On a dissipative dynamics with Hamiltonian structure

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Abstract

In this work, we study a class of dissipative, non-smooth n degree-of-freedom dynamical systems. As the dissipation is assumed to be proportional to the momentum, the dynamics in such systems is conformally symplectic, allowing us to use some of the Hamiltonian structure. We initially show that there exists an integral invariant of the Poincar-Cartan type in such systems. Then, we prove the existence of a generalized Liouvilles Formula for conformally symplectic systems with rigid constraint using the integral invariant. A two degree-of-freedom system is analysed to support the relevance of our results..

Keywords: integral invariant, impact systems, generalized Liouvile formula

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A class of dynamical systems, whose dissipation is proportional to the momentum, has attracted broad attention recently. They are conformally symplectic systems that, though being dissipative, have certain structures similar to the ones in Hamiltonian systems. The existence of non-smoothness in vibro-impact systems, considered in this work, restricts the application of theories from smooth conformally symplectic systems. In this work, we make a first step in addressing non-smooth conformally symplectic systems. We show that there exists an integral invariant of the Poincaré-Cartan type in such systems. By using the integral invariance, we prove the existence of a generalized Liouvilles Formula for conformally symplectic systems with a rigid constraint. We illustrate the applicability of our proofs through an example of such a dynamical system.

1. Introduction

Mechanical systems with impacts have received both practical and theoretical interests in the study of non-linear dynamics [1–3]. Since the impacts introduce new non-linearity and non-smoothness, the dynamics can be quite rich and complicated, such as grazing bifurcation [4–7], Neimark-Sacker bifurcation [8], Hopf-Hopf bifurcation [9]. We refer the reader to the survey [10] for a detailed discussion. In recent years, the conformally symplectic systems are studied by methods usually that appeared in conservative cases.

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Marò and Sorrentino [11] extend the Aubry-Mather theory to conformally symplectic systems. Invariant sets such as Mather sets are still well defined and related to attractors. Moreover, for conformally symplectic systems, classical KAM techniques are also applied to show the existence of attractive tori, see [12, 13]. In fact, it is a natural generalization because the usual action can be modified to adapt the dissipative equations, see Section 2 for more details.

Poincaré-Cartan integral invariant plays an fundamental role in conservative dynamical systems. Basic results in Hamiltonian mechanics can be obtained directly through it, see [14]. It is natural to ask if there exist integral invariants for conformally symplectic systems. In this paper, we derive an integral invariant by modifying the classical action for such systems and prove the existence of it. Then we use the integral invariant to study forced dissipative systems with one-side constraint and obtain the Jacobian determinant formulas for two types of Poincaré maps. Since the existence of the constraint causes the implicity of arrived time at the constraint, one cannot apply the classical Liouville's formula to evaluate the Jacobian determinant of a time periodic map. However, the existence of an integral invariant can overcome this difficulty. In addition, the formulas may be helpful to study the bifurcation of impact systems. For example, as a simple application of the generalized Liouville's formulas, one can infer that Neimark-Sacker bifurcation does not occur in one degree-of-freedom forced systems with impacts, and that the singularity of grazing motion is directly followed by the change of Jacobian. A similar result is obtained in [15] for one degree-of-freedom linear oscillator with impact, though our result has no restriction about forcing and about the number of degrees of freedom. The most natural way for the further applications of our results are the establishment of the dissipative twist map [16, 17], in which attractive periodic orbits, invariant curves and strange chaotic sets can show up.

The remaining of paper is organized as follows. In Section 2, we use the formalization of Lagrangian and Hamiltonian mechanics to deduce the existence of an integral invariant for a class of dissipative systems. We obtain a determinant formula, the generalized Liouville's formula for impact systems using the integral invariant in Section 3. The numerical example of a two-degree-of-freedom system is presented in Section 4 to demonstrate our mathematical result.

2. Integral invariant of the conformally symplectic systems

First we recall the definition of integral invariant. Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $F: \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 vector field of $\mathbb{R}^n$. We assume $F(x) \neq 0$ for any $x \in \mathbb{R}^n$.

