Vertex F-algebra structures on the complex oriented homology of H-spaces

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\textbf{A R T I C L E I N F O}

\textbf{Article history:}
Received 27 August 2021
Received in revised form 10 January 2022
Available online 20 January 2022
Communicated by C.A. Weibel

\textbf{MSC:}
17B69; 55N20

\textbf{Keywords:}
Vertex algebra
Formal group law
Generalized cohomology
H-space

\textbf{A B S T R A C T}

We give a topological construction of graded vertex $F$-algebras by generalizing Joyce’s vertex algebra construction to complex-oriented homology. Given an H-space $X$ with a $BU(1)$-action, a choice of K-theory class, and a complex oriented homology theory $E$, we build a graded vertex $F$-algebra structure on $E_\ast(X)$ where $F$ is the formal group law associated with $E$.

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1. Introduction and results

The algebraic topology of moduli stacks, arising for example in algebraic geometry and gauge theory, is of fundamental importance for the study of invariants. Let \( \mathcal{A} \) be an additive \( \mathbb{C} \)-linear dg-category, whose \( \tau \)-stable objects we wish to classify, for \( \tau \) a stability condition. The category \( \mathcal{A} \) has an associated moduli stack \( \mathcal{M}_\mathcal{A} \) by [19]. In [8], Joyce constructs a graded vertex algebra on the ordinary homology \( H_*(\mathcal{M}_\mathcal{A}) \). Vertex algebras are algebraic structures with origins in conformal field theory which can be regarded as singular commutative rings whose operation \( Y: V \otimes V \rightarrow V((z)) \), the state-to-field correspondence, takes values in Laurent series. This profound algebraic structure is used to describe wall-crossing formulas relating the virtual fundamental classes \( [\mathcal{M}_\mathcal{A}]^{\text{virt}}_\tau, [\mathcal{M}_\mathcal{A}]^{\text{virt}}_\nu \in H_*(\mathcal{M}_\mathcal{A}) \) for different stability conditions. These are powerful tools for computing invariants.

Motivated by physics, many authors currently investigate refined invariants such as \( K \)-theoretic Donaldson–Thomas invariants [5,6,13,18]. Here the virtual classes should be viewed in \( K \)-homology \( K_*(\mathcal{M}_\mathcal{A}) \).

As a first step towards extending wall-crossing formulas to refined invariants, we here extend Joyce’s construction to any generalized (complex oriented) homology theory \( E_* \) with associated formal group law \( F(z,w) \). Our main result constructs a vertex \( F \)-algebra structure on \( E_*(\mathcal{M}_\mathcal{A}) \) in the sense of Li [14].

In addition, our construction of vertex \( F \)-algebra works in greater generality, namely for any topological H-space (i.e. abelian group up to homotopy) with an action of \( BU(1) \).

Let \( E^* \) be a complex oriented generalized cohomology theory with associated formal group law \( F(z,w) \) over its coefficient ring \( R_* \), see §3. As a preliminary result, we present a Laurent-polynomial version of the Conner-Floyd Chern classes (see Definition 3.4) with values in \( E^* \).

**Theorem 1.1.** For every class \( \theta \in K^0(X) \) in the topological \( K \)-theory of topological space \( X \) there is an \( R_* \)-linear transformation

\[
(-) \cap C^F(z,\theta): E_*(X) \rightarrow E_*(X)[[z]][z^{-1}] \quad a \mapsto a \cap C^F_z(\theta), \tag{1.1}
\]

of degree \(-2r \) if \( \theta \) has constant rank \( r \in \mathbb{Z} \), with the following properties:

(a) (Naturality.) For continuous \( f: X' \rightarrow X, \theta \in K^0(X), \) and \( a' \in E_*(X') \)

\[
f_*\left(a' \cap C^F_z(f^*(\theta))\right) = f_*\left(a'\right) \cap C^F_z(\theta). \tag{1.2}
\]

(b) (Direct sums.) For \( \zeta, \theta \in K^0(X) \) and \( a \in E_*(X) \) we have

\[
a \cap C^F_z(\zeta + \theta) = [a \cap C^F_z(\zeta)] \cap C^F_z(\theta). \tag{1.3}
\]

(c) (Normalization.) For a complex line bundle \( L \rightarrow X \) and \( a \in E_*(X) \) we have

\[
a \cap C^F_z(L) = a \cap F(z,c^F_1(L)). \tag{1.4}
\]

More generally, for any \( \theta \in K^0(X) \) we have

\[
a \cap C^F_z(L \otimes \theta) = i_{z,c^F_1(L)}(a \cap C^F_z(L \otimes \theta)). \tag{1.5}
\]

Here, as usual, the variable \( z \) has degree \(-2 \). We prove Theorem 1.1 in §4. The notations used in (1.4) and (1.5) will be explained in Notations 3.9 & 4.1 below.

For our main result, let \( X \) be an \( H \)-space with an operation \( \Phi: X \times X \rightarrow X \) that is associative, commutative, and has a unit \( e \in X \) up to homotopy. Recall that the classifying space \( BU(1) \) for complex line bundles
is an H-space with the tensor product $\mu_{BU(1)}$ and trivial bundle $e_{BU(1)}$. Assume there is an action $\Psi$ of $BU(1)$ on $X$ up to homotopy, meaning $\Psi \circ (id_{BU(1)} \times \Psi) \simeq \Psi \circ (\mu_{BU(1)} \times id_X)$ and $\Psi(e_{BU(1)}, -) \simeq id_X$. Suppose $\Psi(e, -) \simeq e$ is an H-fixed point and $\Phi \circ (\Psi \times \Psi) \circ \delta \simeq \Psi \circ (\Phi \times id_{BU(1)})$, where $\delta(x_1, x_2, g) = (x_1, g, x_2, g)$. The set of connected components $\pi_0(X)$ is a monoid with unit $\Omega = [e]$ and operation $\alpha + \beta = \Phi(e \otimes \beta)$ and we partition $X = \bigsqcup_{\alpha \in \pi_0(X)} X_\alpha$. Write $\Phi_{\alpha, \beta} : X_\alpha \times X_\beta \to X_{\alpha + \beta}$, $\Psi_\alpha : BU(1) \times X_\alpha \to X_\alpha$ for the restrictions. Let $\theta_{\alpha, \beta} \in K^0(X_\alpha \times X_\beta)$ for all $\alpha, \beta$.

