}}$ are non-zero.)

Our main result is Theorem C below. Theorem C concerns p-torsion in the level n and degree 1 component of the de Rham-Witt complex, $W_n\Omega^1_A$, for certain perfectoid rings A and for all positive integers $n \geq 1$. Our proof uses induction on n. The base case uses Theorem A below and the induction step uses Theorem B. Together, these three theorems are the main results of this paper, and we believe that Theorem A and Theorem B are of interest independent of their use in the proof of Theorem C. We mention that both Theorem A and Theorem B hold for all p-torsion-free perfectoid rings (unlike Theorem C). Also, we note that Theorem A and its cotangent complex counterpart, Theorem A.2, do not involve the de Rham-Witt complex.

We now briefly describe these three main results, going through them in the same order in which they appear later in the paper. Our first result concerns p-torsion in the module of Kähler differentials.

Theorem A. *Let A denote a p-torsion-free perfectoid ring. The p-adic Tate module of $\Omega^1_{A/\mathbf{Z}_p}$, denoted $T_p(\Omega^1_{A/\mathbf{Z}_p})$, is a free A-module of rank one.*

See Theorem 3.5 below for the proof of Theorem A. A similar result appears in (Fontaine, 1982, Section 1) for the case that A is the ring of integers in a separable closure of a local field.

We give an example of the sort of considerations that arise when trying to prove Theorem A.

Example 1.1. Let $R = \mathbf{Z}_p[\zeta_p]$; this ring is not perfectoid because the p-power map is not surjective modulo p. Because $d1 = 0 \in \Omega^1_{R/\mathbf{Z}_p}$, it follows directly from the Leibniz rule that $d\zeta_p$ is p-torsion. Proving that $d\zeta_p \neq 0 \in \Omega^1_{R/\mathbf{Z}_p}$ is more subtle. One approach is to use the isomorphism $\mathbf{Z}_p[\zeta_p] \cong \mathbf{Z}_p[x]/(x^{p-1} + x^{p-2} + \cdots + x + 1)$ together with an exact sequence involving Kähler differentials of a quotient ring (see (Matsumura, 1989, Theorem 25.2) for the precise result, or see Remark 1.2 below for a brief summary). Similar considerations can be used for $\Omega^1_{S/\mathbf{Z}_p}$ when $S = \mathbf{Z}_p[\zeta_{p^n}]$ for any $n \geq 1$ or for their union $S = \mathbf{Z}_p[\zeta_{p^\infty}]$. It seems significantly more difficult to extend this style of argument to the case of the p-adic completion $\mathbf{Z}_p[\zeta_{p^\infty}]^\wedge$ (which is a perfectoid ring, unlike the other rings mentioned in this example). Different techniques seem necessary to deal with rings like $\mathbf{Z}_p[\zeta_{p^\infty}]^\wedge$, and our approach is to use the cotangent complex. (For clarification, we briefly point out that $d\zeta_p$ is not an S-module generator of p-torsion in $\Omega^1_{S/\mathbf{Z}_p}$ for any of $S = \mathbf{Z}_p[\zeta_{p^n}]$, $n \geq 2$, or $S = \mathbf{Z}_p[\zeta_{p^\infty}]$ or $S = \mathbf{Z}_p[\zeta_{p^\infty}]^\wedge$. Instead, exactly as in (Hesselholt, 2006), there exists a p-torsion element α such that $(\zeta_p - 1)\alpha = d\zeta_p$, and such an α is a generator of p-torsion.)

Remark 1.2. In Example 1.1 we referenced (Matsumura, 1989, Theorem 25.2); we briefly recall the context of that theorem. Let $R_0 \to R$ denote a ring homomorphism and let $I \subseteq R$ denote an ideal. (What we describe here works in complete generality for commutative rings with unity; the notation need not refer to any specific rings from Example 1.1.) Then (Matsumura, 1989, Theorem 25.2) describes an exact sequence of R/I-modules

$$I/I^2 \to \Omega^1_{R/R_0} \otimes_R (R/I) \to \Omega^1_{(R/I)/R_0} \to 0.$$

We point out two aspects of this sequence. First, notice that the left-most map is not guaranteed to be injective. This is again a reflection of the fact that it is in general difficult to prove elements are non-zero in Kähler differentials (and in the de Rham-Witt complex). Second, notice that if I is a principal ideal generated by a non-zero-divisor, then I/I^2 is isomorphic to R/I. The authors consider this one of the benefits of the requirement in the definition of perfectoid that $\ker \theta$ be a principal ideal; see for example the proof of (Bhatt et al., 2019, Proposition 4.19(2)), which is an essential result in our arguments involving Kähler differentials.

The strategy described in Example 1.1 relies heavily on having an explicit description of $\mathbf{Z}_p[\zeta_p]$. For rings of the generality considered in this paper, we cannot expect such an explicit description. To prove Theorem A for an arbitrary p-torsion-free perfectoid ring, we rely on properties of the cotangent complex, and the authors thank Bhargav Bhatt for suggesting this approach to Theorem A. In particular, our proof of Theorem A is intertwined with the proof of the following. Again, see Theorem 3.5 below for the proof.

Theorem A.2. *Let A denote a p-torsion-free perfectoid ring. The multiplication-by-p map is surjective on $H^{-1}(L_{A/\mathbf{Z}_p})$, where $L_{A/\mathbf{Z}_p}$ denotes the cotangent complex of $\mathbf{Z}_p \to A$.*

Many of the algebraic arguments in this paper are elementary. With the exception of the final section relating our results to algebraic K-theory, our arguments involving the cotangent complex are the most technically advanced part of this paper. On the other hand, the cotangent complex arguments are used only to prove results concerning Kähler differentials, so the reader who is willing to accept these results can understand the paper without the cotangent complex. The authors had originally hoped to bypass the cotangent complex at the expense of considering only p-torsion-free perfectoid rings which are rings of integers in a valued field. Even in this case, technical difficulties arose. To continue the scenario described in Example 1.1, again following Fontaine, we were able to describe p-torsion in $\Omega^1_{\mathbf{Z}_p[\zeta_{p^\infty}]/\mathbf{Z}_p}$, but without using the cotangent complex, we were unable to describe p-torsion in $\Omega^1_{\mathbf{Z}_p[\zeta_{p^\infty}]^\wedge/\mathbf{Z}_p}$ (where $\mathbf{Z}_p[\zeta_{p^\infty}]^\wedge$ denotes the p-adic completion of $\mathbf{Z}_p[\zeta_{p^\infty}]$; perfectoid rings are by definition required to be p-adically complete).

We have already mentioned the difficulty of proving that elements in the module of Kähler differentials are non-zero. We next discuss results, Theorem B and Corollary B.2, which help to prove certain elements in the de Rham-Witt complex are non-zero. The results are based on corresponding results proved by Hesselholt and Madsen; we give a reference to their results after stating Corollary B.2.

The significance of Corollary B.2 in this paper is as follows. Our primary goal is to study p-power torsion in the de Rham-Witt complex. Assume we have a satisfactory understanding of the Kähler differentials (e.g., we have an explicit description of all p-power torsion elements in the module of Kähler differentials), and assume, inductively, that we also have a satisfactory understanding of the level n and degree 1 component of the de Rham-Witt complex. When A is a p-torsion-free perfectoid ring, Corollary B.2 below precisely describes the kernel of the surjective restriction map, $R : W_{n+1}\Omega^1_A \to W_n\Omega^1_A$, and thus enables us to describe the level $n+1$ and degree 1 component of the de Rham-Witt complex in terms of three other components: A, Ω^1_A and $W_n\Omega^1_A$.

Theorem B. *Let p denote an odd prime. Assume A is a ring satisfying one of the following two conditions:*

(1) We have that A is a p-torsion-free perfectoid ring, in the sense of (Bhatt et al., 2018, Definition 3.5); or

(2) We have that $A = \mathcal{O}_K$, where $K/\mathbf{Q}_p$ is an algebraic extension.

Fix an integer $n \geq 1$. Assume $\omega \in \Omega^1_A$ and $a \in A$ are such that $V^n(\omega) + dV^n(a) = 0 \in W_{n+1}\Omega^1_A$. Then there exists $a_0 \in A$ such that $\omega = -da_0$ and $a = p^n a_0$.

The significance of Theorem B is due to the following corollary.

Corollary B.2. *Let p be an odd prime, let A be a ring as in Theorem B, and let $n \geq 1$ be an integer. Then for every integer $n \geq 1$, the following is an exact sequence of $W_{n+1}(A)$-modules:*

$$0 \to A \xrightarrow{(-d,p^n)} \Omega^1_A \oplus A \xrightarrow{V^n \oplus dV^n} W_{n+1}\Omega^1_A \xrightarrow{R} W_n\Omega^1_A \to 0, \tag{1.3}$$

where the $W_{n+1}(A)$-module structure is defined as follows. The $W_{n+1}(A)$-module structure on the left-most A in the sequence is induced by F^n, where F denotes the Witt vector Frobenius. The $W_{n+1}(A)$-module structure on $\Omega^1_A \oplus A$ is defined by

$$y \cdot (\alpha, a) = \big(F^n(y)\alpha - aF^n(dy), F^n(y)a\big), \text{ where } y \in W_{n+1}(A),\ \alpha \in \Omega^1_A,\ a \in A.$$

The $W_{n+1}(A)$-module structure on $W_{n+1}\Omega^1_A$ is the natural one, and the $W_{n+1}(A)$-module structure on $W_n\Omega^1_A$ is induced by restriction.

See Section 4 below for the proof of Theorem B. The remaining portion of the claimed exactness in Corollary B.2 was proved in complete generality (i.e., with no restrictions on the $\mathbf{Z}_{(p)}$-algebra A) by Hesselholt and Madsen; see (Hesselholt and Madsen, 2003, Proposition 3.2.6), and note that the de Rham-Witt complex is a special case of the logarithmic de Rham-Witt complex, attained by taking the multiplicative monoid M (as in Hesselholt-Madsen's notation) to be trivial, $M = \{1\}$. Applying Hesselholt and Madsen's results, the exact sequence in Corollary B.2 follows immediately from Theorem B.

Hesselholt and Madsen's proof of the remaining portion exactness of (1.3) does not involve any notions from topology, and we simply quote their result. For proofs by Hesselholt and Madsen related to Theorem B itself, see (Hesselholt and Madsen, 2003, Proof of Theorem 3.3.8) and (Hesselholt, 2006, Proposition 2.2.1). The argument in (Hesselholt and Madsen, 2003, Proof of Theorem 3.3.8) does use topology. Our two main contributions with respect to Corollary B.2 are proving exactness in the case of p-torsion-free perfectoid rings, and providing an algebraic proof of Theorem B in this generality.

The benefit of having an exact sequence as guaranteed in Corollary B.2 is clear from the analogous situation involving Witt vectors. Recall that, for every ring A and every integer $n \geq 1$, we have a short exact sequence of $W_{n+1}(A)$-modules

$$0 \to A \xrightarrow{V^n} W_{n+1}(A) \overset{R}{\to} W_n(A) \to 0, \tag{1.4}$$

where the $W_{n+1}(A)$-module structures are defined as follows. Let F denote the Witt vector Frobenius and let R denote the restriction map. The module structure on A is induced by $F^n : W_{n+1}(A) \to W_1(A) \cong A$. The module structure on $W_{n+1}(A)$ is the natural one, and the module structure on $W_n(A)$ is induced by restriction $R : W_{n+1}(A) \to W_n(A)$. The sequence (1.4) is always an exact sequence of $W_{n+1}(A)$-modules, for every ring A, and this fact is very useful when making induction arguments. For example, using induction on n and exactness of the sequence in Equation (1.4), one proves that if A is a $\mathbf{Z}_{(p)}$-algebra, then for every positive integer n, the ring $W_n(A)$ is also a $\mathbf{Z}_{(p)}$-algebra; see for example (Hesselholt, 2015, Lemma 1.9).

The sequence (1.3) from Corollary B.2 is analogous to the sequence (1.4), but for general rings A, the sequence (1.3) may not be exact. For example, the sequence (1.3) is never exact if the ring A is an $\mathbf{F}_p$-algebra. Even when the sequence (1.3) is exact, it is typically difficult to prove exactness. (In contrast, the proof of exactness of (1.4) is trivial.) The ring $A = \mathbf{Z}_p$ is the easiest case of Theorem B, but even in this case we are not aware of a simple proof of exactness. (Exactness in this case $A = \mathbf{Z}_p$ follows for example from (Hesselholt and Madsen, 2004, Example 1.2.5), which relies on topology. It was pointed out to the authors by a referee that exactness in this case also follows from Hesselholt's direct construction of the (big) de Rham-Witt complex of the ring of integers $\mathbf{Z}$ in (Hesselholt, 2015, Theorem 6.1).)

The $W_{n+1}(A)$-module structures described in Corollary B.2 can be imposing at first, but they are very natural. For example, consider the degree zero case considered in the sequence (1.4). Why do we use $F^n : W_{n+1}(A) \to A$ to equip A with a $W_{n+1}(A)$-module structure, instead of using, for example, the restriction map, $R^n : W_{n+1}(A) \to A$? The reason of course is the Witt vector formula $xV^n(y) = V^n(F^n(x)y)$; this formula is saying precisely that $A \overset{V^n}{\to} W_{n+1}(A)$ is a $W_{n+1}(A)$-module homomorphism for the proposed module structure using F^n. The case is the same for our module structures described in Corollary B.2. Let $x \in W_{n+1}(A)$, $y \in \Omega^1_A$, and $z \in A$. Using the same formula $xV^n(y) = V^n(F^n(x)y)$ and the Leibniz rule, we compute

$$\begin{aligned} x \cdot (V^n(y) + dV^n(z)) &= V^n(F^n(x)y) + xdV^n(z) \\ &= V^n(F^n(x)y) + d(xV^n(z)) - V^n(z)dx \\ &= V^n(F^n(x)y) + dV^n(F^n(x)z) - V^n(zF^n(dx)) \end{aligned}$$

$$= V^n\Big(F^n(x)y - zF^n(dx)\Big) + dV^n\Big(F^n(x)z\Big).$$

As above, this formula is saying precisely that $\Omega^1_A \oplus A \xrightarrow{V^n + dV^n} W_{n+1}\Omega^1_A$ is a $W_{n+1}(A)$-module homomorphism for the proposed module structure.

We will use Corollary B.2 during our proof by induction of Theorem C. That is our most important application and our original motivation for considering Theorem B. Here we first record several other applications.

Proposition 1.5. *Let A_0 denote any $\mathbf{Z}_{(p)}$-algebra for which there exists a p-torsion-free perfectoid ring A and a ring homomorphism $A_0 \to A$. (For example, any subring of $\mathcal{O}_{\mathbf{C}_p}$ containing $\mathbf{Z}_{(p)}$ is a suitable choice of A_0.) Then $dV^n(1) \in W_{n+1}\Omega^1_{A_0}$ is non-zero for every integer $n \geq 1$.*

Proof. The sequence (1.3) is exact for the p-torsion-free perfectoid ring A by Corollary B.2. This shows that $dV^n(1) \neq 0 \in W_{n+1}\Omega^1_A$. By considering the map $W_{n+1}\Omega^1_{A_0} \to W_{n+1}\Omega^1_A$ induced by functoriality, we see that we must also have $dV^n(1) \neq 0 \in W_{n+1}\Omega^1_{A_0}$. ■

The fact that Proposition 1.5 is not obvious, even in the case $A = \mathbf{Z}_{(p)}$, underscores the difficulty of proving that elements in the de Rham-Witt complex are non-zero.

We briefly point out in the following question that we are unsure how restrictive is the hypothesis from Proposition 1.5.

Question 1.6. Does the hypothesis of Proposition 1.5 hold for every p-torsion-free, p-adically separated ring?

The following provides another basic application of Corollary B.2. Proposition 1.5 is stated mostly as a curiosity. On the other hand, the following, Proposition 1.7, is more important and will be used at several different points later in this paper.

Proposition 1.7. *Let A denote any p-torsion-free $\mathbf{Z}_{(p)}$-algebra for which the sequence (1.3) is exact for all integers $n \geq 1$. Then for all integers $m, n \geq 1$, the map $V^m : W_n\Omega^1_A \to W_{n+m}\Omega^1_A$ is injective.*

Proof. Because $V^m = V \circ \cdots \circ V$ (m total iterations), it suffices to prove the map V is injective. We prove that V is injective using induction on n. When $n = 1$, $W_n\Omega^1_A = \Omega^1_A$, and the result follows from the exactness of the sequence (1.3), using the fact that A is p-torsion free.

Now assume we know that $V : W_n\Omega^1_A \to W_{n+1}\Omega^1_A$ is injective. Consider an element $x \in W_{n+1}\Omega^1_A$ such that $V(x) = 0 \in W_{n+2}\Omega^1_A$. Let R denote the restriction map $R : W_{m+1}\Omega^1_A \to W_m\Omega^1_A$ (for some m). Because $R \circ V = V \circ R$ and $V : W_n\Omega^1_A \to W_{n+1}\Omega^1_A$ is injective by our induction hypothesis, we deduce that $R(x) = 0 \in W_n\Omega^1_A$. Hence by exactness of the sequence (1.3), there exist $\alpha \in \Omega^1_A$ and $a \in A$ such that

$$x = V^n(\alpha) + dV^n(a) \in W_{n+1}\Omega^1_A.$$

Applying V to this element, we get

$$0 = V(x) = V^{n+1}(\alpha) + VdV^n(a) = V^{n+1}(\alpha) + dV^{n+1}(pa) \in W_{n+2}\Omega^1_A.$$

Using (the most difficult part of) exactness of the sequence (1.3), we have that $pa = p^{n+1}a_0$ and $\alpha = -da_0$ for some $a_0 \in A$. Because A is p-torsion free, we have that $a = p^n a_0$. Thus

$$x = V^n(-da_0) + dV^n(p^n a_0) = 0 \in W_{n+1}\Omega^1_A,$$

which completes the proof. ■

Remark 1.8. The analogue of Proposition 1.7 is not true in the classical case that A is an $\mathbf{F}_p$-algebra. For example, if $A = \mathbf{F}_p[x]$ and we consider $dx \in \Omega^1_A \cong W_1\Omega^1_A$, then $dx \neq 0 \in W_1\Omega^1_A$, but $V(dx) = dV(px) = dV(0) = 0 \in W_2\Omega^1_A$, so the Verschiebung map is not injective in this case.

The maps V and F on the de Rham-Witt complex were studied extensively by Illusie in the case of $\mathbf{F}_p$-algebras, and many very explicit formulas are known in that case. For example, the kernel of Verschiebung is computed in (Illusie, 1979, Equation I(3.21.1.4)): if A is a smooth k-algebra for k a perfect field of characteristic p, then for all $n \geq 1$, we have

$$\ker\left(V : W_n\Omega^1_A \to W_{n+1}\Omega^1_A\right) = dV^{n-1}\left(A\right).$$

Our formula above comes from the case $n = 1$. (Our only additional contribution is the remark that $dx \neq 0 \in W_1\Omega^1_A$ when $A = \mathbf{F}_p[x]$.)

Nor does the analogue of Proposition 1.5 hold for the case that A is an $\mathbf{F}_p$-algebra. Indeed, $V(1) = p \in W(A)$ when A is an $\mathbf{F}_p$-algebra, so $dV^n(1) = 0$ for all integers $n \geq 1$.)

We next describe the main result of this paper, Theorem C. It is completely modeled after the work of Hesselholt in (Hesselholt, 2006), where the case of the ring of integers in an algebraic closure of a local field is considered.

Theorem C. *Let A denote a p-torsion-free perfectoid ring containing a compatible system of p-power roots of unity (see Notation 5.1). Then the following hold:*

1. *For all integers $n \geq 1$ and $r \geq 1$, the p^r-torsion $W_n\Omega^1_A[p^r]$ is a free $W_n(A)/p^rW_n(A)$-module of rank one.*
2. *For every integer $n \geq 1$, the p-adic Tate module $T_p\left(W_n\Omega^1_A\right)$ is a free $W_n(A)$-module of rank one.*
3. *The inverse limit $\varprojlim_F T_p\left(W_n\Omega^1_A\right)$ is a free $\varprojlim_F \left(W_n(A)\right) \cong W\left(\varprojlim_{x\mapsto x^p} A/pA\right)$-module of rank one.*

The proofs of these claims are located as follows. Claim (2) follows from Theorem 5.8. Once we know Claim (2), then Claim (1) is a consequence of Lemma 5.7. Lastly, Claim (3) follows from Corollary 5.18. See (Hesselholt, 2006, Theorem B, Proposition 2.3.2, and Proposition 2.4.2) for closely related results.

Remark 1.9. The following are possible choices of A satisfying the conditions of Theorem C: $\mathcal{O}_{\mathbf{C}_p}$, $\mathbf{Z}_p[\zeta_{p^\infty}]^\wedge$, $\mathbf{Z}_p[\zeta_{p^\infty}, p^{1/p^\infty}]^\wedge$, $\mathcal{O}_{\mathbf{C}_p}\langle T^{1/p^\infty}\rangle$. An example of a p-torsion-free perfectoid ring which does not contain the p-power roots of unity is $\mathbf{Z}_p[p^{1/p^\infty}]^\wedge$. Thus our Theorems A and B hold for $\mathbf{Z}_p[p^{1/p^\infty}]^\wedge$, but we are unsure if Theorem C holds for this ring.

Remark 1.10. Our proofs are closely modeled on the work of Hesselholt (Hesselholt, 2006) which concerned the case of the ring of integers in an algebraic closure of a local field, such as $A = \mathcal{O}_{\overline{\mathbf{Q}_p}}$. Notice that $\mathcal{O}_{\overline{\mathbf{Q}_p}}$ is not covered by our Theorem C, because $\mathcal{O}_{\overline{\mathbf{Q}_p}}$ is not p-adically complete and hence not perfectoid. If $K/\mathbf{Q}_p$ is algebraic and $(\mathcal{O}_K)^\wedge$ is a perfectoid ring, then we expect analogues of most of our results (and the referenced results of Hesselholt) to hold for $A = \mathcal{O}_K$, and it should be straightforward to adjust our techniques to this situation. For example, it should follow that the p-adic Tate module $T_p\left(W_n\Omega^1_{\mathcal{O}_K}\right)$ is a free $W_n(\mathcal{O}_K)^\wedge$-module of rank one. We do not consider non-p-adically complete rings in this paper for two reasons. The first reason is that, by restricting our attention to the p-adically complete case, we do not have to reprove results which are essentially automatic in the perfectoid case (such as the kernel of $F : W_{n+1}(A) \to W_n(A)$ being a principal ideal). The second reason is that we are unsure what the correct generality to consider is in the non-p-adically complete setting. We thank Kiran Kedlaya for calling this question to our attention.

One of the main motivations for studying the p-adic Tate modules $T_p\left(W_n\Omega^1_A\right)$ and $\varprojlim_F T_p\left(W_n\Omega^1_A\right)$ comes from topology. The p-adic algebraic K-theory $K(A, \mathbf{Z}_p)$ maps via the trace maps to $\mathrm{TR}^n(A, \mathbf{Z}_p)$ and $\mathrm{TF}(A, \mathbf{Z}_p)$ which are spectra obtained from the cyclotomic structure on the topological Hochschild homology $\mathrm{THH}(A)$. Using results of Hesselholt ((Hesselholt, 2004) and (Hesselholt, 2006)) and their generalizations (Bhatt et al., 2019, Remark 6.6), one can conclude that the homotopy groups $\pi_2\,\mathrm{TR}^n(A, \mathbf{Z}_p)$ and $\pi_2\,\mathrm{TF}(A, \mathbf{Z}_p)$ are isomorphic to the p-adic Tate modules $T_p\left(W_n\Omega^1_A\right)$ and $\varprojlim_F T_p\left(W_n\Omega^1_A\right)$, respectively. Moreover, one can give an explicit description of the trace map $K_2(A, \mathbf{Z}_p) \to \pi_2\,\mathrm{TF}(A, \mathbf{Z}_p)$ using the generator from the proof of Theorem C. For more details see the final Section 8, where we summarize some results on algebraic K-theory and topological cyclic homology and their connections with the main results of this paper. Most of the material in Section 8 is well-known to the experts but we still find it important to give an account since it puts the algebraic objects of this paper in a broader topological context. The readers only interested in algebraic aspects of the de Rham-Witt can safely skip the last section.

There are two key hypotheses on the ring A in Theorem C. The first is that A contain a compatible sequence of p-power roots of unity. The second is that A be a p-torsion-free perfectoid ring. It will be repeatedly evident why the p-power root of unity hypothesis is important for our proof of Theorem C: our constructions make constant use of the elements ζ_{p^n} for varying n.

The other key hypothesis on the ring A in Theorem C is that it be perfectoid. We offer three explanations for why this assumption is convenient.

(1) A first reason is mentioned below in Remark 1.11. That example suggests it is essential that the ring A be infinitely ramified over $\mathbf{Z}_p$, at least in the case of subrings of $\mathcal{O}_{\mathbf{C}_p}$.
(2) A second reason is implicit in Proposition 5.3 below. The condition that A be perfectoid is closely related to the condition that, for all integers $n \geq 1$, the map $F : W_{n+1}(A) \to W_n(A)$ is surjective. For example, the condition that $F : W_2(A) \to W_1(A)$ be surjective is the same as the condition that the p-power Frobenius map is surjective on A/pA.
(3) A third reason was already mentioned above in Remark 1.2, where we commented that, in the usual notation regarding perfectoid rings, it is useful for us that $\ker\theta$ be a principal ideal.

Remark 1.11. Let $K/\mathbf{Q}_p$ be an algebraic extension such that the ramification index of $K/\mathbf{Q}_p$ is *finite*. Let $\mathcal{O}_K$ be its ring of integers. By (Fontaine, 1982, §2), there exists an integer $N \geq 1$ such that $p^N\Omega^1_{\mathcal{O}_K/\mathbf{Z}_p} = 0$, and hence the p-adic Tate module $T_p(\Omega^1_{\mathcal{O}_K})$ is trivial. Thus, the analogue of the level $n = 1$ case of Theorem C is false for $A = \mathcal{O}_K$. (Using induction and an exact sequence analogous to the sequence in Proposition 5.5, it should also follow that $T_p(W_n\Omega^1_{\mathcal{O}_K}) \cong 0$ for all integers $n \geq 1$, and thus the analogue of Theorem C is false for all $n \geq 1$.)

The most difficult part of the proof of Theorem C is constructing suitably compatible $W_n(A)/pW_n(A)$-module generators for the p-torsion $W_n\Omega^1_A[p]$. We construct them using induction on n, and our induction makes repeated use of Corollary B.2. Once one has free $W_n(A)/pW_n(A)$-module generators for the p-torsion in $W_n\Omega^1_A$, it is relatively easy to produce free $W_n(A)/p^rW_n(A)$-module generators for the p^r-torsion, and by choosing everything compatibly, we are able to produce free $W_n(A)$-module generators for the p-adic Tate module of $W_n\Omega^1_A$ and free $\varprojlim_F (W_n(A))$-module generators for $\varprojlim_F T_p(W_n\Omega^1_A)$.

Remark 1.12. Throughout this paper we consider only p-torsion-free perfectoid rings. In the case of characteristic p perfectoid rings, the de Rham-Witt complex is not interesting. Namely, let A denote a perfectoid ring of characteristic p; this condition is equivalent to A being a perfect ring of characteristic p (see, for example, (Bhatt et al., 2018, Example 3.15)). Let $n \geq 1$ denote an integer. Then $W_n\Omega^d_A$ is a $W_n(A)$-module, and hence is p^n-torsion, on one hand, but on the other hand, multiplication by p is surjective on $W_n\Omega^d_A$ for all degrees $d \geq 1$ by Proposition 5.3 below. Thus $W_n\Omega^d_A \cong 0$ for all $d \geq 1$. We have not considered the case of perfectoid rings which are not characteristic p but which do have p-torsion.

Question 1.13. Our construction of Frobenius-compatible generators of the p-torsion in $W_n\Omega^1_A$ is rather indirect for the levels $n \geq 2$. The proof of Theorem C would be simplified considerably if we could find a more direct construction. We now describe one possible approach. For every integer $n \geq 1$, one can check that there is a unique Witt vector $\frac{p^n}{[\zeta_{p^n}]-1} \in W_n(A)$ such that $\frac{p^n}{[\zeta_{p^n}]-1} \cdot ([\zeta_{p^n}]-1) = p^n \in W_n(A)$. The elements $\frac{p^n}{[\zeta_{p^n}]-1}\, d\log[\zeta_{p^{n+1}}]$ seemingly have all the compatibility properties needed to make our proofs work, although we are only able to prove they are p-torsion in the case $n = 1$. Are these elements p-torsion? If not, can they be modified to produce generators for the p-torsion which are more explicit than what we work with in our proof of Theorem C?