Definition 2.1. *Let $g_i: \mathbb{R}^n \to \mathbb{R}$ be a C^1 function. A 1-form $\sum_{i=1}^n g_i(x)dx_i$ is called an integral invariant of a vector field F, if for any two closed orientated loops C_1 and C_2 such that $C_1 - C_2$ is the boundary of an orientated cylinder made of the orbits of the vector field F (see Figure 1), we have*

$$\int_{C_1} \sum_{i=1}^n g_i(x)dx_i = \int_{C_2} \sum_{i=1}^n g_i(x)dx_i.$$

For a C^2 Hamiltonian $H(x, y, t) = \frac{1}{2}|y|^2 + V(x, t)$, where $y = (y_1, \cdots, y_n), t \in \mathbb{R}, |\cdot|$ is Euclidean norm and $V: \mathbb{R}^n \times \mathbb{R}$ is a C^2 function, it is known that the form

$$\omega = \sum_{i=1}^n y_i dx_i - H(x, y, t)dt$$

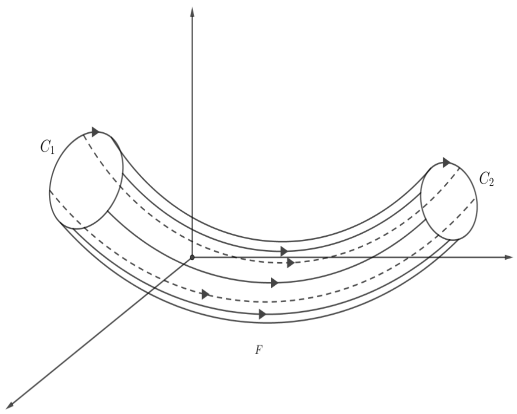

Figure 1: The cylinder made of orbits of F.

is an integral invariant of the extended Hamiltonian vector field

$$(\frac{\partial H}{\partial y}, -\frac{\partial H}{\partial x}, 1) = (y, -\nabla V(x,t), 1)$$

in $\mathbb{R}^{2n+1}$, where $\nabla V(x,t) = \frac{\partial V}{\partial x}(x,t)$.

Now we consider a class of dissipative systems

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\nabla V(x,t) - \lambda y, \\ \dot{t} &= 1, \end{aligned} \tag{1}$$

where $\lambda > 0$ is a constant.

Theorem 2.1. *The 1-form*

$$\omega_\lambda = e^{\lambda t}(\sum_{i=1}^{n} y_i dx_i - H(x,y,t)dt) \tag{2}$$

is an integral invariant for the vector field of equation (1).

Proof. With the notation in Definition 2.1, the 1-form $\sum_{i=1}^{n} g_i(x)dx_i$ is an integral invariant of the vector field F if and only if

$$d(\sum_{i=1}^{n} g_i(x)dx_i)(F(x), \cdot) = 0,$$

where d is the exterior differential operator. See Chapter 9 of [14] or [18] for a proof. By a direct calculation, for any $\xi = (\xi_1, \xi_2, \cdots, \xi_{2n+1}) \in \mathbb{R}^{2n+1}$ we have

$$d\Big(e^{\lambda t}(\sum_{i=1}^{n} y_i dx_i - H dt)\Big)(X_\lambda, \xi) = e^{\lambda t}\Big(\sum_{i=1}^{n}\big(dy_i \wedge dx_i - (\frac{\partial H}{\partial x_i} + \lambda y_i) dx_i \wedge dt - \frac{\partial H}{\partial y_i} dy_i \wedge dt\big)(X_\lambda, \xi)\Big)$$

$$= e^{\lambda t}\sum_{i=1}^{n}\left(\begin{vmatrix} -\lambda y_i - \frac{\partial H}{\partial x_i} & \xi_{i+n} \\ \frac{\partial H}{\partial y_i} & \xi_i \end{vmatrix} - (\frac{\partial H}{\partial x_i} + \lambda y_i)\begin{vmatrix} \frac{\partial H}{\partial y_i} & \xi_i \\ 1 & \xi_{2n+1} \end{vmatrix} - \frac{\partial H}{\partial y_i}\begin{vmatrix} -\lambda y_i - \frac{\partial H}{\partial x_i} & \xi_{i+n} \\ 1 & \xi_{2n+1} \end{vmatrix}\right) = 0,$$

where $X_\lambda = (y, \nabla V(x,t) - \lambda y, 1)$. Hence, we have $d\omega_\lambda(X_\lambda, \cdot) = 0$, which implies the statement of theorem. $\square$

Significantly, such dissipative systems still have the formalization of Hamiltonian and Lagrangian systems. More precisely, let $L(x, \dot{x}, t) = \frac{1}{2}|\dot{x}|^2 - V(x,t)$ and the modified Lagrangian $\tilde{L}$ be defined by

$$\tilde{L}(x, \dot{x}, t) := e^{\lambda t} L(x, \dot{x}, t) = e^{\lambda t}(\frac{1}{2}|\dot{x}|^2 - V(x,t)).$$

Then the *Euler-Lagrange* equation of the modified Lagrangian becomes

$$\frac{d}{dt}\frac{\partial \tilde{L}}{\partial \dot{x}} = \frac{\partial \tilde{L}}{\partial x}.$$

