Strange nonchaotic attractors in a periodically forced

piecewise linear system with noise

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Abstract: The study of strange nonchaotic attractors (SNAs) has been mainly restricted to quasiperiodically forced systems. At present, SNAs have also been uncovered in several periodically forced smooth systems with noise. In this work, we consider periodically forced nonsmooth system, and find that SNAs are created by a small amount of noise. SNAs can be generated in different periodic windows with weak noise perturbation. If the parameter is varied further from the chaotic range, a larger noise intensity is required to induce SNAs. Besides, noise-induced SNAs can be generated by the periodic attractors near the boundary crisis. In addition, with the increasing noise intensity, the intermittency between SNAs and periodic attractors can be induced by transient chaos. The characteristics of SNAs are analyzed by Lyapunov exponent, power spectrum, singular continuous spectrum, spectral distribution functions, and finite time Lyapunov exponent.

Keywords: piecewise linear system, strange nonchaotic attractor, noise, crisis, intermittency

1. Introduction

An SNA has a fractal structure, which is not piecewise differentiable, and for which the Lyapunov exponents are nonpositive [1]. Many experts have found abundant strange nonchaotic dynamical properties in quasiperiodically forced systems [2-4]. In the periodically forced vibro-impact systems, the strange nonchaotic dynamical phenomena are found near points of codimension-2 [5] and codimension-3

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[6] bifurcations. In a nonsmooth system with border-collision bifurcations [7], the different mechanisms for the birth of SNAs are investigated. The results show that the mechanisms of SNAs are more complex than smooth systems. In a quasiperiodically forced piecewise smooth system with Farey tree, the torus-adding bifurcation is interrupted, the strange nonchaotic dynamical phenomena can be observed [8]. SNAs are also found in a quasiperiodically forced single-degree-of freedom gear dynamical system [9]. The numerical experiments show that SNAs are the transition between quasiperiodic motion and chaotic motion. The attractor does wrinkle in the transition which can be regarded as the precursor to SNAs. Papers [10-13] uncover multistability of the coexistence of SNAs and quasiperiodic attractors, which enriches the study of SNAs. The strange and nonchaotic properties of SNAs can be verified by numerical methods such as Lyapunov exponent, phase sensitivity, power spectrum, fractal dimension, spectral distribution functions, rational approximations, and so on [14-16].

The generation mechanisms of SNAs are complicated. Torus-doubling route can generate SNAs by the interruption of torus-doubling bifurcation [17]. Heagy-Hammel route can generate SNAs by the collision of two stable tori with an unstable torus [18]. Blowout route means that the quasiperiodic attractor becomes transversally unstable when the blowout bifurcation occurs, and the attractor evolves into an SNA with the change of the control parameter [19]. In addition, intermittency route is the evolution of quasiperiodic attractors into SNAs by saddle-node bifurcation (Type-I intermittency) or subharmonic bifurcation (Type-III intermittency) [20-21].

Robust SNAs are also found in a random system, which can be induced by a small amount of noise [22]. In a periodically forced noisy FitzHugh-Nagumo neuron model, the strange nonchaotic dynamical phenomena are found, and the properties of SNAs are analyzed by numerical methods [23]. In a periodically forced nonlinear dynamical system, noise-induced logical SNAs are studied, and the robustness of these attractors is tested by logical signal perturbation [24]. In addition, Khovanov et al. [25] studied the influence of noise on SNAs and found that a very weak noise could cause the dynamical complexity of SNAs in a quasiperiodically forced system.

The remaining of this paper is organized as follows. In section 2, we introduce a class of periodically forced nonsmooth dynamical systems with noise that is studied in this work. In section 3, we discuss noise-induced SNAs in different periodic windows. The strange property of SNAs is analyzed by numerical methods. In section 4, we study the evolution of attractors, the interval of SNAs is determined by Lyapunov exponent and singular continuous spectrum. In section 5, it is shown that the periodic-3 attractors near the boundary crisis can evolve into noise-induced SNAs. In addition, we uncover that a small noise can induce the intermittency of SNAs in the system. Finally in section 6 we present the conclusion.

2. A single-degree-of freedom piecewise linear system

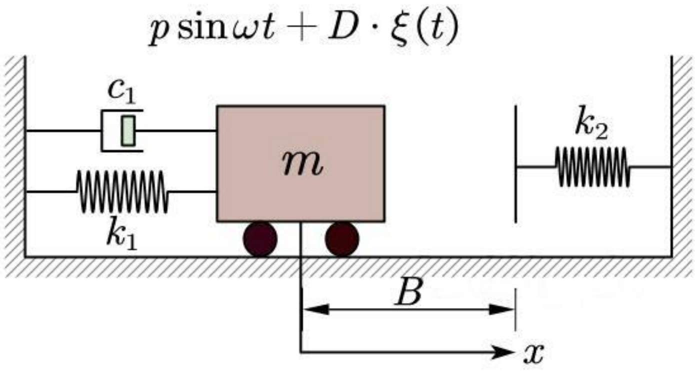


Fig. 1 Schematics of a piecewise linear system.