Remark 1.14. Our proofs require $p \neq 2$. Attaining similar results for $p = 2$ would necessitate substantial changes. For example, even the definition of the de Rham-Witt complex we use is incorrect for $p = 2$ (see (Costeanu, 2008) for the correct definition). The requirement $p \neq 2$ is also used in the first author's paper (Davis, 2019), so the results from (Davis, 2019) which are used in the present paper would also have to be adjusted to allow for $p = 2$.

Notation 1.15. Rings in this paper are commutative and have unity, and ring homomorphisms must send unity to unity. Throughout this paper, $p \geq 3$ is a fixed *odd* prime. When we refer to Witt vectors, we mean p-typical Witt vectors with respect to this prime, and when we refer to a ring being *perfectoid*, it is in the sense of (Bhatt et al., 2018, Definition 3.5), and it is with respect to this same prime p. When we say that ζ_p is a primitive p-th root of unity, we mean that $1 + \zeta_p + \cdots + \zeta_p^{p-1} = 0$.

Here is some more specialized notation. For a perfectoid ring A and an integer $n \geq 2$, we let $z_n \in W_n(A)$ denote a generator of $\ker F^{n-1} : W_n(A) \to W_1(A)$; in the case that we have identified a compatible system of roots of unity of A, then we will usually choose the generator $1 + [\zeta_{p^n}] + \cdots + [\zeta_{p^n}]^{p-1}$. If u is a unit in A (or in $W_n(A)$), we write $d\log u$ for $\frac{du}{u}$ in Ω^1_A (or in $W_n\Omega^1_A$).

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2. Algebraic preliminaries

In this section we gather miscellaneous algebraic results that will be used later. This section is organized so that all the results concerning Witt vectors come at the end. The de Rham-Witt complex does not appear in this section. The reader is advised to skip this entire section and return to it as needed.

We begin with a few standard properties related to the cotangent complex. We use the cotangent complex extensively in Section 3, with regards to analyzing p-power torsion in modules of Kähler differentials. (The cotangent complex is not used in the rest of the paper, so the reader who is willing to accept these results concerning Kähler differentials can skip the arguments involving the cotangent complex.) There are two results concerning the cotangent complex that we will use repeatedly, the Jacobi-Zariski sequence and the Universal coefficient theorem.

Proposition 2.1 (The Jacobi-Zariski sequence (The Stacks Project Authors, 2017, Tag 08QX))**.** *Let $A \to B \to C$ be ring homomorphisms. Then there is an exact triangle in the derived category of C-modules*

$$L_{B/A} \otimes^L_B C \to L_{C/A} \to L_{C/B}, \tag{2.2}$$

where $\otimes^L_B$ denotes the derived tensor product.

Given a chain complex C (cohomologically graded), the notation $C[1]$ denotes the homological shift, that is $H^i(C[1]) = H^{i+1}(C)$. In particular given an R-module G, we denote by $G[-1]$ the chain complex which has only G in cohomological degree 1. In the following, note that the stated condition in Part (1) holds in particular when R is a PID.

Proposition 2.3 (The universal coefficient theorem)**.**

(1) Let R be a ring, let C denote a chain complex of R-modules, and let M denote an R-module. Assume that, for every R-module N and every $j \geq 2$, we have

$$\mathrm{Tor}_j^R(N, M) \cong 0.$$

Then for every $i \in \mathbf{Z}$ we have a short exact sequence of R-modules

$$0 \to H^i(C) \otimes_R M \to H^i(C \otimes^L_R M) \to \mathrm{Tor}_1^R(H^{i+1}(C), M) \to 0,$$

where $\otimes^L_R$ denotes the derived tensor product.

(2) Let C denote a chain complex of abelian groups, and let $C^\wedge$ denote its derived p-completion, which by definition is equivalent to the derived Hom-complex

$$\mathbf{R}\,\mathrm{Hom}(\mathbf{Q}_p/\mathbf{Z}_p[-1], C).$$

Then for every $i \in \mathbf{Z}$ we have a short exact sequence of abelian groups

$$0 \to \mathrm{Ext}^1(\mathbf{Q}_p/\mathbf{Z}_p, H^i(C)) \to H^i(C^\wedge) \to \mathrm{Hom}(\mathbf{Q}_p/\mathbf{Z}_p, H^{i+1}(C)) \to 0.$$

(3) Let C denote a chain complex of R-modules and let G denote an abelian group. For every $i \in \mathbf{Z}$, we have a short exact sequence of R-modules

$$0 \to \mathrm{Ext}^1(G, H^i(C)) \to H^i(\mathbf{R}\,\mathrm{Hom}(G[-1], C)) \to \mathrm{Hom}(G, H^{i+1}(C)) \to 0.$$

In particular, if we let $C^\wedge$ denote the derived p-completion of C, which by definition is equivalent to

$$\mathbf{R}\,\mathrm{Hom}(\mathbf{Q}_p/\mathbf{Z}_p[-1], C),$$

then for every $i \in \mathbf{Z}$, we have a short exact sequence of R-modules

$$0 \to \mathrm{Ext}^1(\mathbf{Q}_p/\mathbf{Z}_p, H^i(C)) \to H^i(C^\wedge) \to \mathrm{Hom}(\mathbf{Q}_p/\mathbf{Z}_p, H^{i+1}(C)) \to 0.$$

Also, for every integer $n \geq 1$ and every $i \in \mathbf{Z}$, we have a short exact sequence of R-modules

$$0 \to H^i(C)/p^n H^i(C) \to H^i(C \otimes^L_{\mathbf{Z}} (\mathbf{Z}/p^n\mathbf{Z})) \to H^{i+1}(C)[p^n] \to 0.$$

Proof. Part (1) follows from (Weibel, 1994, 5.7.6-5.7.8) (see also (Elmendorf et al., 1997, Theorem 4.1)). Though we do not need this, we note that in particular Tor-spectral sequences are strongly convergent for not necessarily bounded complexes. Part (2) and (3) follow from (Rotman, 2009, Theorem 10.85). Note that the latter also holds for not necessarily bounded complexes. ■

We have the following consequence. We will use this result to analyze p-power torsion in the module of Kähler differentials $\Omega^1_{A/\mathbf{Z}_p} \cong H^0(L_{A/\mathbf{Z}_p})$.

Corollary 2.4. *Let A denote a $\mathbf{Z}_p$-algebra. Let $(L_{A/\mathbf{Z}_p})^\wedge$ denote the derived p-completion of $L_{A/\mathbf{Z}_p}$, and let T_p denote the p-adic Tate module. There is a surjective A-module map*

$$H^{-1}\left((L_{A/\mathbf{Z}_p})^\wedge\right) \to T_p(H^0(L_{A/\mathbf{Z}_p})).$$

It is natural in the sense that, if $A \to B$ is a $\mathbf{Z}_p$-algebra map, then the following diagram commutes:

$$\begin{array}{ccc} H^{-1}\left((L_{B/\mathbf{Z}_p})^\wedge\right) & \longrightarrow & T_p(H^0(L_{B/\mathbf{Z}_p})) \\ \uparrow & & \uparrow \\ H^{-1}\left((L_{A/\mathbf{Z}_p})^\wedge\right) & \longrightarrow & T_p(H^0(L_{A/\mathbf{Z}_p})). \end{array}$$

Proof. Note that for any chain complex C and any $i \in \mathbf{Z}$, we have that $\mathrm{Hom}(\mathbf{Q}_p/\mathbf{Z}_p, H^{i+1}(C))$ is naturally isomorphic to the p-adic Tate module $T_p(H^{i+1}(C))$. Now if we take $C = L_{A/\mathbf{Z}_p}$, then by Proposition 2.3, Part (2), we in particular get a surjective map

$$H^{-1}\left((L_{A/\mathbf{Z}_p})^\wedge\right) \to T_p(H^0(L_{A/\mathbf{Z}_p}))$$

satisfying the stated naturality. Moreover, this map is in fact an A-module map. This also follows by naturality, since the cotangent complex is a chain complex of A-modules. ■

When $R \to S$ is a smooth ring map, the cotangent complex is quasi-isomorphic to the module of Kähler differentials, concentrated in degree zero (The Stacks Project Authors, 2017, Tag 08R5). The next few results identify more instances of ring maps $R \to S$ in which the cotangent complex is quasi-isomorphic to the module of Kähler differentials, concentrated in degree zero. These results are closely related to exercises in Bhatt's 2017 Arizona winter school notes (Bhatt, 2017a), with Proposition 2.6 being taken directly from those exercises. We thank the referee for an earlier version of this paper for suggesting a variant of the following lemma.

Lemma 2.5. *Let $R \to S$ be a ring map satisfying the following properties.*

1. *The ring S is flat as an R-module;*
2. *There exists a non-zero divisor-divisor $f \in R$ such that $R[\frac{1}{f}] \to S[\frac{1}{f}]$ is smooth;*
3. *There exists a positive integer n such that S is isomorphic as a ring to $R[x_1, \ldots, x_n]/I$, where I is an ideal in $R[x_1, \ldots, x_n]$ generated by a regular sequence.*

Then there is a quasi-isomorphism $L_{S/R} \cong \Omega^1_{S/R}$.

Proof. Considering the Jacobi-Zariski sequence (2.2) associated to $R \to R[x_1, \dots, x_n] \to R[x_1, \dots, x_n]/I$, we immediately deduce that $H^i(L_{S/R}) \cong 0$ for $i \neq -1, 0$, and that there is an exact sequence

$$0 \to H^{-1}(L_{S/R}) \to I/I^2 \to H^0\left(L_{R[x_1,\dots,x_n]/R} \otimes^L_{R[x_1,\dots,x_n]} (R[x_1, \dots, x_n]/I)\right).$$

It remains to show that $H^{-1}(L_{S/R}) \cong 0$. Note that f is a non-zero-divisor on S, because S/R is flat and f is a non-zero-divisor on R. Because I is generated by a regular sequence, we have that I/I^2 is a free S-module by (The Stacks Project Authors, 2017, Tag 00LN), and hence I/I^2 is f-torsion-free, and hence $H^{-1}(L_{S/R})$ is f-torsion-free.

We finish the proof by showing that $H^{-1}(L_{S/R})$ is f-torsion. By (The Stacks Project Authors, 2017, Tag 08SF), we have a quasi-isomorphism

$$L_{S[\frac{1}{f}]/R[\frac{1}{f}]} \cong L_{S/R} \otimes^L_S S[\frac{1}{f}].$$

Because $S[\frac{1}{f}]/R[\frac{1}{f}]$ is smooth, we deduce that $H^{-1}\left(L_{S/R} \otimes^L_S S[\frac{1}{f}]\right) \cong 0$. Hence, by the universal coefficient theorem Proposition 2.3, using that $S[\frac{1}{f}]$ is a flat S-module, we have that

$$H^{-1}(L_{S/R}) \otimes_S S[\frac{1}{f}] \cong 0.$$

This shows that $H^{-1}(L_{S/R})$ is f-torsion, which completes the proof. ∎

Proposition 2.6 ((Bhatt, 2017a, Exercise 8 and Exercise 12))**.**

(1) Let $L \supset K \supset \mathbf{Q}_p$ denote a tower of algebraic extensions. Then there is a quasi-isomorphism

$$L_{\mathcal{O}_L/\mathcal{O}_K} \cong \Omega^1_{\mathcal{O}_L/\mathcal{O}_K}.$$

(2) Let $\overline{V}$ denote a p-torsion-free $\mathbf{Z}_p$-algebra. Assume that $\overline{V}$ is a valuation ring and that $\operatorname{Frac}\overline{V}$ *is algebraically closed. Then there is a quasi-isomorphism*

$$L_{\overline{V}/\mathbf{Z}_p} \cong \Omega^1_{\overline{V}/\mathbf{Z}_p}.$$

Proof. Proof of (1). The cotangent complex commutes with filtered colimits by (The Stacks Project Authors, 2017, Tag 08S9), so $\operatorname{colim} L_{B_i/A_i} \cong L_{\operatorname{colim}(B_i)/\operatorname{colim}(A_i)}$, so it suffices to assume that $L/K/\mathbf{Q}_p$ are finite extensions. We now check that the conditions of Lemma 2.5 hold, taking $f = p$. Because $\mathcal{O}_K$ is a valuation ring, we have that $\mathcal{O}_L$ is a flat $\mathcal{O}_K$-module because it is torsion-free. Because K/L is a finite extension of characteristic zero fields, the map $\mathcal{O}_K[\frac{1}{p}] \to \mathcal{O}_L[\frac{1}{p}]$ is smooth (see for example (The Stacks Project Authors, 2017, Tag 07ND)). There exists a polynomial $g(x) \in \mathcal{O}_K[x]$ such that $\mathcal{O}_L$ is isomorphic to the quotient ring $\mathcal{O}_K[x]/(g(x))$ (see for example (Neukirch, 1999, Chapter II, Lemma 10.4)). Thus the conditions of Lemma 2.5 hold, and thus we have a quasi-isomorphism $L_{\mathcal{O}_L/\mathcal{O}_K} \cong \Omega^1_{\mathcal{O}_L/\mathcal{O}_K}$.

Proof of (2). We use the following strategy outlined in Bhatt's Arizona winter school notes (Bhatt, 2017a): We use de Jong's alterations theorem to show that every finitely generated $\mathbf{Z}_p$-subalgebra R of $\overline{V}$ is contained in a *regular* $\mathbf{Z}_p$-subalgebra R_1 of $\overline{V}$. Thus $\overline{V}$ is a filtered colimit of such $\mathbf{Z}_p$-subalgebras R_1, and the desired result will follow from quasi-isomorphisms $L_{R_1/\mathbf{Z}_p} \cong \Omega^1_{R_1/\mathbf{Z}_p}$.

Let R denote a finitely generated $\mathbf{Z}_p$-subalgebra of $\overline{V}$. By de Jong's alterations theorem (de Jong, 1996, Theorem 6.5), there exists a regular $\mathbf{Z}_p$-scheme X_1 and a dominant, proper map $X_1 \to \operatorname{Spec} R$ such that the function field of X_1 is a finite extension of $\operatorname{Frac} R$. Because $\operatorname{Frac} R \subseteq \operatorname{Frac}\overline{V}$ and $\operatorname{Frac}\overline{V}$

is algebraically closed, we deduce that the function field of X_1 embeds into $\operatorname{Frac} \overline{V}$. These maps fit into a solid commutative diagram

$$\begin{array}{ccc} X_1 & \longleftarrow & \operatorname{Spec}\left(\operatorname{Frac} \overline{V}\right) \\ \downarrow & \nwarrow & \downarrow \\ \operatorname{Spec} R & \longleftarrow & \operatorname{Spec} \overline{V} \end{array}$$

By the valuative criterion for properness, there exists a map $\operatorname{Spec} \overline{V} \to X_1$ as in the dashed arrow in the diagram, for which the diagram remains commutative. This map $\operatorname{Spec} \overline{V} \to X_1$ factors through some open affine subscheme $\operatorname{Spec} R_1 \subseteq X_1$ with R_1 regular and with the composite $R \to R_1 \to \overline{V}$ injective.

The rings R_1 as produced above (for various rings R) fit into a filtered diagram, with arrows corresponding to commutative diagrams

$$\begin{array}{ccc} R_2 & \longrightarrow & \overline{V} \\ \uparrow & \nearrow & \\ R_1 & & \end{array}$$

Because the ring R above could be any finitely generated $\mathbf{Z}_p$-subalgebra of $\overline{V}$, we find that $\varinjlim R_\alpha$ is a filtered colimit, and that the corresponding map $\varinjlim R_\alpha \to \overline{V}$ is a ring isomorphism (where R_α ranges over the rings R_1 produced above).

We conclude the proof again using Lemma 2.5 as follows. Let R_α be as in the previous paragraph. Because $R_\alpha \to \overline{V}$ is injective and $\overline{V}$ is p-torsion-free, we have that $\mathbf{Z}_p \to R_\alpha$ is flat. By (The Stacks Project Authors, 2017, Tag 00TX), we have that $R_\alpha[1/p]/\mathbf{Q}_p$ is smooth. By (The Stacks Project Authors, 2017, Tag 0E9J), we may assume each map $\mathbf{Z}_p \to R_\alpha$ can be factored as $\mathbf{Z}_p \to \mathbf{Z}_p[x_1, \ldots, x_n] \to \mathbf{Z}_p[x_1, \ldots, x_n]/I \cong R_\alpha$, where I is generated by a regular sequence. Thus the conditions of Lemma 2.5 hold, and so $L_{R_\alpha/\mathbf{Z}_p} \cong \Omega^1_{R_\alpha/\mathbf{Z}_p}$. Because the cotangent complex commutes with filtered colimits by (The Stacks Project Authors, 2017, Tag 08S9), we have that $L_{\overline{V}/\mathbf{Z}_p} \cong \varinjlim L_{R_\alpha/\mathbf{Z}_p} \cong \Omega^1_{\overline{V}/\mathbf{Z}_p}$. ■

Remark 2.7. Bhargav Bhatt has recently communicated to us that, in fact, a significantly stronger version of Proposition 2.6(2) also holds. Namely, if V is a valuation ring and a flat (equivalently, p-torsion-free) $\mathbf{Z}_p$-algebra, then there is a quasi-isomorphism $L_{V/\mathbf{Z}_p} \cong \Omega^1_{V/\mathbf{Z}_p}$. The proof of this stronger result relies on resolution of singularities in characteristic zero and a result of Gabber and Ramero, (Gabber and Ramero, 2003, Theorem 6.5.12). Our proof of Proposition 2.6(2) follows the original strategy suggested by Bhatt in his *Arizona winter school* lectures, (Bhatt, 2017a, Exercise 12). Closely related to our proof is an observation of Elmanto and Hoyois, relying on deep results of Temkin, that appears in the recent paper (Antieau and Datta, 2020, Proposition 4.2.1); we thank Elden Elmanto for calling this result to our attention.

The previous result had a hypothesis requiring a certain ring to be a valuation ring, and in fact, valuation rings play a special role at several points in this paper. One reason is because, if A is a p-torsion-free perfectoid ring containing a compatible sequence of p-power roots of unity, then A contains an isomorphic copy of the valuation ring $\mathbf{Z}_p[\zeta_{p^\infty}]^\wedge$. A second reason is because every p-torsion-free perfectoid ring embeds into a product of perfectoid valuation rings (see Lemma 2.23 and its proof).

We use (Zariski and Samuel, 1960, Chapter VI) as our basic reference on valuation rings, although we write our valuations multiplicatively:

$$v(xy) = v(x)v(y) \text{ for all } x, y.$$

(See (Scholze, 2012, Remark 2.3) for a remark on why Scholze writes his valuations multiplicatively.) We emphasize that our valuation rings are not assumed to be rank one. (For an alternative to Zariski-Samuel, see (Bourbaki, 2006, Chapter VI).)

Theorem 2.8. *Let $K_1 \supseteq K_0$ denote two fields and assume there is a valuation on K_0. Then there is a valuation on K_1 extending the valuation on K_0.*

Proof. This fact is well-known. For example, see (Zariski and Samuel, 1960, Chapter VI, Theorem 5′) for the fact that places can be extended, and (Zariski and Samuel, 1960, Chapter VI, Section 9) for the correspondence between places and valuations. ∎

Proposition 2.9. *Let K_0 denote a field equipped with a valuation, and let $\mathcal{O}_{K_0}$ denote the corresponding valuation ring. Let K_1 denote an algebraic extension of K_0. By Theorem 2.8, the field K_1 can be equipped with a valuation extending the valuation on K_0. The corresponding map on valuation rings $\mathcal{O}_{K_0} \to \mathcal{O}_{K_1}$ is faithfully flat.*

Proof. Every valuation ring is in particular a Prüfer domain, and so $\mathcal{O}_{K_1}$ is flat over $\mathcal{O}_{K_0}$ because it is torsion-free. We will now show that the induced map $\operatorname{Spec} \mathcal{O}_{K_1} \to \operatorname{Spec} \mathcal{O}_{K_0}$ is surjective, which will complete the proof that $\mathcal{O}_{K_0} \to \mathcal{O}_{K_1}$ is faithfully flat. Let v denote the valuation on K_1 extending the given valuation on K_0. The most important preliminary result we will need is the fact that the quotient group $v(K_1)/v(K_0)$ is torsion or, equivalently, for every element $x \in K_1$, there exists some integer $n > 0$ such that $v(x^n) \in v(K_0)$. Let us name this condition (*). We do not prove it; it is stated explicitly as (Zariski and Samuel, 1960, Chapter VI, Section 11, Lemma 1).

Consider any prime ideal $\mathfrak{p}_0 \subseteq \mathcal{O}_{K_0}$. We are trying to show that there exists some prime ideal $\mathfrak{p}_1 \subseteq \mathcal{O}_{K_1}$ such that $\mathfrak{p}_1 \cap \mathcal{O}_{K_0} = \mathfrak{p}_0$. We may assume $\mathfrak{p}_0$ is not the zero ideal. Our construction is based on (Zariski and Samuel, 1960, Chapter VI, Theorem 15). Define

$$\mathfrak{p}_1 := \{y \in \mathcal{O}_{K_1} : v(y)^n = v(x) \text{ for some } n \in \mathbf{Z}_{>0}, x \in \mathfrak{p}_0\}.$$

Using condition (*) above, we see that $\mathfrak{p}_1$ is an ideal in $\mathcal{O}_{K_1}$. It is a proper ideal. Lastly, assume $ab \in \mathfrak{p}_1$. Say $v(a^n b^n) = v(x)$, where $x \in \mathfrak{p}_0$. After possibly raising to a higher power, using again condition (*), we may assume that $v(a^n) = v(c)$ and $v(b^n) = v(d)$ for some $c, d \in \mathcal{O}_{K_0}$. Then $cd/x \in \mathcal{O}_{K_0}^\times$, and hence $cd \in \mathfrak{p}_0$, and hence $c \in \mathfrak{p}_0$ or $d \in \mathfrak{p}_0$, and hence $a \in \mathfrak{p}_1$ or $b \in \mathfrak{p}_1$. This completes the proof that $\mathfrak{p}_1$ is a prime ideal.

It remains to check that $\mathfrak{p}_1 \cap \mathcal{O}_{K_0} = \mathfrak{p}_0$. Clearly $\mathfrak{p}_1 \cap \mathcal{O}_{K_0} \supseteq \mathfrak{p}_0$. Conversely $y \in \mathcal{O}_{K_0}$ and $y \in \mathfrak{p}_1$, then $v(y^n) = v(x)$ for some $x \in \mathfrak{p}_0$ and some $n \in \mathbf{Z}_{>0}$, but then $y^n/x \in \mathcal{O}_{K_0}$, so $y^n \in \mathfrak{p}_0$, so $y \in \mathfrak{p}_0$. This proves the reverse inclusion, $\mathfrak{p}_1 \cap \mathcal{O}_{K_0} \subseteq \mathfrak{p}_0$. This completes the proof that $\mathcal{O}_{K_0} \to \mathcal{O}_{K_1}$ is faithfully flat. ∎

Later in this section we will need to embed a p-torsion-free perfectoid valuation ring into a larger perfectoid valuation ring for which its fraction field is algebraically closed. In order to accomplish this, we will need the next few preliminary results.

Lemma 2.10. *Let K be a characteristic zero field equipped with a valuation v for which $\mathcal{O}_K$ is p-adically separated. The values $v(p), v(p^2), \dots, v(p^n), \dots$ are cofinal in the set of all values of non-zero elements in K, in the sense that for every non-zero element $x \in K$, there exists an integer $n \geq 1$ such that $v(p^n) < v(x)$.*

Proof. If $x \in K \setminus \mathcal{O}_K$, then any value of n will work. Otherwise, because $\mathcal{O}_K$ is p-adically separated, we may choose n such that $x \notin p^n \mathcal{O}_K$. ■

We next refer to the *completion* of a field K with respect to a valuation. There seems to be some subtlety involved in defining the notion of completion for general valued fields (see (Engler and Prestel, 2005, Section 2.4)), but in our case it is easier, because Lemma 2.10 shows that there exists a countable sequence of elements which are cofinal in the value group, so the elements in the completion will correspond to Cauchy sequences $(a_1, a_2, \ldots)$, where the index set is $\mathbb{N}$.

Lemma 2.11. *Let K be a characteristic zero field equipped with a valuation for which $\mathcal{O}_K$ is p-adically separated. Let L denote the valued field which is the completion of K, as in (Engler and Prestel, 2005, Theorem 2.4.3). The corresponding valuation ring $\mathcal{O}_L$ is p-adically complete and separated.*

Proof. We must show that every p-adic Cauchy sequence of elements in $\mathcal{O}_L$ has a unique limit in $\mathcal{O}_L$. The existence of such a limit in L follows immediately from Lemma 2.10 and the construction of L as equivalence classes of Cauchy sequences. If the elements in the Cauchy sequence are all in $\mathcal{O}_L$, then the values of the elements in the Cauchy sequence are all at most $1 = v(1)$. The value of the Cauchy sequence, viewed as an element of L, is either 0 (in which case it is in $\mathcal{O}_L$), or is equal to the value of an entry in the sequence (in which case it is also in $\mathcal{O}_L$). In either case, we see that every p-adic Cauchy sequence of elements in $\mathcal{O}_L$ converges in $\mathcal{O}_L$.

We next show that $\mathcal{O}_L$ is p-adically separated. If the Cauchy sequence corresponds to 0 in L, then there is nothing to check. Otherwise, the value of the Cauchy sequence is equal to the value of one of its non-zero entries; call that entry x. Because $\mathcal{O}_K$ is p-adically separated, we have that $v(p^n) < v(x)$ for some integer $n \geq 1$. Thus the element corresponding to the Cauchy sequence is not in $p^n \mathcal{O}_L$. This shows that $\mathcal{O}_L$ is p-adically separated. ■

Lemma 2.12. *Let L denote a characteristic zero field equipped with a valuation for which $\mathcal{O}_L$ is p-adically separated. Let L' denote an algebraic extension of L. Then $\mathcal{O}_{L'}$ is also p-adically separated.*

Proof. This follows directly from condition (*) that was named in the proof of Proposition 2.9. ■

Lemma 2.13. *Let K be a characteristic zero field equipped with a valuation for which $\mathcal{O}_K$ is p-adically separated, and let L denote the completion of K, as in (Engler and Prestel, 2005, Theorem 2.4.3). If K is algebraically closed, then so is L.*

Proof. Much of this proof is taken verbatim from notes written by Brian Conrad (Conrad, 2008). (Those notes are phrased in terms of absolute values rather than valuations. We have attempted to translate Conrad's proof into the setting of possibly higher rank valued fields.) We first outline the proof strategy. We must show that every non-constant polynomial $f(x)$ in $L[x]$ has a zero in L. We can approximate $f(x)$ by polynomials $f_j(x) \in K[x]$, each of which has a zero $r_j \in K$, because K is algebraically closed. We will show that some subsequence of (r_j) converges to a zero of $f(x)$. Because L is the completion of K, this will imply that some zero of $f(x)$ lies in L, as desired.

More precisely, the following are the key steps. Let Γ denote the value group of K.

(1) There exists $\gamma \in \Gamma$ such that $v(r_j) \leq \gamma$ for all j.
(2) The sequence $f(r_j) \in L$ converges to zero.
(3) Let $\lambda_1, \ldots, \lambda_n$ denote the zeros of $f(x)$ in some field L' which is an algebraic extension of L. We have that the sequence $\left(\min_i v(r_j - \lambda_i)\right)$ approaches zero as j approaches infinity, and hence, because there are only finitely many values of λ_i, some subsequence of (r_j) is a Cauchy sequence converging to λ_k for some $1 \leq k \leq n$. Thus $\lambda_k \in L$, as required.

We now carry out these steps. The proof of (Engler and Prestel, 2005, Theorem 2.4.3) describes the valuation on L, and by Theorem 2.8, we can further extend the valuation on L to a valuation on any field extension L'. If L'/L is algebraic, then $\mathcal{O}_{L'}$ is p-adically separated by Lemma 2.11 and Lemma 2.12.

We may assume our polynomial is separable. Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ denote an arbitrary separable, non-constant polynomial in $L[x]$. It is also convenient for the proof below to assume that $a_0 \neq 0$ and $n \geq 2$. Write $\lambda_1, \ldots, \lambda_n$ for the zeros of $f(x)$ in some field L' which is algebraic over L. We can approximate $f(x)$ by polynomials $f_j(x) = x^n + a_{n-1,j}x^{n-1} + \cdots + a_{0,j} \in K[x]$ satisfying the following two conditions:

(1) If $a_i = 0$, then $a_{i,j} = 0$ for all j.
(2) If $a_i \neq 0$, then $v(a_{i,j} - a_i) < \min(v(a_i), v(p^j))$ for all j.