A direct calculation yields that

$$\frac{d}{dt}(e^{\lambda t}\frac{\partial L}{\partial \dot{x}}) = e^{\lambda t}(\lambda \frac{\partial L}{\partial \dot{x}} + \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}), \qquad \frac{\partial \tilde{L}}{\partial x} = e^{\lambda t}\frac{\partial L}{\partial x},$$

hence we have

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = -\lambda \frac{\partial L}{\partial \dot{x}} + \frac{\partial L}{\partial x}.$$

Let $\partial L / \partial \dot{x} = y = \dot{x}$, then above equation becomes

$$\dot{y} = -\lambda y - \nabla V(x,t),$$

which coincides with (1). The corresponding modified Hamiltonian is obtained by *Legendre* transformation. Let $\tilde{y} = \partial \tilde{L}/\partial \dot{x} = e^{\lambda t} y$. The modified Hamiltonian is given by

$$\begin{aligned}\tilde{H}(x, \tilde{y}, t) &:= \frac{\partial \tilde{L}}{\partial \dot{x}} \cdot \dot{x} - \tilde{L} = e^{\lambda t}\dot{x} \cdot \dot{x} - \frac{1}{2}e^{\lambda t}|\dot{x}|^2 + e^{\lambda t}V(x,t) \\ &= \frac{1}{2e^{\lambda t}}|\tilde{y}|^2 + e^{\lambda t}V(x,t) = e^{\lambda t}H(x,y,t).\end{aligned}$$

One can verify that the corresponding Hamiltonian equation of $\tilde{H}$ coincides with (1) and the 1-form $\tilde{y}dx - \tilde{H}dt$ coincides with ω_λ.

3. A generalized Liouville Formula for the vibro-impact systems

Firstly, we recall the classical Liouville Formula. For a time dependent C^1 system $\dot{x} = F(x,t)$, we usually take the time-T-map to study it. The time-T-map ϕ^T is defined by $\phi^T(x,y) = \phi(x,y,T,0)$, where $\phi(x,y,t,t_0)$ is the solution of (3) with initial condition $x(t_0) = x, y(t_0) = y$. The classical Liouville Formula is

$$\det(D\phi^T) = e^{\int_0^T \mathrm{tr}\frac{\partial \mathrm{F}}{\partial \mathrm{x}}(\mathrm{x(s),s})\mathrm{ds}}.$$

Here and in what follows trA denotes the trace of matrix A.

We consider an n degree-of-freedom system:

$$\begin{aligned}\dot{x} &= y, \\ \dot{y} &= -\lambda y - \nabla k(x,t),\end{aligned} \tag{3}$$

where $x = (x_1, \cdots, x_n), y = (y_1, \cdots, y_n) \in \mathbb{R}^n$, $k : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a C^2 function and $k(x, t+T) = k(x,t)$ for some $T > 0$. Assume that there exists a rigid constraint at $x_1 = 1$ and the impact rule is $y_1 \mapsto -\mu y_1$, where $0 < \mu \leq 1$ is a constant. According to Section 2, the integral invariant of system (3) is $\omega_\lambda = e^{\lambda t}(\sum_{i=1}^n y_i dx_i - H dt)$, where $H(x, y, t) = \frac{1}{2}|y|^2 + k(x, t)$. By the Liouville formula, the Jacobian determinant of time-T-map of (3) is $e^{-\lambda n T}$ if there is no constraint.

To insure the differentiability of ϕ^T, we assume that all the impacts are transversal, i.e., $y_1(t) \neq 0$ when $x_1(t) = 1$. For an orbit with initial condition $x(0) = x, y(0) = y$, if there exists only one transversal impact in the time interval $(0, T)$, then, by the Implicit Function Theorem and the differentiability of solution with respect to initial condition, there exists a neighborhood of (x, y) such that all the orbits with initial position on this neighborhood has one transversal impact in the time interval $(0, T)$. Therefore, our assumption is natural.

Proposition 3.1. *If there exists only one transversal impact during a period, then the Jacobian determinant of time-T-map ϕ^T is equal to $\mu^2 e^{-\lambda n T}$.*

Proof. Let C_0 be any loop at $\{(x, y, 0) \in \mathbb{R}^{2n+1}\}$ and φ_0 be the map from $t = 0$ to $x_1 = 1$ under the flow on extended phase space. Let $C_1 = \varphi C_0$. By (2), ω_λ is an integral invariant of extended vector field $(-\lambda y - \nabla k(x, t), 1)$, then we have

$$\int_{C_0} e^{\lambda t}(\sum_{i=1}^n y_i dx_i - H dt) = \int_{C_1 = \varphi_0 C_0} e^{\lambda t}(\sum_{i=1}^n y_i dx_i - H dt). \tag{4}$$