We consider a piecewise linear system, as shown in Fig. 1. The mass, spring stiffness and damping coefficient are m, k_1 and c_1, respectively. The mass moves in the horizontal direction under the external force $p\sin(\omega t)$ and subject to the random perturbation $D\cdot\xi(t)$. The constraint on the right-hand side of the mass is composed of a linear spring with stiffness k_2. The gap between the constraint and the equilibrium position of the mass is B. The differential equation of the motion can be established as

$$m\ddot{x} + c_1\dot{x} + K(x) = p\sin(\omega t) + D\cdot\xi(t), \tag{1}$$

where

$$K(x)=\begin{cases}k_1 x, & x-B\le 0,\\ k_1 x+k_2(x-B), & x-B>0,\end{cases}$$

and $\xi(t)$ is the Gaussian distributed white noise. The mean and autocorrelation function are as follows (c.f. [23]),

$$\begin{cases}\langle \xi(t)\rangle = 0,\\ \langle \xi(t)\xi(\mathrm{s})\rangle = 2D\delta(t-s),\end{cases} \tag{2}$$

where D is the noise intensity. In this work, the mean is set to 0, and the variance is set to 1. Then the equation (1) can be written as

$$\begin{cases}\dot{x}=y,\\ \dot{y}=(p\sin(\omega t)-c_1 y-K(x)+D\cdot\xi(t))/m.\end{cases} \tag{3}$$

The singular continuous power spectrum can be used to describe the strange property of SNAs [26]. There are three types of power spectra in dynamical systems: discrete power spectrum (periodic or quasiperiodic motion), continuous power spectrum (chaotic motion) and singular continuous power spectrum (strange nonchaotic motion). The discrete power spectrum has δ-peaks at certain frequencies; the continuous power spectrum does not have δ-peaks; singular continuous power spectrum contains both discrete and continuous spectra. According to the definition of the power spectrum, taking the Fourier transform of the orbit of the Poincaré map, we get

$$X(\omega,N)=\sum_{n=1}^{N} x_n e^{i2\pi n\omega}. \tag{4}$$

Then the power spectrum of the attractor can be defined as

$$P=\lim_{N\to\infty}\left|X(\omega,N)/N\right|^2. \tag{5}$$

In addition, the strange property of SNAs can also be characterized by the power-law relation of $X(\omega,N)$. If the attractor is periodic or quasiperiodic, then $\left|X(\omega,N)\right|^2 \sim N^2$. If the attractor is chaotic, then $\left|X(\omega,N)\right|^2 \sim N^1$. If the attractor is an SNA, it has the following relationship

$$|X(\omega, N)|^2 \sim N^k, \tag{6}$$

where $1 < k < 2$.

3. The period-doubling dynamical phenomena with noisy perturbation

3.1 The period-doubling of the system in the absence of noise

We consider the system parameters, (1) $p = 10$, $m = 0.5$, $k_1 = 1.0$, $k_2 = 30$, $c_1 = 0.2$, $B = 0.0001$ and $D = 0$. The excitation frequency ω is taken as the bifurcation parameter, and the bifurcation diagram of the system is shown in Fig. 2. For $\omega = \omega_1 = 3.486$, the corresponding Floquet multipliers of the system are $\lambda_1(\omega_1) = -1.0$ and $\lambda_2(\omega_1) = -0.486$. When ω passes through ω_1, the attractor undergoes a period-doubling bifurcation which converts period-1 to period-2 attractor. For $\omega = \omega_2 = 3.712$, the corresponding Floquet multipliers are $\lambda_1(\omega_2) = -1.0$ and $\lambda_2(\omega_2) = -0.258$. When ω passes through ω_2, the period-doubling bifurcation occurs again, and the period-2 attractor evolves into a period-4 attractor. For $\omega = \omega_3 = 3.797$, the corresponding Floquet multipliers are $\lambda_1(\omega_3) = -1.0$ and $\lambda_2(\omega_3) = -0.071$. When ω passes through ω_3, the period-doubling bifurcation occurs, and the period-4 attractor evolves into a period-8 attractor. When $\omega = \omega_i (i = 1, 2, 3)$, one eigenvalue of the system is -1, and the absolute value of another eigenvalue is less than 1, then the system undergoes period-doubling bifurcations at $\omega = \omega_i (i = 1, 2, 3)$. For $\omega = 3.826$, the system goes into a chaotic state, and the chaotic interval $\omega \in [3.826, 3.868]$. When ω passes through 3.868, the system returns to the periodic state again, which is a period-8 attractor.

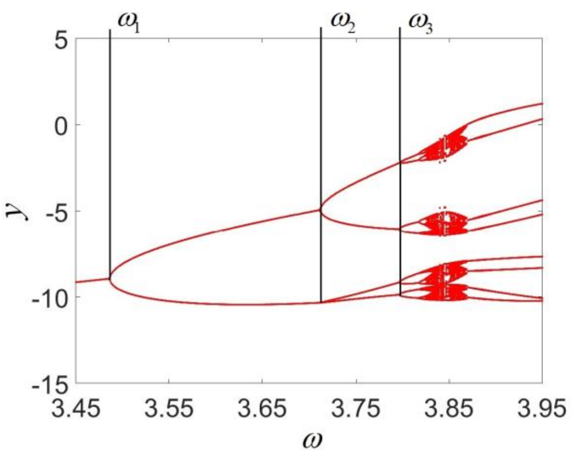


Fig. 2 The bifurcation diagram for the system parameters (1).