These conditions imply that $v(a_i) = v(a_{i,j})$ for all i and all j.

For each j, let $r_j \in K$ denote a root of $f_j(x)$. Because $f_j(r_j) = 0$, we have that

$$v(r_j^n) = v\left(\sum_{i=0}^{n-1} a_{i,j} r_j^i\right) \leq \max_i v(a_{i,j}) v(r_j)^i = \max_i v(a_i) v(r_j)^i.$$

Hence, for each j, there exists some $i(j) \in \mathbf{Z}$ in the range $0 \leq i(j) \leq n-1$ such that

$$v(r_j)^n \leq v(a_{i(j)}) v(r_j)^{i(j)}.$$

Thus

$$v(r_j) \leq \gamma := \max_i v(a_i)^{1/(n-i)}.$$

(Note that, because K is algebraically closed, the group element $v(a_i)^{1/(n-i)}$ makes sense.)

Because f, f_j are monic polynomials of the same degree and because $f_j(r_j) = 0$ by our assumption, we have

$$\begin{aligned} v(f(r_j)) &= v(f(r_j) - f_j(r_j)) \\ &= v\left(\sum_{i=0}^{n-1} (a_i - a_{i,j}) r_j^i\right) \\ &\leq \max_{0 \leq i \leq n-1} v(a_i - a_{i,j}) v(r_j)^i. \end{aligned}$$

We know $v(r_j) \leq \gamma$, so $v(r_j)^i \leq \gamma^i$, and in particular if $\gamma < 1$, then $v(r_j)^i \leq 1$. If on the other hand $\gamma \geq 1$, then $v(r_j)^i \leq \gamma^{n-1}$. In total, we deduce

$$v(f(r_j)) \leq \max_i v(a_i - a_{i,j}) \cdot \max\{1, \gamma^{n-1}\}.$$

By our choice of the coefficients $a_{i,j}$, we have

$$v(f(r_j)) \leq v(p^j) \cdot \max\{1, \gamma^{n-1}\}.$$

By Lemma 2.10, there exists some integer $m \geq 1$ such that $v(p^m) < \frac{1}{\max\{1, \gamma^{n-1}\}}$, and hence $\max\{1, \gamma^{n-1}\} < v(p^{-m})$ and therefore $v(f(r_j)) < v(p^{j-m})$ for all j. Thus the sequence $f(r_j) \in L$ converges to 0.

Recall that we denoted the roots of $f(x)$ by $\lambda_i \in L'$, so $f(x) = \prod_{i=1}^n (x - \lambda_i) \in L'[x]$, where L' is some fixed algebraic extension of L. Thus for all j, we have that

$$\prod_{i=1}^{n} v(r_j - \lambda_i) \leq v(p^j) \cdot \max\{1, \gamma^{n-1}\},$$

and so for all j, we have that

$$\min_i v(r_j - \lambda_i) \leq v(p^j)^{1/n} \cdot \max\{1, \gamma^{(n-1)/n}\}.$$

By the pigeonhole principle, there is some $0 \leq i_0 \leq n-1$ for which the inequality

$$v(r_j - \lambda_{i_0}) \leq v(p^j)^{1/n} \cdot \max\{1, \gamma^{(n-1)/n}\}$$

holds for infinitely many values of j. As in the previous paragraph, some subsequence of $(r_j - \lambda_{i_0})$ converges to 0, and hence some subsequence of (r_j) converges to λ_{i_0}. Because the elements r_j are all in K, and because L is the completion of K, it follows that some subsequence of r_j converges to some element $\lambda \in L$. Because $f(x) \in L[x]$ is a polynomial and $f(r_j)$ converges to 0 in L, we have that $\lambda \in L$ satisfies $f(\lambda) = 0$, as desired. (Alternatively, it could be shown that in fact $\lambda = \lambda_{i_0}$, using the fact that $\mathcal{O}_{L_1}$ is p-adically separated, but that argument requires consideration of the case $\lambda, \lambda_{i_0} \in L_1 \setminus \mathcal{O}_{L_1}$.) This completes the proof that the completion of K is algebraically closed. ■

We next discuss perfectoid rings. We follow the presentation and notation of Bhatt-Morrow-Scholze's (Bhatt et al., 2018, Section 3). Many of these next results also correspond to related results in Hesselholt's (Hesselholt, 2006, Section 1.2). For example, Lemma 2.16 is closely related to (Hesselholt, 2006, Proposition 1.2.3 and Addendum 1.2.4).

Lemma 2.14 ((Bhatt et al., 2018, Lemma 3.2(i)))**.** *Let A denote a ring which is p-adically complete and separated. Define the* tilt *of A, denoted $A^\flat$, to be the ring $A^\flat := \varprojlim_{x \mapsto x^p} A/pA$. The map which reduces each element modulo p,*

$$\varprojlim_{x \mapsto x^p} A \to \varprojlim_{x \mapsto x^p} A/pA = A^\flat,$$

is an isomorphism of multiplicative monoids.

Notation 2.15. For any ring A, we write $\varprojlim_F W_r(A)$ for the ring which is the inverse limit of the diagram

$$\cdots \xrightarrow{F} W_3(A) \xrightarrow{F} W_2(A) \xrightarrow{F} W_1(A),$$

where the transition maps are the finite-length Witt vector Frobenius. For any integer $n \geq 1$, we let $\mathrm{pr}_n : \varprojlim_F W_r(A) \to W_n(A)$ denote the projection.

Lemma 2.16. *Let A denote a ring which is p-adically complete and separated. Let notation be as in Lemma 2.14 and Notation 2.15. There is a unique p-adically continuous ring homomorphism*

$$\pi : W(A^\flat) \to \varprojlim_F W_r(A)$$

such that, for every integer $r \geq 1$ and every $(t^{(0)}, t^{(1)}, \ldots) \in \varprojlim_{x \mapsto x^p} A$ (identified with an element $t \in A^\flat$ as in Lemma 2.14), we have

$$(pr_r \circ \pi)([t]) = [t^{(r)}] \in W_r(A).$$

The map π is an isomorphism of rings.

Proof. See for example (Bhatt et al., 2018, Lemma 3.2), and the two paragraphs following that proof. Uniqueness follows immediately from the fact that every element in $W(A^\flat)$ is a p-adic combination $\sum p^i[t_i]$, where $t_i \in A^\flat$. ■

Definition 2.17. Let A be a ring which is p-adically complete and p-adically separated. For every integer $r \geq 1$, we define $\widetilde{\theta}_r$ to be the ring homomorphism

$$\widetilde{\theta}_r := \mathrm{pr}_r \circ \pi : W(A^\flat) \to W_r(A)$$

from Lemma 2.16. We define the ring homomorphism $\theta : W(A^\flat) \to A$ to be the composite $\widetilde{\theta}_1 \circ F$, where $F : W(A^\flat) \to W(A^\flat)$ is the Witt vector Frobenius.

Remark 2.18. Let $t \in A^\flat$ be arbitrary, and assume t corresponds to $(t^{(0)}, t^{(1)}, \dots) \in \varprojlim_{x \mapsto x^p} A$. We have

$$\theta([t]) = t^{(0)}.$$

Definition 2.19 ((Bhatt et al., 2018, Definition 3.5))**.** We say a ring A is perfectoid if the following three conditions hold:

(1) The ring A is π-adically complete and separated for some element π such that π^p divides p.
(2) The p-power Frobenius $A/pA \to A/pA$ is surjective.
(3) The kernel of $\theta : W(A^\flat) \to A$ (with θ as in Definition 2.17) is a principal ideal.

Example 2.20. Assume A is a p-torsion-free ring containing a primitive p-th root of unity, ζ_p. (As throughout this paper, we assume p is odd.) Then Condition (1) in the definition of *perfectoid* can be replaced with

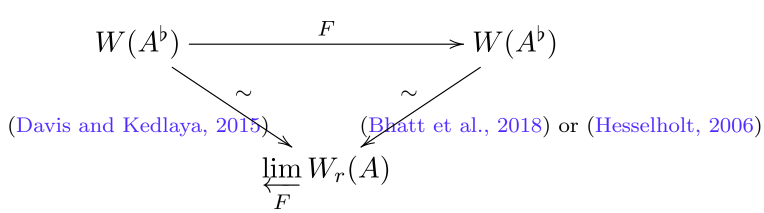
(1′) The ring A is p-adically complete and separated.

Indeed, on one hand, a perfectoid ring is p-adically complete and separated. On the other hand, if the p-power map is surjective modulo p on a p-torsion-free ring A containing ζ_p, then there exists some $\pi, a \in A$ such that

$$\pi^p = (\zeta_p - 1) + pa.$$

If A is p-adically complete, then we deduce that π^p divides $\zeta_p - 1$ (using that $p \neq 2$), and hence π^p divides p. On the other hand, p divides π^{p^2}, so the ring A is π-adically complete and separated.

Remark 2.21. We typically use (Bhatt et al., 2018) as our reference, but many of these properties were studied earlier. For example, the significance of θ was recognized by Fontaine, and the isomorphism between $W(A^\flat)$ and $\varprojlim_F W_r(A)$ appears (in a slightly different context) in Hesselholt's (Hesselholt, 2006, Addendum 1.2.4). An isomorphism $W(A^\flat) \cong \varprojlim_F W_r(A)$ was also studied by the first author and Kedlaya in (Davis and Kedlaya, 2015, Theorem 3.6), but that isomorphism differs from the isomorphisms of Hesselholt and Bhatt-Morrow-Scholze. More precisely, the isomorphism from (Davis and Kedlaya, 2015, Theorem 3.6) is attained from the isomorphism in (Bhatt et al., 2018, Lemma 3.2) by first applying the Witt vector Frobenius automorphism on $W(A^\flat)$, as indicated in the following commutative diagram



The present paper uses the normalizations of Hesselholt and Bhatt-Morrow-Scholze.

In general, a perfectoid ring A and its tilt $A^\flat$ may contain zero divisors, and hence $W(A^\flat)$ may also contain zero divisors. On the other hand, every generator of the ideal $\ker\theta \subseteq W(A^\flat)$ is a non-zero-divisor, as we recall in the next result.

Lemma 2.22 ((Bhatt et al., 2018, Lemma 3.10 and Remark 3.11)). *Let A denote a perfectoid ring, and let $\xi \in W(A^\flat)$ be any generator of $\ker\theta$.*

(1) The element ξ is a non-zero-divisor in $W(A^\flat)$.
(2) The generators ξ are functorial in the following sense. Let B denote another perfectoid ring, with corresponding theta map $\theta_B : W(B^\flat) \to B$. Let $f : A \to B$ denote a ring homomorphism. By functoriality of tilts and Witt vectors, the map f induces a map $W(f^\flat) : W(A^\flat) \to W(B^\flat)$, and the element $W(f^\flat)(\xi) \in W(B^\flat)$ is a generator of $\ker(\theta_B)$.

One of the deeper results concerning perfectoid rings which we will need is the following.

Lemma 2.23. *Every p-torsion-free perfectoid ring A embeds into a product of p-torsion-free perfectoid valuation rings, $\prod \overline{V}_\alpha$, for which each $\operatorname{Frac}\overline{V}_\alpha$ is algebraically closed.*

Proof. It is shown in (Bhatt et al., 2019, Proof of Proposition 4.19) that every perfectoid ring embeds into a product of perfectoid valuation rings. Every valuation ring is either p-torsion-free or is annihilated by p, and because our ring A is p-torsion-free, if A embeds into a product of perfectoid valuation rings $\prod V_\alpha$, it also embeds into the possibly smaller product consisting of only those V_α which are p-torsion-free. (If a maps to 0 in this smaller product, then pa maps to 0 in the original product, and hence $pa = 0$, and hence $a = 0$.) For each α, let $K_\alpha := \operatorname{Frac} V_\alpha$ and let L_α denote the completion (as in (Engler and Prestel, 2005, Theorem 2.4.3)) of an algebraic closure of K_α. Let W_α denote the valuation ring in L_α. By Lemma 2.13, $L_\alpha = \operatorname{Frac} W_\alpha$ is algebraically closed, so it suffices to show that W_α is perfectoid. The ring W_α is p-adically separated by Lemma 2.11 and Lemma 2.12, and W_α is p-adically complete by construction. Let $\pi \in W_\alpha$ denote an element for which $\pi^p = p$. From the fact that L_α is algebraically closed, it's clear that such an element π exists and that the p-power map $W_\alpha/\pi W_\alpha \to W_\alpha/\pi^p W_\alpha$ is surjective. By (Bhatt et al., 2018, Lemma 3.10(ii)), to show that W_α is perfectoid, it suffices to show that this p-power map $W_\alpha/\pi W_\alpha \to W_\alpha/\pi^p W_\alpha$ is also injective, but this is obvious from the fact that W_α is a valuation ring: if $\pi^p \mid w^p$, then $v(\pi)^p \geq v(w)^p$, and so $v(\pi) \geq v(w)$, and so $\pi \mid w$. ∎

We next transition to our algebraic results concerning Witt vectors. A foundational result is the following, which is due to Hesselholt.

Lemma 2.24 ((Hesselholt, 2006, Lemma 1.1.1)). *Let A denote a p-adically complete and p-torsion-free ring. Then for each integer $n \geq 1$, the ring $W_n(A)$ is also p-adically complete and p-torsion-free.*

Most of our Witt vector results in this section are less foundational and more specialized. We briefly indicate the significance of these technical results. Theorem C relates p-torsion in the de Rham-Witt complex to $W_n(A)$-modules of the form $W_n(A)/pW_n(A)$. In almost every situation, it is easier to prove results about $W_n(A)/pW_n(A)$ than to prove results about $W_n\Omega^1_A[p]$. In this section we gather several algebraic results concerning $W_n(A)/pW_n(A)$.

Lemma 2.25. *Let A denote a perfectoid ring. Assume $f : A/pA \to A/pA$ is a map of $W_{n+1}(A)$-modules, where A/pA is equipped with a $W_{n+1}(A)$-module structure via F^n. If there exists a unit $u \in A/pA$ which is in the image of f, then f is an isomorphism.*

Proof. Using that $F^n : W_{n+1}(A) \to W_1(A)$ is surjective, we find that $f(a) = af(1)$ for every $a \in A/pA$, so it suffices to prove that $f(1)$ is a unit in A/pA, and this follows by our assumption, which shows that some multiple of $f(1)$ is a unit. ■

Before stating another Witt vector result, we state a general result about the tilt of a p-adically complete valuation ring.

Lemma 2.26. *Let A denote a p-adically complete and p-adically separated valuation ring. Then the ring $A^\flat$ is a valuation ring.*

Proof. The authors noticed this argument in notes from a course of Bhargav Bhatt (Bhatt, 2017b). We first observe that $A^\flat$ is an integral domain; indeed, elements in $A^\flat$ are uniquely representable by p-power-compatible sequences of elements in A (as opposed to A/pA, see for example (Bhatt et al., 2018, Lemma 3.2(i))), and multiplication on $A^\flat$ corresponds to componentwise multiplication of these sequences. Because A is an integral domain, it is then clear that $A^\flat$ does not have any zero divisors. To prove that $A^\flat$ is a valuation ring, we show that if $x, y \in A^\flat$, then either x divides y or y divides x. Write $x = (x_i)$ where $x_i \in A$ and $x_i^p = x_{i-1}$ and similarly for $y = (y_i)$. Notice that in the valuation ring A, we have that x_i divides y_i if and only if x_i^p divides y_i^p: the forward direction is obvious. For the other direction, assume $x_i^p a = y_i^p$ and $y_i b = x_i$. Then we have $x_i^p a b^p = x_i^p$, and hence (unless $x_i = 0$) b is a unit, and hence x_i divides y_i. The result now follows. ■

The ring $\mathbf{Z}_p[\zeta_{p^\infty}]^\wedge$ plays a special role in this paper, especially in the proof of Theorem C. The next result concerns this ring.

Lemma 2.27. *Let $A_0 = \mathbf{Z}_p[\zeta_{p^\infty}]^\wedge$ and let $n \geq 1$ be an integer. The ring $W_n(A_0)/pW_n(A_0)$ has the following property: If $N \subseteq W_n(A_0)/pW_n(A_0)$ is an ideal and $x, y \in W_n(A_0)/pW_n(A_0)$ are such that $x, y \notin N$ and $x \equiv y \bmod N$, then there exists a unit $u \in W_n(A_0)/pW_n(A_0)$ such that $x = uy$.*

Proof. The ring homomorphism $\widetilde{\theta}_n : W(A_0^\flat) \to W_n(A_0)$ is surjective (see for example (Bhatt et al., 2018, Lemma 3.9(iv))), and hence we have a surjective ring homomorphism $A_0^\flat \cong W(A_0^\flat)/pW(A_0^\flat) \to W_n(A_0)/pW_n(A_0)$. By Lemma 2.26, this means $W_n(A_0)/pW_n(A_0)$ is a quotient of a valuation ring.

Write $x = y + z$, where $z \in N$. Let $x', y' \in A_0^\flat$ denote elements mapping to x, y (respectively) under this surjection $A_0^\flat \twoheadrightarrow W_n(A_0)/pW_n(A_0)$. We find that $z' := x' - y' \in A_0^\flat$ maps to z. We cannot have $\frac{y'}{z'} \in A_0^\flat$, because we know that $y \notin N$, and hence y is not a multiple of z in $W_n(A_0)/pW_n(A_0)$, and hence y' is not a multiple of z' in $A_0^\flat$. Then because $A_0^\flat$ is a valuation ring, we know $\frac{z'}{y'} \in A_0^\flat$, and $(1 + \frac{z'}{y'})y' = x' \in A_0^\flat$. In other words, x' is a multiple of y' in $A_0^\flat$. Reversing the roles of x and y in the argument, we find that y' is also a multiple of x' in $A_0^\flat$. Because $A_0^\flat$ is an integral domain, x' is a unit multiple of y', and hence x is a unit multiple y in $W_n(A_0)/pW_n(A_0)$, as required. ■

Here is a more basic result.

Lemma 2.28. *Let A denote a p-adically complete ring. An element $x \in W_n(A)$ is a unit if and only if its projection $x \in W_n(A)/pW_n(A)$ is a unit. Furthermore, $x \in W_n(A)$ is a unit if and only if its first Witt coordinate is a unit in A.*

Proof. Certainly if x is a unit, then its projection to any quotient ring is a unit. On the other hand, if $x, y, z \in W_n(A)$ satisfy $xy = 1 + pz$, then because $W_n(A)$ is p-adically complete (Lemma 2.24), there exists $u \in W_n(A)$ such that $(1 + pz)u = 1 \in W_n(A)$. Thus $x(yu) = 1 \in W_n(A)$. This shows that if the image of x is a unit in $W_n(A)/pW_n(A)$, then x is a unit in $W_n(A)$.

The proof of the second assertion is similar. If $x \in W_n(A)$ is a unit, then clearly the first Witt coordinate of x is a unit in A. Conversely, if the first Witt coordinate of x is a unit, then

$$x = [x_0] + V(y)$$

for some $y \in W_{n-1}(A)$. If we multiply by the unit $[x_0^{-1}] \in W_n(A)$, we reduce to showing that every Witt vector in $W_n(A)$ of the form $1 + V(y')$ is a unit. This again follows because $W_n(A)$ is p-adically complete, using that $\big(V(y')\big)^m \in p^{m-1}W_n(A)$ for every integer $m \geq 1$. ∎

A similar result is the following.

Lemma 2.29. *Assume A is a p-torsion-free perfectoid ring containing a primitive p^n-th root of unity ζ_{p^n} for some integer $n \geq 1$. (Recall that throughout this paper, $p \neq 2$.) If $y \in W_n(A)$ satisfies $([\zeta_{p^n}] - 1)y = ([\zeta_{p^n}] - 1) \in W_n(A)/pW_n(A)$, then y is a unit in $W_n(A)/pW_n(A)$.*

Proof. Let y_0 denote the first Witt component of y. The condition implies

$$(\zeta_{p^n} - 1)(y_0 - 1) = pb$$

for some $b \in A$. Because A is p-torsion-free, it is also $(\zeta_{p^n} - 1)$-torsion-free, so we have that

$$y_0 \equiv 1 \bmod \frac{p}{\zeta_{p^n} - 1}A.$$

Because A is p-adically complete (since it is perfectoid), it is also $\frac{p}{\zeta_{p^n}-1}$-adically complete. Thus y_0 is a unit in A and thus by Lemma 2.28, y is a unit in $W_n(A)$. ∎

The following result is taken from (Bhatt et al., 2018). See also (Hesselholt, 2006, Proposition 1.2.3) for a related result.

Lemma 2.30 ((Bhatt et al., 2018, Corollary 3.18(i)))**.** *Let A denote a p-torsion-free perfectoid ring containing a compatible system of p-power roots of unity, and let $r > s \geq 1$. The kernel of $F^{r-s} : W_r(A) \to W_s(A)$ is the principal ideal generated by $\sum_{i=0}^{p^s-1}[\zeta_{p^r}]^i$. In particular, the kernel of $F^n : W_{n+1}(A) \to W_1(A)$ is the principal ideal generated by*

$$z_{n+1} = 1 + [\zeta_{p^{n+1}}] + [\zeta_{p^{n+1}}]^2 + \cdots + [\zeta_{p^{n+1}}]^{p-1} \in W_{n+1}(A).$$

Determining whether one Witt vector is divisible by another Witt vector is often difficult. One result we will need in this direction is the following lemma. It is required for our proof of Proposition 5.15. The following result should be compared to (Bhatt et al., 2018, Lemma 3.23).

Lemma 2.31. *Fix an integer $n \geq 1$. Let $A_0 = \mathbf{Z}_p[\zeta_{p^\infty}]^\wedge$. If $x \in W_n(A_0)$ is in the intersection*

$$x \in \bigcap_{s=1}^{\infty} \frac{[\zeta_{p^n}] - 1}{[\zeta_{p^{n+s}}] - 1} W_n(A_0),$$

then $x \in ([\zeta_{p^n}] - 1)W_n(A_0)$.

Proof. We prove this using induction on n. The base case follows by considering valuations in A_0. Assume now the result has been shown for some fixed value $n - 1 \geq 1$. We will prove the result also for n. Thus, assume $x \in W_n(A_0)$ is in the intersection. Again considering valuations, we find that the first Witt component of x must be divisible by $\zeta_{p^n} - 1$, say $x_0 = (\zeta_{p^n} - 1)y_0$. Consider the element $x' := x - ([\zeta_{p^n}] - 1)[y_0] \in W_n(A_0)$. If we can prove that x' is divisible by $[\zeta_{p^n}] - 1$, then it will follow that x is also divisible by $[\zeta_{p^n}] - 1$, and we will be done. We have that x' is in the same intersection, and also $x' = V(z)$ for some $z \in W_{n-1}(A_0)$.

We claim that z is divisible by $\frac{[\zeta_{p^{n-1}}]-1}{[\zeta_{p^{n-1+s}}]-1}$ in $W_{n-1}(A_0)$ for every integer $s \geq 1$. Fix $s \geq 1$. We know that

$$V(z) = \frac{[\zeta_{p^n}] - 1}{[\zeta_{p^{n+s}}] - 1}\, y$$

for some $y \in W_n(A_0)$, and considering the first Witt components, we see that $y = V(y')$ for some $y' \in W_{n-1}(A_0)$, so

$$V(z) = \frac{[\zeta_{p^n}] - 1}{[\zeta_{p^{n+s}}] - 1}\, V(y') = V\left(F\left(\frac{[\zeta_{p^n}] - 1}{[\zeta_{p^{n+s}}] - 1}\right) y'\right) = V\left(\frac{[\zeta_{p^{n-1}}] - 1}{[\zeta_{p^{n-1+s}}] - 1}\, y'\right).$$

This proves the claim that z is divisible by $\frac{[\zeta_{p^{n-1}}]-1}{[\zeta_{p^{n-1+s}}]-1}$ in $W_{n-1}(A_0)$ for every integer $s \geq 1$. Fix $s \geq 1$.

By our induction hypothesis, we have that $z = ([\zeta_{p^{n-1}}] - 1)w$ for some $w \in W_{n-1}(A_0)$. Thus

$$x' = V(z) = V\big(([\zeta_{p^{n-1}}] - 1)w\big) = ([\zeta_{p^n}] - 1)V(w).$$

This proves that x' is divisible by $[\zeta_{p^n}] - 1$ in $W_n(A_0)$, and as explained above, this finishes the proof. ■

We will eventually prove that, when A is a p-torsion-free perfectoid ring containing a compatible sequence of p-power roots of unity, then the p-adic Tate module $T_p(W_n\Omega^1_A)$ is a free $W_n(A)$-module of rank one. The restriction map from level $n+1$ to level n on the Tate modules does *not* correspond to the restriction map on Witt vectors. Instead it will correspond to the map Rz_{n+1} appearing in the following lemma.

Lemma 2.32. *Let A denote a p-torsion-free perfectoid ring containing a compatible system of p-power roots of unity. We have an exact sequence of $W_{n+1}(A)$-modules*

$$0 \to A \xrightarrow{V^n} W_{n+1}(A) \xrightarrow{Rz_{n+1}} W_n(A) \xrightarrow{F^n} A/p^nA \to 0,$$

where the module structures and maps are defined as follows. Both A and A/p^nA are considered as $W_{n+1}(A)$-modules via F^n, and $W_n(A)$ is considered as a $W_{n+1}(A)$-module via restriction. The map Rz_{n+1} denotes the composite given by multiplication by z_{n+1} followed by Frobenius:

$$\psi_n \colon W_{n+1}(A) \xrightarrow{1+[\zeta_{p^{n+1}}]+\cdots+[\zeta_{p^{n+1}}]^{p-1}} W_{n+1}(A) \xrightarrow{R} W_n(A).$$

The map $F^n : W_n(A) \to A/p^nA$ is defined using the isomorphism $W_n(A) \cong W_{n+1}(A)/V^n(A)$.

Proof. It's clear that $V^n(A)$ is in the kernel of Rz_{n+1}. Conversely, if $y \in W_{n+1}(A)$ is in the kernel of Rz_{n+1}, then $\big(1 + [\zeta_{p^{n+1}}] + \cdots + [\zeta_{p^{n+1}}]^{p-1}\big) \cdot y \in V^n(A)$. Because the first n ghost components of $1 + [\zeta_{p^{n+1}}] + \cdots + [\zeta_{p^{n+1}}]^{p-1}$ are not zero divisors (because A is p-torsion-free), we see that the first n ghost components of y are zero, and hence $y \in V^n(A)$, proving the other inclusion.

The map $W_n(A) \cong W_{n+1}(A)/V^n(A) \to A/p^nA$ induced by F^n is surjective (see for example (Bhatt et al., 2018, Lemma 3.9(iv))). It remains to show that its kernel is equal to the image of the map Rz_{n+1}. Because $F^n(1 + [\zeta_{p^{n+1}}] + \cdots + [\zeta_{p^{n+1}}]^{p-1}) = 0 \in W_1(A)$, we see that the image of Rz_{n+1} is contained in the kernel of $W_n(A) \to A/p^nA$. For the reverse inclusion, say $x \in W_{n+1}(A)$ is the lift (under restriction) of some element in the kernel of $W_n(A) \to A/p^nA$. Write $F^n(x) = p^n a$, where $a \in A$. Then $F^n(x - V^n(a)) = 0$, and so by Lemma 2.30, we can find an element $y \in W_{n+1}(A)$ such that

$$\big(1 + [\zeta_{p^{n+1}}] + \cdots + [\zeta_{p^{n+1}}]^{p-1}\big)\, y = x - V^n(a).$$

Applying R to both sides, we get that $R(x)$ is in the image of Rz_{n+1}, as required. ■

3. On p-torsion in the module of Kähler differentials

Our goal in this section is to determine the p^n-torsion in the module of absolute Kähler differentials $\Omega^1_A := \Omega^1_{A/\mathbf{Z}}$ for a p-torsion-free perfectoid ring A. Our strategy is first to consider the cotangent complex $L_{A/\mathbf{Z}_p}$, then to consider the relative Kähler differentials $\Omega^1_{A/\mathbf{Z}_p}$, and then finally (and this is the easiest part) to consider the absolute Kähler differentials Ω^1_A.

Our starting point is a result of Bhatt-Morrow-Scholze which states that when A is a perfectoid ring, the derived p-completion (as defined in Proposition 2.3) of $L_{A/\mathbf{Z}_p}$ is quasi-isomorphic to $A[1]$. We recall this result in Proposition 3.1. (It does not require our usual p-torsion-free assumption.)

