Hausdorff dimension of chaotic attractors in a class of nonsmooth systems

In Memory of Professor Vadim S. Anishchenko

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Abstract

Fractal dimension is a important feature of a chaotic attractors. Generally, the rigorous value of Hausdorff dimension of a chaotic attractor is not easy to compute. In this work, we consider a class of nonsmooth systems. Initially, we determine a set of parameter values in which the systems have a chaotic attractor with an Sinai-Ruelle-Bowen measure. Then we give a lower bound and an upper bound of the Hausdorff dimension of the attractor. Our rigorous analysis shows that the two bounds are equal, and thus the exact formula of the Hausdorff dimension of the attractor is obtained. Moreover, the relationship between the Hausdorff dimension and the parameter values is discussed in terms of the derived formula.

Keywords: chaotic attractor, Hausdorff dimension, SRB measure.

1. Introduction

Edward Lorenz used the phrase "deterministic nonperiodic flow" to describe the first example of what is now known as a strange or chaotic attractor [1]. A chaotic attractor is often a fractal object, whose geometric structure is invariant under the time evolution of the dynamics [2, 3]. Chaotic attractors exist widely in nature and in man-made systems. The study of the properties of chaotic attractors is an important topic in the theory of dynamical systems, though there are other types of invariant sets in dynamical systems that are not attracting, such as strange saddles [4]. Chaotic attractors and invariant sets embedded in the closure of unstable manifolds, while fractal basin boundaries [5] are open invariant sets embedded in stable manifolds. Grebogi, Ott and Yorke [6] studied the interaction of both sets in a phenomenon called crisis. It was shown in [7] that an infinite number of unstable periodic orbits embedded in the support of invariant measures provides the key for the understanding of the multifractal structure of chaotic attractors. Benedicks and Carleson [8] proved that there is a set of positive Lebesgue measure in the control parameters for which the Hénon map has a chaotic attractor. For the parameter values considered in [8], Benedicks and Young [9] proved that the Hénon map has an

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Sinai-Ruelle-Bowen (SRB) measure. Wang and Young further extended the results in [9] to a class of chaotic attractors with one direction of instability [10, 11]. Anishchenko et al [12] studied the effect of noise on the relaxation to an invariant probability measure of nonhyperbolic chaotic attractors.

Non-smoothness is an important feature for the complexity of dynamical systems, see e.g. [13–17]. In the non-smooth case, even those dissipative piecewise linear systems can have chaotic attractors. The Lozi map [18] and the geometric Lorenz map are classical models of nonsmooth maps. Misiurewicz [19] proved that there exist chaotic attractors with topological mixing for the Lozi map in some parameter regions, and that the attractor is the closure of the unstable manifold of hyperbolic fixed point in the first quadrant. Cao and Liu [20] and Baptista et al. [21] studied the structure of chaotic attractor and its basin of attraction for the Lozi map. Rychlik [22], Collet and Levy [23], and Young [24] proved that Lozi maps and Lozi-like maps have SRB measures. Pesin [25] constructed a broader class of maps allowing for high dimensionality and unbounded derivatives (including the geometric Lorenz map), and studied the topological and ergodic properties of the chaotic attractors. The corresponding class of chaotic attractors (called generalized hyperbolic attractors) contains the well known generalized Lozi attractors, geometric Lorenz attractors, and Belykh attractors.

Chaotic attractors have often a complicated fractal structure [7]. One way to characterize this property is to determine their dimension. Box-counting dimension and Hausdorff dimension are most often used in dynamical systems theory. The box-counting dimension is easier to compute numerically, but the Hausdorff dimension is much harder though more satisfying from a theoretical point of view, see e.g. [26–28]. Young [29] has established a formula which relates the Hausdorff dimension to the measure entropy and Lyapunov exponents for surface diffeomorphisms. Ishii [30] estimated the Hausdorff dimension of the Lozi attractors.