3.2 SNAs in different periodic windows with noisy perturbation

For $D \neq 0$, the dynamical properties of the system change due to noisy perturbation. ω is taken as the control parameter to study the dynamical properties in different periodic windows. For ω=3.6 and $D = 0.013$, the period-2 attractor becomes unstable, and the period-2 attractor evolves into a noise-induced SNA, as shown in Fig. 3(a). The largest Lyapunov exponent is nonpositive ($\lambda_{max} = -0.175$), as shown in Fig. 4(a). For ω=3.75 and $D = 0.0115$, the period-4 components merge together, and it evolves into a noise-induced SNA, as shown in Fig. 3(c). The corresponding largest Lyapunov exponent is negative ($\lambda_{max} = -0.13$), as shown in Fig. 4(b). For ω=3.88 and $D = 0.0085$, four adjacent components merge together, the period-8 attractor evolves into noise-induced SNAs, as shown in Fig. 3(e), with the largest Lyapunov exponent $\lambda_{\max} = -0.02$, as shown in Fig. 4(c). Snapshot attractors can resolve the strange geometry of noise-induced SNAs formed by a largest number of trajectories [22, 24, 27]. The blow-up parts of Figs. 3(a), 3(c) and 3(e) show that the points are randomly distributed, these snapshot attractors are formed by 100000 trajectories, which are apparently fractal as shown in Figs. 3(b), 3(d) and 3(f). Summarizing, if the parameter is varied further from the chaotic area, the larger noise intensity is required to induce SNAs.

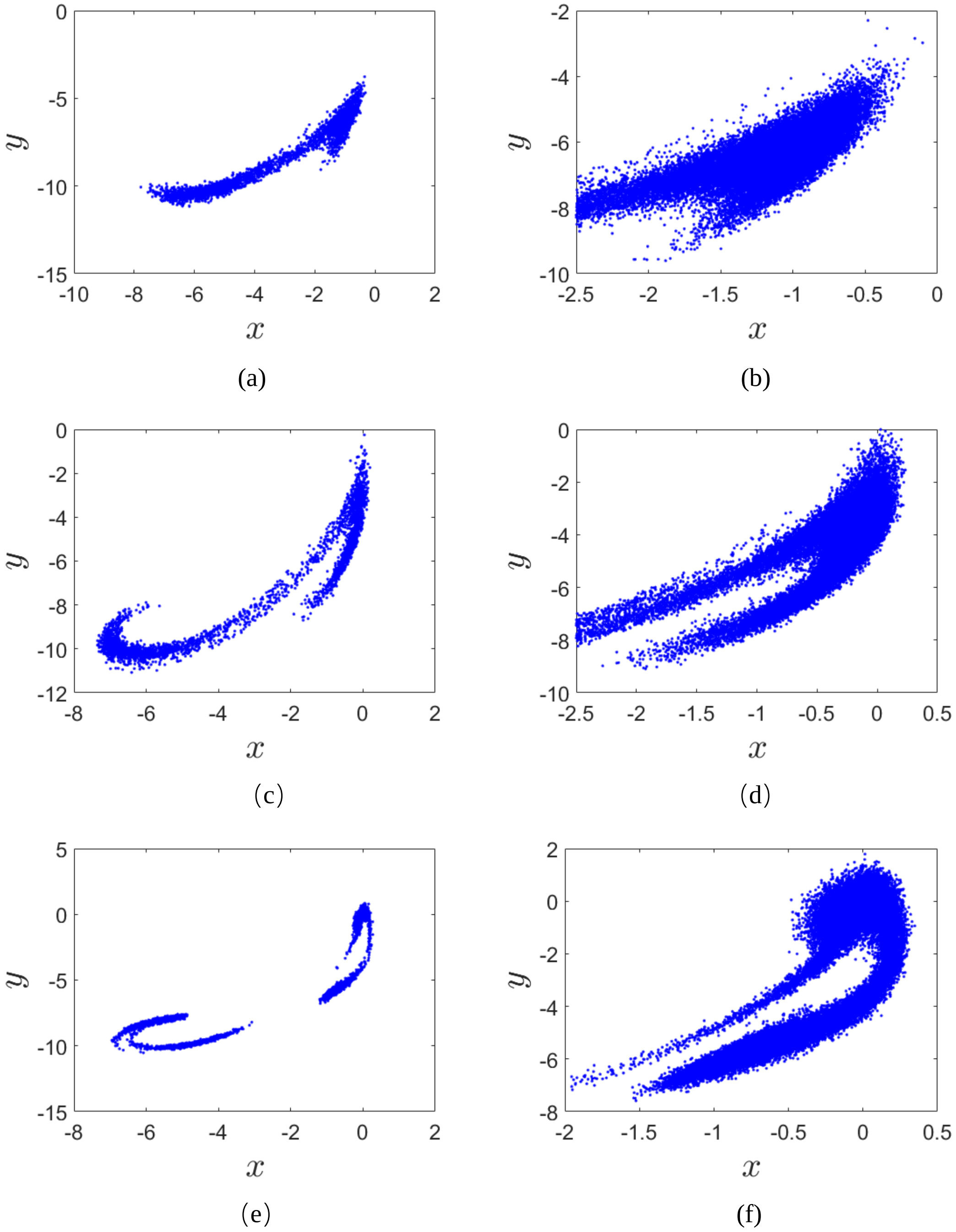


Fig. 3 The phase diagram in the (x, y) plane. (a) ω=3.6; (c) ω=3.75; (e) ω=3.88, (b), (d) and (f) are blow-up parts of (a), (c) and (e).