Let $\omega_0 = \omega_\lambda|_{t=0}$ and $\omega_1 = \omega_\lambda|_{x_1=1}$, then $\omega_0 = \sum_{i=1}^n y_i dx_i$, $\omega_1 = e^{\lambda t}(\sum_{i=2}^n y_i dx_i - H|_{x_1=1} dt)$ and we have

$$\int_{C_0} \omega_0 = \int_{C_1} \omega_1 = \int_{C_0} \varphi_0^* \omega_1, \tag{5}$$

where $*$ is the pull back of differential forms. Let D_0 be the domain in $t = 0$ with boundary C_0, then by Green's formula and (5) we have

$$\iint_{D_0} d\omega_0 = \iint_{D_0} d\varphi_0^* \omega_1 = \iint_{D_0} \varphi_0^* d\omega_1. \tag{6}$$

Since D_0 is arbitrary, we must have $d\omega_0 = \varphi_0^* d\omega_1$. Hence,

$$\underbrace{d\omega_0 \wedge d\omega_0 \wedge \cdots \wedge d\omega_0}_{n \text{ times}} = \underbrace{\varphi_0^* d\omega_1 \wedge \varphi_0^* d\omega_1 \wedge \cdots \wedge \varphi_0^* d\omega_1}_{n \text{ times}} = \varphi_0^*(\underbrace{d\omega_1 \wedge d\omega_1 \wedge \cdots \wedge d\omega_1}_{n \text{ times}}).$$

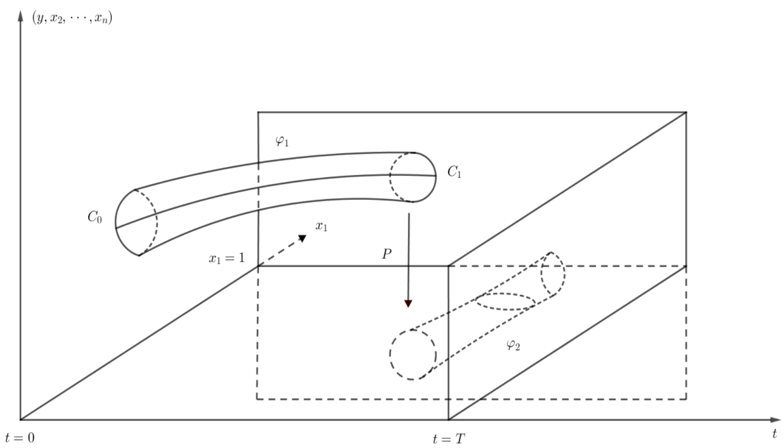

Figure 2: The time-T-map with one impact.

According to the definition of ω_0 and ω_1, we have

$$\underbrace{d\omega_0 \wedge d\omega_0 \wedge \cdots \wedge d\omega_0}_{n \text{ times}} = n! dy_1 \wedge dx_1 \wedge dy_2 \wedge dx_2 \wedge \cdots \wedge dy_n \wedge dx_n,$$

$$\underbrace{d\omega_1 \wedge d\omega_1 \wedge \cdots \wedge d\omega_1}_{n \text{ times}} = -n! e^{\lambda nt} y_1 dy_1 \wedge dt \wedge dy_2 \wedge dx_2 \wedge dy_3 \wedge dx_3 \wedge \cdots \wedge dy_n \wedge dx_n.$$

Let $P(x, y, t) = (x, -\mu y_1, \cdots, y_n, t)$ and A be a $2n$ dimension volume on $t = 0$. Then,

$$\int_A d\omega_0 \wedge \cdots \wedge d\omega_0 = \int_{\varphi_0 A} d\omega_1 \wedge \cdots \wedge d\omega_1, \tag{7}$$

$$\int_{P\varphi_0 A} d\omega_1 \wedge \cdots \wedge d\omega_1 = \int_{\varphi_0 A} P^*(d\omega_1 \wedge \cdots \wedge d\omega_1) = \mu^2 \int_{\varphi_0 A} d\omega_1 \wedge \cdots \wedge d\omega_1. \tag{8}$$

Denote by φ_1 the map from $x_1 = 1$ to $t = T$ under the flow on extended phase space. Let $\omega_T = \omega_\lambda|_{t=T}$. With the same calculation as (4), (5) and (6), we get the following identities:

$$\varphi_1^* d\omega_T = d\omega_1, \ \varphi_1^*(\underbrace{d\omega_T \wedge \cdots \wedge d\omega_T}_{n \text{ times}}) = \underbrace{d\omega_1 \wedge \cdots \wedge d\omega_1}_{n \text{ times}},$$

where $d\omega_T \wedge \cdots \wedge d\omega_T = e^{\lambda nT} d\omega_0 \wedge \cdots \wedge d\omega_0$. Then according to (8) and (7) we have

$$\begin{aligned} \int_{\varphi_1 P\varphi_0 A} d\omega_T \wedge \cdots \wedge d\omega_T &= \int_{P\varphi_0 A} \varphi_1^*(d\omega_T \wedge \cdots \wedge d\omega_T) \\ &= \int_{P\varphi_0 A} d\omega_1 \wedge \cdots \wedge d\omega_1 \\ &= \mu^2 \int_{\varphi_0 A} d\omega_1 \wedge \cdots \wedge d\omega_1 \\ &= \mu^2 \int_A d\omega_0 \wedge \cdots \wedge d\omega_0. \end{aligned} \tag{9}$$