The purpose of this work is to study the Hausdorff fractal dimension in the following class of nonsmooth systems:

$$f = f_{a,b} : \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} s(x) - ax + by \\ x \end{pmatrix},$$

where

$$s(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ -1 & \text{if } x < 0. \end{cases}$$

The map f is different from the geometric Lorenz map, since the contracting fibers are not parallel to its y-axis singularity sets; there is no global contracting foliation here. The symbolic dynamics of the map f is studied in [31].

The remaining of this paper is organized as follows. In Section 2, we construct a chaotic attractor of the map f. Then we determine a parameter region in which f admits an SRB measure whose support set is contained in the closure of the unstable manifolds of the hyperbolic fixed points. In Section 3, an upper bound of the Hausdorff dimension of the chaotic attractor is estimated in terms of its box-counting dimension. Moreover, by the SRB measure we determine a lower bound of its Hausdorff dimension. We find that the lower bound and the upper bound are equal, and thus we obtain the exact formula of the Hausdorff dimension of the attractor.

2. Chaotic attractors

In this Section, we shall show that our map has a chaotic attractor for a positive Lebesgue measure set of parameters values.

2.1. Assumptions and notations

Since the map f is discontinuous in the y-axis, some definitions used here are slightly different from the ones used for smooth systems. Let Q be a point whose orbits does not intersect the y-axis, then we still call the sets $W_Q^s = \{P| \lim_{n\to+\infty} dis(f^n(Q), f^n(P)) = 0\}$ and $W_Q^u = \{P| \lim_{n\to+\infty} dis(f^{-n}(Q), f^{-n}(P)) = 0\}$ the stable and unstable manifold of Q, respectively, though they are broken lines.

We shall use various assumptions on the parameters a and b. For convenience we list them below. See Figure 1.

H1: $a > 0, 0 < b < 1$;

H2: $a > b + 1$;

H3: $a^2 + 4b < 4$;

H4: $\sqrt{2}a + b > 2$.

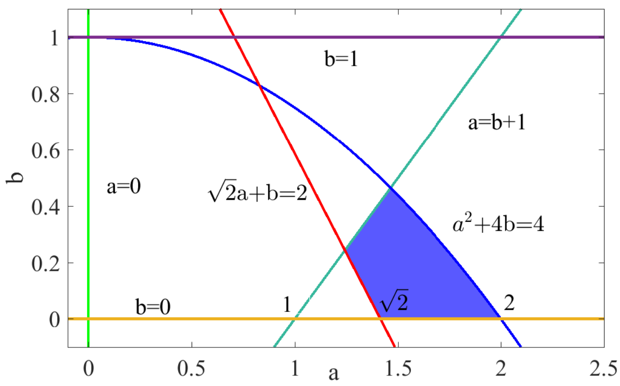


Figure 1: Parameter region.

2.2. Existence of a attractor

When a, b satisfy H1 and H2, f has a hyperbolic fixed point $X = (\frac{1}{1+a-b}, \frac{1}{1+a-b})$ in the first quadrant and a hyperbolic fixed point $Y = (\frac{-1}{1+a-b}, \frac{-1}{1+a-b})$ in the third quadrant. The eigenvalues of the Jacobian matrix of f are $\lambda_u = \frac{-a-\sqrt{a^2+4b}}{2}$ and $\lambda_s = \frac{-a+\sqrt{a^2+4b}}{2}$. One half of the local unstable manifold of X intersects with x-axis for the first time at point A, one half of the local unstable manifold of Y intersects with x-axis for the first time at point B. We shall construct a "trapping region" by using these local unstable manifolds. Let G denote the region $Af(A)Bf(B)$, as shown in Figure 2, the y-axis divides G into two sub-regions $G_1 := \{(x, y) \in G|x \geq 0\}$ and $G_2 := \{(x, y) \in G|x < 0\}$.