**Theorem 1.2.** Given $(X, \Phi, e, \Psi)$ as above, suppose the following identities hold for all $\alpha, \beta, \gamma \in \pi_0(X)$:

\[
(\Phi_{\alpha, \beta} \times id_{X_\gamma})^*(\theta_{\alpha + \gamma, \beta}) = \pi_{\alpha, \gamma}^*(\theta_{\alpha, \gamma}) + \pi_{\alpha, \beta}^*(\theta_{\beta, \gamma}), \tag{1.6}
\]

\[
(id_{X_\alpha} \times \Phi_{\beta, \gamma})^*(\theta_{\alpha, \beta + \gamma}) = \pi_{\alpha, \beta}^*(\theta_{\alpha, \beta}) + \pi_{\alpha, \gamma}^*(\theta_{\alpha, \gamma}), \tag{1.7}
\]

\[
(\Psi_{\alpha} \times id_{X_\beta})^*(\theta_{\alpha, \beta}) = \pi_{BU(1)}(\mathcal{L}) \otimes \pi_{\alpha, \beta}^*(\theta_{\alpha, \beta}), \tag{1.8}
\]

\[
(id_{X_\alpha} \times \Psi_{\beta})^*(\theta_{\alpha, \beta}) = \pi_{BU(1)}(\mathcal{L})^\vee \otimes \pi_{\alpha, \beta}^*(\theta_{\alpha, \beta}), \tag{1.9}
\]

\[
\theta|_{X_\alpha \times \{\Omega\}} = 0, \quad \theta|_{\{\Omega\} \times X_\beta} = 0, \tag{1.10}
\]

\[
\sigma^*(\theta_{\alpha, \beta}) = (\theta_{\alpha, \beta})^\vee. \tag{1.11}
\]

Here $\sigma$ swaps the factors of $X_\alpha \times X_\beta$ and $\mathcal{L} \to BU(1)$ is the universal line bundle with dual $\mathcal{L}^\vee$. With the $F$-shift operator $D(z)$ of (3.3) below, the graded $R_*$-module

\[
V_* = \bigoplus_{\alpha \in \pi_0(X)} E_{*-\text{rk} \theta_{\alpha, \alpha}}(X_\alpha) \tag{1.12}
\]

is a graded nonlocal vertex $F$-algebra $(V_*, D, \Omega, Y)$ with state-to-field correspondence

\[
Y(a, z)b = (\Phi_{\alpha, \beta})_* (D_{\alpha}(z) \otimes id_{E_\alpha(X_\beta)}) \left[(a \otimes b) \cap C_z^E(\theta_{\alpha, \beta})\right]. \tag{1.13}
\]

Similarly, the graded $R_*$-module

\[
\overline{V}_* = \bigoplus_{\alpha \in \pi_0(X)} E_{*-2\text{rk} \theta_{\alpha, \alpha}}(X_\alpha) \tag{1.14}
\]

becomes a graded vertex $F$-algebra $(\overline{V}_*, D, \Omega, \overline{Y})$, where

\[
\overline{Y}(a, z)b = (\Phi_{\alpha, \beta})_* (D_{\alpha}(z) \otimes id_{E_\alpha(X_\beta)}) \left[(a \otimes b) \cap \overline{C}_z^F(\theta_{\alpha, \beta})\right] \tag{1.15}
\]

uses the operation of degree $-4\text{rk} \theta_{\alpha, \beta}$ defined by

\[
c \cap \overline{C}_z^F(\theta_{\alpha, \beta}) = [c \cap C_z^F(\theta_{\alpha, \beta})] \cap C_\epsilon z(\sigma^*(\theta_{\beta, \alpha})), \quad c \in E_*(X_\alpha \times X_\beta).\]

Here $\epsilon((z))$ is the inverse for $F$ (see §2). The proof of Theorem 1.2 is given in §5.

As a special case, our result applies to the topological realization $X = \mathcal{M}_A^{\text{top}}$ of a moduli stack. Taking direct sums in the additive category defines $\Phi$ making $\mathcal{M}_A$ into an H-space. Moreover, scaling morphism by $U(1)$ defines an operation $\Psi$ of the quotient stack $[\ast // U(1)]$, endowing $\mathcal{M}_A^{\text{top}}$ with the required action of $BU(1) = [\ast // U(1)]^{\text{top}}$. As shown in Proposition 3.3 below, this action yields an $F$-shift operator $D(z)$. The $K$-theory classes $\theta_{\alpha, \beta}$ are given by the Ext-complexes in the dg-category $\mathcal{A}$, which satisfy (1.6)–(1.11). In geometric examples, one may wish to incorporate signs $\epsilon_{\alpha, \beta}$ into (1.15). These are related to orientations, see [8, §8.3]. The orientation problems were solved in the series [9–11]. For simplicity, we ignore this additional data here and set up a symmetrized construction without signs.
2. Formal groups laws and vertex $F$-algebras

In the section, we will keep everything general and assume the following setup. Later, the data $R_*$ and $F(z, w)$ will arise naturally from a complex oriented cohomology theory, see §3, and $V_*$ will be constructed from an H-space as in (1.12).

Notation 2.1.

- $R_*$ a graded commutative ring with unit. Write $R^+$ for the same ring with the reverse grading, $R^n = R_{-n}$, $n \in \mathbb{Z}$, and $R$ for the ring with the grading removed
- $V_*$ a graded module over $R_*$
- $z, w$ variables of degree $-2$
- $F(z, w)$ a graded formal group law over $R_*$
- $V[[z]]$ the formal power series $\sum_{i=0}^{\infty} a_i z^i$; a ring when $V = R$
- $V((z))$ the $R_*$-module of Laurent series $\sum_{i = -\infty}^{\infty} a_i z^i$ with its partially defined product. The fact that $V((z))$ is not a ring frequently causes confusion.
- The meromorphic series $V[[z]][z^{-1}]$; a ring when $V = R$.
- $i_{z, w}: V[[z, w]][z^{-1}, w^{-1}, F(z, w)^{-1}] \to V((z, w))$ expands $F(z, w)^{-N}$, see Notation 2.4. We have $i_{z, w}(V[[z, w]][F(z, w)^{-1}]) \subset V((z))[w]$.
- $(-1)^a$ means $(-1)^{\text{degree}(a)}$

Definition 2.2. A graded formal group law over $R_*$ is a formal power series $F(z, w) = \sum_{i, j \geq 0} F_{ij} z^i w^j \in R[[z, w]]$ with $F_{ij} \in R_{2i + 2j - 2}$ satisfying

$$F(z, w) = F(w, z), \quad F(z, 0) = z, \quad F(F(z, w), v) = F(z, F(w, v)). \quad (2.1)$$

There exists a unique power series $\iota \in R[[z]]$ with $F(z, \iota(z)) = 0$, the inverse. Note that $\iota(\iota(z)) = z$ and $\iota(F(\iota(z), w)) = F(z, \iota(w))$.

Example 2.3.

(i) The additive formal group law $\mathbb{G}_a$ over $\mathbb{Z}$ (in degree zero) is defined by $F(z, w) = z + w$, and the inverse is $\iota(z) = -z$.

(ii) The multiplicative formal group law $\mathbb{G}_m$ over $\mathbb{Z}$ is defined by $F(z, w) = z + w + zw$ and has $\iota(z) = (1 + z)^{-1} - 1 = -z + z^2 - z^3 + \cdots$.

(iii) There is a universal formal group law $\mathbb{G}_a$ over the Lazard ring $R_L$ generated by variables $F_{ij}$ subject to the relations contained in (2.1).