Due to $d\omega_T \wedge \cdots \wedge d\omega_T = e^{\lambda nT} d\omega_0 \wedge \cdots \wedge d\omega_0$, by (9) we have

$$\int_{\varphi_1 P\varphi_0 A} d\omega_0 \wedge \cdots \wedge d\omega_0 = \mu^2 e^{-\lambda nT} \int_A d\omega_0 \wedge \cdots \wedge d\omega_0. \tag{10}$$

Since $\phi^T = \varphi_1 P \varphi_0$, we obtain

$$\begin{aligned} \int_{\phi^T A} dy_1 \wedge dx_1 \wedge \cdots \wedge dy_n \wedge dx_n &= \int_A \det D\phi^T dy_1 \wedge dx_1 \wedge \cdots \wedge dy_n \wedge dx_n \\ &= \mu^2 e^{-\lambda nT} \int_A dy_1 \wedge dx_1 \wedge \cdots \wedge dy_n \wedge dx_n. \end{aligned} \tag{11}$$

Since A is arbitrary, we have the determinant Liouville's formula, $\det(D\phi^T) = \mu^2 e^{-\lambda nT}$. □

Remark 1. *Observe that Proposition 3.1 does not rely on the periodicity of (3). Therefore, if there exist p times transversal impacts in a period, then*

$$detD\phi^T = \mu^2 e^{-\lambda n(t_1 - 0)} \times \mu^2 e^{-\lambda n(t_2 - t_1)} \times \cdots \times \mu^2 e^{T - t_{p-1}} = \mu^{2p} e^{-\lambda nT},$$

where t_i is the time which impact occurs.

Remark 2. *For one degree-of-freedom systems, we can conclude that there is no Neimark-Sacker bifurcation because a pair of conjugate eigenvalues of fixed points for time-T-map does not cross the unit circle on the characteristic plane provided that $\lambda > 0$.*

Remark 3. *We can straightforwardly infer the singularity of grazing motion from the Jacobian formula, since the Jacobian determinant of time-T-map without impact is $e^{-\lambda nT}$ by the generalized Liouville Formula.*

We can also choose the impact section $\Pi = \{(x, y, s) \in \mathbb{R}^n \times \mathbb{R}^n \times S^1 : x_1 = 1\} \cong \mathbb{R}^{2n-1} \times S^1$ as the Poincaré section. Let $P : \Pi \to \Pi$ be defined by

$$P(1, x_2, \cdots, x_n, y_1, \cdots, y_n, s) = (1, x_2, \cdots, x_n, -\mu y_1, \cdots, y_n, s).$$

Assume that there exists an orbit with twice successive transversal impacts whose states after impact are $(x^0, y^0, t_0), (x^1, y^1, t_1) \in \Pi$. Then we have

$$P\phi(x^0, y^0, t_1, t_0) = (x^1, y^1, t_1).$$

According to the Implicit Function Theorem, there exists a neighborhood U of (x^0, y^0, t_0) on Π such that $P\phi(x, y, s(x, y, t), t)$ is well defined, where $s(x, y, t)$ is the time such that

$$\phi(x, y, s(x, y, t), t) \in \Pi.$$

Let $K(x, y, t) = P\phi(x, y, s(x, y, t), t)$ defined on U. For simplicity, we ignore the first coordinate. Like in Proposition 3.1, we can use integral invariant to calculate the determinant of DK.

Proposition 3.2. *Let $K(x_2, \cdots, x_n, y_1, \cdots, y_n, s) = (x'_2, \cdots, x'_n, y'_1, \cdots, y'_n, s')$, then the determinant of $DK(x_2, \cdots, x_n, y_1, \cdots, y_n, s)$ is $\mu^2 e^{\lambda(s-s')} y_1/y'_1$.*

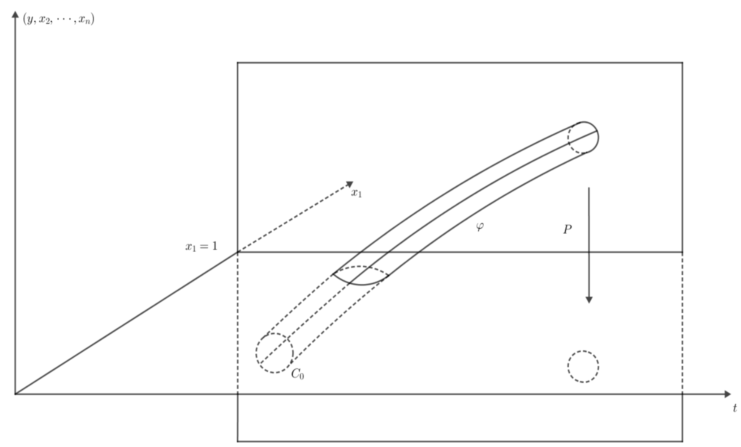

Figure 3: The map K on extended phase space.