Proposition 3.1 ((Bhatt et al., 2019, Proposition 4.19(2)))**.** *Let A denote a perfectoid ring. The derived p-completion of $L_{A/\mathbf{Z}_p}$ is naturally quasi-isomorphic to $\ker\theta/(\ker\theta)^2[1]$, where $\theta : W(A^\flat) \to A$ is Fontaine's map. By choosing a generator ξ of $\ker\theta$, we can identify $\ker\theta/(\ker\theta)^2[1]$ with $A[1]$. Hence the derived p-completion of $L_{A/\mathbf{Z}_p}$ is quasi-isomorphic to $A[1]$ and this quasi-isomorphism depends on the choice of ξ.*

An immediate corollary of this is the following.

Proposition 3.2. *Let A denote a p-torsion-free perfectoid ring and let $n \geq 1$ denote an integer. Then we have a quasi-isomorphism*

$$L_{A/\mathbf{Z}_p} \otimes^L_{\mathbf{Z}_p} \mathbf{Z}/p^n\mathbf{Z} \simeq (A/p^nA)[1].$$

Proof. Let M denote the derived p-completion of $L_{A/\mathbf{Z}_p}$. On one hand, we have

$$L_{A/\mathbf{Z}_p} \otimes^L_{\mathbf{Z}_p} \mathbf{Z}/p^n\mathbf{Z} \cong M \otimes^L_{\mathbf{Z}_p} \mathbf{Z}/p^n\mathbf{Z}.$$

On the other hand, by Proposition 3.1, we have

$$M \cong A[1].$$

The claimed result now follows directly from the universal coefficient theorem (Proposition 2.3). ■

We immediately deduce the following consequence concerning p-power torsion in the module of relative Kähler differentials.

Corollary 3.3. *Let A denote a p-torsion free perfectoid ring and let $n \geq 1$ be an integer. Then we have a short exact sequence of A-modules*

$$0 \to H^{-1}(L_{A/\mathbf{Z}_p})/p^n H^{-1}(L_{A/\mathbf{Z}_p}) \to A/p^nA \to \Omega^1_{A/\mathbf{Z}_p}[p^n] \to 0.$$

Proof. Again from the universal coefficient theorem (Proposition 2.3), we know there is a short exact sequence

$$0 \to H^{-1}(L_{A/\mathbf{Z}_p}) \otimes_{\mathbf{Z}_p} \mathbf{Z}/p^n\mathbf{Z} \to H^{-1}(L_{A/\mathbf{Z}_p} \otimes^L_{\mathbf{Z}_p} \mathbf{Z}/p^n\mathbf{Z}) \to \mathrm{Tor}_1^{\mathbf{Z}_p}(H^0(L_{A/\mathbf{Z}_p}), \mathbf{Z}/p^n\mathbf{Z}) \to 0.$$

Using Proposition 3.2 to replace the middle term with A/p^nA, the result follows. ■

Remark 3.4. Since the quasi-isomorphism $(L_{A/\mathbf{Z}_p})^\wedge \simeq A[1]$ in Proposition 3.1 depends on the choice of ξ, we must be careful regarding any desired functoriality properties of maps deduced from this quasi-isomorphism. For example, the A-module homomorphism

$$A/p^nA \to \Omega^1_{A/\mathbf{Z}_p}[p^n]$$

appearing in Corollary 3.3 is *not* functorial in A. However, we will see below that this map $A/p^nA \to \Omega^1_{A/\mathbf{Z}_p}[p^n]$ is an isomorphism and is determined uniquely if we moreover fix $\xi \in W(A^\flat)$ such that ξ is a generator for the kernel of Fontaine's map $\theta : W(A^\flat) \to A$. Notice that, because ξ is a non-zero-divisor, any two such generators differ by a unit $u \in W(A^\flat)$, and it will follow that the two corresponding isomorphisms $A/p^nA \to \Omega^1_{A/\mathbf{Z}_p}[p^n]$ will differ by the unit $\theta(u) \in A$.

The most important result in this section is the following.

Theorem 3.5. *Let A denote a p-torsion-free perfectoid ring.*

(1) Multiplication by p is surjective on $\Omega^1_{A/\mathbf{Z}_p}$.

(2) For every integer $n \geq 1$, we have an isomorphism of A-modules

$$T_p(\Omega^1_{A/\mathbf{Z}_p})/p^nT_p(\Omega^1_{A/\mathbf{Z}_p}) \cong \Omega^1_{A/\mathbf{Z}_p}[p^n].$$

(3) For every integer $n \geq 1$, the map from Corollary 3.3

$$A/p^nA \to \Omega^1_{A/\mathbf{Z}_p}[p^n]$$

is an isomorphism of A-modules.

(4) We have an isomorphism of A-modules

$$A \xrightarrow{\sim} T_p(\Omega^1_{A/\mathbf{Z}_p})$$

that, when reduced modulo p^n for any integer $n \geq 1$, induces the isomorphism from Part (3).

(5) We have an isomorphism of A-modules

$$H^{-1}(L_{A/\mathbf{Z}_p})/pH^{-1}(L_{A/\mathbf{Z}_p}) \cong 0.$$

Proof. Part (1) is true more generally for any ring A with the property that the p-power map is surjective modulo p. Namely, it suffices to prove that an element of the form da is divisible by p, where $a \in A$ is arbitrary, and this follows after writing $a = a_0^p + pa_1$ and applying the Leibniz rule.

Part (2) is true more generally for any module M on which multiplication by p is surjective. Namely, the natural projection map

$$T_p(M) \to M[p^n]$$

is surjective because any p^n-torsion element can be extended to an element in $T_p(M)$, and so it suffices to show that the kernel of this projection consists precisely of those elements which are divisible by p^n. Thus let $(m_1, m_2, \ldots) \in T_p(M)$, where $m_i \in M[p^i]$, and assume $m_n = 0$. Consider the element $(m_{n+1}, m_{n+2}, \ldots)$; clearly $p^n(m_{n+1}, m_{n+2}, \ldots) = (m_1, m_2, \ldots)$, and moreover $(m_{n+1}, m_{n+2}, \ldots)$ is in the Tate module, because $m_n = 0$ shows that $m_{n+i} \in M[p^i]$ for all integers $i \geq 1$.

We next prove Part (3) in a special case. Let $\overline{V}$ denote a p-torsion-free perfectoid ring which is moreover a valuation ring with $\operatorname{Frac}\overline{V}$ algebraically closed. By Proposition 2.6, we have $H^{-1}(L_{\overline{V}/\mathbf{Z}_p}) \cong 0$, and hence from Corollary 3.3, we have a $\overline{V}$-module isomorphism $\overline{V}/p^n\overline{V} \xrightarrow{\sim} \Omega^1_{\overline{V}/\mathbf{Z}_p}[p^n]$. This proves Part (3) in this special case. Before proving this part in general, we consider Part (4).

Let A be an arbitrary p-torsion-free perfectoid ring. By Corollary 2.4 and Proposition 3.1, we have a surjective A-module homomorphism

$$A \twoheadrightarrow T_p(\Omega^1_{A/\mathbf{Z}_p}),$$

and for any integer $n \geq 1$, we have an induced map

$$A/p^nA \twoheadrightarrow T_p(\Omega^1_{A/\mathbf{Z}_p})/p^nT_p(\Omega^1_{A/\mathbf{Z}_p}) \cong \Omega^1_{A/\mathbf{Z}_p}[p^n].$$

Moreover, this map agrees with the map in Corollary 3.3. The reason is that the universal coefficient sequence in Proposition 2.3(3) is natural in G, and hence the diagram

$$\begin{array}{ccc} H^{-1}((L_{A/\mathbf{Z}_p})^\wedge) & \longrightarrow & T_p(\Omega^1_{A/\mathbf{Z}_p}) \\ \downarrow & & \downarrow \\ H^{-1}((L_{A/\mathbf{Z}_p}) \otimes^L_{\mathbf{Z}} \mathbf{Z}/p^n\mathbf{Z}) & \longrightarrow & \Omega^1_{A/\mathbf{Z}_p}[p^n] \end{array}$$

commutes. Here the vertical maps are induced by (derived and underived) mod p^n reduction. We also note that the identification $H^{-1}((L_{A/\mathbf{Z}_p}) \otimes^L_{\mathbf{Z}} \mathbf{Z}/p^n) \cong A/p^nA$ which is used to define the map $A/p^nA \to \Omega^1_{A/\mathbf{Z}_p}[p^n]$ in Corollary 3.3 is determined by the identification $H^{-1}((L_{A/\mathbf{Z}_p})^\wedge) \cong A$ which itself up to unit only depends on the choice the generator of $\ker\theta$ (see (Bhatt et al., 2019, Proposition 4.19)). Now in the case $A = \overline{V}$, this map is the isomorphism of the previous paragraph. Thus the map $\overline{V} \to T_p(\Omega^1_{\overline{V}/\mathbf{Z}_p})$ is an inverse limit of isomorphisms, and hence is an isomorphism. This proves Part (4) in the case of $\overline{V}$.

We now return to the case that A is an arbitrary p-torsion-free perfectoid ring, and consider an injective ring homomorphism $A \to \prod \overline{V}_\alpha$, as in Lemma 2.23. We would like to construct a commutative diagram

$$\begin{array}{ccc} \prod \overline{V}_\alpha & \longrightarrow & \prod T_p(\Omega^1_{\overline{V}_\alpha/\mathbf{Z}_p}) \\ \uparrow & & \uparrow \\ A & \longrightarrow & T_p(\Omega^1_{A/\mathbf{Z}_p}) \end{array}$$

The problem here is that the map $A \to T_p(\Omega^1_{A/\mathbf{Z}_p})$ is only unique after the choice of a generator of $\ker\theta$. Let $f_\alpha : A \to \overline{V}_\alpha$ denote the given embedding to $\prod \overline{V}_\alpha$ composed with the projection onto the α factor. By Corollary 2.4 we have a commutative diagram

$$\begin{array}{ccc} H^{-1}((L_{\overline{V}_\alpha/\mathbf{Z}_p})^\wedge) & \longrightarrow & T_p(\Omega^1_{\overline{V}_\alpha/\mathbf{Z}_p}) \\ {\scriptstyle (f_\alpha)_*}\uparrow & & \uparrow{\scriptstyle (f_\alpha)_*} \\ H^{-1}((L_{A/\mathbf{Z}_p})^\wedge) & \longrightarrow & T_p(\Omega^1_{A/\mathbf{Z}_p}) \end{array}$$

Further, following (Bhatt et al., 2019, Proposition 4.19), we also have a commutative diagram with horizontal arrows isomorphisms

$$\begin{array}{ccccc} \ker\theta_{\overline{V}_\alpha}/(\ker\theta_{\overline{V}_\alpha})^2 & \xrightarrow{\cong} & H^{-1}((L_{\overline{V}_\alpha/W(\overline{V}^\flat_\alpha)})^\wedge) & \xrightarrow{\cong} & H^{-1}((L_{\overline{V}_\alpha/\mathbf{Z}_p})^\wedge) \\ {\scriptstyle (f_\alpha)_*}\uparrow & & {\scriptstyle (f_\alpha)_*}\uparrow & & {\scriptstyle (f_\alpha)_*}\uparrow \\ \ker\theta_A/(\ker\theta_A)^2 & \xrightarrow{\cong} & H^{-1}((L_{A/W(A^\flat)})^\wedge) & \xrightarrow{\cong} & H^{-1}((L_{A/\mathbf{Z}_p})^\wedge) \end{array}$$

Let $\xi \in W(A^\flat)$ be a generator of the principal ideal $\ker\theta_A$. The key observation is that the image of ξ under the map induced by functoriality, $W(f^\flat_\alpha)(\xi)$, is a generator of $\ker\theta_{\overline{V}_\alpha}$ in $W(\overline{V}^\flat_\alpha)$; see Lemma 2.22. The elements ξ and $W(f^\flat_\alpha)(\xi)$ are non-zero-divisors (see again Lemma 2.22) and hence determine isomorphisms $\ker\theta_{\overline{V}_\alpha}/(\ker\theta_{\overline{V}_\alpha})^2 \cong \overline{V}_\alpha$ and $\ker\theta_A/(\ker\theta_A)^2 \cong A$. This compatible choice of generators allows us to fit the latter isomorphisms in a commutative diagram

$$\begin{array}{ccc} \overline{V}_\alpha & \xrightarrow{\cong} & \ker\theta_{\overline{V}_\alpha}/(\ker\theta_{\overline{V}_\alpha})^2 \\ {\scriptstyle f_\alpha}\uparrow & & \uparrow{\scriptstyle (f_\alpha)_*} \\ A & \xrightarrow{\cong} & \ker\theta_A/(\ker\theta_A)^2 \end{array}$$

By combining the last three commutative diagrams, we obtain, for every α, a commutative diagram

$$\begin{array}{ccc} \overline{V}_\alpha & \longrightarrow & T_p(\Omega^1_{\overline{V}_\alpha/\mathbf{Z}_p}) \\ {\scriptstyle f_\alpha}\uparrow & & \uparrow{\scriptstyle (f_\alpha)_*} \\ A & \longrightarrow & T_p(\Omega^1_{A/\mathbf{Z}_p}) \end{array}$$

Here the horizontal arrows are the morphisms obtained using Corollary 2.4 and Proposition 3.1 (and, which we again emphasize, depend up to a unit on the choice of a generator of $\ker\theta$ in $W(A^\flat)$). Now passing to the products we attain the desired commutative diagram

$$\begin{array}{ccc} \prod\overline{V}_\alpha & \longrightarrow & \prod T_p(\Omega^1_{\overline{V}_\alpha/\mathbf{Z}_p}) \\ \uparrow & & \uparrow \\ A & \longrightarrow & T_p(\Omega^1_{A/\mathbf{Z}_p}) \end{array}$$

The left vertical map and the top horizontal map are injective, so the map $A \to T_p(\Omega^1_{A/\mathbf{Z}_p})$ is injective, and we have already remarked that it is surjective. This proves Part (4) in general, and reducing this isomorphism modulo p^n proves Part (3) in general. Finally, Part (5) follows in general from Corollary 3.3 and the fact that $A/p^nA \to \Omega^1_{A/\mathbf{Z}_p}[p^n]$ is an isomorphism. ■

Theorem 3.5 indicates that, for every p-torsion-free perfectoid ring A, multiplication by p on $\Omega^1_{A/\mathbf{Z}_p}$ is far from being injective. We next give a complementary result, which indicates a situation where multiplication by p on Kähler differentials is injective.

Lemma 3.6. *Let k denote a perfect ring of characteristic p. The multiplication-by-p map*

$$p : \Omega^1_{W(k)/\mathbf{Z}_p} \to \Omega^1_{W(k)/\mathbf{Z}_p}$$

is an isomorphism of $W(k)$-modules.

Proof. The proof is the same as the proof of (Davis, 2019, Proposition 2.7) (which concerned absolute Kähler differentials), simply by replacing every occurrence of $\mathbf{Z}$ in that proof with $\mathbf{Z}_p$. The result is also a consequence of (Bhatt et al., 2018, Lemma 3.14). ■

When A is a p-torsion-free perfectoid ring, we know from Theorem 3.5 that there exists $\alpha \in \Omega^1_{A/\mathbf{Z}_p}$ which freely generates $\Omega^1_{A/\mathbf{Z}_p}[p^n]$ as an A/p^nA-module. The following two results concern identifying such a generator α. The more explicit of the two results, Corollary 3.8, requires that A contain a compatible system of p-power roots of unity. The same condition appears in Theorem C.

Proposition 3.7. *Let A denote a p-torsion-free perfectoid ring. Let $\xi \in W(A^\flat)$ denote a generator of $\ker\theta$. Let $\alpha_n \in \Omega^1_{W(A^\flat)/\mathbf{Z}_p}$ denote the unique element such that $p^n\alpha_n = d\xi$. Then $\Omega^1_{A/\mathbf{Z}_p}[p^n]$ is generated as an A/p^nA-module by the image of α_n under the map $\theta : \Omega^1_{W(A^\flat)/\mathbf{Z}_p} \to \Omega^1_{A/\mathbf{Z}_p}$.*

Proof. Uniqueness of the element α_n follows from Lemma 3.6. From the ring homomorphisms $\mathbf{Z}_p \to W(A^\flat) \xrightarrow{\theta} A$, the Jacobi-Zariski sequence (2.2) associates an exact triangle in the derived category $D(A)$

$$L_{W(A^\flat)/\mathbf{Z}_p} \otimes^L_{W(A^\flat)} A \to L_{A/\mathbf{Z}_p} \to L_{A/W(A^\flat)}.$$

From the associated long exact sequence in cohomology, using that θ is surjective with kernel generated by a non-zero divisor (so $\ker(\theta)/\ker(\theta)^2 \cong A$), there is an exact sequence

$$\cdots \longrightarrow A \longrightarrow \Omega^1_{W(A^\flat)/\mathbf{Z}_p} \otimes_{W(A^\flat)} A \longrightarrow \Omega^1_{A/\mathbf{Z}_p} \longrightarrow 0.$$

We then form a double complex

$$\begin{array}{ccccccc} \cdots \longrightarrow & A & \longrightarrow & \Omega^1_{W(A^\flat)/\mathbf{Z}_p} \otimes_{W(A^\flat)} A & \longrightarrow & \Omega^1_{A/\mathbf{Z}_p} & \longrightarrow 0 \\ & \uparrow{\scriptstyle p^n} & & \uparrow{\scriptstyle -p^n} & & \uparrow{\scriptstyle p^n} & \\ \cdots \longrightarrow & A & \longrightarrow & \Omega^1_{W(A^\flat)/\mathbf{Z}_p} \otimes_{W(A^\flat)} A & \longrightarrow & \Omega^1_{A/\mathbf{Z}_p} & \longrightarrow 0. \end{array}$$

The horizontal maps $A \to \Omega^1_{W(A^\flat)/\mathbf{Z}_p} \otimes_{W(A^\flat)} A$ send $1 \mapsto d\xi \otimes 1$ (see for example (Matsumura, 1989, Theorem 25.2)). Multiplication by p^n is an isomorphism on $\Omega^1_{W(A^\flat)/\mathbf{Z}_p}$ (by Lemma 3.6) and hence also on $\Omega^1_{W(A^\flat)/\mathbf{Z}_p} \otimes_{W(A^\flat)} A$. Considering the two spectral sequences associated to this double complex, we must have a surjective map $A/p^nA \to \Omega^1_{A/\mathbf{Z}_p}[p^n]$ given by $1 \mapsto \theta(\alpha_n)$. This completes the proof. ■

Corollary 3.8. *Let A denote a p-torsion-free perfectoid ring and assume furthermore that A contains a compatible system of p-power roots of unity ζ_{p^n}. There exists an element $\alpha \in \Omega^1_{A/\mathbf{Z}_p}$ such that $(\zeta_p - 1)\alpha = \mathrm{d}\log\zeta_p$, and any such element is p-torsion and freely generates $\Omega^1_{A/\mathbf{Z}_p}[p]$ as an A/pA-module.*

Proof. We first prove that there is at least one such element α satisfying all the listed conditions, and then prove that any element $\alpha \in \Omega^1_{A/\mathbf{Z}_p}$ satisfying $(\zeta_p - 1)\alpha = d\log\zeta_p$ is automatically p-torsion and a free A/pA-module generator of $\Omega^1_{A/\mathbf{Z}_p}[p]$.

Let $\varepsilon \in A^\flat$ consist of the p-power compatible system $(\zeta_{p^n})_{n\geq 0}$. One generator of $\ker\theta$ is $\xi := 1 + [\varepsilon^{1/p}] + [\varepsilon^{1/p}]^2 + \cdots + [\varepsilon^{1/p}]^{p-1}$ (see (Bhatt et al., 2018, Example 3.16)). Define

$$\alpha_1 := \sum_{m=1}^{p-1} m[\varepsilon^{1/p}]^m \, d\log[\varepsilon^{1/p^2}] \in \Omega^1_{W(A^\flat)/\mathbf{Z}_p}.$$

Note that $p\alpha_1 = d\xi$. By Proposition 3.7, we know that $\theta(\alpha_1) \in \Omega^1_{A/\mathbf{Z}_p}$ freely generates $\Omega^1_{A/\mathbf{Z}_p}[p]$ as an A/pA-module. On the other hand,

$$\theta(\alpha_1) = \sum_{m=1}^{p-1} m\zeta_p^m \, d\log\zeta_{p^2} \in \Omega^1_{A/\mathbf{Z}_p}.$$

Because
$$(\zeta_p - 1)\sum_{m=1}^{p-1} m\zeta_p^m = p,$$
this shows that the element $\theta(\alpha_1)$ satisfies all the conditions of the statement.

Now let $\alpha \in \Omega^1_{A/\mathbf{Z}_p}$ denote an arbitrary element which satisfies $(\zeta_p - 1)\alpha = d\log\zeta_p$. From
$$p\alpha = \frac{p}{\zeta_p - 1}(\zeta_p - 1)\alpha = \frac{p}{\zeta_p - 1}(\zeta_p - 1)\theta(\alpha_1) = p\theta(\alpha_1) = 0,$$
we see that α is p-torsion. Because $\theta(\alpha_1)$ generates the p-torsion, we have $\alpha = a\theta(\alpha_1)$ for some $a \in A$, and we must furthermore have
$$(\zeta_p - 1)a \equiv \zeta_p - 1 \bmod pA,$$
because $(\zeta_p - 1)a\theta(\alpha_1) = (\zeta_p - 1)\theta(\alpha_1)$. By Lemma 2.29, we then have that a is a unit in A/pA, which completes the proof. ■

In Theorem B, in addition to considering p-torsion-free perfectoid rings, we also consider rings of integers in finite extensions of $\mathbf{Q}_p$. The following algebraic result enables us to relate such rings to the perfectoid ring $\mathcal{O}_{\mathbf{C}_p}$.

Proposition 3.9. *Let K denote an algebraic extension of $\mathbf{Q}_p$ and let $\overline{K}$ denote an algebraic closure of K.*

(1) The natural map
$$\Omega^1_{\mathcal{O}_K/\mathbf{Z}_p} \to \Omega^1_{\mathcal{O}_{\overline{K}}/\mathbf{Z}_p}$$
is injective.

(2) For every integer $n \geq 1$, the natural map
$$\Omega^1_{\mathcal{O}_{\overline{K}}/\mathbf{Z}_p}[p^n] \to \Omega^1_{\mathcal{O}_{\mathbf{C}_p}/\mathbf{Z}_p}[p^n]$$
is an isomorphism.

Proof. We know $H^{-1}(L_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}) \cong 0$ by Proposition 2.6, so the map
$$\Omega^1_{\mathcal{O}_K/\mathbf{Z}_p} \otimes_{\mathcal{O}_K} \mathcal{O}_{\overline{K}} \hookrightarrow \Omega^1_{\mathcal{O}_{\overline{K}}/\mathbf{Z}_p}$$
is injective by the Jacobi-Zariski sequence (2.2). We know $\mathcal{O}_K \to \mathcal{O}_{\overline{K}}$ is faithfully flat by Proposition 2.9, so the map $\Omega^1_{\mathcal{O}_K/\mathbf{Z}_p} \to \Omega^1_{\mathcal{O}_K/\mathbf{Z}_p} \otimes_{\mathcal{O}_K} \mathcal{O}_{\overline{K}}$ is injective by (Matsumura, 1989, Theorem 7.5). Thus the composite of these two maps $\Omega^1_{\mathcal{O}_K/\mathbf{Z}_p} \to \Omega^1_{\mathcal{O}_{\overline{K}}/\mathbf{Z}_p}$ is also injective. This completes the proof of the first statement.

We now prove the second statement. Let R denote either of the rings $\mathcal{O}_{\overline{K}}$ or $\mathcal{O}_{\mathbf{C}_p}$. In both cases, $H^{-1}(L_{R/\mathbf{Z}_p}) \cong 0$, again by Proposition 2.6. By Proposition 2.3, in both cases we have
$$H^{-1}\Big(L_{R/\mathbf{Z}_p} \otimes^L_{\mathbf{Z}_p} (\mathbf{Z}/p^n\mathbf{Z})\Big) \cong \Omega^1_{R/\mathbf{Z}_p}[p^n].$$
By flat base change, we have
$$H^{-1}\Big(L_{R/\mathbf{Z}_p} \otimes^L_{\mathbf{Z}_p} (\mathbf{Z}/p^n\mathbf{Z})\Big) \cong H^{-1}(L_{(R/p^nR)/(\mathbf{Z}/p^n\mathbf{Z})});$$

see for example (The Stacks Project Authors, 2017, Tag 08QQ). The second result now follows from the fact that

$$\mathcal{O}_{\overline{K}}/p^n\mathcal{O}_{\overline{K}} \cong \mathcal{O}_{\mathbf{C}_p}/p^n\mathcal{O}_{\mathbf{C}_p}.$$

■

This concludes our treatment of the relative Kähler differentials, $\Omega^1_{A/\mathbf{Z}_p}$. We are ultimately interested in the p-power torsion in the module of absolute Kähler differentials, $\Omega^1_A := \Omega^1_{A/\mathbf{Z}}$. Although Ω^1_A is much larger than $\Omega^1_{A/\mathbf{Z}_p}$, their p^n-torsion modules are isomorphic.

Lemma 3.10. *Let A denote a p-torsion-free $\mathbf{Z}_p$-algebra for which the multiplication-by-p map*

$$H^{-1}(L_{A/\mathbf{Z}_p}) \xrightarrow{p} H^{-1}(L_{A/\mathbf{Z}_p})$$

is surjective. (For example, A could be any p-torsion-free perfectoid ring; see Theorem 3.5.) Then for every integer $n \geq 1$, the natural map

$$\Omega^1_A[p^n] \to \Omega^1_{A/\mathbf{Z}_p}[p^n]$$

is an isomorphism.

Proof. Consider the double complex

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & H^{-1}(L_{A/\mathbf{Z}_p}) & \longrightarrow & \Omega^1_{\mathbf{Z}_p/\mathbf{Z}} \otimes_{\mathbf{Z}_p} A & \longrightarrow & \Omega^1_{A/\mathbf{Z}} & \longrightarrow & \Omega^1_{A/\mathbf{Z}_p} & \longrightarrow & 0 \\
 & {\scriptstyle p^n}\big\uparrow & & {\scriptstyle -p^n}\big\uparrow & & {\scriptstyle p^n}\big\uparrow & & {\scriptstyle -p^n}\big\uparrow & & \\
\cdots \longrightarrow & H^{-1}(L_{A/\mathbf{Z}_p}) & \longrightarrow & \Omega^1_{\mathbf{Z}_p/\mathbf{Z}} \otimes_{\mathbf{Z}_p} A & \longrightarrow & \Omega^1_{A/\mathbf{Z}} & \longrightarrow & \Omega^1_{A/\mathbf{Z}_p} & \longrightarrow & 0.
\end{array}$$

Each of the four displayed vertical maps is surjective, and in fact, multiplication-by-p is an isomorphism on $\Omega^1_{\mathbf{Z}_p/\mathbf{Z}} \otimes_{\mathbf{Z}_p} A$ (see for example (Hesselholt and Madsen, 2003, Lemma 2.2.4) or (Davis, 2019, Proposition 2.7)). Because the rows of this complex are exact, the two spectral sequences associated to this double complex must both converge to zero. Consider the spectral sequence attained by first taking cohomology along the columns. The E_1-page of this spectral sequence will be

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \Omega^1_{A/\mathbf{Z}}/p^n\Omega^1_{A/\mathbf{Z}} & \longrightarrow & \Omega^1_{A/\mathbf{Z}_p}/p^n\Omega^1_{A/\mathbf{Z}_p} & \longrightarrow & 0 \\
 & & & & & & & & & \\
\cdots \longrightarrow & H^{-1}(L_{A/\mathbf{Z}_p})[p^n] & \longrightarrow & 0 & \longrightarrow & \Omega^1_{A/\mathbf{Z}}[p^n] & \longrightarrow & \Omega^1_{A/\mathbf{Z}_p}[p^n] & \longrightarrow & 0.
\end{array}$$

Because of the two zeros at the top-left of the diagram, we deduce that $\Omega^1_{A/\mathbf{Z}}[p^n] \to \Omega^1_{A/\mathbf{Z}_p}[p^n]$ is an isomorphism, as required. ■

We end this section with the following corollary. It is the key result used in the base case of our inductive proof of Theorem C.