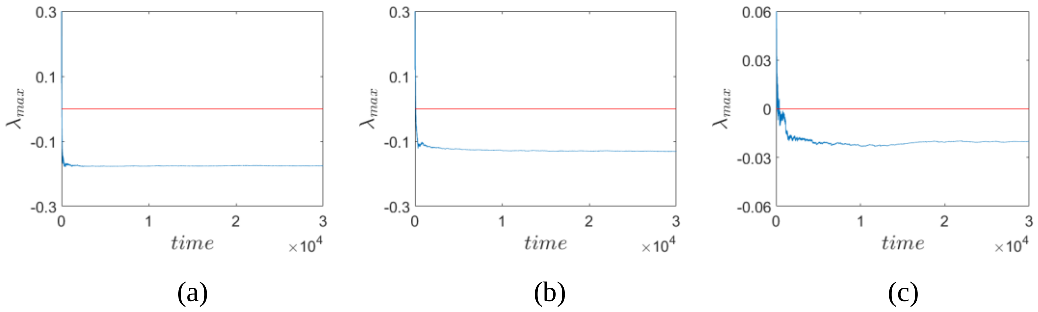


Fig. 4 The largest Lyapunov exponent. (a) ω=3.6; (b) ω=3.75; (c) ω=3.88.

3.3 Determining the strange property of SNAs

The singular continuous power spectrum contains discrete and continuous components, which can reflect the strangeness of the attractor [22]. The special power spectrum plays a very important role in characterizing SNAs, because SNAs exhibit a dynamical property between regular and irregular. In order to study the singular continuous power spectrum, ω=3.75 and $D = 0.0115$ are taken to verify the strange property of SNAs (Fig. 3(c)). The power spectrum P of $X(\omega, N)$ is shown in Fig. 5(a). The power spectrum is continuous, but there are many δ peaks, indicating the strange property of the attractor.

On the basis of studying the power spectrum, the spectrum distribution function method [4] can also be used to verify the strange property of SNAs. In the diagram of power spectrum (Fig. 5(a)), we get the number of peak values which is greater than a constant value σ, which obeys the standard power-law relation for SNAs $N(\sigma) \approx \sigma^{\beta}$ ($-2 < \beta < -1$) [22]. The parameters ω=3.75 and $D = 0.0115$ (Fig. 3(c)) are taken to study the power-law relation. In Fig. 5(b), the scaling exponent $\beta = -1.91$ satisfies the power-law relation of SNAs, indicating that the attractor is strange.

We consider the SNA in Fig. 3(c), the scaling exponent $k = 1.17$ can be obtained by calculating the power-law relation $|X(\omega, N)|^2 \sim N^k$, the scaling exponent k satisfies the power-law relation for large *N*, as shown in Fig. 5(c). In general, strange nonchaotic dynamical phenomena exist in the transition region from quasiperiodic motion to chaotic motion [9, 28]. In terms of the power-law relation of $X(\omega, N)$, its exponent k is also in the transition region between periodic attractors and chaotic attractors. The spectral trajectory in the complex plane of $X(\omega, N)$ shows the fractal structure, which can explain again the strange property of SNAs, as shown in Fig. 5(b) [22].

To characterize further that the property of SNAs, we take ω=3.75 and

$D = 0.0115$. Figure 6 is the finite time Lyapunov exponents for P(100, λ), the distribution of the finite time Lyapunov exponents shows that it is present mostly in the negative region. A feature is that the distribution P(100, λ) picks up a tail which extends into the $\lambda > 0$ region when the attractor is an SNA.