Notation 2.4. It follows from (2.1) that for a general formal group law

$$F(z, w) = z + w + O(zw), \quad \iota(z) = -z + O(z^2).$$

Write $F(z, w) = z(1 + w/z + wG(z, w))$ and expand using the binomial theorem

$$i_{z, w} F(z, w)^n = \sum_{k=0}^{\infty} \binom{n}{k} z^{n-k} w^k (1 + zG(z, w))^k \in R[[w]][(z)], \quad n \in \mathbb{Z}. \quad (2.2)$$

As the $k$-th summand has $w$-degree $\geq k$, this converges as a formal power series. Define $i_{w, z} F(z, w)^n \in R[[z]][(w)]$ by expanding $F(z, w) = w(1 + z/w + zG(z, w))$ similarly. We extend $i_{z, w}$ and $i_{w, z}$ to $V[[z, w]][z^{-1}, w^{-1}][F(z, w)^{-1}]$ by linearity.
Note that \( i_{z,w}F(z,w)^{-n} \cdot F(z,w)^n = 1 \) and \( i_{w,z}F(w,z)^{-n} \cdot F(z,w)^n = 1 \) for all \( n \geq 0 \). For every \( P(z,w) = \sum_{n \geq 0} a_n(\zeta)F(z,w)^n \in \mathbb{V}[\zeta, w][\zeta^{-1}, w^{-1}]F(z,w)^{-1} \) we thus have
\[
F(z,w)^N(i_{z,w}P(z,w) - i_{w,z}P(z,w)) = 0. \tag{2.3}
\]

**Definition 2.5.** Let \( V \) be a graded \( R_\ast \)-module and \( F \) a graded formal group law over \( R_\ast \). An \( F \)-shift operator is a graded \( R_\ast \)-linear map \( D(z) : V \to V[\zeta] \) with
\[
D(0) = \text{id}_V, \quad D(z) \circ D(w) = D(F(z,w)). \tag{2.4}
\]

**Example 2.6.** Let \( R_\ast = \mathbb{Q} \), \( V = \mathbb{Q}[w] \). Then \( D(z)(f(w)) = e^{z\frac{dw}{2\pi}}f(w) \) defines a \( \mathbb{G}_a \)-shift operator. The relation \( D(z)(f(w)) = f(z + w) \) motivates the terminology.

We now define vertex \( F \)-algebras. For \( F = \mathbb{G}_a \) we recover ordinary vertex algebras, see Frenkel–Ben-Zvi [3], Frenkel–Lepowsky–Meurman [4], and Kac [12].

**Definition 2.7.** Let \( F(z,w) \) be a graded formal group law over \( R_\ast \). A graded nonlocal vertex \( F \)-algebra is a graded \( R_\ast \)-module \( V \), a vacuum vector \( \Omega \in V_0 \), an \( F \)-shift operator \( D(z) \), and a graded \( R_\ast \)-linear state-to-field correspondence
\[
V \otimes_R V \to V[[\zeta]][\zeta^{-1}], \quad a \otimes b \mapsto Y(a, z)b, \tag{2.5}
\]

satisfying the following axioms:

(a) **Vacuum and creation:** \( Y(a, z)\Omega \) is holomorphic for all \( a \in V \) and
\[
Y(a, z)\Omega|_{z=0} = a, \tag{2.6}
\]
\[
Y(\Omega, z) = \text{id}_V. \tag{2.7}
\]

(b) **\( F \)-translation covariance:** for all \( a \in V \) we have
\[
Y(D(w)(a), z) = i_{z,w}Y(a, F(z,w)), \tag{2.8}
\]
\[
D(z)\Omega = \Omega. \tag{2.9}
\]

(c) **Weak \( F \)-associativity:** for all \( a, b, c \in V \) there exists \( N \geq 0 \) with
\[
F(z,w)^{N}Y(Y(a, z)b, w)c = F(z,w)^{N}i_{z,w}Y(a, F(z,w))Y(b, w)c. \tag{2.10}
\]

A graded nonlocal vertex \( F \)-algebra is a graded vertex \( F \)-algebra if, in addition,
\[
Y(a, z)b = (-1)^{ab}D(z) \circ Y(b, e(z))a, \quad \text{for all } a, b \in V. \tag{2.11}
\]

**Remark 2.8.** It is a consequence of (2.6)–(2.11) that for all \( a, b, c \in V \) there exists \( N \geq 0 \) with
\[
(z - w)^N Y(a, z)Y(b, w)c = (-1)^{ab}(z - w)^N Y(b, w)Y(a, z)c. \tag{2.12}
\]

So our definitions agree with those given by Li [14] in the ungraded case.
3. Complex oriented cohomology and Chern classes

Let $E^*$ be a generalized cohomology theory, see for example Rudyak [16, Ch. II, §3]. Thus, for every pair $A \subset X$ of topological spaces there is defined a graded abelian group $E^*(X, A)$. Continuous maps $f: (X, A) \to (X', A')$ induce homomorphisms $f^*: E^*(X', A') \to E^*(X, A)$ that depend only on the homotopy class of $f$. For a pointed space $x_0 \in X$ write $\tilde{E}^*(X) = E^*(X, \{x_0\})$ for reduced cohomology. The smash product of $(X, x_0)$ and $(Y, y_0)$ is the quotient $X \wedge Y = (X \times Y)/(X \cup Y)$ with one-point union $X \vee Y = (X \times \{y_0\}) \cup (\{x_0\} \times Y)$ collapsed to become the new base-point. As part of the structure, $E^*$ comes equipped with natural suspension isomorphisms $\sigma_X: \tilde{E}^*(X) \to \tilde{E}^*(X \wedge S^1)$.

Suppose $E^*$ is a multiplicative generalized cohomology theory. Then there is a bilinear cross product $\boxtimes: E^*(X, A) \otimes E^*(Y, B) \to E^*(X \times Y, X \times B \cup A \times Y)$ and units $1_X \in E^0(X)$, both natural. If we let $R_*= E_*(pt)$ be the coefficient ring, then $R^* = E^*(pt)$ for the reverse grading, which is the reason for this convention in Notation 2.1. Pulling the cross product back along the diagonal makes $E^*(X)$ a graded commutative unital $R^*$-algebra for the cup product ‘$\cup$’ over $R^*$. Dually, there is a homological cross product that in particular makes $E_*(X)$ a graded module over $R_*$. There is a cap product

$$E_a(X) \otimes_R E^b(X) \longrightarrow E_{a-b}(X), \quad a \otimes \varphi \mapsto a \cap \varphi$$

which is $R_*$-linear, unital $a \cap 1 = a$, and natural $f_*(a \cap f^*(\varphi')) = f_*(a) \cap \varphi'$, where $f: X \to X'$ and $\varphi' \in E^b(X')$. See Rudyak [16] for further properties.

**Definition 3.1.** The suspension isomorphism shows that $\tilde{E}^*(\mathbb{C}P^1) \cong \tilde{E}^*(S^2) \cong R^{*-2}$ is a free $R_*$-module on a single generator. A multiplicative cohomology theory $E^*$ is complex orientable if $i^*: \tilde{E}^*(\mathbb{C}P^\infty) \to \tilde{E}^*(\mathbb{C}P^1)$ is surjective, where $\mathbb{C}P^\infty \cong \operatorname{colim}_m \mathbb{C}P^m$ and $i: \mathbb{C}P^1 \to \mathbb{C}P^\infty$. A complex orientation is a choice of $\xi_0 \in \tilde{E}^2(\mathbb{C}P^\infty)$ such that $i^*(\xi_0)$ generates the $R_*$-module $\tilde{E}^*(\mathbb{C}P^1)$.