Corollary 3.11. *Assume A is a p-torsion-free perfectoid ring containing a compatible system of p-power roots of unity. There exists an element $\alpha \in \Omega^1_A$ satisfying $(\zeta_p - 1)\alpha = \mathrm{d}\log\zeta_p$, and given any such element, it is p-torsion and the map $A/pA \xrightarrow{\alpha} \Omega^1_A[p]$ given by $a \mapsto a\alpha$ is an isomorphism of A-modules. Writing $A_0 := \mathbf{Z}_p[\zeta_{p^\infty}]^\wedge$, we may furthermore find one such α in the image of the natural map $\Omega^1_{A_0} \to \Omega^1_A$.*

Proof. Consider any $\alpha \in \Omega^1_A$ satisfying $(\zeta_p - 1)\alpha = d\log\zeta_p$. (For example, we can take $\alpha = \sum_{m=1}^{p-1} m\zeta_p^m \, d\log\zeta_{p^2} = \frac{p}{\zeta_p - 1} d\log\zeta_{p^2}$.) We do not yet know that such an element is p-torsion, but $p\alpha$ is certainly p-torsion, since $p\, d\log\zeta_p = d\log 1 = 0$. By Lemma 3.10, we know that the map $\Omega^1_A[p] \to \Omega^1_{A/\mathbf{Z}_p}[p]$ is injective. Under this map, $p\alpha \mapsto 0 \in \Omega^1_{A/\mathbf{Z}_p}[p]$ by Corollary 3.8, so $p\alpha = 0 \in \Omega^1_A[p]$, so α is p-torsion in Ω^1_A.

The composition

$$A/pA \xrightarrow{\alpha} \Omega^1_A[p] \to \Omega^1_{A/\mathbf{Z}_p}[p]$$

is an isomorphism by Corollary 3.8. The second map in this composition is an isomorphism by Lemma 3.10, and hence the first map is also an isomorphism.

Our example of α at the beginning of the proof was in the image of $\Omega^1_{A_0} \to \Omega^1_A$, which justifies the final assertion of the corollary, i.e., we can find such an α which is in the image of $\Omega^1_{A_0}$. ∎

4. Proof of Theorem B and Corollary B.2

Most of the exactness asserted in Corollary B.2 holds in much more generality than the cases covered by Corollary B.2, thanks to the following result of Hesselholt and Madsen. (We state the result for the de Rham-Witt complex, whereas their result concerns the logarithmic de Rham-Witt complex. Notice that the de Rham-Witt complex is a special case of the logarithmic de Rham-Witt complex, attained by taking the trivial monoid $M = \{1\}$ when specifying the log ring (A, M).)

Proposition 4.1 ((Hesselholt and Madsen, 2003, Proposition 3.2.6))**.** *Let A denote a p-torsion-free $\mathbf{Z}_{(p)}$-algebra. For a fixed integer $n \geq 1$, recall the sequence (1.3) of $W_{n+1}(A)$-modules*

$$0 \to A \xrightarrow{(-d, p^n)} \Omega^1_A \oplus A \xrightarrow{V^n \oplus dV^n} W_{n+1}\Omega^1_A \xrightarrow{R} W_n\Omega^1_A \to 0,$$

where the module structure is the same as in Corollary B.2. This sequence is exact at all slots, except that possibly the segment $A \xrightarrow{(-d,p^n)} \Omega^1_A \oplus A \xrightarrow{V^n \oplus dV^n} W_{n+1}\Omega^1_A$ is not exact. Furthermore, we have $\operatorname{im}\big(-d, p^n\big) \subseteq \ker\big(V^n \oplus dV^n\big)$.

The statement of Theorem B is the inclusion $\ker\big(V^n \oplus dV^n\big) \subseteq \operatorname{im}\big(-d, p^n\big)$.

The results in the previous section immediately imply Part (1) of Theorem B and Corollary B.2. With a little more effort, we will use Part (1) to deduce Part (2) of Theorem B.

Proof of Theorem B and Corollary B.2, Part (1). It was shown in (Davis, 2019, Section 6) that the sequence (1.3) is exact if A is a p-torsion-free perfectoid ring and, moreover, there exists a p-torsion element $\alpha \in \Omega^1_A$ with annihilator equal to pA. Theorem 3.5 and Lemma 3.10 guarantee the existence of such an element α. ∎

We now prove Part (2) of Theorem B and Corollary B.2. A version of this result for the log de Rham-Witt complex can be found in (Hesselholt and Madsen, 2003, Proposition 3.2.6 and Proof of Theorem 3.3.8).

The sequence (1.3) is exact for $B = \mathcal{O}_{\mathbf{C}_p}$ by Part (1), because $\mathcal{O}_{\mathbf{C}_p}$ is a p-torsion-free perfectoid ring. To prove exactness for $A = \mathcal{O}_K$, where K is an algebraic extension of $\mathbf{Q}_p$, we will use the following result.

Proposition 4.2. *Let A denote a p-torsion-free $\mathbf{Z}_{(p)}$-algebra such that there exists a p-torsion-free ring $B \supseteq A$ with the following properties.*

(1) The sequence (1.3) is exact for the ring B for all integers $n \geq 1$.

(2) For all integers $n \geq 1$, we have $A \cap p^n B = p^n A$.
(3) For all integers $n \geq 1$, the natural map $\Omega^1_A[p^n] \to \Omega^1_B[p^n]$ is injective.

Then the sequence (1.3) is also exact for the ring A for all integers $n \geq 1$.

Proof. We need to show that if $\alpha \in \Omega^1_A$ and $a \in A$ are such that $V^n(\alpha) + dV^n(a) = 0 \in W_{n+1}\Omega^1_A$, then there exists $a_0 \in A$ such that $p^n a_0 = a$ and $-da_0 = \alpha$. By exactness of the sequence for B, there exists an element $b_0 \in B$ such that $p^n b_0 = a$ and $-db_0 = \alpha \in \Omega^1_B$. By our assumption that $A \cap p^n B = p^n A$, we deduce that there at least exists $a_1 \in A$ such that $p^n a_1 = a$; we will be finished after we show that $-da_1 = \alpha \in \Omega^1_A$.

We know

$$V^n(\alpha) + dV^n(a) = 0 \in W_{n+1}\Omega^1_A$$

and

$$V^n(-da_1) + dV^n(p^n a_1) = 0 \in W_{n+1}\Omega^1_A,$$

and because $p^n a_1 = a$, we have

$$V^n(\alpha + da_1) = 0 \in W_{n+1}\Omega^1_A.$$

Applying F^n to both sides of this last equation, we have that $\alpha + da_1 \in \Omega^1_A[p^n]$. We also have $V^n(\alpha + da_1) = 0 \in W_{n+1}\Omega^1_B$. Because V^n is injective on Ω^1_B by Proposition 1.7, we have that $\alpha + da_1 = 0 \in \Omega^1_B$. Thus the element $\alpha + da_1$ is simultaneously p^n-torsion in Ω^1_A and is also in the kernel of $\Omega^1_A \to \Omega^1_B$. By our assumption (3), we have that $\alpha + da_1 = 0 \in \Omega^1_A$, as required. ■

We can now prove Part (2) of Theorem B and Corollary B.2.

Proof of Theorem B and Corollary B.2, Part (2). Let $A = \mathcal{O}_K$ and $B = \mathcal{O}_{\mathbf{C}_p}$. It suffices show that the conditions of Proposition 4.2 are satisfied for this choice of A and B. Because B is a p-torsion-free perfectoid ring, we saw at the beginning of this section that the sequence (1.3) is exact for B. We also have $\mathcal{O}_K \cap p^n \mathcal{O}_{\mathbf{C}_p} = p^n \mathcal{O}_K$ (for example, by considering the valuations on $\mathcal{O}_K$ and $\mathcal{O}_{\mathbf{C}_p}$).

We now verify Condition (3). First note that $H^{-1}(L_{A/\mathbf{Z}_p}) \cong 0$ and $H^{-1}(L_{B/\mathbf{Z}_p}) \cong 0$ by Proposition 2.6. Although our desired statement concerns absolute Kähler differentials, by Lemma 3.10, it suffices to show that for every integer $n \geq 1$, the map

$$\Omega^1_{A/\mathbf{Z}_p}[p^n] \to \Omega^1_{B/\mathbf{Z}_p}[p^n]$$

is injective. This was proved in Proposition 3.9. ■

We end this section with an application of exactness from Corollary B.2. (Two other applications were given in the introduction; see Proposition 1.5 and Proposition 1.7.) The following result gives conditions on rings $A \to B$ under which the induced map $W_n\Omega^1_A \to W_n\Omega^1_B$ is injective.

Corollary 4.3. *Let $A \subseteq B$ be p-torsion-free rings, and assume the following conditions are met.*
(1) For all integers $n \geq 1$, we have $A \cap p^n B = p^n A$.
(2) For all integers $n \geq 1$, the sequence (1.3) is exact for both A and B.
(3) The induced map $\Omega^1_A \to \Omega^1_B$ is injective.

Then for all integers $n \geq 1$, the induced map

$$W_n\Omega^1_A \to W_n\Omega^1_B$$

is injective.

Proof. We prove this using induction on the level n. The base case $n = 1$ is precisely condition (3). Now assume the result holds for some fixed value of n. Let $h_n : A \to \Omega^1_A \oplus A$ be given by $h_n(a) = (-da, p^n a)$, and similarly for the ring B. Consider the double-complex of $W_{n+1}(A)$-modules arising from the sequences (1.3),

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \left(\Omega^1_B \oplus B\right)/h_n(B) & \longrightarrow & W_{n+1}\Omega^1_B & \longrightarrow & W_n\Omega^1_B & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \left(\Omega^1_A \oplus A\right)/h_n(A) & \longrightarrow & W_{n+1}\Omega^1_A & \longrightarrow & W_n\Omega^1_A & \longrightarrow & 0. \end{array}$$

These are short exact sequences because we are assuming the sequence (1.3) is exact for both A and B. From our induction hypothesis, we know the right-hand vertical map is injective. The horizontal sequences are exact, so by the snake lemma, it suffices to show that the left-hand vertical map is injective. Assume $(\omega, a) \in \Omega^1_A \oplus A$ maps to an element $(-db, p^n b) \in \Omega^1_B \oplus B$. Thus the element a is in $A \cap p^n B$, so there exists $a_0 \in A$ such that $a = p^n a_0 = p^n b$. Because B is p-torsion-free, we know that in fact $a_0 = b$. Thus the differentials $\omega, -da_0 \in \Omega^1_A$ are equal in Ω^1_B. Because the map $\Omega^1_A \to \Omega^1_B$ is injective by our assumption, in fact $\omega = -da_0 \in \Omega^1_A$. Thus $(\omega, a) = 0 \in \left(\Omega^1_A \oplus A\right)/h_n(A)$, so the left-hand vertical map is injective, as required. ■

Example 4.4. These conditions of Corollary 4.3 are not so easy to verify in practice, but only because it is difficult to verify the "base case" of level $n = 1$, i.e., that $\Omega^1_A \to \Omega^1_B$ is injective. (That is clearly a necessary condition.) The conditions are satisfied, for example, for the rings $A = \mathcal{O}_K$, $B = \mathcal{O}_L$, when $\mathbf{Q}_p \subseteq K \subseteq L$ is a tower of algebraic extensions. In particular, Condition (3) in this case follows from Proposition 3.9.

5. On p-power-torsion in the de Rham-Witt complex

This section contains the main results of the paper, including the proof of Theorem C from the introduction. These results are valid for a ring A satisfying the following assumptions.

Notation 5.1. Let A denote a p-torsion-free perfectoid ring which contains a sequence $(1, \zeta_p, \zeta_{p^2}, \ldots)$ of p^n-th roots of unity, compatible in the sense that $\zeta_{p^{n+1}}^p = \zeta_{p^n}$ for each integer $n \geq 1$, and where ζ_p is a primitive p-th root of unity in the sense that ζ_p satisfies $1 + \zeta_p + \cdots + \zeta_p^{p-1} = 0$. We fix a choice of such elements. Let $z_n := 1 + [\zeta_{p^n}] + \cdots + [\zeta_{p^n}^{p-1}] \in W_n(A)$.

Remark 5.2. We list here a few observations related to Notation 5.1. First of all, for A as in Notation 5.1, the choices of ζ_{p^n} for $n \geq 1$ determine a preferred ring homomorphism $A_0 \to A$, where $A_0 := \mathbf{Z}_p[\zeta_{p^\infty}]^\wedge$. Because A is p-torsion-free, that map $A_0 \to A$ is injective, and so A_0 may be considered as a subring of A. Our formulation of "primitive" p-th root of unity is taken from (Bhatt et al., 2018, Example 3.16); note that it is a strictly stronger requirement than requiring $\zeta_p^p = 1$ and $\zeta_p \neq 1$. For example, $(\zeta_p, 1) \in A_0 \times A_0$ satisfies this latter condition but not the condition of Notation 5.1. Lastly, it is convenient to notice that $z_n = \frac{[\zeta_{p^{n-1}}]-1}{[\zeta_{p^n}]-1} \in W_n(A)$. This latter formulation is well-defined because $[\zeta_{p^n}] - 1$ is a non-zero-divisor in $W_n(A)$, as can be seen by considering ghost components and using that A is p-torsion-free.

This section concerns p-torsion in $W_n\Omega^1_A$, for rings A satisfying Notation 5.1 above. Before considering p-torsion, we consider the image of multiplication by p; this image can be analyzed much more easily and in more generality. We say a ring A is *Witt-perfect* (at p) if for every integer $n \geq 1$, the

Witt vector Frobenius $F : W_{n+1}(A) \to W_n(A)$ is surjective; see (Davis and Kedlaya, 2014, Section 5). All perfectoid rings are Witt-perfect, by (Bhatt et al., 2018, Lemma 3.9(iv)), but Witt-perfect rings are not required to be p-adically complete.

Proposition 5.3. *Let A denote a Witt-perfect ring. (For example, as explained above, the ring A could be a perfectoid ring.) Then for all integers $n \geq 1$ and $d \geq 1$, the multiplication-by-p map,*

$$p : W_n\Omega^d_A \to W_n\Omega^d_A,$$

is surjective.

Proof. Note that any element in $W_n\Omega^d_A$ can be written as a sum of terms $x dy_1 \cdots dy_d$, where $x, y_i \in W_n(A)$. Because A is Witt-perfect, we can find $y_1' \in W_{n+1}(A)$ such that $F(y_1') = y_1$. But then we have

$$x dy_1 dy_2 \cdots dy_d = x dF(y_1') dy_2 \cdots dy_d = px F(dy_1') dy_2 \cdots dy_d.$$

This shows that every element in $W_n\Omega^d_A$ is a multiple of p. ∎

Our arguments in this section analyze the structure of $W_n\Omega^1_A[p]$ using induction on the level, n. We have $W_1\Omega^1_A \cong \Omega^1_A$ (for every $\mathbf{Z}_{(p)}$-algebra A, see (Hesselholt and Madsen, 2004, Theorem D and the first sentence of the proof of Proposition 5.1.1)), and so the base case of our induction will rely heavily on the results from Section 3. To relate levels n and $n+1$, we use Corollary B.2, as in the proof of the following result. Our original motivation for considering Theorem B was to enable these sorts of arguments.

Proposition 5.4. *Let A denote any p-torsion-free perfectoid ring and let $r \geq 1$ be an integer.*

(1) We have an exact sequence of $W_{n+1}(A)$-modules,

$$W_{n+1}\Omega^1_A[p^r] \xrightarrow{R} W_n\Omega^1_A[p^r] \to A/p^{\min(r,n)}A \to 0,$$

where the $W_{n+1}(A)$-module structure on $A/p^{\min(r,n)}A$ is induced by F^n and where the $W_{n+1}(A)$-module structure on $W_n\Omega^1_A[p^r]$ is induced by restriction.

(2) Set $N^r := \ker\left(R : W_{n+1}\Omega^1_A[p^r] \to W_n\Omega^1_A[p^r]\right)$. We have an exact sequence of $W_{n+1}(A)$-modules,

$$0 \to \Omega^1_A[p^r] \xrightarrow{V^n} N^r \to p^{\max(r-n,0)}A/p^rA \to 0,$$

where the $W_{n+1}(A)$-module structures on $\Omega^1_A[p^r]$ and on $p^{\max(r-n,0)}A/p^rA$ are induced by F^n.

Proof. Consider the double complex of $W_{n+1}(A)$-modules

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & A & \xrightarrow{(-d,p^n)} & \Omega^1_A \oplus A & \xrightarrow{(V^n,dV^n)} & W_{n+1}\Omega^1_A & \xrightarrow{R} & W_n\Omega^1_A & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle -p^r} & & \uparrow{\scriptstyle p^r} & & \uparrow{\scriptstyle -p^r} & & \uparrow{\scriptstyle p^r} & & \\
0 & \longrightarrow & A & \xrightarrow{(-d,p^n)} & \Omega^1_A \oplus A & \xrightarrow{(V^n,dV^n)} & W_{n+1}\Omega^1_A & \xrightarrow{R} & W_n\Omega^1_A & \longrightarrow & 0.
\end{array}$$

Because the rows are exact by Corollary B.2, both spectral sequences associated to this double complex must converge to 0.

Consider the spectral sequence with E_1 page attained by taking cohomology along the columns. Using in particular Proposition 5.3, the E_2 page of this spectral sequence has the following form:

$$\begin{array}{cccccc} 0 & p^{\max(r-n,0)}A/p^rA & A/p^{\min(r,n)}A & 0 & 0 & 0 \\ & \searrow & \searrow & \searrow & & \\ 0 & 0 & \ker(V^n) & N^r/(\operatorname{im} V^n) & W_n\Omega^1_A[p^r]/R(W_{n+1}\Omega^1_A[p^r]) & 0. \end{array}$$

All these d_2 maps must be isomorphisms of $W_{n+1}(A)$-modules, and so the results follow. ∎

A consequence of the last proof is the following. A key observation is the similarity between the exact sequence appearing in Proposition 5.5 and the exact sequence from Lemma 2.32.

Proposition 5.5. *Let A denote a p-torsion-free perfectoid ring. For every integer $n \geq 1$, we have an exact sequence of $W_{n+1}(A)$-modules*

$$0 \to T_p(\Omega^1_A) \xrightarrow{V^n} T_p(W_{n+1}\Omega^1_A) \xrightarrow{R} T_p(W_n\Omega^1_A) \xrightarrow{g} A/p^nA \to 0,$$

where the $W_{n+1}(A)$-module structure on $T_p(W_n\Omega^1_A)$ is induced by restriction, and where the $W_{n+1}(A)$-module structures on $T_p(\Omega^1_A)$ and A/p^nA are induced by F^n. (We do not specify the map g.)

Proof. We use the exact sequences from Proposition 5.4 in the case $r \geq n$. The double complex used to obtain the exact sequences for p^{r+1} maps to the double complex for p^r via the identity on the top row and p on the bottom row. This induces a map of spectral sequences and hence we get commutative diagrams with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N^r & \longrightarrow & W_{n+1}\Omega^1_A[p^r] & \xrightarrow{R} & W_n\Omega^1_A[p^r] & \longrightarrow & A/p^nA & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle p} & & \uparrow{\scriptstyle p} & & \uparrow{\scriptstyle p} & & \| & & \\ 0 & \longrightarrow & N^{r+1} & \longrightarrow & W_{n+1}\Omega^1_A[p^{r+1}] & \xrightarrow{R} & W_n\Omega^1_A[p^{r+1}] & \longrightarrow & A/p^nA & \longrightarrow & 0 \end{array}$$

and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^1_A[p^r] & \xrightarrow{V^n} & N^r & \longrightarrow & A/p^nA & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle p} & & \uparrow{\scriptstyle p} & & \uparrow{\scriptstyle p} & & \\ 0 & \longrightarrow & \Omega^1_A[p^{r+1}] & \xrightarrow{V^n} & N^{r+1} & \longrightarrow & A/p^nA & \longrightarrow & 0. \end{array}$$

From this last diagram we obtain an isomorphism

$$T_p(\Omega^1_A) \xrightarrow{V^n} \varprojlim_r N^r.$$

Furthermore, the maps $\Omega^1_A[p^{r+1}] \xrightarrow{p} \Omega^1_A[p^r]$ are surjective and the tower

$$\cdots \xrightarrow{p} A/p^nA \xrightarrow{p} A/p^nA$$

satisfies the Mittag-Leffler condition, so using the $\varprojlim$-$\varprojlim^1$ exact sequence (see for example (Weibel, 1994, Section 3.5 and Proposition 3.5.7)), we deduce that $\varprojlim^1 N^r = 0$.

Let I^r denote the image

$$\operatorname{im}\left(R : W_{n+1}\Omega^1_A[p^r] \to W_n\Omega^1_A[p^r]\right).$$

Because multiplication by p maps $W_{n+1}\Omega^1_A[p^{r+1}]$ surjectively onto $W_{n+1}\Omega^1_A[p^r]$ (using Proposition 5.3), it follows immediately that multiplication by p maps I^{r+1} surjectively onto I^r. We deduce for later that $\varprojlim^1_r I^r = 0$. These $\varprojlim^1$ computations will be used below.

We can split the first commutative diagram above into two commutative diagrams with exact rows:

$$\begin{CD} 0 @>>> N^r @>>> W_{n+1}\Omega^1_A[p^r] @>R>> I^r @>>> 0 \\ @. @AApA @AApA @AApA @. \\ 0 @>>> N^{r+1} @>>> W_{n+1}\Omega^1_A[p^{r+1}] @>R>> I^{r+1} @>>> 0 \end{CD}$$

and

$$\begin{CD} 0 @>>> I^r @>>> W_n\Omega^1_A[p^r] @>>> A/p^nA @>>> 0 \\ @. @AApA @AApA @| @. \\ 0 @>>> I^{r+1} @>>> W_n\Omega^1_A[p^{r+1}] @>>> A/p^nA @>>> 0. \end{CD}$$

Altogether we obtain two short exact sequences of inverse systems. Taking into account that $\varprojlim^1_r N^r = 0$ and $\varprojlim^1_r I^r = 0$ as remarked above, again using the $\varprojlim$-$\varprojlim^1$ exact sequence, we obtain exact sequences

$$0 \longrightarrow \varprojlim_r N^r \longrightarrow T_pW_{n+1}\Omega^1_A \xrightarrow{R} \varprojlim_r I^r \longrightarrow 0$$

and

$$0 \longrightarrow \varprojlim_r I^r \longrightarrow T_pW_n\Omega^1_A \xrightarrow{g} A/p^nA \longrightarrow 0.$$

Now splicing these exact sequences together and using the isomorphism $T_p(\Omega^1_A) \xrightarrow{V^n} \varprojlim_r N^r$, we obtain the desired exact sequence

$$0 \longrightarrow T_p(\Omega^1_A) \xrightarrow{V^n} T_pW_{n+1}\Omega^1_A \xrightarrow{R} T_pW_n\Omega^1_A \xrightarrow{g} A/p^nA \longrightarrow 0.$$

■

Lemma 5.6. *Let A denote any p-torsion-free $\mathbf{Z}_{(p)}$-algebra for which the sequence (1.3) is exact and let $n \geq 1$ be an integer. We have an inclusion $W_{n+1}\Omega^1_A[p] \cap \ker R \subseteq \ker F$. In particular, if A is a p-torsion-free perfectoid ring and if $x, y \in T_p(W_{n+1}\Omega^1_A)$ are such that $R(x) \equiv R(y) \bmod pT_p(W_n\Omega^1_A)$, then $F(x) \equiv F(y) \bmod pT_p(W_n\Omega^1_A)$.*

Proof. Let $x \in W_{n+1}\Omega^1_A[p] \cap \ker R$. The fact that $x \in \ker R$ implies, using exactness of (1.3), that there exist $\alpha \in \Omega^1_A$ and $a \in A$ such that

$$x = V^n(\alpha) + dV^n(a).$$

To prove $W_{n+1}\Omega^1_A[p] \cap \ker R \subseteq \ker F$, we wish to show that if $px = 0$, then $F(x) = 0$. Using standard identities within the de Rham-Witt complex, we wish to show that if $px = 0$, then $V^{n-1}(p\alpha) + dV^{n-1}(a) = 0$. We compute

$$0 = px = V^n(p\alpha) + dV^n(pa) = V^n(p\alpha) + V\left(dV^{n-1}(a)\right) = V\Big(V^{n-1}(p\alpha) + dV^{n-1}(a)\Big).$$

Because V is injective by Proposition 1.7,

$$0 = V^{n-1}(p\alpha) + dV^{n-1}(a).$$

This proves the first assertion. The second assertion about Tate modules follows directly because, as in the proof of Theorem 3.5, we have $T_p(W_{n+1}\Omega^1_A)/pT_p(W_{n+1}\Omega^1_A) \cong W_{n+1}\Omega^1_A[p]$. ∎

We will soon prove that $T_p(W_n\Omega^1_A)$ is a free $W_n(A)$-module of rank one for rings A as in Notation 5.1. It turns out there is essentially no difference between proving this and proving that the p-torsion $W_n\Omega^1_A[p]$ is a free $W_n(A)/pW_n(A)$-module of rank one. This is the content of the following lemma.

Lemma 5.7. *Let A be a p-torsion-free perfectoid ring, and let $n \geq 1$ be an integer. If $\alpha \in T_p(W_n\Omega^1_A)$ freely generates $T_p(W_n\Omega^1_A)$ as a $W_n(A)$-module, then the projection of α to $W_n\Omega^1_A[p^r]$, written $\alpha^{(r)}$, freely generates $W_n\Omega^1_A[p^r]$ as a $W_n(A)/p^rW_n(A)$-module. Conversely, if $\alpha \in T_p(W_n\Omega^1_A)$ is such that $\alpha^{(1)}$ freely generates $W_n\Omega^1_A[p]$ as a $W_n(A)/pW_n(A)$-module, then α freely generates $T_p(W_n\Omega^1_A)$ as a $W_n(A)$-module.*

Proof. The first statement follows from the short exact sequence

$$0 \to T_p(W_n\Omega^1_A) \xrightarrow{p^r} T_p(W_n\Omega^1_A) \to W_n\Omega^1_A[p^r] \to 0.$$

(See the proof of Theorem 3.5.) The second statement follows by showing, level-by-level, that multiplication by $\alpha^{(r)}$ induces an isomorphism of $W_n(A)/p^rW_n(A)$-modules $W_n(A)/p^rW_n(A) \to W_n\Omega^1_A[p^r]$, using induction and the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & W_n\Omega^1_A[p] & \longrightarrow & W_n\Omega^1_A[p^r] & \xrightarrow{p} & W_n\Omega^1_A[p^{r-1}] & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \alpha^{(1)}} & & \uparrow{\scriptstyle \alpha^{(r)}} & & \uparrow{\scriptstyle \alpha^{(r-1)}} & & \\
0 & \longrightarrow & W_n(A)/pW_n(A) & \xrightarrow{p^{r-1}} & W_n(A)/p^rW_n(A) & \longrightarrow & W_n(A)/p^{r-1}W_n(A) & \longrightarrow & 0.
\end{array}$$

(Notice that $W_n(A)$ is p-torsion-free because A is p-torsion-free, and so the lower-left map given by multiplication by p^{r-1} is indeed injective.) Using that $W_n(A)$ is p-adically complete (Lemma 2.24), it follows that multiplication by α induces an isomorphism of $W_n(A)$-modules

$$\varprojlim_r W_n(A)/p^rW_n(A) \xrightarrow{\alpha} T_p(W_n\Omega^1_A).$$

This completes the proof of the second statement. ∎

Theorem 5.8. *Let A be a ring as in Notation 5.1. For every integer $n \geq 1$, the p-adic Tate module $T_p(W_n\Omega^1_A)$ is a free $W_n(A)$-module of rank one. Furthermore, there exists $\alpha_n \in T_p(W_n\Omega^1_A)$ which is a generator and such that the projection of α_n to $W_n\Omega^1_A[p]$, written $\alpha_n^{(1)}$, satisfies*

$$([\zeta_{p^n}] - 1)\alpha_n^{(1)} = \mathrm{d}\log[\zeta_p] \in W_n\Omega^1_A[p].$$

Proof. We prove a stronger result using induction on n. Let $A_0 = \mathbf{Z}_p[\zeta_{p^\infty}]^\wedge$.

- For every integer $n \geq 1$, there exists $\alpha_{0,n} \in T_p(W_n\Omega^1_{A_0})$ such that

$$([\zeta_{p^n}] - 1)\alpha_{0,n}^{(1)} = d\log[\zeta_p] \in W_n\Omega^1_{A_0}[p],$$

  and such that, under the map induced by functoriality, $T_p(W_n\Omega^1_{A_0}) \to T_p(W_n\Omega^1_A)$, the image of $\alpha_{0,n}$ freely generates $T_p(W_n\Omega^1_A)$ as a $W_n(A)$-module. We write α_n for this generator of $T_p(W_n\Omega^1_A)$.