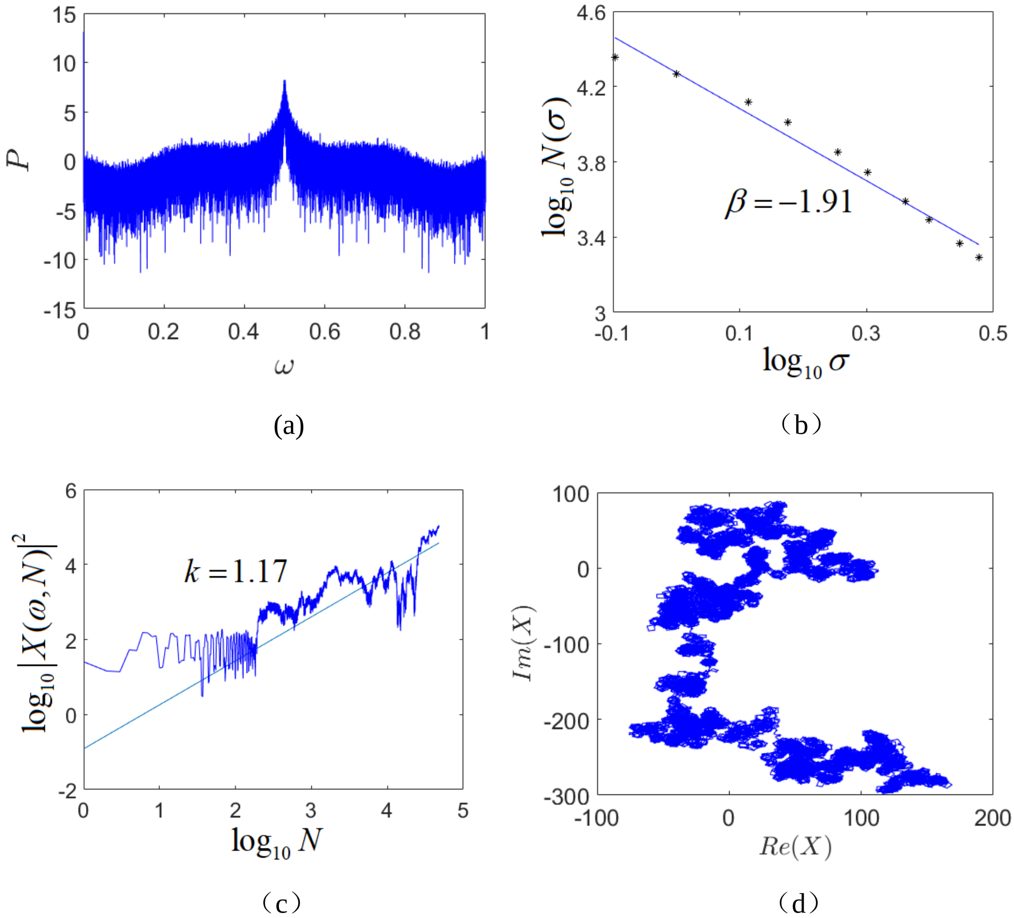


Fig. 5 For ω=3.75 and $D = 0.0115$, (a) power spectrum, (b) spectrum distribution function, (c) singular continuous spectrum, (d) the fractal structure of trajectories in the complex (ReX, ImX) plane.

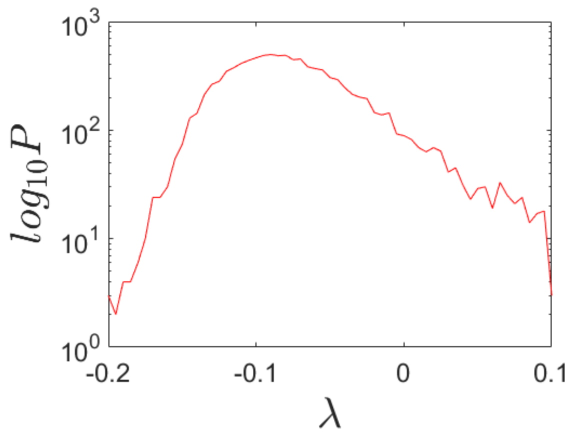
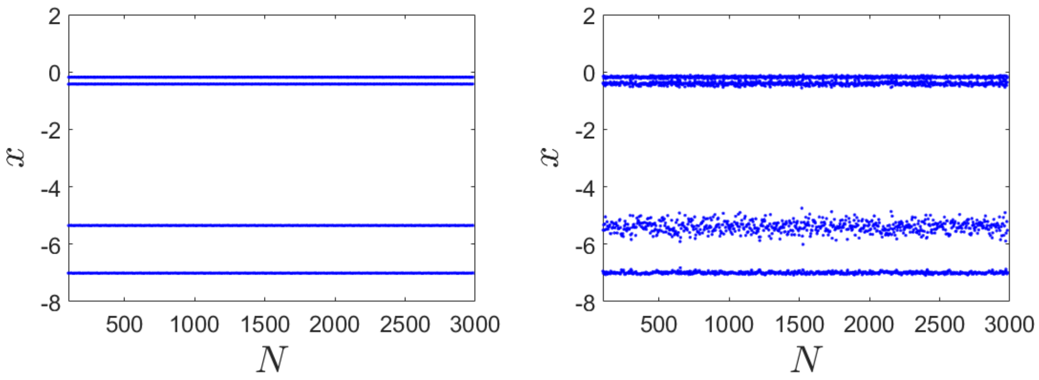


Fig. 6 For ω=3.75 and $D = 0.0115$, the distribution of the finite time Lyapunov exponents.

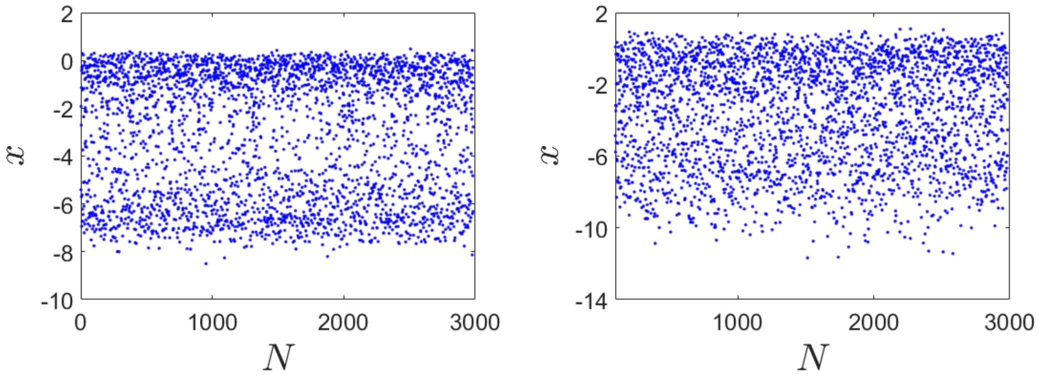
4. The evolution of attractors

By setting ω=3.75, the evolution process of periodic attractors into noise-induced SNAs is studied. In the absence of noise ($D = 0$), the attractor is a period-4 attractor, as shown in Fig. 7(a). When the noise intensity D is equal to 0.005, the attractor is a noisy period-4, as shown in Fig. 7(b). For $D = 0.0115$, the attractor completely loses smoothness, and the period-4 components merge together. The periodic attractor evolves into an SNA, as shown in Fig. 7(c). As the noise intensity D exceeds 0.038, the largest Lyapunov exponent is positive, and the attractor changes from an SNA to a chaotic attractor. For example, we take $D = 0.04$, the attractor is chaotic, as shown in Fig. 7(d).