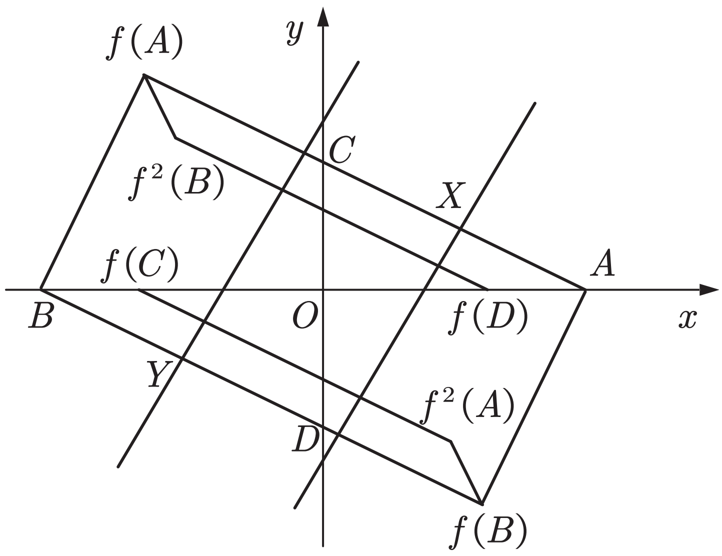


Figure 2: A trapping region of f.

Remark 2.1. *There are various definitions of an attractor, see for instance [32]. A main difference involved is that we require all points in the neighborhood to approach it. One of those definitions is required to have a trapping region, i.e., a region which is mapped with its closure into its interior. However, for a discontinuous map such a definition of trapping region is too strict. We use the definition given by Milnor [33] which only requires it to attract points in a set of positive Lebesgue measure.*

Lemma 2.1. *If a, b satisfy* H1-H3 *then $f(G) \subset G$.*

Proof. We consider the region G_1. The image of the point C is the point A. Since point C lies above point D, point $f(D)$ lies to the left-hand side of point A. Since point $f(B)$ is in the right half-plane, point$f^2(B)$ shall lie in the upper half-plane. If a, b satisfy $a^2 + 4b < 4$ (see appendix A) then point $f^2(B)$ lies to the right of the line $Bf(A)$. Besides, we know that f is a homeomorphism on both the left and right half-plane and that the segment $f^2(B)f(D)$ does not intersect segment $Af(A)$, therefore $f^2(B)$ lies below the segment $Af(A)$. Thus $f^2(B)$ belongs to the region G. As a consequence, G_1 is mapped into G. By the symmetry of the map f, G_2 is also mapped into G. Therefore, we have $f(G) \subset G$. □

Let a, b be as in Lemma 2.1. Let

$$\Lambda = \mathrm{cl}\Big(\bigcap_{n=0}^{\infty} f^n(G)\Big),$$

where $\mathrm{cl}(\cdot)$ denotes the closure of a set. Since Λ contains the unstable manifolds of the hyperbolic fixed points of f, these unstable manifolds being broken line segments, Λ is a chaotic attractor. The basin of attraction of the map f is shown in Figure 3 in yellow color.

2.3. SRB measure

Ergodic theory studies the evolution of long time behavior of typical orbits of dynamical systems based on invariant measures. Roughly speaking, an invariant measure

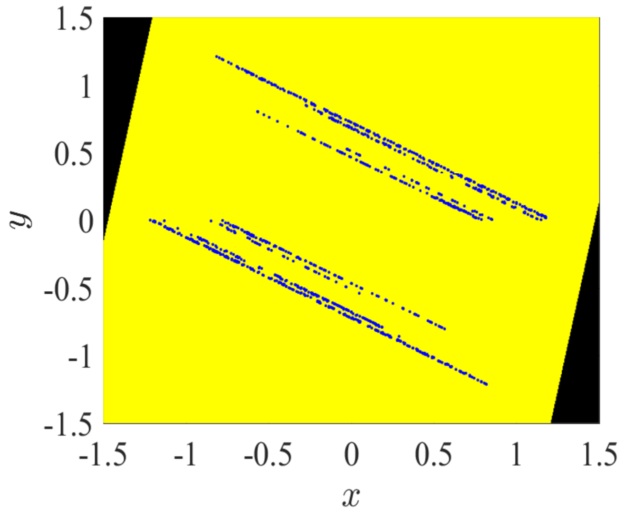


Figure 3: The attractor of f for $a = 1.5$, $b = 0.3$, where the yellow region is the basin of attraction.