We prove the base case. Because multiplication by p is surjective on $\Omega^1_{A_0}$, given any element $x \in \Omega^1_{A_0}[p]$, there exists $\alpha_{0,1} \in T_p(\Omega^1_{A_0})$ with $\alpha_{0,1}^{(1)} = x$. Thus, by Corollary 3.11, there exists $\alpha_{0,1} \in T_p(\Omega^1_{A_0})$ satifying all the listed properties, with the exception that we do not yet know its image $\alpha_1 \in T_p(\Omega^1_A)$ is a generator for $T_p(\Omega^1_A)$ as an A-module. As yet, we only know (again by Corollary 3.11), that $\alpha_1^{(1)}$ is a generator for the p-torsion. But then α_1 is a generator for the Tate module $T_p(\Omega^1_A)$ by Lemma 5.7.

Now inductively assume the result has been proved for some fixed value of n, and let $\alpha_{0,n}$ and α_n denote the corresponding elements. Considering the $W_{n+1}(A_0)$-module structure on the terms in the exact sequence from Proposition 5.5, we see that $g(R(z_{n+1})\alpha_{0,n}) = F^n(z_{n+1})g(\alpha_{0,n}) = 0 \in A_0/p^nA_0$, so there must exist $\alpha'_{0,n+1} \in T_p(W_{n+1}\Omega^1_{A_0})$ such that $R(\alpha'_{0,n+1}) = R(z_{n+1})\alpha_{0,n}$. Fix one such element $\alpha'_{0,n+1}$ and its image $\alpha'_{n+1} \in T_p(W_{n+1}\Omega^1_A)$. We have the following commutative diagram:

$$
\begin{array}{ccc}
T_p(W_{n+1}\Omega^1_{A_0}) & \longrightarrow & T_p(W_{n+1}\Omega^1_A) \\
{\scriptstyle R}\downarrow & & \downarrow{\scriptstyle R} \\
T_p(W_n\Omega^1_{A_0}) & \longrightarrow & T_p(W_n\Omega^1_A)
\end{array}
\qquad\qquad
\begin{array}{ccc}
\alpha'_{0,n+1} & \longmapsto & \alpha'_{n+1} \\
\downarrow & & \downarrow \\
R(z_{n+1})\alpha_{0,n} & \longmapsto & R(z_{n+1})\alpha_n
\end{array}
$$

In particular, $R(\alpha'_{n+1}) = R(z_{n+1})\alpha_n$.

By patching together the exact sequences from Proposition 5.5 and Lemma 2.32, we form a commutative diagram in which the rows are exact sequences of $W_{n+1}(A)$-modules:

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & T_p(\Omega^1_A) & \xrightarrow{V^n} & T_p(W_{n+1}\Omega^1_A) & \xrightarrow{R} & T_p(W_n\Omega^1_A) & \xrightarrow{g} & A/p^nA & \longrightarrow & 0 \\
 & & {\scriptstyle F^n(\alpha'_{n+1})}\uparrow & & {\scriptstyle \alpha'_{n+1}}\uparrow & & {\scriptstyle \alpha_n}\uparrow & & {\scriptstyle ?}\uparrow & & \\
0 & \longrightarrow & A & \xrightarrow{V^n} & W_{n+1}(A) & \xrightarrow{Rz_{n+1}} & W_n(A) & \xrightarrow{F^n} & A/p^nA & \longrightarrow & 0
\end{array}
\tag{5.9}
$$

By our induction hypothesis, the vertical map given by α_n is an isomorphism. There exists a unique vertical map as in the dashed arrow. By a diagram chase, that vertical map is surjective, and viewing it as a map of A/p^nA-modules, we see that it is also an isomorphism.

Notice that $([\zeta_{p^{n+1}}] - 1)z_{n+1} = [\zeta_{p^n}] - 1$, and so

$$R(d\log[\zeta_p]) = R(([\zeta_{p^{n+1}}] - 1)\alpha'^{(1)}_{0,n+1}) \in W_n\Omega^1_{A_0}[p].$$

Then by Lemma 5.6, we have

$$F(d\log[\zeta_p]) = F\left(([\zeta_{p^{n+1}}] - 1)\alpha'^{(1)}_{0,n+1}\right) \in W_n\Omega^1_{A_0}[p],$$

and therefore, applying F^{n-1} to both sides, we have

$$d\log\zeta_p = (\zeta_p - 1)\,F^n\left(\alpha'^{(1)}_{0,n+1}\right) \in \Omega^1_{A_0}[p],$$

and by functoriality, we have

$$d\log\zeta_p = (\zeta_p - 1)\,F^n\left(\alpha'^{(1)}_{n+1}\right) \in \Omega^1_A[p].$$

Again using Corollary 3.11, we know that $F^n(\alpha'^{(1)}_{n+1})$ freely generates $\Omega^1_A[p]$ as an A/pA-module, and hence $F^n(\alpha'_{n+1})$ freely generates $T_p(\Omega^1_A)$ as an A-module by Lemma 5.7. Thus the left-hand vertical map in the diagram (5.9) is also an isomorphism. It follows by the five lemma that the remaining vertical map, given by multiplication by α'_{n+1}, is also an isomorphism. This shows that $T_p(W_{n+1}\Omega^1_A)$ is a free $W_{n+1}(A)$-module of rank one. We must still construct the element $\alpha_{0,n+1}$ from $\alpha'_{0,n+1}$.

We recall that we have elements $\alpha_{0,n}$ and $\alpha'_{0,n+1}$ satisfying

$$([\zeta_{p^n}]-1)\alpha^{(1)}_{0,n} = d\log[\zeta_p] \in W_n\Omega^1_{A_0}[p].$$

and

$$R(\alpha'_{0,n+1}) = R(z_{n+1})\alpha_{0,n}.$$

We claim that there exists a unit $u \in W_{n+1}(A_0)/pW_{n+1}(A_0)$ such that

$$([\zeta_{p^{n+1}}]-1)u\alpha'^{(1)}_{0,n+1} = d\log[\zeta_p] \in W_{n+1}\Omega^1_{A_0}[p].$$

Because $d\log[\zeta_p]$ is p-torsion and because $\alpha'^{(1)}_{0,n+1}$ freely generates the p-torsion in $W_{n+1}\Omega^1_{A_0}$ by Lemma 5.7, we know there exists a (unique) element $x \in W_{n+1}(A_0)/pW_{n+1}(A_0)$ such that $d\log[\zeta_p] = x\alpha'^{(1)}_{0,n+1}$. Let

$$J := \ker\left(W_{n+1}(A_0)/pW_{n+1}(A_0) \xrightarrow{Rz_{n+1}} W_n(A_0)/pW_n(A_0)\right).$$

Note that $J \subseteq W_{n+1}(A_0)/pW_{n+1}(A_0)$ is an ideal. Because

$$R(x\alpha'^{(1)}_{0,n+1}) = R(([\zeta_{p^{n+1}}]-1)\alpha'^{(1)}_{0,n+1}) = d\log[\zeta_p] \in W_n\Omega^1_{A_0}[p],$$

we have

$$x \equiv ([\zeta_{p^{n+1}}]-1) \bmod J.$$

These elements x and $[\zeta_{p^{n+1}}]-1$ are not themselves in the ideal J, because $d\log[\zeta_p] \neq 0 \in W_n\Omega^1_{A_0}[p]$ (the element $d\log[\zeta_p]$ is non-zero because its restriction to $W_1\Omega^1_{A_0}$ is non-zero by Corollary 3.8). Thus, by Lemma 2.27 (which applies because we are considering the subring $A_0 \subseteq A$), there exists a unit $u \in W_{n+1}(A_0)/pW_{n+1}(A_0)$ such that $x = u([\zeta_{p^{n+1}}]-1)$. Write u also for a unit $u \in W_{n+1}(A_0)$ which lifts $u \in W_{n+1}(A_0)/pW_{n+1}(A_0)$; any lift to $W_{n+1}(A_0)$ is in fact a unit by Lemma 2.28. Let $\alpha_{0,n+1} := u\alpha'_{0,n+1} \in T_p(W_{n+1}\Omega^1_{A_0})$ and write $\alpha_{n+1} := u\alpha'_{n+1} \in T_p(W_{n+1}\Omega^1_A)$ for its image. Because α'_{n+1} freely generates $T_p(W_{n+1}\Omega^1_A)$, the same is true for this unit multiple α_{n+1}, and by construction it is the image of an element $\alpha_{0,n+1}$ satisfying $([\zeta_{p^{n+1}}]-1)\alpha^{(1)}_{0,n+1} = d\log[\zeta_p]$. This completes the induction. ■

We proved in Theorem 5.8 the existence of a generator of $T_p(W_n\Omega^1_A)$ as a free rank one $W_n(A)$-module. The next result gives a condition for identifying generators.

Corollary 5.10. *Let A be a ring as in Notation 5.1 and let $n \geq 1$ be an integer. Let $\alpha_n \in T_p(W_n\Omega^1_A)$ be any element satisfying $([\zeta_{p^n}]-1)\alpha^{(1)}_n = \mathrm{d}\log[\zeta_p] \in W_n\Omega^1_A[p]$. Then α_n freely generates $T_p(W_n\Omega^1_A)$ as a $W_n(A)$-module.*

Proof. Fix $\alpha \in T_p(W_n\Omega^1_A)$ such that $([\zeta_{p^n}]-1)\alpha^{(1)} = d\log[\zeta_p] \in W_n\Omega^1_A[p]$ and such that α freely generates $T_p(W_n\Omega^1_A)$ as a $W_n(A)$-module; such an element α exists by Theorem 5.8. Now let α_n be as in the statement of the corollary. There exists a unique $y \in W_n(A)$ such that $\alpha_n = y\alpha$, and we know that

$$([\zeta_{p^n}]-1)y\alpha \equiv ([\zeta_{p^n}]-1)\alpha \bmod pT_p(W_n\Omega^1_A).$$

Therefore

$$([\zeta_{p^n}]-1)y \equiv ([\zeta_{p^n}]-1) \bmod pW_n(A).$$

So y projects to a unit in $W_n(A)/pW_n(A)$ by Lemma 2.29. Thus y is a unit in $W_n(A)$ by Lemma 2.28. Because α was a generator, then $\alpha_n = y\alpha$ is also a generator. This completes the proof. ∎

We also have the following consequence.

Corollary 5.11. *Let A be a ring as in Notation 5.1, let $A_0 = \mathbf{Z}_p[\zeta_{p^\infty}]^\wedge$, and let $n \geq 1$ be an integer. The natural map induced by functoriality $T_p(W_n\Omega^1_{A_0}) \to T_p(W_n\Omega^1_A)$ induces an isomorphism of $W_n(A)$-modules*

$$W_n(A) \otimes_{W_n(A_0)} T_p(W_n\Omega^1_{A_0}) \xrightarrow{\sim} T_p(W_n\Omega^1_A).$$

Proof. Let $\alpha_{0,n} \in T_p(W_n\Omega^1_{A_0})$ be as in Corollary 5.10 (such an element exists by Theorem 5.8). The map $W_n(A)\otimes_{W_n(A_0)}W_n(A_0) \to W_n(A)\otimes_{W_n(A_0)}T_p(W_n\Omega^1_{A_0})$ sending $x\otimes y \mapsto x\otimes y\alpha_{0,n}$ is an isomorphism. The composition

$$W_n(A) \to W_n(A) \otimes_{W_n(A_0)} W_n(A_0) \to W_n(A) \otimes_{W_n(A_0)} T_p(W_n\Omega^1_{A_0}) \to T_p(W_n\Omega^1_A)$$

is an isomorphism by Corollary 5.10, and hence the result follows. ∎

The elements produced in Theorem 5.8 are not canonical. We next describe canonical generators. Our description of these generators is modeled after Hesselholt's (Hesselholt, 2006, Theorem B). For rings A as in Notation 5.1, there is an obvious element in $T_p(W_n\Omega^1_A)$; namely, for the component in $W_n\Omega^1_A[p^r]$, we take the element $d\log[\zeta_{p^r}]$. We refer to the corresponding element as $d\log[\zeta_p^{(\infty)}] \in T_p(W_n\Omega^1_A)$. Notice that our way of writing this element does not indicate the level n. These elements, for various n, are all compatible under the restriction map (as well as the Frobenius map). This compatibility under restriction is the key observation to identifying a canonical generator of $T_p(W_n\Omega^1_A)$. This generator $\alpha_n^{(\infty)}$ will satisfy

$$([\zeta_{p^n}] - 1)\alpha_n^{(\infty)} = d\log[\zeta_p^{(\infty)}] \in T_p(W_n\Omega^1_A);$$

notice that this is similar to the condition of Theorem 5.8, but the condition of Theorem 5.8 was only a condition modulo p. Proving the existence of such an element $\alpha_n^{(\infty)}$ seems rather delicate and will require several preliminary results.

Lemma 5.12. *Fix an integer $n \geq 1$. Let A and z_{n+1} be as in Notation 5.1. We have that $\beta \in T_p(W_n\Omega^1_A)$ is in the image of the restriction map*

$$R : T_p(W_{n+1}\Omega^1_A) \to T_p(W_n\Omega^1_A),$$

if and only if there exists $\alpha \in T_p(W_n\Omega^1_A)$ such that $R(z_{n+1})\alpha = \beta$.

Proof. The "if" direction was shown in the proof of Theorem 5.8; it was used to produce the element $\alpha'_{0,n+1}$. We now prove the "only if" direction. We know $T_p(W_n\Omega^1_A) \cong W_n(A)$ as $W_n(A)$-modules or, equivalently, as $W_{n+1}(A)$-modules via restriction. Fix one such isomorphism, and let $\alpha_n \in T_p(W_n\Omega^1_A)$ correspond to $1 \in W_n(A)$. Let $g : T_p(W_n\Omega^1_A) \to A/p^nA$ be the $W_{n+1}(A)$-module map indicated in Proposition 5.5. Because g is surjective, we know $g(\alpha_n)$ is a unit in A/p^nA.

Return now to our element $\beta \in T_p(W_n\Omega^1_A)$ which is in the image of restriction. We have $\beta = R(x)\alpha_n$ for some $x \in W_{n+1}(A)$. Because β is in the image of restriction, by exactness of the sequence from Proposition 5.5, we have $g(\beta) = 0 \in A/p^nA$. Thus, considering the $W_{n+1}(A)$-module structure, we find

$$g\left(R(x)\alpha_n\right) = F^n(x)g(\alpha_n) = 0 \in A/p^nA.$$

We have already observed that $g(\alpha_n)$ is a unit in A/p^nA, so $F^n(x) = 0 \in A/p^nA$. By Lemma 2.32, this means precisely that $R(x) = R(z_{n+1}y)$ for some $y \in W_{n+1}(A)$. For our desired element α, we may then take $R(y)\alpha_n$. ■

Lemma 5.13. *Let A denote a ring as in Notation 5.1. Fix integers $n, s \geq 1$. If $\beta \in T_p(W_n\Omega^1_A)$ is in the image of the restriction map*

$$R^s : T_p(W_{n+s}\Omega^1_A) \to T_p(W_n\Omega^1_A),$$

then there exists $\alpha \in T_p(W_n\Omega^1_A)$ such that

$$R^s(z_{n+s}) \cdots R^2(z_{n+2})R(z_{n+1})\alpha = \beta.$$

Proof. We prove this result using induction on s. The base case $s = 1$ is precisely Lemma 5.12. Now assume the result has been shown for some fixed value of s, i.e., assume the claimed result holds for that value of s and for all values of $n \geq 1$. We will prove that the claimed result holds also for $s + 1$ and all values of $n \geq 1$.

We know that for all integers $n \geq 1$, there exists $\alpha_n \in T_p(W_n\Omega^1_A)$ such that α_n freely generates $T_p(W_n\Omega^1_A)$ as a $W_n(A)$-module. It clearly suffices to show

$$R^{s+1}(\alpha_{n+s+1}) = R^{s+1}(z_{n+s+1}) \cdots R^2(z_{n+2})R(z_{n+1})\alpha$$

for some $\alpha \in T_p(W_n\Omega^1_A)$. By the induction hypothesis (applied to the values s and $n+1$), we know

$$R^s(\alpha_{n+s+1}) = R^s(z_{n+s+1}) \cdots R(z_{n+2})\alpha'$$

for some $\alpha' \in T_p(W_{n+1}\Omega^1_A)$. We also know

$$R(\alpha') = R(z_{n+1})\alpha$$

for some $\alpha \in T_p(W_n\Omega^1_A)$ by Lemma 5.12. The desired result follows immediately from the two previous displayed equations. ■

Proposition 5.14. *Let A denote a ring as in Notation 5.1. Fix an integer $n \geq 1$. There exists $\alpha \in T_p(W_n\Omega^1_A)$ such that*

$$([\zeta_{p^n}] - 1)\alpha = \mathrm{d}\log[\zeta_p^{(\infty)}].$$

Proof. By functoriality, it suffices to prove this result for the ring $A_0 = \mathbf{Z}_p[\zeta_{p^\infty}]^\wedge$. Choose $\alpha_n \in T_p(W_n\Omega^1_{A_0})$ which freely generates $T_p(W_n\Omega^1_{A_0})$ as a $W_n(A_0)$-module. Then there exists some unique $x \in W_n(A_0)$ such that

$$x\alpha_n = d\log[\zeta_p^{(\infty)}].$$

By Lemma 2.31, it suffices to show, for every integer $s \geq 1$, that x is a multiple of $\frac{[\zeta_{p^n}]-1}{[\zeta_{p^{n+s}}]-1}$. Notice that for every integer $s \geq 1$, we have

$$R^s\left(d\log[\zeta_p^{(\infty)}]\right) = d\log[\zeta_p^{(\infty)}],$$

where the $d\log[\zeta_p^{(\infty)}]$ on the left is considered as an element in $T_p(W_{n+s}\Omega^1_{A_0})$, and where the $d\log[\zeta_p^{(\infty)}]$ on the right is considered as an element of $T_p(W_n\Omega^1_{A_0})$. Thus our desired result follows immediately from Lemma 5.13 and the fact that

$$R^s(z_{n+s}) \cdots R^2(z_{n+2})R(z_{n+1}) = \frac{[\zeta_{p^n}] - 1}{[\zeta_{p^{n+s}}] - 1} \in W_n(A_0).$$

This completes the proof. ■

The following result is one of the most important results of this section. Its proof contains no new ideas; it simply requires assembling the results attained earlier. This result should be compared to (Hesselholt, 2006, Theorem B); notice that Hesselholt's proof uses techniques from topology (see especially (Hesselholt, 2006, Section 3)).

Theorem 5.15. *Let A denote a ring as in Notation 5.1. For each integer $n \geq 1$, there exists a unique element $\alpha_n^{(\infty)} \in T_p(W_n\Omega^1_A)$ with $([\zeta_{p^n}] - 1)\alpha_n^{(\infty)} = \mathrm{d}\log[\zeta_p^{(\infty)}]$. This element freely generates $T_p(W_n\Omega^1_A)$ as a $W_n(A)$-module. For each $n \geq 1$, these elements satisfy $F(\alpha_{n+1}^{(\infty)}) = \alpha_n^{(\infty)}$ and $R(\alpha_{n+1}^{(\infty)}) = R(z_{n+1})\alpha_n^{(\infty)}$.*

Proof. We first claim that if α and $\alpha' \in T_p(W_n\Omega^1_A)$ are such that

$$([\zeta_{p^n}] - 1)\alpha = d\log[\zeta_p^{(\infty)}] \qquad \text{and} \qquad ([\zeta_{p^n}] - 1)\alpha' = d\log[\zeta_p^{(\infty)}],$$

then $\alpha = \alpha'$. Choose an isomorphism of $W_n(A)$-modules $T_p(W_n\Omega^1_A) \cong W_n(A)$; this is possible by Theorem 5.8. Let $x, y \in W_n(A)$ correspond to α, α', respectively, under our chosen isomorphism. Our assumption implies

$$([\zeta_{p^n}] - 1)x = ([\zeta_{p^n}] - 1)y \in W_n(A),$$

but we then find $x = y$ because $[\zeta_{p^n}] - 1$ is not a zero divisor in $W_n(A)$. We already know such an element $\alpha \in T_p(W_n\Omega^1_A)$ exists by Proposition 5.14, and now that we know it is unique, we name it $\alpha_n^{(\infty)}$. It freely generates $T_p(W_n\Omega^1_A)$ as a $W_n(A)$-module by Corollary 5.10.

It remains to check the stated compatibilities under Frobenius and restriction. Applying Frobenius to both sides of

$$([\zeta_{p^{n+1}}] - 1)\alpha_{n+1}^{(\infty)} = d\log[\zeta_p^{(\infty)}],$$

we find that $F(\alpha_{n+1}^{(\infty)})$ satisfies

$$F([\zeta_{p^{n+1}}] - 1)F(\alpha_{n+1}^{(\infty)}) = ([\zeta_{p^n}] - 1)F(\alpha_{n+1}^{(\infty)}) = d\log[\zeta_p^{(\infty)}].$$

Thus by uniqueness, we have $F(\alpha_{n+1}^{(\infty)}) = \alpha_n^{(\infty)}$.

Similarly, applying restriction to both sides of

$$([\zeta_{p^{n+1}}] - 1)\alpha_{n+1}^{(\infty)} = d\log[\zeta_p^{(\infty)}] \in T_p(W_{n+1}\Omega^1_A),$$

we find that $R(\alpha_{n+1}^{(\infty)})$ satisfies

$$([\zeta_{p^{n+1}}] - 1)R(\alpha_{n+1}^{(\infty)}) = d\log[\zeta_p^{(\infty)}] \in T_p(W_n\Omega^1_A).$$

On the other hand, $\alpha_n^{(\infty)}$ satisfies

$$([\zeta_{p^n}] - 1)\alpha_n^{(\infty)} = d\log[\zeta_p^{(\infty)}],$$

so

$$([\zeta_{p^{n+1}}] - 1)R(z_{n+1})\alpha_n^{(\infty)} = d\log[\zeta_p^{(\infty)}].$$

The fact that $R(\alpha_{n+1}^{(\infty)}) = R(z_{n+1})\alpha_n^{(\infty)}$ follows because $[\zeta_{p^{n+1}}] - 1$ is not a zero divisor in $W_n(A)$. ■

We next record a few easy consequences of Theorem 5.8 and Theorem 5.15.

Corollary 5.16. *Let A denote a ring as in Notation 5.1. Then for all integers $n, r \geq 1$, we have that $W_n\Omega^1_A[p^r]$ is a free $W_n(A)/p^rW_n(A)$-module of rank one*

Proof. This follows immediately from Theorem 5.8 and Lemma 5.7. ∎

Corollary 5.17. *Let A denote a ring as in Notation 5.1 and let $r, n \geq 1$ be integers. The Frobenius map $F : W_{n+1}\Omega^1_A[p^r] \to W_n\Omega^1_A[p^r]$ is surjective. Also the Frobenius map $F : T_p(W_{n+1}\Omega^1_A) \to T_p(W_n\Omega^1_A)$ is surjective.*

Proof. These follow immediately from the fact that $F : W_{n+1}(A) \to W_n(A)$ is surjective, and the fact from Proposition 5.15 that Frobenius maps a generator of $T_p(W_{n+1}\Omega^1_A)$ to a generator of $T_p(W_n\Omega^1_A)$. ∎

The following corollary is analogous to (Hesselholt, 2006, Proposition 2.4.2); see also the paragraph immediately preceding that proposition. The proof of (Hesselholt, 2006, Proposition 2.4.2) uses topology.

Corollary 5.18. *Let A denote a ring as in Notation 5.1. Consider $\varprojlim_F T_p\left(W_n\Omega^1_A\right)$ as a $W(A^\flat)$-module via the ring isomorphism $\varprojlim_F W_n(A) \cong W(A^\flat)$ from Lemma 2.16. For each integer $n \geq 1$, let $\alpha_n^{(\infty)} \in T_p(W_n\Omega^1_A)$ be the element specified in Theorem 5.15, and let $\alpha \in \varprojlim_F T_p\left(W_n\Omega^1_A\right)$ be the sequence of elements*

$$\alpha = (\alpha_1^{(\infty)}, \alpha_2^{(\infty)}, \ldots) \in \varprojlim_F T_p\left(W_n\Omega^1_A\right).$$

Let $\varepsilon = (1, \zeta_p, \zeta_{p^2}, \ldots) \in A^\flat$. The following properties hold.

1. *The element α is the unique element satisfying $([\varepsilon] - 1)\alpha = (\mathrm{d}\log[\zeta_p^{(\infty)}])$.*
2. *The $W(A^\flat)$-module $\varprojlim_F T_p\left(W_n\Omega^1_A\right)$ is a free module of rank one, generated by α.*

Proof. Notice that $\widetilde{\theta}_r([\varepsilon]) = [\zeta_{p^r}] \in W_r(A)$. The stated properties follow immediately from Theorem 5.15. ∎

Corollary 5.19. *Let A denote a ring as in Notation 5.1 and let $r \geq 1$ denote an integer. The projection map*

$$\varprojlim_F T_p\left(W_n\Omega^1_A\right) \to T_p\left(W_r\Omega^1_A\right)$$

is a surjective map of $W(A^\flat)$-modules, where the $W(A^\flat)$-module structure on $T_p\left(W_r\Omega^1_A\right)$ is defined via $\widetilde{\theta}_r$.

Proof. This follows immediately from Corollary 5.18 and the fact that, for each integer $r \geq 1$, the projection map $\widetilde{\theta}_r : W(A^\flat) \cong \varprojlim_F W_n(A) \to W_r(A)$ is surjective. ∎

Let $(x_1, x_2, \ldots)$ denote an arbitrary element in $\varprojlim_F T_p(W_n\Omega^1_A)$, where for each integer $n \geq 1$, we have $x_n \in T_p(W_n\Omega^1_A)$. Let $R : T_p(W_{n+1}\Omega^1_A) \to T_p(W_n\Omega^1_A)$ denote the restriction map. The sequence $(R(x_2), R(x_3), \ldots)$ is still Frobenius-compatible, and we again use R to denote the corresponding map $(x_1, x_2, \ldots) \mapsto (R(x_2), R(x_3), \ldots)$. Over the next few results, we seek to describe the elements which are fixed by this map R.

The map R is not a $W(A^\flat)$-module map for the module structure defined in Corollary 5.18, but we do have the following structure.