The presence of the permanent cycle $\xi_0|_{\mathbb{C}P^m}$ implies that the Atiyah–Hirzebruch spectral sequence $H^p(\mathbb{C}P^m; E^q(pt)) \Longrightarrow E^{p+q}(\mathbb{C}P^m)$ collapses, see Adams [1, p. 42]. Hence we have canonical isomorphisms

$$E^*(\mathbb{C}P^m) \cong R[\xi_0]/(\xi_0^{m+1}), \quad E^*(\mathbb{C}P^\infty) \cong \lim E^*(\mathbb{C}P^m) \cong R[\xi_0].$$

More generally, let $P \to X$ be a bundle of projective spaces $\mathbb{C}P^m$ and suppose that $w \in E^*(P)$ restricts on every fiber $P_x$ to generators $1_{P_x}, w|_{P_x}, \ldots, w^m|_{P_x}$ of the $R_*$-module $E^*(P_x)$. Then Dold’s theorem implies that $E^*(P)$ is a free $E^*(X)$-module on $1_P, w, \ldots, w^m$, see [2, (7.4)]. In particular,

$$E^*(X \times \mathbb{C}P^\infty) \cong E^*(X)[\xi_0], \quad E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong R[\pi_1^0(\xi_0), \pi_2^0(\xi_0)].$$

**Definition 3.2.** Let $\xi_0$ be a complex orientation of $E^*$. Write $\mathcal{L} \to \mathbb{C}P^\infty$ for the universal complex line bundle with $\mathcal{L}|_L = L$. Recall that $\mathbb{C}P^\infty = BU(1)$ is an H-space with operation a classifying map $\mu_{\mathbb{C}P^\infty}: \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ of the tensor product $\pi_1^0(\mathcal{L}) \otimes \pi_2^0(\mathcal{L})$ and unit $t_0$ the trivial line bundle. The associated formal group law $F = \sum_{i,j \geq 0} F_{ij} z^i w^j$ is defined by the expansion

$$\mu_{\mathbb{C}P^\infty}(\xi_0) = \sum_{i,j \geq 0} F_{ij} \xi_0^i \xi_0^j, \quad F_{ij} \in R^{2-2i-2j} = R_{2i+2j-2}. \quad (3.2)$$

As in [1, p. 42] the homology $E_*(\mathbb{C}P^\infty)$ is the free $R_*$-module on the dual generators $t_n$, $n \geq 0$, of degree $2n$ characterized by $\langle t_n, \xi_0^m \rangle = \delta_n^m$.

**Proposition 3.3.** Let $(E^*, \xi_0)$ be a complex oriented cohomology theory with associated formal group law $F(z, w)$. Suppose $\Psi: BU(1) \times X \to X$ satisfies the axioms for a group action of the H-space $BU(1)$ on $X$ up to homotopy. Then
\[ D(z)(a) = \sum_{k \geq 0} \Psi_s(t_k \boxtimes a) z^k, \quad a \in E_s(X), \quad (3.3) \]

defines an F-shift operator on \( E_s(X) \).

**Proof.** Since \( \Psi(t_0, x) = x \) is neutral, \( D(0) = \text{id}_{E_s(X)} \). Define coefficients \( F^n_{ij} \) by \( F(z, w)^n = \sum_{i,j \geq 0} F^n_{ij} z^i w^j \). Then \( (\mu_{\mathbb{C}P^\infty})_*(t_i \boxtimes t_j) = \sum_{n \geq 0} F^n_{ij} t_n \), and so

\[
D(z) \circ D(w)(a) = \sum_{i,j \geq 0} \Psi_s(t_i \boxtimes \Psi_s(t_j \boxtimes a)) z^i w^j \\
= \sum_{i,j \geq 0} \Psi_s((\mu_{\mathbb{C}P^\infty})_*(t_i \boxtimes t_j) \boxtimes a) z^i w^j \\
= \sum_{i,j,n \geq 0} \Psi_s(t_n \boxtimes a) F^n_{ij} z^i w^j = D(F(z, w)). \quad \square
\]

**Definition 3.4.** Let \( V \to X \) be a complex vector bundle of rank \( n \) with zero section \( 0_X \). The bundle of projective spaces \( \mathbb{P}(V) = (V \setminus 0_X)/\mathbb{C}^* \) carries a tautological line bundle \( L_V \to \mathbb{P}(V) \) with \( L_V|_L = L \). Its classifying map \( f^{L_V}_E : \mathbb{P}(V) \to \mathbb{C}P^\infty \) is unique up to homotopy. Define \( w = f^{L_V}_E(\xi_E) \) using the complex orientation. By the above, \( E^*(\mathbb{P}(V)) \) is a free \( E^*(X) \)-module with basis \( 1_{\mathbb{P}(V)}, w, \ldots, w^{n-1} \). The Conner–Floyd Chern classes are defined by expanding \( w^n \) in this basis:

\[ c^E_0(V) = 1, \quad 0 = \sum_{i=0}^{n} (-1)^i c^E_i(V) \cdot w^{n-i}, \quad c^E_i(V) = 0 \quad (\forall i > n) \quad (3.4) \]

Naturality under pullback is obvious. There is a Whitney sum formula [2, p. 47]

\[ c^E_k(V \oplus W) = \sum_{i=0}^{k} c^E_i(V)c^E_{k-i}(W). \quad (3.5) \]

For complex line bundles \( L_L \to \mathbb{P}(L) \) is isomorphic to \( L \to X \) so \( c^E_1(L) = f_L^*(\xi_E) \) for the classifying map \( f_L \) of \( L \). In particular,

\[ c^E_1(L_1 \otimes L_2) = F(c^E_1(L_1), c^E_1(L_2)). \quad (3.6) \]

Moreover, \( c^E_1(\mathbb{C}) = 0 \) as \( \xi_E \) is reduced. Hence \( c^E_1(\mathbb{C}^N) = 0 \) for every trivial bundle.

**Example 3.5.** Ordinary cohomology \( E^* = H^* \) has a complex orientation \( \xi_H \) in \( H^2(\mathbb{C}P^\infty) = \lim H^2(\mathbb{C}P^m) \) that is Poincaré dual to the fundamental class \( [\mathbb{C}P^{m-1}] \in H_{m-2}(\mathbb{C}P^m) \) with orientation of \( \mathbb{C}P^{m-1} \) fixed by the complex structure. We obtain the ordinary Chern classes, and \( c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2) \) implies \( F_H = \mathbb{G}_a \).

**Example 3.6.** Topological \( K \)-theory \( E^* = K^* \) on compact spaces is the group completion of isomorphism classes of complex vector bundles. Write \( L_m = L|m_{\mathbb{C}P^m} \) for the tautological complex line bundle over \( \mathbb{C}P^m \), \( \mathbb{C} \) for the trivial bundle, and \( [L_m], 1 \in K^0(\mathbb{C}P^m) \) for their classes in \( K \)-theory. The classes \( [L_m] \) are compatible under restriction and define a complex orientation \( \xi_K \in K^2(\mathbb{C}P^\infty) = \lim K^2(\mathbb{C}P^m) \). Here \( F_K = \mathbb{G}_m \) is the multiplicative formal group law, as

\[ \mu^*([L] - 1) = [\mu^*([L]) - 1] = [\pi_1^*([L]) \otimes \pi_2^*([L])] - 1 = \mathbb{G}_m([L] - 1, [L] - 1). \]