describes the relative frequency of certain parts of phase space being visited by typical orbits. We are interested particularly in those invariant measures which are relevant to physical observations. A class of physical measures, called SRB measures, plays an important role in the study of particular dissipative systems that exhibit chaotic behavior. The SRB measures offer a mechanism for explaining how local instability on attractors can produce coherent stochastic-like behavior for orbits starting from sets with positive Lebesgue measures in the basin. (c.f.[35]).

We begin by recalling the definition of an SRB measure.

Definition 2.1. *An f-invariant Borel probability measure μ is called an SRB measure if*
(1) f has no zero Lyapunov exponents μ-a.e.;
(2) the conditional measures of μ on local unstable manifolds are absolutely continuous with respect to the Lebesgue measures on these manifolds.

When a, b satisfy conditions H1 and H2, it is clear that f has a hyperbolic split at any point p whose orbit does not intersect the y-axis since Df is a constant matrix. There is an invariant splitting:

$$T_pR^2 = E_p^u \oplus E_p^s$$

of the tangent space such that for all $n \geq 0$

$$\|Df^n(p)v\| = |\lambda_u|^n\|v\| \quad \forall v \in E_p^u$$

and

$$\|Df^n(p)v\| = |\lambda_s|^n\|v\| \quad \forall v \in E_p^s.$$

Thus, the unstable manifolds are broken line segments having the same slope.

Since the map f is a Lozi type map, the proof of the existence of an SRB measure is standard, see e.g. [34]. We only need to determine a parameter region in which f has such a measure. Except for the assumptions H1-H3, the following expanding condition on the unstable manifold is required.

H1': There exist positive integers N and K with $|\lambda_u|^N > K$ such that if γ is a sufficiently short segment of the unstable manifold, then $f^N(\gamma)$ has $\leq K$ smooth components.

The condition H4' is imposed to ensure there are abundant local unstable manifolds. If $|\lambda_u| = \frac{a+\sqrt{a^2+4b}}{2} > \sqrt{2}$ (simplified to H4: $\sqrt{2}a + b > 2$), then f satisfies the condition H4' for $K = 2$ and $N = 2$.

Let γ_0 be a smooth piece of the unstable manifold of the fixed point X, and let m_0 be the arc-length measure of γ_0. Under the assumptions H1-H4, consider

$$\nu_n = \frac{1}{n} \sum_{j=0}^{n-1} f_*^j(m_0),$$

where $f_*^j(m_0)(U) = m_0(f^{-j}(U))$ for any measurable set U. Let ν be a limit of ν_n in the weak-star topology. It is clear that ν is f-invariant. To construct an SRB measure of f, we use the approach given in [34] to catch a positive fraction of ν with absolutely continuous conditional measures on local unstable manifolds. Thus, we obtain the following

Theorem 2.1. *When a, b satisfy* H1-H4*, f has an SRB measure μ whose supported set is contained in the chaotic attractor Λ.*

Remark 2.2. *An attractor having an SRB measure is usually called a generalized hyperbolic attractor (c.f. [25]).*

3. Hausdorff dimension of the chaotic attractor

We recall first some definitions and results used in what follows. Let A be a set and $B_r(x)$ be the ball of radius r centered at $x \in A$. For $d > 0$ and $\varepsilon > 0$, let

$$H_d(\varepsilon, A) = \inf_{A \subseteq \cup_j B_{r_j}(x_j), \sup_j r_j < \varepsilon} \sum_j r_j^d.$$

The Hausdorff dimension of the set A is defined as:

$$HD(A) = \inf \left\{ d | \lim_{\varepsilon \to 0} H_d(\varepsilon, A) = 0 \right\}.$$

The box counting dimension of a set A is defined as follows. For a positive number r, let $N_A(r)$ be the smallest number of boxes of width r needed to cover the set A. We call

$$d_C(A) = \lim_{r \to \infty} \frac{\ln N_A(r)}{\ln(1/r)}.$$

the box-counting dimension of set A, see e.g. [2, 28]. For any bounded set A, we have $HD(A) \leq d_C(A)$.