Lemma 5.20. *Let A denote a ring as in Notation 5.1. Let R denote the map*

$$R : \varprojlim_F T_p(W_n\Omega^1_A) \to \varprojlim_F T_p(W_n\Omega^1_A),$$

defined by sending $(x_1, x_2, x_3, \ldots) \mapsto (R(x_2), R(x_3), \ldots)$. *Let* $\varphi : W(A^\flat) \to W(A^\flat)$ *denote the Witt vector Frobenius ring automorphism, and let* φ^{-1} *denote its inverse. For an arbitrary element* $t \in W(A^\flat)$ *and* $x \in \varprojlim_F T_p(W_n\Omega^1_A)$, *define the product* $tx \in \varprojlim_F T_p(W_n\Omega^1_A)$ *using the isomorphism* $W(A^\flat) \cong \varprojlim_F W_n(A)$. *We have*

$$R(tx) = \varphi^{-1}(t)R(x).$$

Proof. Let $t \in W(A^\flat)$ correspond to $(t_1, t_2, \ldots) \in \varprojlim_F W_n(A)$ under the isomorphism from Lemma 2.16. Then

$$R(tx) = (R(t_2x_2), R(t_3, x_3), \ldots) = \big(R(t_2), R(t_3), \ldots\big)R(x),$$

so we reduce to checking that $(R(t_2), R(t_3), \ldots) \in \varprojlim_F W_n(A)$ corresponds to $\varphi^{-1}(t) \in W(A^\flat)$. For this last claim, see the last sentence of (Bhatt et al., 2018, Lemma 3.2). ∎

Proposition 5.21. *Let* A *denote a ring as in Notation 5.1. Let* $\alpha \in \varprojlim_F T_p(W_n\Omega^1_A)$ *denote the* $W(A^\flat)$*-module generator described in Corollary 5.18. Let* $R : \varprojlim_F T_p(W_n\Omega^1_A) \to \varprojlim_F T_p(W_n\Omega^1_A)$ *denote the map described in Lemma 5.20. Finally, let* $\varepsilon \in A^\flat$ *denote the element* $(1, \zeta_p, \zeta_{p^2}, \ldots)$. *We have the following equality of* $\mathbf{Z}_p$*-modules,*

$$\Big\{x \in \varprojlim_F T_p(W_n\Omega^1_A) \colon R(x) = x\Big\} = \Big\{y\alpha \in \varprojlim_F T_p(W_n\Omega^1_A) \colon y \in W(A^\flat) \text{ satisfies } \varphi(y) = \frac{[\varepsilon^p] - 1}{[\varepsilon] - 1}\, y\Big\}.$$

Proof. We first find an element $t \in W(A^\flat)$ such that $R(\alpha) = t\alpha$. By definition of α, using the notation of Corollary 5.18, we have

$$\begin{aligned} R(\alpha) &= (R(\alpha_2^{(\infty)}), R(\alpha_3^{(\infty)}), \ldots) \in \varprojlim_F T_p(W_n\Omega^1_A) \\ &= (R(z_2)\alpha_1^{(\infty)}, R(z_3)\alpha_2^{(\infty)}, \ldots) \in \varprojlim_F T_p(W_n\Omega^1_A) \\ &= (\frac{[\zeta_p] - 1}{[\zeta_{p^2}] - 1}\alpha_1^{(\infty)}, \frac{[\zeta_{p^2}] - 1}{[\zeta_{p^3}] - 1}\alpha_2^{(\infty)}, \ldots) \in \varprojlim_F T_p(W_n\Omega^1_A) \\ &= \left(\frac{[\varepsilon] - 1}{[\varepsilon^{1/p}] - 1}\right)\alpha \in \varprojlim_F T_p(W_n\Omega^1_A). \end{aligned}$$

Using this preliminary calculation, it is easy to complete the proof. Namely, each element $x \in \varprojlim_F T_p(W_n\Omega^1_A)$ can be written uniquely in the form $y\alpha$ for some $y \in W(A^\flat)$. We have $R(x) = x$ if and only if

$$\varphi^{-1}(y)\left(\frac{[\varepsilon] - 1}{[\varepsilon^{1/p}] - 1}\right)\alpha = y\alpha,$$

where we have used Lemma 5.20 to express $R(y\alpha)$ in terms of $R(\alpha)$. Because α is a free generator and because φ is an automorphism, this last equality holds if and only if

$$y\left(\frac{[\varepsilon^p] - 1}{[\varepsilon] - 1}\right) = \varphi(y),$$

as claimed. ∎

The following result gives a concrete description of the $\mathbf{Z}_p$-module $\{x \in \varprojlim_F T_p(W_n\Omega^1_A) \colon R(x) = x\}$, but we are not able to find a similar result in the same generality as Notation 5.1, so in the following proposition, in addition to the assumptions of Notation 5.1, we assume the ring A is an integral domain. The exact same result certainly does not hold in general; for example, it will not hold for $A = \mathbf{Z}_p[\zeta_{p^\infty}]^\wedge \times \mathbf{Z}_p[\zeta_{p^\infty}]^\wedge$, where the $\mathbf{Z}_p$-module in question will be isomorphic to $\mathbf{Z}_p \times \mathbf{Z}_p$. The proof of Proposition 5.22 is based on (Hesselholt, 2006, Corollary 1.3.3).

Proposition 5.22. *Let A be as in Notation 5.1, and assume furthermore that A is an integral domain. The elements $y \in W(A^\flat)$ satisfying the equation*

$$\varphi(y) = \frac{[\varepsilon^p]-1}{[\varepsilon]-1} y$$

are precisely the elements of the form $c([\varepsilon]-1)$ for $c \in \mathbf{Z}_p$. In other words, the $\mathbf{Z}_p$-module

$$\left\{ x \in \varprojlim_F T_p(W_n\Omega^1_A) \colon R(x) = x \right\}$$

is a free $\mathbf{Z}_p$-module of rank one, generated by

$$(\mathrm{d}\log[\zeta_p^{(\infty)}], \mathrm{d}\log[\zeta_p^{(\infty)}], \dots) \in \varprojlim_F T_p(W_n\Omega^1_A),$$

where the element $\mathrm{d}\log[\zeta_p^{(\infty)}] \in T_p(W_n\Omega^1_A)$ was defined in the paragraph after Corollary 5.10.

Proof. Recall that for a characteristic p ring R, the Witt vector Frobenius map $W_{n+1}(R) \to W_n(R)$ induces the map $\varphi : W_n(R) \to W_n(R)$ which sends $(r_1, r_2, \dots, r_n) \mapsto (r_1^p, r_2^p, \dots, r_n^p)$. It suffices to show, for every integer $n \geq 1$, that

$$\left\{ y \in W_n(A^\flat) \colon \varphi(y) = \frac{[\varepsilon^p]-1}{[\varepsilon]-1} y \right\} = \left\{ x([\varepsilon]-1) \in W_n(A^\flat) \colon x \in W_n(\mathbf{F}_p) \right\}.$$

The inclusion $\supseteq$ is obvious, because $\varphi(x) = x$ for all $x \in W_n(\mathbf{F}_p)$. We prove equality using induction on $n \geq 1$. In the base case, $n = 1$, we consider

$$\left\{ y \in A^\flat \colon y^p - \frac{\varepsilon^p - 1}{\varepsilon - 1} y = 0 \right\}.$$

Notice that, because we assumed that A was an integral domain, we know further that $A^\flat$ is an integral domain (for example, this is clear by considering the isomorphism of multiplicative monoids $A^\flat \cong \varprojlim_{x \to x^p} A$). We already have p solutions to the above equation, and because $A^\flat$ is an integral domain, there can be no other solutions. This proves the base case.

Now assume the equality is known for the case of some fixed $n \geq 1$, and assume

$$y \in W_{n+1}(A^\flat) \text{ is such that } \varphi(y) = \frac{[\varepsilon^p]-1}{[\varepsilon]-1} y.$$

By our induction hypothesis, we know that

$$y = x([\varepsilon]-1) + V^n(z), \text{ for some } x \in W_{n+1}(\mathbf{F}_p), z \in A^\flat.$$

We are finished if we show that there exists $a \in \mathbf{F}_p$ such that $V^n(z) = ([\varepsilon]-1)V^n(a) \in W_{n+1}(A^\flat)$. Using our assumption on y, we find

$$\varphi(V^n(z)) = \frac{[\varepsilon^p]-1}{[\varepsilon]-1} V^n(z) \in W_{n+1}(A^\flat).$$

Thus

$$z^p = \frac{\varepsilon^{p^{n+1}} - 1}{\varepsilon^{p^n} - 1} z \in A^\flat.$$

The values $z = a(\varepsilon^{p^n} - 1)$ for $a \in \mathbf{F}_p$ provide p solutions, and hence all solutions (again using that $A^\flat$ is an integral domain). This completes the proof. ■

Remark 5.23. Proposition 5.22 can be generalized. Let A be a ring as in Notation 5.1. Using algebraic K-theory and topological cyclic homology, it follows from (Clausen et al., 2018, Corollary 6.9) and (Anschütz and Le Bras, 2020, Corollary 6.5) that the $\mathbf{Z}_p$-module

$$\left\{x \in \varprojlim_F T_p(W_n\Omega^1_A) \colon R(x) = x\right\}$$

is isomorphic to the p-adic Tate module $T_p(A^\times)$. We do not know an algebraic proof of this fact. More details are discussed in Remark 8.6 below.

6. Higher degrees

The analysis is easier in degrees $d \geq 2$. The main result in this section, Proposition 6.3, is modeled after (Hesselholt, 2006, Proposition 2.2.1).

Lemma 6.1. *Assume that R is a ring, that M is an R-module, and that $r \in R$ is such that multiplication by r is surjective on M. Then for every integer $d \geq 2$, multiplication by r is an isomorphism on the exterior power $\Lambda^d_R M$.*

Proof. Let $M[r^\infty]$ denote the submodule of M consisting of all elements which are annihilated by a power of r. We have a short exact sequence of R-modules

$$0 \to M[r^\infty] \to M \to R[1/r] \otimes_R M \to 0.$$

For any integer $n \geq 1$, let $T^n_R(M) := M \otimes_R M \otimes_R \cdots \otimes_R M$, with n total M terms. Because tensor product is right exact, we obtain an exact sequence of R-modules

$$M[r^\infty] \otimes_R T^{d-1}_R(M) \to M \otimes_R T^{d-1}_R(M) \to R[1/r] \otimes_R M \otimes_R T^{d-1}_R(M) \to 0.$$

On the other hand, the left-hand term is zero because multiplication by r is surjective on M. This shows that multiplication by r is an isomorphism on the R-module $T^d_R(M)$. It follows immediately that multiplication by r is surjective on the exterior power $\Lambda^d_R M$. It remains to show that multiplication by r is injective on $\Lambda^d_R M$.

Assume $\sum r m_{1i} \wedge \cdots \wedge m_{di} = 0 \in \Lambda^d_R M$. We must show that $\sum m_{1i} \wedge \cdots \wedge m_{di} = 0 \in \Lambda^d_R M$. By definition of $\Lambda^d_R M$ (see for example (The Stacks Project Authors, 2017, Tag 00DM)), we know that

$$\sum r m_{1i} \otimes \cdots \otimes m_{di} = \sum a_j \cdot x_j \otimes x_j \cdot b_j \in T^d_R(M),$$

for some suitable $a_j, b_j \in T_R(M)$ and $x_j \in M$. Choose y_j such that $r y_j = x_j$. Then we have

$$\sum r m_{1i} \otimes \cdots \otimes m_{di} = \sum a_j \cdot r y_j \otimes r y_j \cdot b_j \in T^d_R(M),$$

Because, as we saw above, multiplication by r is injective on $T^d_R(M)$, we have

$$\sum m_{1i} \otimes \cdots \otimes m_{di} = \sum r a_j \cdot y_j \otimes y_j \cdot b_j \in T^d_R(M),$$

Thus $\sum m_{1i} \wedge \cdots \wedge m_{di} = 0 \in \Lambda^d_R M$, as required. ■

Corollary 6.2. *Let A denote a p-torsion-free perfectoid ring and let $d \geq 2$ be an integer. The multiplication-by-p map $\Omega^d_A \xrightarrow{p} \Omega^d_A$ is an isomorphism of A-modules.*

Proof. Because multiplication by p is surjective on Ω^1_A, this follows immediately from Lemma 6.1. ■

Proposition 6.3. *Let A denote a p-torsion-free perfectoid ring and let $d \geq 2$ and $n \geq 1$ be integers. There is a short exact sequence of $W_{n+1}(A)$-modules*

$$0 \to \Omega^d_A \xrightarrow{V^n} W_{n+1}\Omega^d_A \to W_n\Omega^d_A \to 0,$$

where the $W_{n+1}(A)$-module structure on Ω^d_A is defined via F^n and where the $W_{n+1}(A)$-module structure on $W_n\Omega^d_A$ is defined via restriction.

Proof. By (Hesselholt and Madsen, 2003, Proposition 3.2.6), we always have an exact sequence

$$\Omega^d_A \oplus \Omega^{d-1}_A \xrightarrow{V^n + dV^n} W_{n+1}\Omega^d_A \to W_n\Omega^d_A \to 0,$$

so it suffices to show that the map $V^n : \Omega^d_A \to W_{n+1}\Omega^d_A$ is injective, and that the image of $V^n : \Omega^d_A \to W_{n+1}\Omega^d_A$ is the same as the image of $V^n + dV^n : \Omega^d_A \oplus \Omega^{d-1}_A \to W_{n+1}\Omega^d_A$.

To see injectivity, notice that $p^n = F^n \circ V^n$ on Ω^d_A, and by Corollary 6.2, multiplication by p^n is injective on Ω^d_A (this is one of two places where we use that $d > 1$), so we have that V^n is also injective on Ω^d_A.

To see the claim about the image, it suffices to show that for every $\alpha \in \Omega^{d-1}_A$, there exists $\alpha' \in \Omega^d_A$ such that $dV^n(\alpha) = V^n(\alpha') \in W_{n+1}\Omega^d_A$. Because multiplication by p is surjective on Ω^{d-1}_A (this is the other place where we use that $d > 1$), we can write $\alpha = p^n\alpha_0$. Thus $dV^n(\alpha) = V^n(d\alpha_0)$, and so we may take $\alpha' = d\alpha_0$. ∎

Corollary 6.4. *Let A denote a p-torsion-free perfectoid ring and let $d \geq 2$ and $n \geq 1$ be integers. Multiplication by p on $W_n\Omega^d_A$ is an isomorphism of $W_n(A)$-modules.*

Proof. This follows easily from the above results using the five lemma and induction on n. ∎

7. Results on the logarithmic de Rham-Witt complex

In this section, we consider the logarithmic de Rham-Witt complex, $W.\Omega^\bullet_{(A,M)}$, which is defined by Hesselholt-Madsen in (Hesselholt and Madsen, 2003, Proposition 3.2.2) as the initial object in the category of log Witt complexes. We prove that, when A is a p-torsion-free perfectoid ring that is also a valuation ring, and when $M = A \setminus \{0\}$, then the logarithmic de Rham-Witt complex is the same as the usual de Rham-Witt complex. Many of the ideas in this section were based on (Hesselholt and Madsen, 2003, Proof of Proposition 2.2.2) and conversations with Lars Hesselholt. We are also grateful to the referee for an earlier version of this paper, for showing us that our original results, which were isomorphisms in the degree one part of these Witt complexes, could be extended to isomorphisms in all degrees. Several of our arguments in this section were suggested to us by the referee.

The goal of this section is to prove the following.

Theorem 7.1. *Let A denote a p-torsion-free perfectoid ring, and assume further that A is a valuation ring. Let $M = A \setminus \{0\}$. Because $W.\Omega^\bullet_{(A,M)}$ is a Witt complex, we have a natural map*

$$W.\Omega^\bullet_A \to W.\Omega^\bullet_{(A,M)}.$$

This map is an isomorphism of Witt complexes over A. For every integer $n \geq 1$, the corresponding map

$$W_n\Omega^1_A \to W_n\Omega^1_{(A,M)}$$

is an isomorphism of $W_n(A)$-modules.

We will prove Theorem 7.1 below.

Remark 7.2. There are many immediate consequences of this theorem. For example, for all integers $n \geq 1$ and $d \geq 0$, the natural map

$$W_n\Omega^d_A \to W_n\Omega^d_{(A,A\setminus\{0\})}$$

is an isomorphism of $W_n(A)$-modules (still requiring that A be a p-torsion-free perfectoid ring and a valuation ring). As another example, by Theorem 7.1 and Corollary B.2, the analogue of the exact sequence (1.3) is also exact for the log ring $(A, A \setminus \{0\})$.

There are two major hypotheses in Theorem 7.1: that A be a perfectoid ring, and that A be a valuation ring. Briefly, the perfectoid hypothesis is used to guarantee certain p-power roots, as in Lemma 7.3, and the valuation ring hypothesis is used to guarantee certain divisibiliity properties, as in Lemma 7.4. We now include these preliminary lemmas.

Lemma 7.3. *Let A denote a p-torsion-free perfectoid valuation ring. Then for every element $a \in A$, there exists a unit $u \in A^\times$ and an element $\omega = (\omega^{(0)}, \omega^{(1)}, \ldots) \in \varprojlim_{x\mapsto x^p} A$ such that $ua = \omega^{(0)}$.*

Proof. We adapt the proof of the "moreover" assertion in (Bhatt et al., 2018, Lemma 3.9). Let $a \in A$ be an arbitrary non-zero element, and let $\pi \in A$ be as in the definition of perfectoid. Because A is π-adically separated, we can write $a = \pi^m x$ for some integer $m \geq 0$ and some $x \in A$ such that $x \notin \pi A$. It suffices to prove the claim for the special cases $a = \pi$ and $a = x$ separately. The case $a = \pi$ is explicitly stated in (Bhatt et al., 2018, Lemma 3.9). As in that proof, we can find $\eta \in \varprojlim_{x\mapsto x^p} A$ and $y \in A$ with

$$\eta^{(0)} = x + \pi p y = x(1 + \frac{\pi}{x} p y);$$

note that $\frac{\pi}{x} \in A$ because A is a valuation ring and $x \notin \pi A$. Because A is p-adically complete, the element $1 + \frac{\pi}{x} p y$ is a unit in A, which completes the proof. ∎

We next give a preliminary result about divisibility of Witt vectors. Even though we have strong divisibility properties in the ring A, deducing corresponding properties in the Witt vectors $W_n(A)$ is more difficult.

Lemma 7.4. *Let A denote a p-torsion-free perfectoid valuation ring. Fix an integer $n \geq 1$. For every non-zero $a \in A$, there exists an integer $N \geq 1$ and a Witt vector $x \in W_n(A)$ such that*

$$[a]x = p^N \in W_n(A).$$

For a fixed value of N, the corresponding value of x is unique.

Proof. The uniqueness is obvious because the ghost components of $[a]$ are non-zero. We prove existence by proving the following stronger result using induction on n.

- Let $n \geq 1$ be an integer. Let $a \in A$ be non-zero. There exists an integer $N \geq 1$ (depending on n and on a) such that, if $y \in W_n(A)$ has all ghost components in $p^N A$, then there exists $x \in W_n(A)$ such that $[a]x = y \in W_n(A)$.

The base case $n = 1$ follows immediately from the fact that A is a valuation ring which is p-adically separated. Now inductively assume the result has been proven for some fixed value of n. We prove the result for $n+1$. Let $a \in A$ be non-zero, and let N be such that, if $y' \in W_n(A)$ has all ghost components in $p^N A$, then y' is divisible by $[a^p]$. We claim that if $y \in W_{n+1}(A)$ has all ghost components in $p^{2N} A$, then y is divisible by $[a]$ in $W_{n+1}(A)$. Thus, assume we have such a y, and let $p^N z_0$ denote the first Witt component of y. We have that $z_0 \in p^N A$, so $z_0 = a x_0$ for some $x_0 \in A$. Then we have

$$y - [p^N x_0][a] = V(z') \in W_{n+1}(A)$$

for some $z' \in W_n(A)$, and note that all the ghost components of z' are in $p^N A$. Thus

$$z' = [a^p]z'' = F([a])z'' \in W_n(A)$$

for some $z'' \in W_n(A)$. Thus

$$y = [a]\Big([p^N x_0] + V(z'')\Big) \in W_{n+1}(A).$$

This completes the induction. ■

We now consider logarithmic differentials. For any log ring (A, M), with map of monoids $\alpha : M \to A$ (where A is viewed as a multiplicative monoid), and for any integer $n \geq 1$, we consider $(W_n(A), M)$ as a log ring using the monoid map $m \mapsto [\alpha(m)]_n \in W_n(A)$. (Usually we are not careful about specifying the level in our Tecihmüller notation, i.e., we usually write $[m]$ instead of $[m]_n$, but in the following arguments we often are switching levels using Frobenius or Restriction, so we have attempted to be careful about indicating the level.) In the case considered in this section, that $M = A \setminus \{0\}$ and α is the inclusion, we will commit a slight abuse of notation and write $d\log[m]_n$ instead of $d\log m$, to remind ourselves about the presence of this Teichmüller lift. Our main task in the remainder of this section is, for each element $m \in A \setminus \{0\}$, to identify an element in $W_n\Omega^1_A$ (not $W_n\Omega^1_{(A,A\setminus\{0\})}$) which will correspond to $d\log[m]_n$.

Remark 7.5. In this remark we motivate the formula appearing in Part (2) of Lemma 7.6 below. The element $d\log[z]_n$ should be thought of as $\frac{d[z]_n}{[z]_n}$. Now assume that $[y]_n^{p^N} = [u]_n[m]_n$. Then taking $d\log$ of both sides, we have

$$p^N \, d\log[y]_n = d\log[u]_n + d\log[m]_n, \text{ so } \frac{p^N}{[y]_n} d[y]_n - [u^{-1}]_n d[u]_n = d\log[m]_n,$$

assuming the element $\frac{p^N}{[y]_n}$ makes sense. This informal argument motivates the formula

$$xd[y]_n - [u^{-1}]_n d[u]_n \in W_n\Omega^1_A$$

appearing below.

We will eventually equip $W.\Omega^\bullet_A$ with the structure of a log Witt complex over $(A, A\setminus\{0\})$. The point of the following lemma is to identify candidates for the elements $d\log[m]_n \in W_n\Omega^1_A$ for $m \in A \setminus \{0\}$.

Lemma 7.6. *(1) Let A denote a p-torsion-free perfectoid ring, and assume further that A is a valuation ring. Let $m \in A \setminus \{0\}$ be arbitrary, and let $n \geq 1$ be an integer. Let $[m]_n \in W_n(A)$ denote the Teichmüller lift of m in $W_n(A)$. There exists an integer $N \geq 1$ and elements $u \in A^\times$, $y \in A$, $x \in W_n(A)$ such that*

$$[m]_n \mid p^N \in W_n(A)$$
$$y^{p^N} = um \in A$$
$$x[y]_n = p^N \in W_n(A).$$

(2) Keep notation as in the previous part. The element

$$xd([y]_n) - [u^{-1}]_n d([u]_n) \in W_n\Omega^1_A$$

depends only on m and n; it is independent of the choice of N, u, y, x.

Proof. Proof of (1). The existence of N follows from Lemma 7.4. The existence of u and y follows from Lemma 7.3. The existence of x is then obvious.

Proof of (2). This part of the proof is more involved. We assume the quadruples N_1, u_1, y_1, x_1 and N_2, u_2, y_2, x_2 are as in the statement. We first reduce to the case that $N_1 = N_2$. Assume $z^p = u_3 y_1$ (using Lemma 7.3), and hence $z^{p^{N_1+1}} = u_3^{p^{N_1}} u_1 m$. We then have that $N_1 + 1, u_3^{p^{N_1}} u_1, z, px_1[u_3^{-1} z^{p-1}]_n$ is also a quadruple N, u, y, x as in the statement. We compute

$$\begin{aligned} x_1 d([y_1]_n) - [u_1^{-1}]_n d([u_1]_n) &= x_1 d([u_3^{-1}]_n [z]_n^p) - [u_1^{-1}]_n d([u_1]_n) \\ &= x_1 [u_3^{-1}]_n p [z]_n^{p-1} d([z]_n) - [u_3]_n^{-2} x_1 [z]_n^p d([u_3]_n) - [u_1^{-1}]_n d([u_1]_n) \\ &= px_1 [u_3^{-1} z^{p-1}]_n d([z]_n) - [u_3^{-p^{N_1}} u_1^{-1}]_n d([u_3^{p^{N_1}}]_n [u_1]_n). \end{aligned}$$

Thus we can always replace a quadruple involving N with a quadruple involving a larger value of N without changing the proposed expression $xd([y]_n) - [u^{-1}]_n d([u]_n) \in W_n \Omega_A^1$. Thus we may assume $N_1 = N_2$.

We now assume the quadruples N, u_1, y_1, x_1 and N, u_2, y_2, x_2 are as in the statement. Because A is a valuation ring, we can without loss of generality assume $y_2 = y_1 v$ for some $v \in A$. Raising both sides to the p^n power and cancelling out m from both sides, we find $u_2/u_1 = v^{p^N} \in A$; in particular, $v \in A$ is a unit. Because

$$x_1 [y_1]_n = x_2 [y_2]_n = x_2 [y_1]_n [v]_n \in W_n(A)$$

and because $[y_1]_n$ is a non-zero-divisor in $W_n(A)$ (since it divides the non-zero-divisor $p^{N_1} \in W_n(A)$), we have that $x_1 [v]_n^{-1} = x_2 \in W_n(A)$.

We compute

$$\begin{aligned} x_2 d([y_2]_n) - [u_2^{-1}]_n d([u_2]_n) &= x_1 [v]_n^{-1} d([v]_n [y_1]_n) - [u_2^{-1}]_n d([u_2]_n) \\ &= x_1 d([y_1]_n) + x_1 [y_1]_n [v]_n^{-1} d([v]_n) - [u_2^{-1}]_n d([u_2]_n) \\ &= x_1 d([y_1]_n) + p^N [v]_n^{-1} d([v]_n) - [u_2^{-1}]_n d([u_2]_n) \\ &= x_1 d([y_1]_n) - [u_1^{-1}]_n d([u_1]_n). \end{aligned}$$

∎

We are now ready to prove Theorem 7.1. We again thank the anonymous referee of an earlier version of this paper for providing several of these arguments.

Proof of Theorem 7.1. It suffices to equip $W.\Omega_A^\bullet$ with the structure of a log Witt complex over (A, M), because we will then have natural maps between these initial objects

$$W.\Omega_A^\bullet \to W.\Omega_{(A,M)}^\bullet \to W.\Omega_A^\bullet \to W.\Omega_{(A,M)}^\bullet$$

such that the two-map compositions are equal to the identity map.

The main difficulty is to define, for all integers $n \geq 1$, a suitable map

$$d\log : M \to W_n \Omega_A^1.$$

We define $d\log$ using the formula given in Part 2 of Lemma 7.6; that same lemma shows that the formula is well-defined. It remains to check that the given map satisfies all required properties to make $W.\Omega_A^\bullet$ a log Witt complex over (A, M).

We first check that $d\log(m_1 m_2) = d\log(m_1) + d\log(m_2)$. Choose an integer N such that $[m_1]_n \mid p^N$ and $[m_2]_n \mid p^N$ in $W_n(A)$, and using Lemma 7.3, choose $u_1, u_2 \in A^\times$ and $y_1, y_2 \in A$ and $x_1, x_2 \in W_n(A)$

such that $y_1^{p^{2N}} = u_1m_1$, $y_2^{p^{2N}} = u_2m_2$, $x_1[y_1]_n = p^N \in W_n(A)$ and $x_2[y_2]_n = p^N \in W_n(A)$. Notice that $(y_1y_2)^{p^{2N}} = u_1u_2m_1m_2 \in A$ and $x_1x_2[y_1y_2]_n = p^{2N} \in W_n(A)$. Thus, on one hand,

$$d\log(m_1m_2) = x_1x_2d([y_1y_2]_n) - [u_1u_2]_n^{-1}d([u_1u_2]_n).$$

On the other hand, using $p^Nx_1[y_1]_n = p^{2N}$ and $p^Nx_2[y_2]_n = p^{2N}$, we have

$$d\log(m_1) + d\log(m_2) = p^Nx_1d([y_1]_n) + p^Nx_2d([y_2]_n) - [u_1]_n^{-1}d([u_1]_n) - [u_2]_n^{-1}d([u_2]_n).$$

It follows that $d\log(m_1m_2) = d\log(m_1) + d\log(m_2)$.

It is easy to see that the restriction of $d\log([m]_{n+1})$ is $d\log([m]_n)$: namely, simply find the necessary elements N, u, y, x for $d\log([m]_{n+1})$, and notice that we can use the elements $N, u, y, R(x)$ for $d\log([m]_n)$.

We next check that $F(d\log([m]_{n+1})) = d\log([m]_n)$. Choose $N \geq 0$ so that $[m]_{n+1} \mid p^N \in W_{n+1}(A)$, and choose $u \in A^\times$ and $y \in A$ such that $y^{p^N} = um \in A$. Lastly, choose $x \in W_{n+1}(A)$ such that $x[y]_{n+1} = p^N \in W_{n+1}(A)$. We then compute

$$\begin{aligned} F(d\log([m]_{n+1})) &= F\Big(xd([y]_{n+1}) - [u]_{n+1}^{-1}d([u]_{n+1})\Big) \\ &= F(x)[y]_n^{p-1}d([y]_n) - [u]_n^{-1}d([u]_n). \end{aligned}$$

On the other hand, we have $[m]_n \mid p^N \in W_n(A)$ and we still have $y^{p^N} = um \in A$. If we apply F to both sides of $x[y]_{n+1} = p^N \in W_{n+1}(A)$, we have $F(x)[y]_n^{p-1}[y]_n = p^N \in W_n(A)$. Thus

$$d\log([m]_n) = F(x)[y]_n^{p-1}d([y]_n) - [u]_n^{-1}d([u]_n).$$

This completes the proof that $F(d\log([m]_{n+1})) = d\log([m]_n)$.

