SPECTRAL SYNTHESIS IN THE MULTIPLIER ALGEBRA OF A $C_0(X)$-ALGEBRA

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Abstract. Let A be a $C_0(X)$-algebra with continuous map ϕ from $\mathrm{Prim}(A)$, the primitive ideal space of A, to a locally compact Hausdorff space X. Then the multiplier algebra $M(A)$ is a $C(\beta X)$-algebra with continuous map $\overline{\phi} : \mathrm{Prim}(M(A)) \to \beta X$ extending ϕ. For $x \in \mathrm{Im}(\phi)$, let $J_x = \bigcap\{P \in \mathrm{Prim}(A) : \phi(P) = x\}$ and $H_x = \bigcap\{Q \in \mathrm{Prim}(M(A)) : \overline{\phi}(Q) = x\}$. Then $J_x \subseteq H_x \subseteq \tilde{J}_x$, the strict closure of J_x in $M(A)$. Thus H_x is strictly closed if and only if $H_x = \tilde{J}_x$, and the 'spectral synthesis' question asks when this happens. In this paper it is shown that, for σ-unital A, H_x is strictly closed for all $x \in \mathrm{Im}(\phi)$ if and only if J_x is locally modular for all $x \in \mathrm{Im}(\phi)$ and ϕ is a closed map relative to its image. Various related results are obtained.

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1. Introduction

Let A be a C*-algebra with multiplier algebra $M(A)$ [2, 9]. The ideal structure of $M(A)$ is typically much more complicated than that of A and has been widely studied by a number of authors [1, 16, 23, 26, 27, 31, 32]. In [6, 7] we investigated certain aspects of the ideal structure of M(A) which arise from a $C_0(X)$-algebra structure on the algebra A. The notion of a $C_0(X)$-algebra was introduced by Kasparov [22] following extensive earlier work on continuous and upper semi-continuous fields (see, for example, [10, 17, 19, 25, 34]). As noted in [7, Section 1], every C^*-algebra is a $C_0(X)$-algebra, typically in many ways. $C_0(X)$-algebras have been studied in [3, 8, 13, 14, 15, 20, 28]. In [6, Section 1], we saw that if A is a $C_0(X)$-algebra then corresponding to each $x \in X$ there are two natural ideals H_x and $\tilde{J}_x$ in $M(A)$ which one might hope would be equal but in fact need not be so. The aim of this paper is to characterize the equality $H_x = \tilde{J}_x$ in terms of A and X (without reference to $M(A)$).

Recall that A is a $C_0(X)$-*algebra* if there is a continuous map ϕ, called the *base map*, from $\mathrm{Prim}(A)$, the primitive ideal space of A with the hull-kernel topology, to the locally compact Hausdorff space X [35, Proposition C.5]. We will use X_ϕ to denote the image of ϕ in X. If A is a $C_0(X)$-algebra then $M(A)$ is a $C(\beta X)$-algebra with continuous map $\overline{\phi} : \mathrm{Prim}(M(A)) \to \beta X$ (the Stone-Čech compactification of X) extending ϕ (see Section 2).

For $x \in X_\phi$, let J_x be the closed ideal of A defined by $J_x = \bigcap\{P \in \mathrm{Prim}(A) : \phi(P) = x\}$ and let H_x be the closed ideal of $M(A)$ defined by $H_x = \bigcap\{Q \in \mathrm{Prim}(M(A)) : \overline{\phi}(Q) = x\}$. Let $\tilde{J}_x$ be the strict closure of J_x in $M(A)$. Then $J_x \subseteq H_x \subseteq \tilde{J}_x$ and hence H_x is strictly closed if and only if $H_x = \tilde{J}_x$ (see Proposition 2.3). The 'spectral synthesis' question asks for conditions on A and X characterizing when H_x is strictly closed. It was shown in [7], for instance, that if A is stable and σ-unital then for $x \in X_\phi$, H_x is strictly closed if and only if x is a P-point in X_ϕ.

In this paper we return to the question in a more general context. The main result (Corollary 3.7) is that if A is a σ-unital $C_0(X)$-algebra with base map ϕ then H_x is strictly closed for all $x \in X_\phi$ if and only if J_x is locally modular for all $x \in X_\phi$ and ϕ is a closed map relative to its image. Here J_x is said to be *locally modular* if whenever Q lies in the boundary in $\mathrm{Prim}(A)$ of $H(x) = \{P \in \mathrm{Prim}(A) : P \supseteq J_x\}$ then there exists a neighbourhood V of Q in $\mathrm{Prim}(A) \setminus U(x)$ (where $U(x)$ is the interior of $H(x)$) such that $A/\ker V$ is a unital C^*-algebra.

If A is separable then the same characterization is valid for spectral synthesis at a point $x \in X_\phi$ (Corollary 3.10), namely H_x is strictly closed if and only if J_x is locally modular and ϕ is a locally closed map at x. For general σ-unital $C_0(X)$-algebras this condition is close to characterizing spectral synthesis at a point (see Theorem 3.6) but does not quite succeed (Example 4.5), and we have to leave the problem open.

The structure of the paper is as follows. Section 2 gives basic information on $C_0(X)$-algebras and spectral synthesis. Section 3 contains some of the main results of the paper, described above. Section 4 looks more closely at pointwise spectral synthesis for a σ-unital $C_0(X)$-algebra A and identifies the points in X_ϕ which are difficult to deal with. The main result is the characterization of pointwise spectral synthesis in the case when A is a continuous $C_0(X)$-algebra (Theorem 4.6).