Proof. Let $\varphi = \phi(x, y, s(x, y, t), t)$. Since ω_λ is an integral invariant in the extended phase space, for any loop C_0 contained in U we have

$$\int_{C_0} e^{\lambda t} (\sum_{i=1}^{n} y_i dx_i - H dt) = \int_{\varphi C_0} e^{\lambda t} (\sum_{i=1}^{n} y_i dx_i - H dt). \tag{12}$$

Let D_0 be the domain with boundary C_0. It follows (12) that

$$\iint_{D_0} d\omega_1 = \iint_{\varphi D_0} d\omega_1 = \iint_{D_0} \varphi^* d\omega_1,$$

where $\omega_1 = \omega_\lambda|_{x_1=1}$. Therefore, we have

$$d\omega_1 = \varphi^* d\omega_1, \ \underbrace{d\omega_1 \wedge \cdots \wedge d\omega_1}_{n \text{ times}} = \varphi^*(\underbrace{d\omega_1 \wedge \cdots \wedge d\omega_1}_{n \text{ times}}), \tag{13}$$

where $d\omega_1 \wedge \cdots \wedge d\omega_1 = -n!e^{\lambda nt} y_1 dy_1 \wedge dt \wedge dy_2 \wedge dx_2 \wedge dy_3 \wedge dx_3 \wedge \cdots \wedge dy_n \wedge dx_n$. Let A be any $2n$ dimension volume contained in U. Since $K = P\varphi$, by definition of Π and (13) we have

$$\begin{aligned}
\int_{K(A)} d\omega_1 \wedge \cdots \wedge d\omega_1 &= \int_{\varphi(A)} P^*(d\omega_1 \wedge \cdots \wedge d\omega_1) \\
&= -\mu^2 \int_{\varphi(A)} n!e^{\lambda nt} y_1 dy_1 \wedge dt \wedge dy_2 \wedge dx_2 \wedge dy_3 \wedge dx_3 \wedge \cdots \wedge dy_n \wedge dx_n \\
&= \mu^2 \int_{\varphi(A)} d\omega_1 \wedge \cdots \wedge d\omega_1 = \mu^2 \int_A \varphi^*(d\omega_1 \wedge \cdots \wedge d\omega_1) \\
&= \mu^2 \int_A d\omega_1 \wedge \cdots \wedge d\omega_1. \qquad (14)
\end{aligned}$$

Moreover, since

$$\int_{K(A)} d\omega_1 \wedge \cdots \wedge d\omega_1 = \int_A -n!e^{\lambda nt'} y_1' \det DK dy_1 \wedge dt \wedge dy_2 \wedge dx_2 \wedge dy_3 \wedge dx_3 \wedge \cdots \wedge dy_n \wedge dx_n,$$

it together with (14) yield

$$\begin{aligned}
&\mu^2 \int_A -n!e^{\lambda nt} y_1 dy_1 \wedge dt \wedge dy_2 \wedge dx_2 \wedge dy_3 \wedge dx_3 \wedge \cdots \wedge dy_n \wedge dx_n \\
&\qquad = \int_A -n!e^{\lambda nt'} y_1' \det DK dy_1 \wedge dt \wedge dy_2 \wedge dx_2 \wedge dy_3 \wedge dx_3 \wedge \cdots \wedge dy_n \wedge dx_n.
\end{aligned}$$

Since A is arbitrary, we have $\det DK = \mu^2 e^{-n(t'-t)} y_1/y_1'$. □

Remark 4. *For a T-periodic solution of (3) with one transversal impact during a period, the determinant formula in Proposition 3.1 is coincident with that in Proposition 3.2, though they are based on different Poincaré maps. Moreover, the Jacobian determinant tends to infinity as an orbit tends to the grazing motion.*