We next check that $d(d\log(m)) = 0 \in W_n\Omega_A^2$ for all $m \in M$ and all $n \geq 1$. Choose $N \geq 0$ such that $[m]_n \mid p^N \in W_n(A)$ and choose $N_1 \geq N$ such that $[m]_{n+N} \mid p^{N_1} \in W_{n+N}(A)$. Find $y_1 \in A$, $u_1 \in A^\times$ such that $y_1^{p^{N_1}} = u_1m$, and find $x_1 \in W_{n+N}(A)$ such that $p^{N_1} = x_1[y_1]_{n+N}$. Applying F^Nd to the equality $p^{N_1} = x_1[y_1]_{n+N}$, we find

$$0 = F^Nd(x_1[y_1]_{n+N}) = F^N(x_1)F^N(d([y_1]_{n+N})) + F^N([y_1]_{n+N})F^N(d(x_1))$$

and so

$$-[y_1]_n^{p^N}F^N(d(x_1)) = F^N(x_1)[y_1]_n^{p^N-1}d([y_1]_n).$$

By our choice of y_1, u_1, x_1 and our formula for $d\log(m)$, we have

$$d\log[m]_{n+N} = x_1d([y_1]_{n+N}) - [u_1^{-1}]_{n+N}d([u_1]_{n+N}),$$

and so

$$d\log[m]_n = F^N(x_1d([y_1]_{n+N})) - F^N([u_1^{-1}]_{n+N}d([u_1]_{n+N}))$$

and furthermore

$$d(d\log[m]_n) = dF^N(x_1d([y_1]_{n+N})) = p^NF^N\Big(d(x_1)d([y_1]_{n+N})\Big) = p^N[y_1]_n^{p^N-1}F^N\Big(d(x_1)\Big)d([y_1]_n).$$

We have $y_1 \mid m \in A$, so $[y_1]_n \mid [m]_n \in W_n(A)$, and we have $[m]_n \mid p^N \in W_n(A)$, so we can write $w[y_1]_n = p^N \in W_n(A)$ for some $w \in W_n(A)$. Thus

$$d(d\log[m]_n) = w[y_1]_n^{p^N}F^N\Big(d(x_1)\Big)d([y_1]_n).$$

We saw above that $[y_1]_n^{p^N}F^N\Big(d(x_1)\Big)$ is a multiple of $d([y_1]_n)$, and so $d(d\log[m]_n) = 0$, as required. ■

8. Connections to algebraic K-theory and topological Hochschild and cyclic homology

Algebraic K-theory provides one of the motivations for studying the p-adic Tate module $T_p\left(W_n\Omega^1_A\right)$ and the generators α and $\alpha_n^{(\infty)}$. We describe these connections in this section following mainly (Hesselholt, 2006). We will assume that the reader has some familiarity with algebraic K-theory; the readers only interested in algebraic aspects of the de Rham-Witt complex can safely skip this section. Most of the results below are well-known to the experts and some of them are more general than we state here (see (Hesselholt, 2006), (Bhatt et al., 2019), (Clausen et al., 2018)). Hence we do not claim any originality in this section. We only put the results of Sections 5 and 7 in a topological context.

Let V be a complete discrete valuation ring with quotient field K of characteristic 0 and with a perfect residue field of odd characteristic p. In (Hesselholt, 2006), Hesselholt studies the p-adic Tate module of $W_n\Omega^1_{(\overline{V},\overline{M})}$, where $\overline{V}$ is the ring of integers of the algebraic closure $\overline{K}$ and $\overline{M} = \overline{V} \setminus \{0\}$. He shows that $T_p\left(W_n\Omega^1_{(\overline{V},\overline{M})}\right)$ is a free $W_n(\overline{V})^\wedge$-module of rank 1 on a certain generator $\alpha_{n,\varepsilon}$. It turns out that one can describe the image of the trace map from the p-adic algebraic K-theory group $K_2(\overline{K},\mathbf{Z}_p)$ to $T_p\left(W_n\Omega^1_{(\overline{V},\overline{M})}\right)$ in terms of this element $\alpha_{n,\varepsilon}$ (Hesselholt, 2006). In particular, one can understand the image of a certain Bott class $\beta_\varepsilon \in K_2(\overline{K},\mathbf{Z}_p)$ using the element $\alpha_{n,\varepsilon}$. We note that this Bott class corresponds to the classical Bott class in complex K-theory by results of Suslin (Suslin, 1983), (Suslin, 1984) which show that $ku^\wedge \simeq K(\overline{K})^\wedge$, where ku is the connective complex topological K-theory spectrum. In this section we show that $\alpha_{n,\varepsilon}$ is in fact a special case of the element constructed in Theorem 5.15 and we compute the image of the Bott class in a more general setting of p-torsion-free perfectoid rings containing a compatible system of p-power roots of unity.

The main tools which connect $K_2(\overline{K},\mathbf{Z}_p)$ with $T_p\left(W_n\Omega^1_{(\overline{V},\overline{M})}\right)$ are the Bökstedt trace and the cyclotomic trace as constructed by (Bökstedt et al., 1993). These are maps from algebraic K-theory to invariants called *topological Hochschild homology* and *topological cyclic homology*, respectively. We give now a very brief overview of these objects.

Given a commutative ring A, the topological Hochschild homology spectrum $\mathrm{THH}(A)$ is defined as the tensor construction $S^1 \otimes A$ which is isomorphic to the geometric realization of the cyclic bar construction of A over the sphere spectrum. For various equivalent definitions of $\mathrm{THH}(A)$ and the equivalences between them, see (Bökstedt, 1986), (Shipley, 2000), (Angeltveit et al., 2018), (Nikolaus and Scholze, 2018), (Dotto et al., 2019). The classical Dennis trace map

$$K(A) \to \mathrm{HH}(A)$$

going from algebraic K-theory of A to Hochschild homology can be refined to the map of spectra

$$\mathrm{trace}: K(A) \to \mathrm{THH}(A),$$

known as the Bökstedt trace. The spectrum $\mathrm{THH}(A)$ has an action of the circle group S^1 by definition and also has a structure of a *p-cyclotomic spectrum* in the sense of (Nikolaus and Scholze, 2018). More precisely it comes equipped with an S^1-equivariant map $\phi_{\mathrm{cycl}}\colon \mathrm{THH}(A) \to \mathrm{THH}(A)^{tC_p}$, where $(-)^{tC_p}$ denotes the Tate construction (see (Greenlees and May, 1995) and (Nikolaus and Scholze, 2018)). This map is referred to as the *cyclotomic Frobenius*. Let $\mathrm{THH}(A,\mathbf{Z}_p)$ denote the p-completion of $\mathrm{THH}(A)$, also referred to as the p-adic THH. We recall from (Nikolaus and Scholze, 2018) and (Bhatt et al., 2019), the following notations:

$$\mathrm{TC}^-(A,\mathbf{Z}_p) = \mathrm{THH}(A,\mathbf{Z}_p)^{hS^1}$$

and

$$\mathrm{TP}(A, \mathbf{Z}_p) = \mathrm{THH}(A, \mathbf{Z}_p)^{tS^1}.$$

Here t again stands for the Tate construction and h for the homotopy fixed points. Using the equivalence $(\mathrm{THH}(A)^{tC_p})^{hS^1} \simeq \mathrm{TP}(A, \mathbf{Z}_p)$ (see (Nikolaus and Scholze, 2018, Lemma II.4.2)), one gets a map

$$\phi_{\mathrm{cycl}}^{hS^1}\colon \mathrm{TC}^-(A, \mathbf{Z}_p) \to \mathrm{TP}(A, \mathbf{Z}_p).$$

Additionally one has the canonical map from homotopy fixed points to the Tate construction

$$\mathrm{can}\colon \mathrm{TC}^-(A, \mathbf{Z}_p) \to \mathrm{TP}(A, \mathbf{Z}_p).$$

One then defines the p-adic topological cyclic homology TC via the fiber sequence of spectra (see (Nikolaus and Scholze, 2018))

$$\mathrm{TC}(A, \mathbf{Z}_p) \longrightarrow \mathrm{TC}^-(A, \mathbf{Z}_p) \xrightarrow{\mathrm{can} - \phi_{\mathrm{cycl}}^{hS^1}} \mathrm{TP}(A, \mathbf{Z}_p).$$

One can also consider $\mathrm{THH}(A)$ as a genuine S^1-equivariant spectrum with respect to the family of finite subgroups and take the derived fixed points $\mathrm{THH}(A)^{C_{p^{n-1}}}$. This spectrum is denoted by $\mathrm{TR}^n(A)$. We note that $\mathrm{TR}^n(A)$ spectra can be also constructed using the cyclotomic Frobenius as in (Nikolaus and Scholze, 2018, Theorem II.4.10). The spectra $\mathrm{TR}^n(A)$ for various values of n are related by morphisms

$$F : \mathrm{TR}^{n+1}(A) \to \mathrm{TR}^n(A)$$

and

$$V : \mathrm{TR}^n(A) \to \mathrm{TR}^{n+1}(A).$$

The map F is induced by the fixed points inclusion and V is the transfer. Moreover, the cyclotomic structure induces an S^1-equivariant map (with respect residual S^1-actions)

$$R : \mathrm{TR}^{n+1}(A) \to \mathrm{TR}^n(A).$$

These maps induce obvious maps on graded homotopy rings $\pi_* \mathrm{TR}^n(A)$, $n \geq 1$ (denoted by the same letters). Moreover, the circle action induces the differential

$$d : \pi_* \mathrm{TR}^n(A) \to \pi_* \mathrm{TR}^n(A)[-1].$$

It follows from (Hesselholt and Madsen, 1997, Theorem 2.3) that there is natural isomorphism $\lambda_n^0\colon \pi_0 \mathrm{TR}^n(A) \cong W_n(A)$. Now the results of (Hesselholt and Madsen, 2004) imply that for any $\mathbf{Z}_{(p)}$-algebra A, where the prime p is odd,

$$(\pi_* \mathrm{TR}^\bullet(A), R, F, V, d, \lambda_\bullet^0)$$

forms a Witt complex and hence there is a unique map of Witt complexes from the de Rham-Witt complex over A to the latter Witt complex:

$$\lambda_\bullet^* : W_\bullet\Omega_A^* \to \pi_* \mathrm{TR}^\bullet(A).$$

A theorem of Hesselholt shows that in fact the map

$$\lambda_\bullet^1 : W_\bullet\Omega_A^1 \to \pi_1 \mathrm{TR}^\bullet(A)$$

is an isomorphism (for p an odd prime) (Hesselholt, 2004).

The trace map $\mathrm{trace} : K(A) \to \mathrm{THH}(A)$ is S^1-invariant and has refinements

$$\mathrm{trace} : K(A) \to \mathrm{TR}^n(A)$$

and

$$\text{trace}: K(A,\mathbf{Z}_p) \to \mathrm{TC}(A,\mathbf{Z}_p).$$

It follows from (Bhatt et al., 2019, Proposition 7.17), (Anschütz and Le Bras, 2020, Corollary 6.5) and (Clausen et al., 2018, Corollary 6.9) the canonical maps

$$K_2(A,\mathbf{Z}_p) \xrightarrow{\cong} T_p(K_1(A)) \xleftarrow{\cong} T_p(A^\times)$$

are isomorphisms. Here the first map is the map from the universal coefficient sequence for $K(A)$ (see Theorem 2.3 which is stated for chain complexes but works similarly for spectra) and the second is induced by the homomorphism $A^\times \to GL(A)$. By naturality we have a commutative diagram

$$\begin{array}{ccc} K_2(A,\mathbf{Z}_p) & \xrightarrow{\cong} & T_p(K_1(A)) \\ \Big\downarrow{\scriptstyle \text{trace}} & & \Big\downarrow{\scriptstyle T_p(\text{trace})} \\ \pi_2 \,\mathrm{TR}^n(A,\mathbf{Z}_p) & \longrightarrow & T_p(\pi_1 \,\mathrm{TR}^n(A)). \end{array}$$

We also recall a result of Geisser and Hesselholt (Geisser and Hesselholt, 1999, Lemma 4.2.3), which states that the composite

$$A^\times \longrightarrow K_1(A) \xrightarrow{\text{trace}} \pi_1 \,\mathrm{TR}^\bullet(A) \cong W_\bullet \Omega^1_A$$

is given by $a \mapsto d\log[a] = [a]^{-1}d[a]$.

Let A be a p-torsion-free perfectoid ring containing a compatible system of p-power roots of unity

$$\varepsilon = (1, \zeta_p, \zeta_{p^2}, \zeta_{p^3}, \dots).$$

We define the Bott class $\beta_\varepsilon \in K_2(A,\mathbf{Z}_p)$ to be the element corresponding to ε under the isomorphism $K_2(A,\mathbf{Z}_p) \cong T_p(A^\times)$.

Proposition 8.1. *Let A be a p-torsion-free perfectoid ring containing a compatible system of p-power roots of unity*

$$\varepsilon = (1, \zeta_p, \zeta_{p^2}, \zeta_{p^3}, \dots).$$

Then the composite

$$K_2(A,\mathbf{Z}_p) \xrightarrow{\text{trace}} \pi_2 \,\mathrm{TR}^n(A,\mathbf{Z}_p) \longrightarrow T_p(\pi_1 \,\mathrm{TR}^n(A)) \cong T_p(W_n\Omega^1_A)$$

sends the Bott class β_ε to $([\zeta_{p^n}]-1)\alpha_n^{(\infty)}$.

Proof. By definition of β_ε and the commutative square above, it suffices to compute the image of ε under the composite

$$T_pA^\times \longrightarrow T_pK_1(A) \xrightarrow{T_p(\text{trace})} T_p(\pi_1 \,\mathrm{TR}^\bullet(A)) \cong T_p(W_\bullet\Omega^1_A).$$

Now by (Geisser and Hesselholt, 1999, Lemma 4.2.3), it follows that ε goes to $d\log[\zeta_p^{(\infty)}]$. By Theorem 5.15, the latter is equal to $([\zeta_{p^n}]-1)\alpha_n^{(\infty)}$. ∎

Let $\mathrm{TF}(A)$ denote the homotopy inverse limit $\mathrm{holim}_F \,\mathrm{TR}^n(A)$. The trace maps trace : $K(A) \to \mathrm{TR}^n(A)$ assemble into a trace map trace : $K(A) \to \mathrm{TF}(A)$.

Corollary 8.2. *Let A be a p-torsion-free perfectoid ring containing a compatible system of p-power roots of unity*

$$\varepsilon = (1, \zeta_p, \zeta_{p^2}, \zeta_{p^3}, \dots).$$

Then the composite

$$K_2(A, \mathbf{Z}_p) \xrightarrow{\text{trace}} \pi_2 \operatorname{TF}(A, \mathbf{Z}_p) \longrightarrow \varprojlim_F T_p(\pi_1 \operatorname{TR}^n(A)) \cong \varprojlim_F T_p(W_n\Omega^1_A)$$

sends the Bott class β_ε to $([\varepsilon] - 1)\alpha$, where α is the generator from Corollary 5.18.

Proof. By Theorem 5.15 the map $F : T_p(W_{n+1}\Omega^1_A) \to T_p(W_n\Omega^1_A)$ sends $\alpha^{(\infty)}_{n+1}$ to $\alpha^{(\infty)}_n$ and the diagram

$$\begin{array}{ccccccc} K_2(A, \mathbf{Z}_p) & \xrightarrow{\text{trace}} & \pi_2 \operatorname{TR}^{n+1}(A, \mathbf{Z}_p) & \longrightarrow & T_p(\pi_1 \operatorname{TR}^{n+1}(A)) & \xleftarrow{\cong} & T_p(W_{n+1}\Omega^1_A) \\ & \searrow^{\text{trace}} & \downarrow F & & \downarrow F & & \downarrow F \\ & & \pi_2 \operatorname{TR}^n(A, \mathbf{Z}_p) & \longrightarrow & T_p(\pi_1 \operatorname{TR}^n(A)) & \xleftarrow{\cong} & T_p(W_n\Omega^1_A) \end{array}$$

commutes. This implies the desired result. ■

It turns out that α and $\alpha^{(\infty)}_n$ in fact determine polynomial generators of $\operatorname{TF}(A, \mathbf{Z}_p)$ and $\operatorname{TR}^n(A, \mathbf{Z}_p)$. It follows from (Nikolaus and Scholze, 2018, Lemma II.4.9) and (Bhatt et al., 2019, Proposition 6.5, Remark 6.6) that $\pi_* \operatorname{TR}^n(A, \mathbf{Z}_p)$ is a polynomial algebra in one variable of degree 2 over $W_n(A)$. In particular $\pi_2 \operatorname{TR}^n(A, \mathbf{Z}_p)$ isomorphic to $W_n(A)$ as a $W_n(A)$-module, and hence the canonical surjective map

$$\pi_2 \operatorname{TR}^n(A, \mathbf{Z}_p) \to T_p(\pi_1 \operatorname{TR}^n(A)) \cong T_p(W_n\Omega^1_A)$$

is an isomorphism since the target $T_p(W_n\Omega^1_A)$ is also a free $W_n(A)$-module on one generator by Theorem 5.15. Consider the inverse

$$T_p(W_n\Omega^1_A) \xrightarrow{\cong} \pi_2 \operatorname{TR}^n(A, \mathbf{Z}_p).$$

Since $T_p(W_n\Omega^1_A)$ is a free $W_n(A)$-module on the generator $\alpha^{(\infty)}_n$, we get that the map

$$W_n(A)[\alpha^{(\infty)}_n] \xrightarrow{\cong} \pi_* \operatorname{TR}^n(A, \mathbf{Z}_p)$$

is an isomorphism of graded rings. Now again using $F\alpha^{(\infty)}_{n+1} = \alpha^{(\infty)}_n$, one obtains the following corollary (see (Bhatt et al., 2019, Remark 6.6)).

Corollary 8.3. *Let A be a p-torsion-free perfectoid ring containing a compatible system of p-power roots of unity*

$$\varepsilon = (1, \zeta_p, \zeta_{p^2}, \zeta_{p^3}, \dots).$$

Then the canonical map

$$\pi_2 \operatorname{TF}(A, \mathbf{Z}_p) \to \varprojlim_F T_p(\pi_1 \operatorname{TR}^n(A)) \cong \varprojlim_F T_p(W_n\Omega^1_A).$$

is an isomorphism. This isomorphism induces an isomorphism of graded rings

$$W(A^\flat)[\alpha] \xrightarrow{\cong} \pi_* \operatorname{TF}(A, \mathbf{Z}_p).$$

Proof. The F maps induce surjections on homotopy groups. Hence by the Milnor sequence $\pi_* \operatorname{TF}(A, \mathbf{Z}_p) \cong \varprojlim_F \pi_* \operatorname{TR}^n(A, \mathbf{Z}_p)$. The rest follows from the isomorphism of Lemma 2.16 $W(A^\flat) \cong \varprojlim_F W_n(A)$. ■

Remark 8.4. The latter results together with (Bhatt et al., 2018, Lemma 3.13) (referred to as Tor-independence in (Bhatt et al., 2018)) in fact imply the following: Let $A \to B$ denote a ring homomorphism, where both A and B are p-torsion free perfectoid rings with compatible systems of p-power roots of unity. Assume that the homomorphism preserves these systems. Then for any $1 \leq n \leq r$,

$$\mathrm{TR}^n(A, \mathbf{Z}_p) \otimes^L_{\mathrm{TR}^r(A,\mathbf{Z}_p)} \mathrm{TR}^r(B, \mathbf{Z}_p) \simeq \mathrm{TR}^n(B, \mathbf{Z}_p),$$

where $\mathrm{TR}^n(A, \mathbf{Z}_p)$ is a $\mathrm{TR}^r(A, \mathbf{Z}_p)$-module via F. (This uses that $\mathrm{TF}(-, \mathbf{Z}_p)/(\sum_{i=0}^{p^n-1} [\varepsilon]^i) \simeq \mathrm{TR}^n(-, \mathbf{Z}_p)$.) In fact (Bhatt et al., 2018, Lemma 3.13) has two versions: one has

$$W_n(A) \otimes^L_{W_r(A)} W_r(B) \simeq W_n(B),$$

where the $W_n(A)$ is considered as a $W_r(A)$-module via F or R. It turns out that using the results from Section 5, we also get the following: Let $W_n\Omega^1_A$ be equipped with a $W_r(A)$-module structure via either Frobenius or restriction. Then the induced map on derived p-completions

$$\left(W_n\Omega^1_A \otimes^L_{W_r(A)} W_r(B)\right)^\wedge \to \left(W_n\Omega^1_B\right)^\wedge$$

is a quasi-isomorphism. To see this, it suffices to prove that

$$\left(\mathbf{F}_p \otimes^L_{\mathbf{Z}} W_n\Omega^1_A\right) \otimes^L_{W_r(A)} W_r(B) \xrightarrow{\simeq} \mathbf{F}_p \otimes^L_{\mathbf{Z}} W_n\Omega^1_B.$$

As usual, we replace $\mathbf{F}_p$ with the complex $\cdots \to 0 \to \mathbf{Z} \xrightarrow{p} \mathbf{Z} \to 0 \to \cdots$. Because multiplication by p is surjective on both $W_n\Omega^1_A$ and $W_n\Omega^1_B$, we reduce to showing that the following is a quasi-isomorphism

$$W_n\Omega^1_A[p] \otimes^L_{W_r(A)} W_r(B) \xrightarrow{\simeq} W_n\Omega^1_B[p],$$

where the de Rham-Witt groups are viewed as complexes concentrated in degree -1. By our earlier results, we reduce to showing that the following is a quasi-isomorphism

$$W_n(A)/pW_n(A) \otimes^L_{W_r(A)} W_r(B) \xrightarrow{\simeq} W_n(B)/pW_n(B).$$

Because A and B (and hence $W_n(A)$ and $W_n(B)$) are p-torsion free, we reduce to showing that the following is a quasi-isomorphism

$$\left(\mathbf{F}_p \otimes^L_{\mathbf{Z}} W_n(A)\right) \otimes^L_{W_r(A)} W_r(B) \xrightarrow{\simeq} \mathbf{F}_p \otimes^L_{\mathbf{Z}} W_n(B).$$

The result now follows from (Bhatt et al., 2018, Lemma 3.13).

The topological result at the beginning of this remark recovers the F-versions of the algebraic equivalences on π_0 and π_2. We do not know what is the topological analogue of the equivalences involving the map R.

Finally, we explain the connection to the results of (Hesselholt, 2006). Let V be a complete discrete valuation ring with quotient field K of characteristic 0 and with a perfect residue field of odd characteristic p. In (Hesselholt, 2006), Hesselholt studies the p-adic Tate module of $W_n\Omega^1_{(\overline{V},\overline{M})}$, where $\overline{V}$ is the ring of integers of the algebraic closure $\overline{K}$ and $\overline{M} = \overline{V} \setminus \{0\}$. He shows that $T_pW_n\Omega^1_{(\overline{V},\overline{M})}$ is a free $W_n(\overline{V})^\wedge$-module of rank 1 on certain generator $\alpha_{n,\varepsilon}$.

The ring $\overline{V}^\wedge$ satisfies the conditions of Theorem 5.15 and Theorem 7.1. Hence the canonical maps

$$T_pW_n\Omega^1_{(\overline{V},\overline{M})} \longrightarrow T_pW_n\Omega^1_{(\overline{V}^\wedge,\overline{M}^\wedge)} \xleftarrow{\cong} T_pW_n\Omega^1_{\overline{V}^\wedge}$$

are module maps between rank 1 free $W_n(\overline{V})^\wedge \cong W_n(\overline{V}^\wedge)$-modules and the right hand map is an isomorphism. Now by (Hesselholt, 2006, Theorem B, Lemma 2.4.1, Proposition 2.4.2), there is a generator $\alpha_{n,\varepsilon}$ of the left hand side such that

$$([\zeta_{p^n}]-1)\alpha_{n,\varepsilon} = d\log[\zeta_p^{(\infty)}].$$

Since the the latter zig-zag consists of maps of $W_n(\overline{V}^\wedge)$-modules, we see that by Theorem 5.15 the generator $\alpha_{n,\varepsilon}$ on the left is mapped to $\alpha_n^{(\infty)}$ and hence the first map in the zig-zag is an isomorphism too. Finally, we conclude that the generator α_ε constructed in (Hesselholt, 2006, Proposition 2.4.2) corresponds to α from Corollary 5.18 under the isomorphism

$$\varprojlim_F T_pW_n\Omega^1_{(\overline{V},\overline{M})} \xrightarrow{\cong} \varprojlim_F T_pW_n\Omega^1_{(\overline{V}^\wedge,\overline{M}^\wedge)} \xleftarrow{\cong} \varprojlim_F T_pW_n\Omega^1_{\overline{V}^\wedge}.$$

Remark 8.5. It follows from the results of Suslin that $K_*(\overline{K},\mathbf{Z}_p) \cong \mathbf{Z}_p[\beta_\varepsilon]$. We also note that $K_2(\overline{K},\mathbf{Z}_p) \cong K_2(\overline{V}^\wedge,\mathbf{Z}_p)$ and that the Bott class corresponds to the Bott class in $K_2(\overline{V}^\wedge,\mathbf{Z}_p)$. Hence $\beta_\varepsilon \in K_2(A,\mathbf{Z}_p)$ corresponding to $\varepsilon = (1,\zeta_p,\zeta_{p^2},\zeta_{p^3},\dots)$ under the isomorphism $K_2(A,\mathbf{Z}_p) \cong T_pA^\times$, generalizes the Bott class of (Hesselholt, 2006), (Suslin, 1983) and (Suslin, 1984) to any p-torsion-free perfectoid ring A containing a compatible system of p-power roots of unity. We note that $T_pA^\times \cong \mathbf{Z}_p$ with ε a generator if additionally A is an integral domain but not in general.

Remark 8.6. We also point out how the generators $\alpha_n^{(\infty)}$ are related to the recent result of (Anschütz and Le Bras, 2020). Let A be a p-torsion-free perfectoid ring containing a compatible system of p-power roots of unity. Theorem 6.4 of (Anschütz and Le Bras, 2020) computes the composition

$$T_pA^\times \cong K_2(A,\mathbf{Z}_p) \xrightarrow{\text{trace}} \pi_2\,\mathrm{TC}(A,\mathbf{Z}_p)$$

which is an isomorphism by (Clausen et al., 2018, Corollary 6.9) and (Anschütz and Le Bras, 2020, Corollary 6.5). It follows from (Bhatt et al., 2019, Section 6), the definition of TC and (Anschütz and Le Bras, 2020, Section 6) that

$$\pi_2\,\mathrm{TC}(A,\mathbf{Z}_p) \cong \left\{ y \in W_n(A^\flat)\colon \varphi(y) = \frac{[\varepsilon^p]-1}{[\varepsilon]-1}y \right\},$$

and under this isomorphism the latter composite is given by the q-logarithm:

$$\log_q([x]) = \sum_{n=1}^{\infty}(-1)^{n-1}q^{-n(n-1)/2}\frac{([x]-1)([x]-q)([x]-q^2)\cdots([x]-q^{n-1})}{[n]_q} \in W(A^\flat),$$

where $q=[\varepsilon] \in W(A^\flat)$ and $[n]_q = \frac{q^n-1}{q-1}$. Recall that there is a cofiber sequence (Nikolaus and Scholze, 2018, Theorem II.4.10)

$$\mathrm{TC}(A,\mathbf{Z}_p) \longrightarrow \mathrm{TF}(A,\mathbf{Z}_p) \xrightarrow{R-1} \mathrm{TF}(A,\mathbf{Z}_p).$$

The diagram

$$\begin{array}{ccc} T_pA^\times \cong K_2(A,\mathbf{Z}_p) & \xrightarrow{\text{trace}} & \pi_2\,\mathrm{TC}(A,\mathbf{Z}_p) \\ & \searrow^{\text{trace}} & \downarrow \\ & & \pi_2\,\mathrm{TF}(A,\mathbf{Z}_p) \end{array}$$

commutes. Under the isomorphism $\pi_2 \operatorname{TF}(A, \mathbf{Z}_p) \cong \varprojlim_F T_p(W_n\Omega^1_A)$, the vertical map in this diagram corresponds to the inclusion

$$\left\{ y \in W_n(A^\flat) \colon \varphi(y) = \frac{[\varepsilon^p]-1}{[\varepsilon]-1} y \right\} \cong \left\{ y\alpha \in \varprojlim_F T_p(W_n\Omega^1_A) \colon y \in W(A^\flat) \text{ satisfies } \varphi(y) = \frac{[\varepsilon^p]-1}{[\varepsilon]-1}\, y \right\} =$$
$$\left\{ x \in \varprojlim_F T_p(W_n\Omega^1_A) \colon R(x) = x \right\} \subset \varprojlim_F T_p(W_n\Omega^1_A).$$

(See Proposition 5.21) If we now plug in $\varepsilon = x$ in the above formula, we get $\log_q([\varepsilon]) = [\varepsilon] - 1$ and hence we recover that the composite

$$T_pA^\times \cong K_2(A, \mathbf{Z}_p) \xrightarrow{\text{trace}} \pi_2 \operatorname{TC}(A, \mathbf{Z}_p) \longrightarrow \pi_2 \operatorname{TF}(A, \mathbf{Z}_p) \cong \varprojlim_F T_p(W_n\Omega^1_A)$$

sends ε to $([\varepsilon] - 1)\alpha$. This is an alternative way of computing the image of the Bott class in $\varprojlim_F T_p(W_n\Omega^1_A)$.

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