For a complex vector bundle \( V \to X \) of rank \( n \) one has \( \pi^*(V) = L \oplus L^\dagger \) over the projectivization \( \pi : \mathbb{P}(V) \to X \) and \( L^\dagger \). The formal power series \( \Lambda_t([V]) = 1 + [V]t + [\Lambda^2 V]t^2 + \ldots \in \mathbb{K}^0(X)[[t]] \) has inverse \( \Lambda_{-t}([V]) \), so \( \Lambda_t([V] - [W]) = \Lambda_t([V])\Lambda_{-t}([W]) \). As \( [L^\dagger]^n = \pi^*([V]) - [\mathcal{L}] \) has rank \( n - 1 \), the \( n \)-th coefficient of \( \Lambda_t([L^\dagger]) = \Lambda_t([\pi^*([V])])\Lambda_{-t}([\mathcal{L}]) = 0 \) is \( [\Lambda^n(L^\dagger)] = \sum_{p=0}^{n}(-1)^{n-p}[\Lambda^p([V])][\mathcal{L}]^{n-p} \). Putting \( [\mathcal{L}] = w + 1 \) and comparing to \( (3.4) \), \( c^K_i(V) = \sum_{p=0}^{i}(-1)^{i+p}(n-p) \Lambda^p([V]) \).
Example 3.7. As in Quillen [15], complex cobordism Ω^n_*(X) for X a smooth manifold is the set of smooth maps \( f : Z \to X \) of codimension \( \dim X - \dim Z = n \) with a complex structure the stable normal bundle, modulo cobordism. The complex orientation \( \xi_\Omega \in \Omega^n_1(C\mathbb{P}^\infty) = \lim_{\to} \Omega^n_\mathbb{Z}(\mathbb{CP}^m) \) is given by \( \mathbb{CP}^{m-1} \to \mathbb{CP}^m \), and \( \mathbb{CP}^{m-1} \) is the zero set of a section of \( L_m^* \). So for complex line bundles \( c^1_\Omega(L) \) is represented by the zero set \( s^{-1}(0) \) of a generic section \( s : X \to L \). The formal group law is the universal law \( G_n \), see Adams [1, Part I, §8].

Lemma 3.8. Let \( V \to X \) be a complex vector bundle over a finite CW complex. Then each of the Conner–Floyd Chern classes \( c^F_L(V) \) is nilpotent.

Proof. There is a finite open cover \( X = \bigcup_{\lambda=1}^N U_\lambda \) with \( U_\lambda \) contractible and \( V|U_\lambda \) trivial. From the long exact sequence of the pair \( (X, U_\lambda) \) we see that we may lift \( c^F_L(V) \) along \( j_\lambda^* : E^{2i}(X, U_\lambda) \to E^{2i}(X) \) to a class \( x_\lambda \in E^{2i}(X, U_\lambda) \). The diagram

\[
\begin{align*}
\prod_{\lambda=1}^N E^{2i}(X, U_\lambda) \xrightarrow{\cup} E^{2iN}(X, \bigcup_{\lambda=1}^N U_\lambda) = E^{2iN}(X, X) &= \{0\} \\
\prod_{\lambda=1}^N j_\lambda^* \xrightarrow{\cup} E^{2i}(X) \xrightarrow{j^*} E^{2iN}(X)
\end{align*}
\]

commutes by naturality of \( \{U\} \), so \( c^F_L(V)^N = \prod_{\lambda=1}^N j_\lambda^*(x_\lambda) = j^*(\prod_{\lambda=1}^N x_\lambda) = 0 \). □

Notation 3.9. When \( X \) is a finite CW complex, it follows that we may substitute \( w \) by \( c^F_L(L) \) in the formal group law \( F(z, w) \). To define the right hand side of (1.4) also for infinite CW complexes \( X \), let \( \{X_i \mid i \in I\} \) be the direct system of finite subcomplexes \( X_i \subset X \) ordered by inclusion. The pro-group E-cohomology is the inverse limit \( \hat{E}^*(X) = \lim_{\to} E^*(X) \). The family of all restrictions \( F(z, c^F_L(L_{X_i})) \) determines an element we write \( F(z, c^F_L(L)) \in \hat{E}^*(X)[[z]] \). As homology and direct limits commute, see [17, Prop. 7.53], we have \( E_*(X) = \text{colim} E_*(X_i) \) and therefore a well-defined cap product \( E_*(X) \otimes \hat{E}^*(X) \to E_*(X) \). This defines (1.4) in general.

4. Proof of Theorem 1.1

Step 1: Vector bundles over finite CW complexes. For a complex line bundle \( L \to X \) over a finite CW complex \( X \) define \( C^F_L(L) = F(z, c^F_L(L)) \). For \( V \to X \) a rank \( n \) complex vector bundle we proceed by the splitting principle. As in Definition 3.4 over the projectivization \( p : \mathbb{P}(V) \to X \) we can split off a line bundle from \( p^*(V) \) and \( p^* : E^*(Y) \to E^*(X) \) is injective. Iterating, we find \( q : Y \to X \) and line bundles \( L_1, \ldots, L_n \to Y \) with \( L_1 \oplus \cdots \oplus L_n = q^*(V) \) and \( q^* : E^*(Y) \to E^*(X) \) is injective. By (3.5), the class \( q^*(c^F_L(V)) \) is the \( k \)-th elementary symmetric polynomial in the Chern roots \( c^F_L(L_1), \ldots, c^F_L(L_1) \). As the expression

\[
F(z, c^F_L(L_1)) \cup \cdots \cup F(z, c^F_L(L_n)) = q^*(C^F_L(V)) \tag{4.1}
\]

is a symmetric polynomial in the Chern roots, the fundamental theorem of symmetric polynomials implies it has a (unique) preimage \( C^F_L(V) \) in \( E^*(X)[[z]] \). The map (1.1) is obtained by combining the class \( C^F_L(V) \) with the cap product

\[
\cap : E_*(X) \otimes E^*(X)[[z]] \to E_*(X)[[z]].
\]

(a) For naturality, let \( f : X' \to X \) and use the pullback \( Q : Y' = X' \times_X Y \to X' \) with its canonical map \( F: Y' \to Y \) to split \( V' = f^*(V) \) as \( Q^*(V') \cong F^*q^*(V) \cong F^*(L_1) \oplus \cdots \oplus F^*(L_n) \). Naturality of the
Conner–Floyd Chern classes implies that the pullback $F^*q^*(C^E_z(V)) = Q^*f^*(C^E_z(V))$ of (4.1) along $F$ is $Q^*C^E_z(V')$. Thus,

$$C^E_z(f^*(V)) = f^*(C^E_z(V)).$$  \hspace{1cm} (4.2)

(b) Let $V, W \to X$ be vector bundles. Pick $q: Y \to X$ such that both $q^*(V) = L_1 \oplus \cdots \oplus L_n$ and $q^*(W) = S_1 \oplus \cdots \oplus S_m$ split into line bundles with $q^*$ injective. Then $q^*C^E_z(V)$ equals (4.1), $q^*C^E_z(W) = F(z, c^E_1(S_1)) \cup \cdots \cup F(z, c^E_1(S_m))$, and

$$q^*C^E_z(V \oplus W) = F(z, c^E_1(L_1)) \cup \cdots \cup F(z, c^E_1(S_m)) = q^*C^E_z(V) \cup q^*C^E_z(W).$$

Hence

$$C^E_z(V \oplus W) = C^E_z(V) \cup C^E_z(W).$$  \hspace{1cm} (4.3)

This proves that cap product with $C^E_z(V)$ satisfies Theorem 1.1(a)&(b). Part (c) holds by construction. For (d), in the case of line bundles the operation $(-) \cap F(z, c^E_1(L)) = \sum_{i,j \geq 0} F_{ij} z^i [(-) \cap c^E_1(L)^j]$ has degree $-2$, as $F_{ij} \in R_{2i+2j-2}$. It then follows from (4.1) that in general $(-) \cap C^E_z(V)$ has degree $-2 \text{rk}(V)$.