In Fig. 8 we plot the largest lyapunov exponent with the change of noise intensity D. It shows that a periodic-4 attractor becomes gradually unstable gradually. In the early stage the noise intensity is small, the periodic attractor does not become an SNA, because the scaling exponent of the singular continuous spectrum is not between 1 and 2, the periodic attractor evolves into an approximately periodic attractor. When the noise intensity D is equal to 0.007, the scaling exponent of the singular continuous spectrum starts to lie in the interval $(1, 2)$. Therefore, the parameter interval corresponding to SNAs can be determined ($0.007 \le D \le 0.037$), as shown in Fig. 8.



(a) (b)



(c) (d)

Fig. 7 The phase diagram in the (N, x) plane; (a) $D = 0$, (b) $D = 0.005$,

(c) $D = 0.0115$, (d) $D = 0.04$.

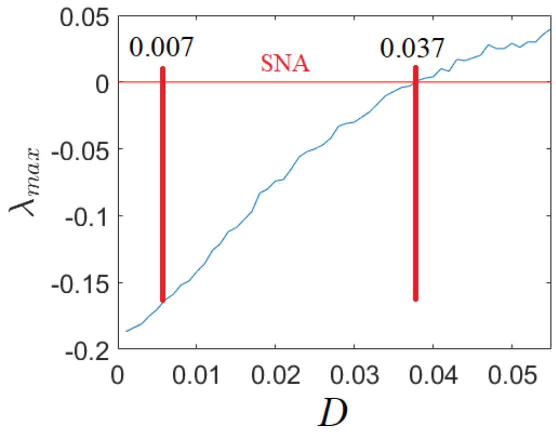


Fig. 8 The largest lyapunov exponent.

5. Strange nonchaotic dynamics near boundary crisis

5.1 Crisis and transient chaos of the system in the absence of noise

Transient chaos is the form of chaos due to nonattracting chaotic sets in the phase space [29]. Now we take the system parameters, (2) $p = 15$, $m = 1$, $k_1 = 1.0$, $k_2 = 30$, $c_1 = 0.3$, $B = 0.2$ and $D = 0$, and ω is taken as the control parameter. The bifurcation diagram of the system is shown in Fig. 9. For $\omega = \omega_1^{'} = 4.27$, the Floquet multipliers of the system are $\lambda_1(\omega_1^{'}) = 0.414$ and $\lambda_2(\omega_1^{'}) = 1.0$, $\lambda_2(\omega_1^{'})$ is 1 as expected. Therefore, the pitchfork bifurcation occurs, and the period-2 attractor evolves into a period-4 attractor. In addition, we get two unstable solutions by the shooting method, two unstable periodic orbits are shown by the two dotted lines in Fig. 9. For $\omega = \omega_2^{'} = 4.272$, the Floquet multipliers of the system are $\lambda_1(\omega_2^{'}) = 0.171$ and $\lambda_2(\omega_2^{'}) = 1.0$. The attractor undergoes a pitchfork bifurcation which converts period-4 attractor to period-2 attractor, but the attractor changes immediately to a period-4 attractor with the change of ω, as shown in Fig. 9. As ω is increased to 4.282, the system goes into a chaotic state. For $\omega = 4.29$, the merging crisis (MC) occurs, two pairs of symmetrical chaotic attractors collide with unstable periodic orbits, and they merge to two larger chaotic attractors. For $\omega = 4.296$, the chaotic attractor collide with an unstable periodic orbit, and the system undergoes boundary crisis (BC). At this event, the chaotic attractor disappears suddenly, as shown in Fig. 8. In addition, for $\omega \in [4.309,\ 4.323]$, the system has is a transient chaos, the asymptotic state is the period-3 attractor, as shown in Fig. 9.

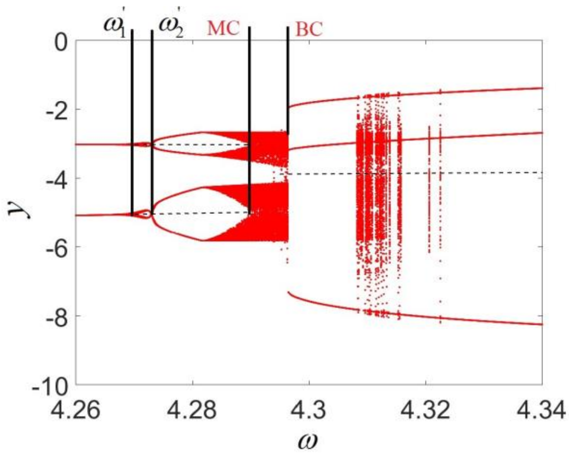
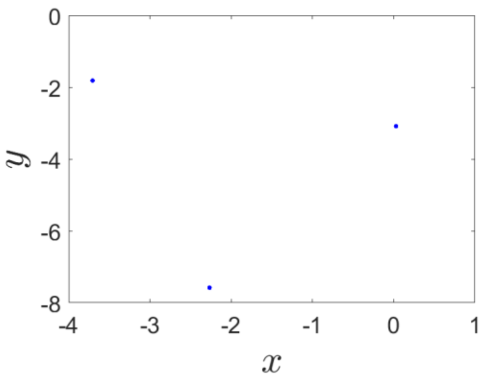


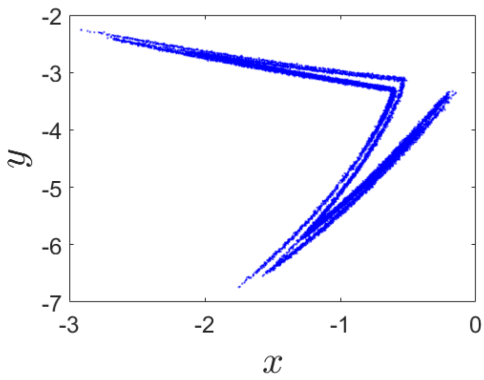
Fig. 9 The bifurcation diagram for the system parameters (2).