4. An example

We consider the two-degree-of-freedom system:

$$\begin{aligned}
\dot{x}_1 &= y_1, \\
\dot{y}_1 &= \frac{1}{m_1}\big(-2k_1(x_1 - x_2) - 4k_3(x_1 - x_2)^3 - \lambda y_1 + A\cos(t)\big), \\
\dot{x}_2 &= y_2, \\
\dot{y}_2 &= \frac{1}{m_2}\big(2k_1(x_1 - x_2) - 2k_2 x_2 + 4k_3(x_1 - x_2)^3 - \lambda y_2\big),
\end{aligned} \tag{15}$$

where $m_1, m_2, k_1, k_2, k_3, A, \lambda$ are constants. The Lagrangian of (15) is

$$L(x_1, x_2, \dot{x}_1, \dot{x}_2, t) = \frac{1}{2}(m_1\dot{x}_1^2 + m_2\dot{x}_2^2) - \big(k_1(x_1 - x_2)^2 + k_2x_2^2 + k_3(x_1 - x_2)^4 - Ax_1\cos(t)\big).$$

Assume there exists a constraint at $x_1 = d_0 > 0$, and the impact rule is $y_1(t) \to -ry_1(t)$.

Numerically, we can construct two types of Poincaré maps by numerical integration. The perturbations of initial states give the interpolation data such that usual interpolation methods yields the numerical derivative along different directions. Therefore, we can get the numerical Jacobian matrices with two orthogonal perturbations. The transversality insures that this method is reasonable. We can use the classical Newton's iteration to search for fixed points of Poincaré maps. In the following we apply this method to (15).

Take the following parameter values:

$$m_1 = 1,\ m_2 = 1,\ k_1 = 0.5,\ k_2 = 0.5,$$
$$k_3 = 0.0625,\ A = 1.7,\ d_0 = 0.1,\ r = 0.8.$$

According to the formula in Proposition 3.1 and Remark 1, the Jacobian determinant of the time-2π-map is $0.8^{2p}e^{-4\lambda\pi}$, where p is the number of impact times during a period. The periodic solutions of (15) with parameter λ are described by Figure 4. During

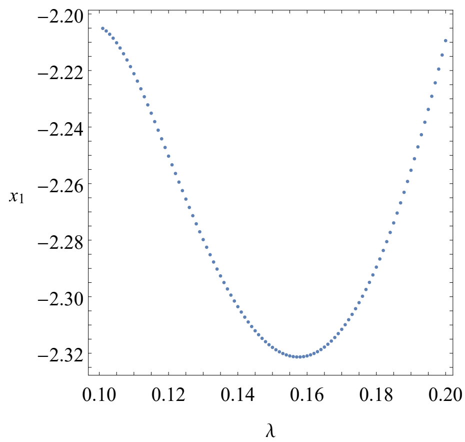

Figure 4: Periodic solutions projected on x_1 as λ varies

a period when λ ranges from 0.1 to 0.2, the periodic solutions of Figure 4 has two impacts. For example, for $\lambda = 0.1$ and initial condition $x_1(0) = -2.20442313, y_1(0) = 3.27870623, x_2(0) = -0.95874086, y_2(0) = 4.30083286$, the projection of the periodic solution of (15) on (x_1, y_1)-plane is presented in Figure 5.

By numerical interpolation we can evaluate the Jacobian determinant of time-2π-map with different initial conditions. The Jacobian determinant of time-2π-map corresponding to different periodic solutions and the graph of $0.8^4e^{-4\lambda\pi}$ are shown in Figure 6. As for the impact section as a Poincaré section, we consider the initial condition $x_1(s) = 1, y_1(s) = -1, x_2(s) = 2, y_2(s) = 2$. Then, according to Proposition 3.2, the determinant of DK is $-0.8^2e^{-4\lambda(s'-s)}/y_1'$, where s' is the next time such that $x_1(s') = 1$ and y_1' is the velocity of x_1 after impact. Take $s = 0$ and $\lambda = 0.1$. By numerical integration, we have

$$K(-1, 2, 2, 0) = (-0.842834, 2.29237, -0.255496, 0.321739).$$

Then

$$\det DK(-1, 2, 2, 0) = 0.8^2e^{-4\times 0.1\times 0.321739}/0.842834 \approx 0.71209862,$$

whose value is extremely close to the numerical value of the Jacobian determinant 0.71209863.