(e) Let $V \to X$ be a vector bundle, $L \to X$ a complex line bundle, and suppose $q^*(V)$ splits as above. Then $q^*(L \otimes V) = (q^*(L) \otimes L_1) \oplus \cdots \oplus (q^*(L) \otimes L_n)$ and so

$$q^*C^E_z(L \otimes V) = F(z, c^E_1(q^*(L) \otimes L_1)) \cup \cdots \cup F(z, c^E_1(q^*(L) \otimes L_n))$$

$$= F(z, F(q^*c^E_1(L), c^E_1(L_1))) \cup \cdots \cup F(z, F(q^*c^E_1(L), c^E_1(L_n)))$$

$$= F(F(z, q^*c^E_1(L)), c^E_1(L_1)) \cup \cdots \cup F(F(z, q^*c^E_1(L)), c^E_1(L_n))$$

$$= q^*C^E_{F(z, c^E_1(L))}(V).$$

Hence

$$C^E_z(L \otimes V) = C^E_{F(z, c^E_1(L))}(V).$$  \hspace{1cm} (4.4)

**Step 2: Extension to K-theory of finite CW complexes.** So far, we have constructed a homomorphism $C^E_z : (\text{Vect}(X), \oplus) \to (E^*(X)[\mathbb{Z}], \cup)$ on the monoid of complex vector bundles $V \to X$ up to isomorphism over a finite CW complex. We claim that every class $C^E_z(V)$ is invertible in the larger ring $E^*(X)[[\mathbb{Z}][z^{-1}]]$. Indeed, there exists a vector bundle $W \to X$ with $V \oplus W \cong \mathbb{C}^N$ trivial and therefore $C^E_z(V) \cup C^E_z(W) = C^E_z(\mathbb{C}^N) = F(z, c^E_1(\mathbb{C}))^N = z^N$. As $X$ is a finite CW complex, its topological $K$-theory is the group completion of $\text{(Vect}(X), \oplus)$ whose universal property allows us to uniquely extend the homomorphism to $C^E_z : K^0(X) \to (E^*(X)[[\mathbb{Z}][z^{-1}], \cup)$. It is easy to check that properties (a)–(d) continue to hold.

**Notation 4.1.** As $X$ is a finite CW complex, we may write $\theta = [V] - [\mathbb{C}^\ell]$. Expand $C^E_z(V) = \sum_{n \geq 0} C_n(V) z^n$. Then

$$C^E_z(\theta) = \sum_{n \geq 0} C_n(V) z^{n-\ell}. \hspace{1cm} (4.5)$$

In Notation 2.1 we have defined $i_{z,w}(F(z, w)^{-\ell} \sum_{n \geq -\ell} C_n(V) F(z, w)^n)$ as a holomorphic series in $w$ which we can substitute by the nilpotent $c^E_1(L)$, see Lemma 3.8. This defines $i_{z,c^E_1(L)} C^E_{F(z, c^E_1(L))}(\theta) \in E^*(X)[[\mathbb{Z}][z^{-1}]]$ for finite $X$. When $X$ is infinite, the classes for the restrictions of $\theta$ to all finite subcomplexes $X_i \subset X$ define $i_{z,c^E_1(L)} C^E_{F(z, c^E_1(L))}(\theta) \in \hat{E}(X)(\mathbb{Z})$ in pro-group $E$-cohomology, see Notation 3.9.
We prove (e). As just seen, $C^E_z(L) = F(z, c^E_1(L))$ is invertible in $E^*(X)[z][z^{-1}]$. Therefore $i_{z,a}F(z, c^E_1(L))^n = F(z, c^E_1(L))^n$ for all $n \in \mathbb{Z}$. Using Notation 4.1, we have

$$C^E_z(L \otimes \theta) = C_z(L \otimes V)C_z(L)^{-\ell}$$

$$= C_{F(z,cF(L))}(V)F(z, c^E_1(L))^{-\ell}$$

$$= \sum_{n \geq 0} C_n(V)F(z, c^E_1(L))^{n-\ell} = i_{z,cF(L)}C^E_{F,z,cF(L)}(\theta).$$

**Step 3: Infinite complexes.** Let $\{X_i \mid i \in I\}$ be the direct system of finite subcomplexes of a CW complex $X$ ordered by inclusion. Write $\iota(i) : X_i \subseteq X$ and $\iota(i,j) : X_i \subseteq X_j$ for the inclusions. For $\theta \in K^0(X)$, Step 2 yields for each $i \in I$ a map

$$E_*(X_i) \xrightarrow{\cap C_z(\iota(i)^*) \theta} E_*(X_i)[z][z^{-1}] \xrightarrow{\iota(i)_*} E_*(X)[z][z^{-1}].$$

By naturality, $\iota(i,j)_*(a) \cap C^E_z(\iota(j)^* \theta) = \iota(i)_*(a \cap C^E_z(\iota(i)^* \theta))$ so the maps (4.6) determine a homomorphism $E_*(X) \cong \operatorname{colim} E_*(X_i) \rightarrow E_*(X)[z][z^{-1}]$ on the colimit, using that homology and direct limits commute, see [17, Prop. 7.53]. Equivalently, the restrictions $C^E_z(\theta|_{X_i})$ define a class $C^E_z(\theta) \in E^*(X)(\mathbb{Z})$ in pro-group $E$-cohomology. Using the cap product $E_*(X) \otimes E^*(X)(\mathbb{Z}) \rightarrow E_*(X)(\mathbb{Z})$ we can define $(-) \cap C^E_z(\theta) : E_*(X) \rightarrow E_*(X)(\mathbb{Z})$ which, a priori, has a larger codomain.

Finally, properties (a)–(e) pass to the limit.

**Step 5: General topological spaces.** By the CW approximation theorem, there is a CW complex $X'$ with a weak homotopy equivalence $f : X' \rightarrow X$. Then

$$a \cap C_z(\theta) = f_*(f^{-1}_*(a) \cap C_z(f^*(\theta)))$$

is well-defined, since this equation holds for a homotopy equivalence $f : X' \rightarrow X$ by (1.2). With this definition, the properties (a)–(e) carry over to $X$. □

5. **Proof of Theorem 1.2**

We verify Definition 2.7(a)–(c) for the graded module $V_* = \bigoplus E_{* - rk \theta_{\alpha, \alpha}}(X_\alpha)$, vacuum vector $\Omega = e_*(1)$, $F$-shift operator (3.3), and state-to-field correspondence (1.13). Here, $e : \text{pt} \rightarrow X_0$ is the H-space unit and $1 \in E_0(\text{pt}) = R^0$.