5.2 SNAs near the boundary crisis with noise disturbance

According to the bifurcation diagram in Fig. 9, we take $\omega = 4.3$ and $D = 0$ (near the boundary crisis), the system has a periodic-3 attractor, as shown in Fig. 10(a). We consider the velocity and displacement with noisy perturbation. If the noise is present, it can generate noise-induced SNAs. Take $D = 0.007$, the attractor has a fractal structure, as shown in Fig. 10(b), and the largest Lyapunov exponent $\lambda_{\max}$ is equal to -0.138, as shown in Fig. 11, so the attractor is nonchaotic. In addition, the power spectrum, spectrum distribution function and the singular continuous spectrum are used to verify the strange property of the attractor (Fig. 10(b)). Many δ-peaks can be observed in the power spectrum, which is discrete and continuous. The power spectrum has periodic and chaotic components, so it validates that SNA is a special dynamical phenomenon between periodicity and chaos, as shown in Fig. 12(a). It satisfies the power-law relation $N(\sigma) \approx \sigma^{\beta}$ by calculating the spectrum distribution functions. The scaling exponent β ($\beta = -1.38$) is between -2 and -1, and the attractor is between ordered and disordered, as shown in Fig. 12(b). Since SNAs have singular continuous power spectrum, another evidence of singular continuous spectrum can be provided by calculating the Fourier transform $|X(\omega, N)|$. The power-law relation $|X(\omega, N)|^2 \sim N^{1.31}$ is seen in Fig. 12(c), which indicates that the attractor is strange. In addition, the fractal structure of attractor can be observed in the complex plane $(\mathrm{Re}\,X,\ \mathrm{Im}X)$, indicating the strange property of SNAs, as shown in Fig. 12(d). Summarizing, the attractor is an SNA in the set of parameters.



（a） $D = 0$



（b） $D = 0.082$

Fig. 10 For ω=4.3, the phase diagram in the (x, y) plane.

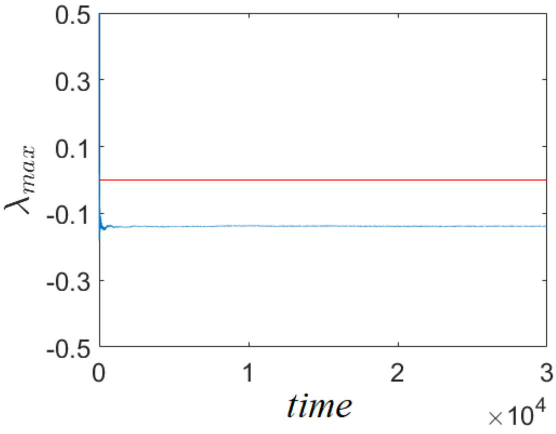


Fig. 11 For ω=4.3 and $D=0.007$, the largest Lyapunov exponent.

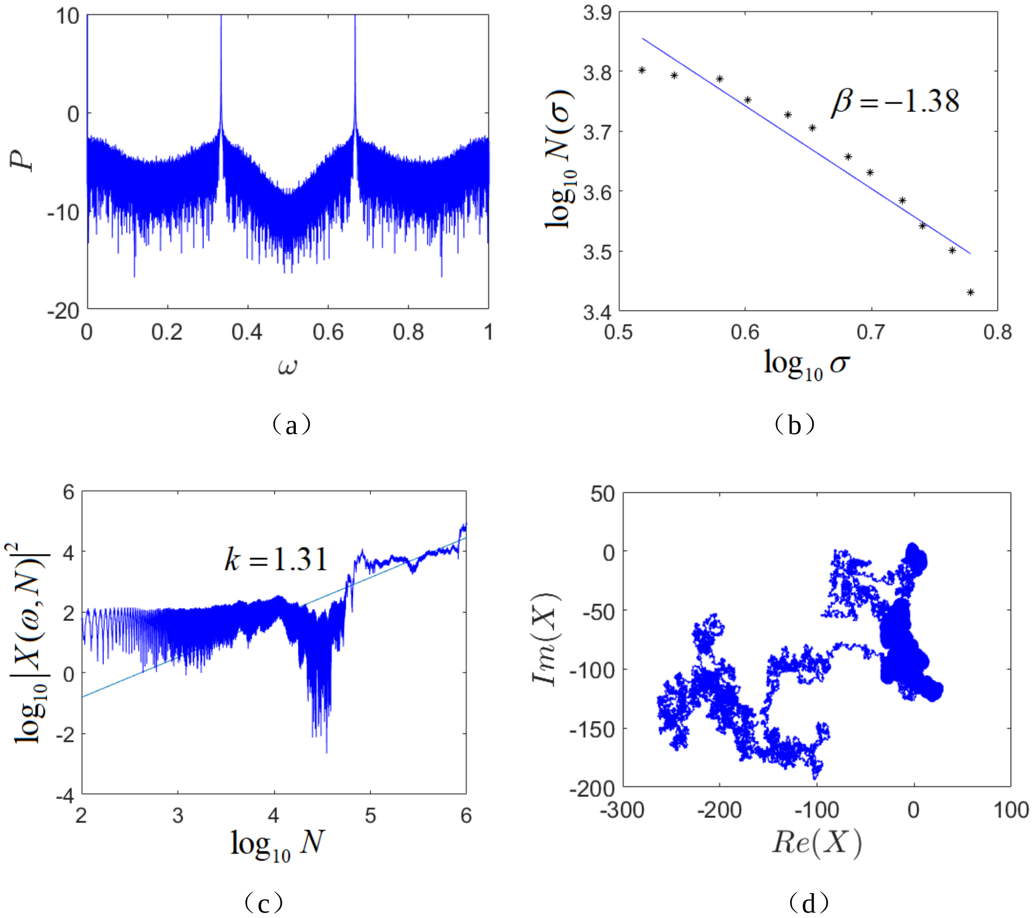


Fig. 12 For ω=4.3 and $D=0.007$, (a) power spectrum, (b) spectral distribution function, (c) singular continuous spectrum, (d) the fractal structure of trajectories in the complex plane.