Now, we compute the box-counting dimension of the chaotic attractor Λ, which gives an upper bound of the Hausdorff dimension of Λ. The following method has been used by Tél [37] to estimate the box-counting dimension of the chaotic attractor of a similar discontinuous piecewise linear map. Let Q and w be the area and width of the region G, respectively. After n iterations of f, the area of the region $f^n(G)$ is Qb^n , the width of each band of $f^n(G)$ is ωq^n, where $q = \frac{b}{|\lambda^u|}$. Taking small squares of side $\varepsilon_n \approx q^n$, the region

$f^n(G)$ can be covered by such squares. Thus, we obtain the box-counting dimension of the attractor:

$$d_C(\Lambda) = \lim_{n\to\infty} \frac{\ln N_n}{\ln(1/\varepsilon_n)} = 1 + \frac{\ln(a+\sqrt{a^2+4b}) - \ln 2}{\ln(a+\sqrt{a^2+4b}) - \ln(2b)}. \tag{1}$$

Before estimating the Hausdorff dimension of the chaotic attractor, we compute the Lyapunov exponents of the map f.

Lemma 3.1. *Denote by Ω the support of the SRB measure μ. For μ-almost every point $p \in \Omega$, the Lyapunov exponents of the map f are*

$$\begin{aligned} \lambda^+ &= \ln \frac{a+\sqrt{a^2+4b}}{2} > 0, \\ \lambda^- &= \ln \frac{-a+\sqrt{a^2+4b}}{2} > 0. \end{aligned}$$

Proof. For μ -a.e. $p \in \Omega$, there is an invariant splitting

$$T_p R^2 = E_p^u \oplus E_p^s$$

of the tangent space such that for all $n \geq 0$

$$\|Df^n(p)v\| = \left(\frac{a+\sqrt{a^2+4b}}{2}\right)^n \|v\| \quad \forall v \in E_p^u$$

and

$$\|Df^n(p)v\| = \left(\frac{\sqrt{a^2+4b}-a}{2}\right)^n \|v\| \quad \forall v \in E_p^s.$$

Besides, we have

$$\lambda^+(p) = \lim_{n\to\infty} \frac{1}{n} \ln \frac{\|Df^n(p)v\|}{\|v\|}, \quad 0 \neq v \in E_z^u$$

and

$$\lambda^-(p) = \lim_{n\to\infty} \frac{1}{n} \ln \frac{\|Df^n(p)v\|}{\|v\|}, \quad 0 \neq v \in E_z^s.$$

This completes the proof of the lemma. □

Pesin [25] proved that a generalized hyperbolic attractor can be decomposed into a number of countable ergodic components plus a component with zero measure in terms of the SRB measure, and the restriction of the map f to each ergodic component is invariant. Our estimate for the lower bound of the Hausdorff dimension of Λ is based on Pesin's entropy formula (c.f. [36]) and Young's formula for the Hausdorff dimension given in [29].