Writing $|a|_V = |a| + \operatorname{rk} \theta_{\alpha, \alpha}$ for the shifted degree, we have

$$|Y(a, z)b|_V = |Y(a, z)b| + \operatorname{rk} \theta_{\alpha, \alpha, \alpha, \alpha} = (|a| - \operatorname{rk} \theta_{\alpha, \alpha})(|b| - \operatorname{rk} \theta_{\beta, \beta}) = |a|_V \cdot |b|_V,$$

for $a \in E_{* - rk \theta_{\alpha, \alpha}}(X_\alpha)$, $b \in E_{* - rk \theta_{\beta, \beta}}(X_\beta)$, so that $Y$ preserves the grading of $V_*$. (a) Let $a \in E_*(X_\alpha)$, $b \in E_*(X_\beta)$. As $e$ is a fixed point, $\Psi_*(t_k \boxtimes \Omega) = 0$ for $k > 0$ and $\Psi_*(t_0 \boxtimes \Omega) = \Omega$. Hence $D(z)\Omega = \Omega$. Let $\varphi = (e, \text{id}_{X_\alpha}) : X_\beta \rightarrow X_\alpha \times X_\beta$. Then

$$(\Omega \boxtimes b) \cap C^E_z(\theta_{\Omega, \beta}) = \varphi_*(b) \cap C^E_z(\theta_{\Omega, \beta})$$

$$= \varphi_*(b \cap \varphi^* C^E_z(\theta_{\Omega, \beta})) = \varphi_*(b \cap 1) = \Omega \boxtimes b,$$

and so $Y(\Omega, z)b = (\Phi_{\Omega, \beta})_* (D(z)\Omega \boxtimes b) = b$, proving (2.7). Similarly,

$$Y(a, z)\Omega = (\Phi_{\alpha, \Omega})_* (D(z) \boxtimes \text{id}_{X_\Omega})(a \boxtimes \Omega) = D(z)(a).$$
is holomorphic with $D(0)(a) = a$ for $z = 0$, proving (2.6).

(b) We have already shown $D(z)\Omega = \Omega$. To prove (2.8), we first need a lemma.

**Lemma 5.1.** For the universal complex line bundle $L \rightarrow \mathbb{C}P^n$ and $n \in \mathbb{Z}$

$$
\sum_{k \geq 0} t_k \cap i_{z,c}F(z,c^E_L)w^k = \sum_{\ell \geq 0} t_\ell i_{z,w}F(z,w)^nw^\ell.
$$

Moreover, for all $a \in E_*(X_\alpha)$, $b \in E_*(X_\beta)$ we have

$$(D_\alpha(w)a \boxtimes b) \cap C_z^E(\theta_{\alpha,\beta}) = (D_\alpha(w) \times \text{id}_{X_\beta})(a \boxtimes b) \cap C_{F(z,w)}^E(\theta_{\alpha,\beta}),$$
and

$$(a \boxtimes D_\beta(w)b) \cap C_z^E(\theta_{\alpha,\beta}) = (\text{id}_{X_\alpha} \times D_\beta(w))(a \boxtimes b) \cap C_{F(z,w)}^E(\theta_{\alpha,\beta}).$$

**Proof.** Introduce the expansion $i_{z,w}F(z,w)^n = \sum_{i,j \geq 0} F_{ij}z^iw^j$. Then

$$
t_k \cap i_{z,c}F(z,c^E_L)w^n = \sum_{i,j \geq 0} F_{ij}z^iw^j.\quad(5.4)
$$

where $t_k = 0$ for $k < 0$. Summing (5.4) over all $k$, the summands with $k < j$ vanish, so we may restrict the sum to $k \geq j$ and reindexing by $\ell = k - j$ gives (5.1):

$$
\sum_{i,j \geq 0} \sum_{\ell \geq 0} F_{ij}z^i w^j t_\ell w^{\ell} = \sum_{\ell \geq 0} t_\ell i_{z,w}F(z,w)^nw^{\ell}
$$

For (5.2) we compute

$$(D_\alpha(w) \boxtimes a) \cap \alpha C_z^E(\theta_{\alpha,\beta}) = \sum_{k \geq 0} (\Psi_{\alpha} \times \text{id}_{X_\beta})(t_k \boxtimes a \boxtimes b) \cap C_{z}^E(\theta_{\alpha,\beta})w^k
$$

$$= (\Psi_{\alpha} \times \text{id}_{X_\beta})(t_k \boxtimes a \boxtimes b) \cap (\Psi_{\alpha} \times \text{id}_{X_\beta})^*C_z^E(\theta_{\alpha,\beta})w^k
$$

$$= (\Psi_{\alpha} \times \text{id}_{X_\beta})(t_k \boxtimes a \boxtimes b) \cap C_z^E(\theta_{\alpha,\beta})w^k
$$

$$= (\Psi_{\alpha} \times \text{id}_{X_\beta})(t_k \boxtimes a \boxtimes b) \cap C_{z,c}^E(\theta_{\alpha,\beta})w^k
$$

$$= (\Psi_{\alpha} \times \text{id}_{X_\beta})(t_k \boxtimes a \boxtimes b) \cap C_{F(z,w)}^E(\theta_{\alpha,\beta})w^k
$$

For (5.3) we similarly use (1.9) which replaces $\epsilon^E_L$ by its formal inverse $\epsilon^E_L$ above, so the same argument with $F(z,\epsilon(w))$ in place of $F(z,w)$ gives (5.3). \(\Box\)

It is now easy to verify (2.8): Let $a \in E_*(X_\alpha)$, $b \in E_*(X_\beta)$. Then

$$Y(D_\alpha(w)a,z)b = \sum_{k \geq 0} (\Phi_{\alpha,\beta})(D_\alpha(z) \boxtimes \text{id}_{b})(D_\alpha(w)a \boxtimes b) \cap C_z^E(\theta_{\alpha,\beta})
$$

$$= (\Phi_{\alpha,\beta})(D_\alpha(z)D_\alpha(w) \boxtimes \text{id}_b)(a \boxtimes b) \cap C_{F(z,w)}^E(\theta_{\alpha,\beta})
$$

$$= i_{z,w}Y(a,F(z,w))b.
$$

(c) Firstly, $\Phi \circ (\Psi \times \Psi) \circ \delta \simeq \Psi \circ (\Phi \times \text{id}_{BU(1)})$ and $\Delta_\ast(t_j) = \sum_{i+j=k} t_i \boxtimes t_j$ imply

$$D_{\alpha+\beta}(z)(\Phi_{\alpha,\beta}) = (\Phi_{\alpha,\beta})(D_\alpha(z) \boxtimes D_\beta(z)).$$