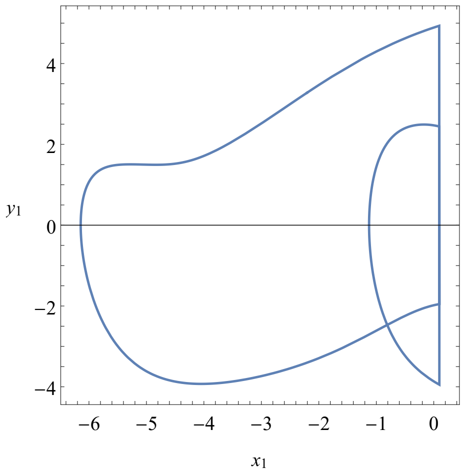

Figure 5: Periodic solution with two impacts when $\lambda = 0.1$

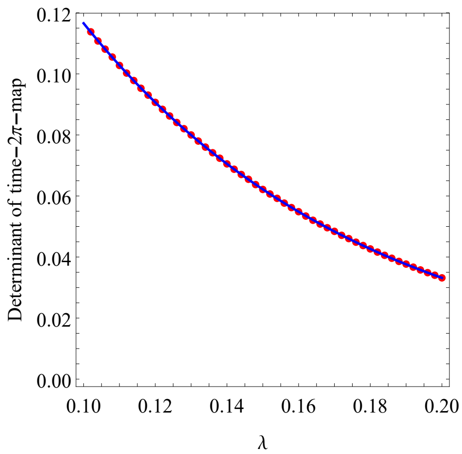

Figure 6: Theoretical and numerical determinants as λ varies: the blue line is the graph of $0.8^4 e^{-4\lambda\pi}$ and the red dots are the numerical Jacobian with 50 points for each value of λ. They are on top of each other.

5. Concluding remarks

In this work, a class of vibro-impact system, the non-smooth conformally symplectic systems, is investigated mathematically. We show the existence of an integral invariant of Poincaré-Cartan type and a generalized Liouville Formula for such systems. The results are illustrated using a two degree-of-freedom vibro-impact system. The most natural way for further applications are the establishment of the dissipative twist map [16], in which attractive periodic orbits, invariant curves and strange chaotic sets can be present in the dynamics [16, 17]. For the non-smooth conformally symplectic systems, this would entail further study.

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Reference

[1] R. A. Ibrahim. *Vibro-impact Dynamics: Modeling, Mapping and Applications.* Springer, Berlin, Heidelberg, 2009.

[2] M. di Bernardo, C. J. Budd, A.R. Champneys, and P. Kowalczyk. *Piecewise-Smooth Dynamical Systems: Theory and Applications.* Springer, London, 2008.

[3] J.-O. Aidanpää, H. H. Shen and R. B. Gupta. Stability and bifurcations of a stationary state for an impact oscillator. *Chaos*, 4(4):621–630, 1994.

[4] A. B. Nordmark. Non-periodic motion caused by grazing incidence in an impact oscillator. *Journal of Sound Vibration*, 145(2):279-297, 1991.

[5] D. R. J. Chillingworth. Dynamics of an impact oscillator near a degenerate graze. *Nonlinearity*, 23(11):2723, 2010.

[6] J. Shen, Y. R. Li, and Z. D. Du. Subharmonic and grazing bifurcations for a simple bilinear oscillator. *International Journal of Non-Linear Mechanics*, 60:70-82, 2014.

[7] W. Chin, E. Ott, H. E. Nusse and C. Grebogi. Grazing bifurcations in forced impact oscillators. *Physical Review E*, 50(6):4427–4444, 1994.

[8] G. W. Luo and J. H. Xie. Hopf bifurcation of a two-degree-of-freedom vibro-impact system. *Journal of Sound and Vibration*, 213(3):391-408, 1998.

[9] J. H. Xie and W. C. Ding. Hopf-Hopf bifurcation and invariant torus T^2 of a vibro-impact system. *International Journal of Non Linear Mechanics*, 40(4):531-543, 2005.

[10] O. Makarenkov and J. S. W. Lamb. Dynamics and bifurcations of nonsmooth systems: A survey. *Physica D*, 241(22):1826-1844, 2012.

[11] S. Marò and A. Sorrentino. Aubry–Mather theory for conformally symplectic systems. *Communications in Mathematical Physics*, 354(2):775-808, 2017.

[12] R. C. Calleja, A. Celletti, and R. de la Llave. A KAM theory for conformally symplectic systems: efficient algorithms and their validation. *Journal of Differential Equations*, 255(5):978-1049, 2013.

[13] R. C. Calleja, A. Celletti, and R. de la Llave. Local behavior near quasi-periodic solutions of conformally symplectic systems. *Journal of Dynamics and Differential Equations*, 25(3):821-841, 2013.

[14] V. I. Arnold. *Mathematical Methods of Classical Mechanics.* Springer, New York, Second edition, 1989.

[15] C. Budd, F. Dux, and A. Cliffe. The effect of frequency and clearance variations on single-degree-of-freedom impact oscillators. *Journal of Sound and Vibration*, 184(3):475-502, 1995.

[16] M. Casdagli. Rotational chaos in dissipative systems. *Physica D*, 29(3):365–386, 1988.

[17] K. Hockett and P. Holmes. Josephson's junction, annulus maps, Birkhoff attractors, horseshoes and rotation sets. *Ergodic Theory and Dynamical Systems*, 6(2):205–239, 1986.

[18] A. Chenciner. *From Euler-Lagrange Equations to Twist Maps*. Tsinghua, 2018.