5.3 The intermittency of SNAs

In general, there are three types of intermittency [30-31], which are generated by saddle-node bifurcation, subcritical Hopf bifurcation and subcritical period-doubling bifurcation. However, there is a special type of intermittency, which is called crisis-induced intermittency [32]. Such an intermittency is caused by chaotic attractors colliding with an unstable periodic orbit. In this subsection, we uncover that noise perturbation can produce intermittency of SNAs in the velocity and displacement with noisy perturbation.

In Figure 9, we take ω=4.31 in the interval corresponding to intermittency. In the absence of noise ($D = 0$), the system exhibits transient chaos. For $N \in [0,\ 1273]$, the system is in a chaotic transient, and the trajectory asymptotes to a period-3 attractor, as shown in Fig. 13(a). In the presence of noise, the period-3 attractor gradually loses its stability with the increase of noise intensity and evolves into noisy period-3 attractors. As the noise intensity D is equal to 0.03, there is an intermittency dynamical phenomenon in the system. In the (N, x)-plane, the noisy period-3 attractors and SNAs alternate. After the transient chaos disappears, the intermittency phenomenon of two attractors appears, as shown in Fig. 13(b). Figure 14 is the largest Lyapunov exponent corresponding to ω=4.31 and $D = 0.03$, the largest Lyapunov exponents of noisy period-3 attractors and SNAs are less than zero. When the largest Lyapunov exponent is less than zero, there is a fluctuating phenomenon, showing that the largest Lyapunov exponent switches between the noisy periodic attractor and SNAs. This implies that the types of the attractor change with time, and the SNAs coexists with the noisy period-3 attractors, resulting in intermittency dynamics.

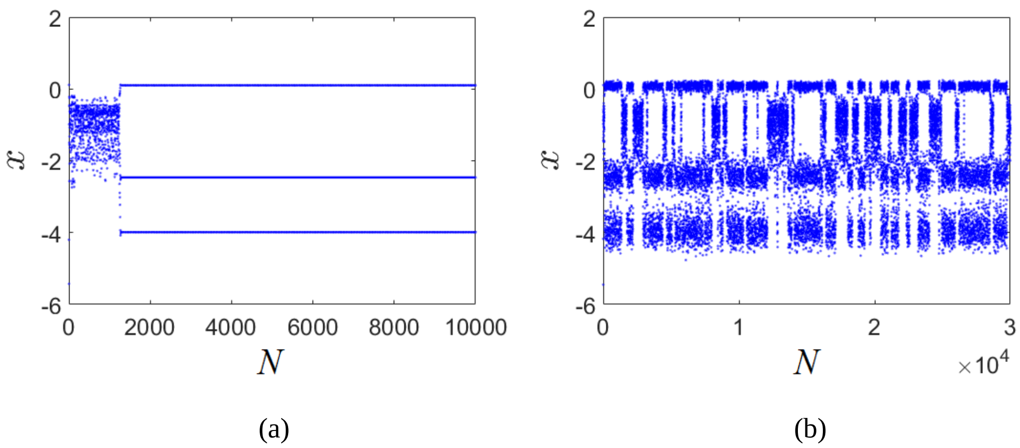


Fig. 13 For ω=4.31, the phase diagram in the (N, x) plane, (a) $D = 0$, (b) $D = 0.03$.

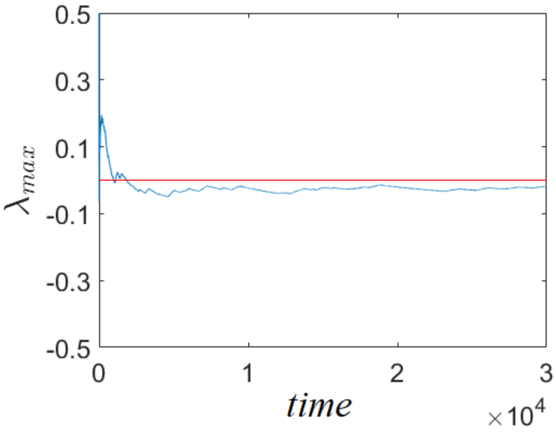


Fig. 14 For ω=4.31 and $D = 0.03$, the largest Lyapunov exponent.

6. Conclusion

In this work, a periodically forced single-degree-of-freedom piecewise linear system with noise is considered. It is found that periodic attractors with noise can induce SNAs. If the parameter is varied further from the chaotic regime, a larger noise intensity is required to induce SNAs. The evolution process of SNAs is uncovered, and the interval of SNAs is obtained by using Lyapunov exponent and singular continuum spectrum. With the increasing noise intensity, the period-3 attractors near the boundary crisis may evolve into SNAs, and the intermittency between SNAs and the periodic attractor may take place after transient chaos.

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