(5.5)
Let \( a \in E_*(X_\alpha) \), \( b \in E_*(X_\beta) \), \( c \in E_*(X_\gamma) \). On the one hand

\[
Y(Y(a, z)b, w)c = (\Phi_{\alpha+\gamma})(\mathcal{D}_{\alpha+\beta}(w) \boxtimes \text{id}_\gamma)
\]

\[
[(\Phi_{\alpha,\beta})(\mathcal{D}_{\alpha}(z) \boxtimes \text{id}_\beta)]((a \boxtimes b) \cap C^E_z(\theta_{\alpha,\beta})) \boxtimes c \cap C^E_w(\theta_{\alpha+\beta})
\]

\[
\overset{(\ref{5.5})}{=} (\Phi_{\alpha+\beta,\gamma})(\mathcal{D}_{\alpha}(w) \boxtimes \mathcal{D}_{\beta}(w) \boxtimes \text{id}_\gamma)
\]

\[
[(\mathcal{D}_{\alpha}(z) \boxtimes \text{id}_\beta \boxtimes \text{id}_\gamma)]((a \boxtimes b \boxtimes c) \cap C^E_z(\theta_{\alpha,\beta}) \cap (\Phi_{\alpha,\beta} \times \text{id}_\gamma)^*(C^E_w(\theta_{\alpha+\beta}))
\]

\[
\overset{(\ref{2.4}), (\ref{5.5})}{=} (\Phi_{\alpha+\beta,\gamma})(\mathcal{D}_{\alpha}(w)\mathcal{D}_{\alpha}(z) \boxtimes \mathcal{D}_{\beta}(w) \boxtimes \text{id}_\gamma)
\]

\[
[(a \boxtimes b \boxtimes c) \cap C^E_z(\theta_{\alpha,\beta}) \cap i_{w,z}C^E_{F(w,z)}(\theta_{\alpha,\gamma}) \cap C^E_w(\theta_{\beta,\gamma})]
\]

and on the other hand

\[
i_{z,w}Y(a, F(z, w))Y(b, w)c = i_{z,w}(\Phi_{\alpha,\beta+\gamma})(\mathcal{D}_{\alpha}(F(z, w)) \boxtimes \text{id}_{\beta+\gamma})
\]

\[
[(a \boxtimes (\Phi_{\beta,\gamma})(\mathcal{D}_{\beta}(w) \boxtimes \text{id}_\gamma))((b \boxtimes c) \cap C^E_w(\theta_{\beta,\gamma}))] \cap C^E_{F(z, w)}(\theta_{\alpha,\beta+\gamma})
\]

\[
\overset{\text{(5.5), (1.7)}}{=} i_{z,w}(\Phi_{\alpha,\beta+\gamma})(\text{id}_\alpha \boxtimes \Phi_{\beta,\gamma})(\mathcal{D}_{\alpha}(F(z, w)) \boxtimes \text{id}_\beta \boxtimes \text{id}_\gamma)
\]

\[
[(\text{id}_\alpha \boxtimes \mathcal{D}_{\beta}(w) \boxtimes \text{id}_\gamma)((a \boxtimes b \boxtimes c) \cap C^E_w(\theta_{\beta,\gamma})) \cap C^E_{F(z, w)}(\theta_{\alpha,\beta}) \cap C^E_{F(z, w)}(\theta_{\alpha,\gamma})]
\]

\[
\overset{\text{(2.4), (5.3)}}{=} (\Phi_{\alpha,\beta+\gamma})(\text{id}_\alpha \boxtimes \Phi_{\beta,\gamma})(\mathcal{D}_{\alpha}(w)\mathcal{D}_{\alpha}(z) \boxtimes \mathcal{D}_{\beta}(w) \boxtimes \text{id}_\gamma)
\]

\[
[(a \boxtimes b \boxtimes c) \cap C^E_w(\theta_{\beta,\gamma}) \cap C^E_z(\theta_{\alpha,\beta}) \cap i_{z,w}C^E_{F(z, w)}(\theta_{\alpha,\gamma})]
\]

As \( Y(Y(a, z)b, w)c \) and \( Y(a, F(z, w))Y(b, w)c \) are both expansions in negative powers of \( F(z, w) \) of the same series in different variables, there exist some \( N \gg 0 \) with \( F(z, w)^NY(Y(a, z)b, w)c = F(z, w)^ny(Y(a, F(z, w))Y(b, w)c, \) see \( \text{(2.3)} \).

The same calculations show that \( \text{(1.15)} \) is a nonlocal vertex \( F \)-algebra and that the state-to-field correspondence \( Y(a, z)b \) preserves the degree shifted by \( 2\chi(\alpha, \alpha) \). It remains to prove \( (-1)^{ab}\overline{Y}(a, z)b = \mathcal{D}_{\alpha+\beta}(z)\overline{Y}(b, \iota(z))a \). Notice \( \sigma^*(\overline{C}^E_z(\theta_{\alpha,\beta})) = \overline{C}^E_{\iota(z)}(\theta_{\alpha,\alpha}) \) for the swap \( \sigma: X_\beta \times X_\alpha \rightarrow X_\alpha \times X_\beta \). Using \( \Phi_{\beta,\alpha} \simeq \Phi_{\alpha,\gamma} \) we find

\[
\mathcal{D}_{\alpha+\beta}(z)\overline{Y}(b, \iota(z))a = \mathcal{D}_{\alpha+\beta}(z)(\Phi_{\alpha,\beta})(\mathcal{D}_{\beta}(\iota(z)) \boxtimes \text{id}_\alpha)((b \boxtimes a) \cap \overline{C}^E_{\iota(z)}(\theta_{\alpha,\beta}))
\]

\[
= \mathcal{D}_{\alpha+\beta}(z)(\Phi_{\alpha,\beta})(\text{id}_\alpha \boxtimes \mathcal{D}_{\beta}(\iota(z)))\sigma((b \boxtimes a) \cap \sigma^*\overline{C}^E_z(\theta_{\alpha,\beta}))
\]

\[
\overset{\text{(5.5)}}{=} (\Phi_{\alpha,\beta})(\mathcal{D}_{\alpha}(z) \boxtimes \text{id}_\beta)[\sigma((b \boxtimes a) \cap \overline{C}^E_z(\theta_{\alpha,\beta}))]
\]

\[
= (-1)^{ab}(a \boxtimes b) \cap C^E_z(\theta_{\alpha,\beta})
\]

\( \square \)

**Remark 5.2.** For the additive formal group law \( \mathbb{G}_a \) and ordinary homology, this was shown by Joyce [8, Thm. 3.14]. When \( X \) is the derived category of a finite quiver or of certain smooth projective complex varieties, then taking \( F(X, Y) = X + Y \) in \( \text{(1.15)} \) gives a (super) lattice vertex algebra [7, Thm. 5.7] [8, Thm. 5.19].

**Remark 5.3.** A similar construction applies to \( H \)-spaces \( X \) with \( BO(1) \)-actions, the classifying space for real line bundles, and homology with \( \mathbb{Z}_2 \)-coefficients. Since \( H^*(BO(1)) = \mathbb{Z}_2[\xi] \) there is a shift operator \( \mathcal{D}(u): H_4(X; \mathbb{Z}_2) \rightarrow H_4(X; \mathbb{Z}_2)[u] \) for \( u \) a variable of degree \( -1 \). One can then build, just as in Theorem 1.1, an operator \( (-) \cap W_\alpha(\theta) \) of degree \( -\text{rk}_\alpha \theta_{\alpha,\beta} \), where \( \theta_{\alpha,\beta} \in KO(X_\alpha \times X_\beta) \), with normalization \( a \cap W_\alpha(L) = a \cap (u + w_1(L)) \) for the first Stiefel–Whitney class of a real line bundle \( L \rightarrow X \). Then \( Y(a, z)b = (\Phi_{\alpha,\beta})(\mathcal{D}_{\alpha}(u) \boxtimes \text{id}_\beta)\[(a \boxtimes b) \cap W_\alpha(\theta_{\alpha,\beta})]\) makes \( V = H_*(X; \mathbb{Z}_2) \) into a vertex algebra over \( \mathbb{Z}_2 \).
Acknowledgements

The authors thank Dominic Joyce for many discussions and suggestions. They also thank Mikhail Kapranov, Kobi Kremnitzer, Sven Meinhardt, and Konrad Voelkel for helpful conversations.

References