Theorem 3.1. *Suppose that a, b satisfy* H1-H4. *The Hausdorff dimension of the chaotic attractor Λ is*

$$DH(\Lambda) = 1 + \frac{\ln(a+\sqrt{a^2+4b}) - \ln 2}{\ln(a+\sqrt{a^2+4b}) - \ln(2b)}. \tag{2}$$

Proof. By the properties of SRB measures, there exist measurable sets $\Lambda_1, \Lambda_2, \ldots$ such that
(1) for each k, $f(\Lambda_k) = \Lambda_k$ and $\mu(\Lambda_k) > 0$;

(2) $\mu(\cup_k \Lambda_k) = 1$;
(3) for each k, (f, μ_k) is ergodic, where $\mu_k = \mu|\Lambda_k$.
Take any ergodic component Λ_k and normalize μ_k to a probability measure $\hat{\mu_k}$. By Young's formula (see [24]), we have

$$DH(\Lambda_k) = h_{\hat{\mu_k}}(f)\left(\frac{1}{\lambda^+} - \frac{1}{\lambda^-}\right),$$

where $h_{\hat{\mu_k}}(f)$ is the measure entropy of f on Λ_k. By Pesin's entropy formula, see e.g. [36],

$$h_{\hat{\mu_k}}(f) = \lambda^+.$$

Thus, we obtain

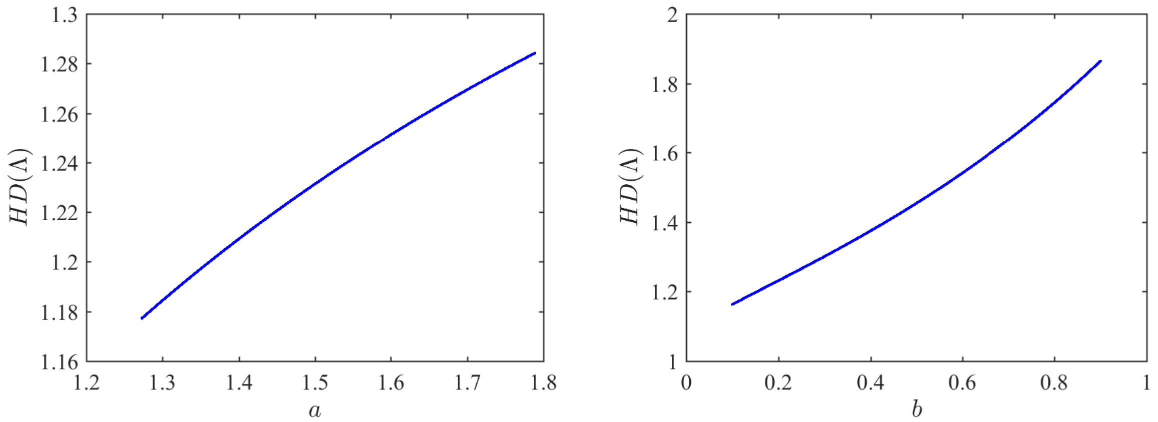
$$DH(\Lambda_k) = 1 - \frac{\lambda^+}{\lambda^-}.$$

Since Λ_k is a subset of Λ, we have $DH_(\Lambda_k) \leq DH(\Lambda)$. Using the conclusion in Lemma 3.1, we obtain

$$DH(\Lambda) \geq 1 + \frac{\ln(a + \sqrt{a^2 + 4b}) - \ln 2}{\ln(a + \sqrt{a^2 + 4b}) - \ln(2b)}. \quad (3)$$

Combining (1) and (3) yields (2). □

By formula (2), we see that $DH(\Lambda)$ is a strictly monotonically increasing function of a for fixed b, and that $DH(\Lambda)$ is a strictly monotonically increasing function of b for fixed a. Moreover, $DH(\Lambda) \to 1$ as $b \to 0$. See Figure 4.



(a) $b = 0.2$ (b) $a = 1.5$

Figure 4: Hausdorff dimension of the chaotic attractors of the map f: (a) as the parameter a varies, and (b) as the parameter b varies.

4. Conclusions

Generally, the Hausdorff dimension of chaotic attractors is not easy to compute. In this work we study the Hausdorff dimension of a chaotic attractor in a class of nonsmooth systems. By deriving the box-counting dimension and SRB measure, the exact formula of the Hausdorff dimension of the chaotic attractor is obtained. Using the derived formula, we discuss the relationship between the Hausdorff dimension of the chaotic attractor and the parameter values of the system and find that the Hausdorff dimension of the chaotic attractor depends on both the expansion rate of the unstable manifold and the dissipation rate of the map. The results of this work may provide a paradigmatic example for verifying the accuracy of the algorithm for calculating the fractal dimension of the chaotic attractors.

Acknowledgments

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Appendix

The hyperbolic fixed point of the map f in the first quadrant is

$$X = \left(\frac{1}{1+a-b}, \frac{1}{1+a-b}\right).$$

The eigenvalue of the Jacobian matrix of the map f with largest modulus is

$$\lambda_u = \frac{-a-\sqrt{a^2+4b}}{2}.$$

Thus, the equation of the line $Af(A)$ is

$$y = \frac{1}{\lambda_u}\left(x - \frac{1}{1+a-b}\right) + \frac{1}{1+a-b}.$$

By the above equation and the symmetry of the map f, a direct computation yields

$$A = \left(\frac{1-\lambda_u}{1+a-b}, 0\right),$$

$$f(A) = \left(\frac{1+a\lambda_u-b}{1+a-b}, \frac{1-\lambda_u}{1+a-b}\right),$$

$$f(B) = \left(-\frac{1+a\lambda_u-b}{1+a-b}, -\frac{1-\lambda_u}{1+a-b}\right),$$

$$f^2(A) = \left(\frac{-1-2a+2b+ab-a^2\lambda_u-b\lambda_u}{1+a-b}, \frac{1+a\lambda_u-b}{1+a-b}\right).$$

The equation of the line $Af(B)$ is

$$y = \frac{1-\lambda_u}{2-b+(a-1)\lambda_u}\left(x - \frac{1-\lambda_u}{1+a-b}\right). \tag{4}$$

Let $\hat{y} = (f^2(A))_y = \frac{1+a\lambda_u-b}{1+a-b}$, where $(f^2(A))_y$ is the ordinate of the point $f^2(A)$, we then obtain

$$\hat{x} = \frac{1+a\lambda_u-b}{1+a-b}\left(\frac{2-b+(a-1)\lambda_u}{1-\lambda_u}\right) + \frac{1-\lambda_u}{1+a-b}. \tag{5}$$

The condition that the point $f^2(A)$ lies to the left of line $Af(B)$ is $\hat{x} - \left(f^2(A)\right)_x > 0$, i.e,

$$\frac{1+a\lambda_u-b}{1+a-b}\left(\frac{2-b+(a-1)\lambda_u}{1-\lambda_u}\right)+\frac{1-\lambda_u}{1+a-b}-\frac{-1-2a+2b+ab-a^2\lambda_u-b\lambda_u}{1+a-b}>0, \quad (6)$$

where $\left(f^2(A)\right)_x$ denotes the abscissa of the point $f^2(A)$. Since $1+a-b>0$, the inequality (6) can be simplified as

$$\frac{(a-1+b)\lambda_u^2+\left(-a^2+(-1+b)a+4-4b\right)\lambda_u+(-2+b)a-4+5b-b^2}{-1+\lambda_u}>0. \quad (7)$$

Since $-1+\lambda_u<0$, the inequality (7) is simplified as

$$(a-1+b)\lambda_u^2+\left(-a^2+(-1+b)a+4-4b\right)\lambda_v+(-2+b)a-4+5b-b^2<0. \quad (8)$$

Due to $\lambda_u = \frac{-(a+\sqrt{a^2+4b})}{2}$, we have

$$\left(a^2-2+2b\right)\sqrt{a^2+4b}+a^3+(-4+4b)a-4+4b<0. \quad (9)$$

Set $\sqrt{a^2+4b}=u$. Then we obtain

$$\frac{1}{2}(-2+u)(a+u+2)^2<0, \quad (10)$$

which is equivalent to $-2+u<0$, i.e., $-2+\sqrt{a^2+4b}<0$. Therefore, we have

$$a^2+4b<4.$$

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