In Sections 5 and 6, we restrict to the important special case when ϕ is the complete regularization map for $\mathrm{Prim}(A)$ and the connecting order $\mathrm{Orc}(A)$ is finite. In Section 5, we show that, in this case, if A is σ-unital then the local modularity of J_x implies that the complete regularization map ϕ is locally closed at x. Hence if A is also separable then H_x is strictly closed if and only if J_x is locally modular (Corollary 5.4). In Section 6, we show that if J_x is locally modular then either J_x does not contain the centre of A or the hull $H(x)$ of J_x in $\mathrm{Prim}(A)$ must have non-empty interior—an unusual property unless $H(x)$ is a clopen set (Corollary 6.3).

Note added in revision.

After this paper was submitted for publication, we learned that David McConnell (private communication) had independently obtained Proposition 2.6(iii), Proposition 2.6(i) (in the case where ϕ is the complete regularization map), and versions of Corollary 4.3, Corollary 4.4 and Theorem 4.6 with somewhat stronger hypotheses.

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2. General $C_0(X)$-algebras

In this section we gather some information about $C_0(X)$-algebras and establish the basic facts about spectral synthesis (Proposition 2.6). For $C_0(X)$-algebras we follow the terminology of [6].

Let A be a $C_0(X)$-algebra with base map $\phi : \mathrm{Prim}(A) \to X$, and recall that $X_\phi = \mathrm{Im}(\phi)$. Then X_ϕ is completely regular; and if A is σ-unital, X_ϕ is σ-compact and hence normal [7, Section 2]. For $x \in X_\phi$, set $J_x = \bigcap\{P \in \mathrm{Prim}(A) : \phi(P) = x\}$, and for $x \in X \setminus X_\phi$, set $J_x = A$. For $a \in A$, the function $x \to \|a + J_x\|$ $(x \in X)$ is upper semi-continuous [35, Proposition C.10]. The $C_0(X)$-algebra A is said to be *continuous* if, for all $a \in A$, the norm function $x \to \|a + J_x\|$ $(x \in X)$ is continuous. By Lee's theorem [35, Proposition C.10 and Theorem C.26], this happens if and only if the base map ϕ is open.

Let J be a proper, closed, two-sided ideal of a C^*-algebra A. The quotient map $q_J : A \to A/J$ has a canonical extension $\tilde{q_J} : M(A) \to M(A/J)$. We define a proper, closed, two-sided ideal $\tilde{J}$ of $M(A)$ by

$$\tilde{J} = \ker \tilde{q_J} = \{b \in M(A) : ba, ab \in J \text{ for all } a \in A\}.$$

The following proposition was proved in [6, Proposition 1.1].

Proposition 2.1. *Let J be a proper, closed, two-sided ideal of a C^*-algebra A. Then*
(i) *$\tilde{J}$ is the strict closure of J in $M(A)$;*
(ii) *$\tilde{J} \cap A = J$;*
(iii) *if $P \in \mathrm{Prim}(A)$ then $\tilde{P}$ is primitive (and hence is the unique ideal in $\mathrm{Prim}(M(A))$ whose intersection with A is P);*
(iv) *$\tilde{J} = \bigcap\{\tilde{P} : P \in \mathrm{Prim}(A)$ and $P \supseteq J\}$ and for all $b \in M(A)$*

$$\|b + \tilde{J}\| = \sup\{\|b + \tilde{P}\| : P \in \mathrm{Prim}(A) \text{ and } P \supseteq J\};$$

(v) *$(A + \tilde{J})/\tilde{J}$ is an essential ideal in $M(A)/\tilde{J}$.*

Furthermore, the map $P \mapsto \tilde{P}$ $(P \in \mathrm{Prim}(A))$ maps $\mathrm{Prim}(A)$ homeomorphically onto a dense, open subset of $\mathrm{Prim}(M(A))$ [29, 4.1.10]. For $S \subseteq \mathrm{Prim}(A)$, we write $\tilde{S} = \{\tilde{P} : P \in S\}$. In view of Proposition 2.1(ii), $(A + \tilde{J})/\tilde{J}$ is canonically isomorphic to A/J. If A/J is unital then $(A + \tilde{J})/\tilde{J}$ is a unital essential ideal of $M(A)/\tilde{J}$ and therefore equal to $M(A)/\tilde{J}$.

Now suppose that A is a $C_0(X)$-algebra, $x \in X$ and $a \in A$. If A/J_x is unital, the spectrum of $a+J_x$ (in A/J_x) coincides with the spectrum of $a+\tilde{J}_x$ in $M(A)/\tilde{J}_x$ by the previous remark. If A/J_x is non-unital, the spectrum of $a + J_x$ (in the unitization of A/J_x) is equal to the spectrum of $a + \tilde{J}_x$ in $(A + \tilde{J}_x)/\tilde{J}_x + \mathbb{C}(1 + \tilde{J}_x)$ and hence in $M(A)/\tilde{J}_x$ [12, 1.3.10(ii)].

The following proposition was proved in [6, Proposition 1.2].

Proposition 2.2. *Let A be a $C_0(X)$-algebra with base map ϕ. Then ϕ has a unique extension to a continuous map $\overline{\phi} : \mathrm{Prim}(M(A)) \to \beta X$ such that $\overline{\phi}(\tilde{P}) = \phi(P)$ for all $P \in \mathrm{Prim}(A)$. Hence $M(A)$ is a $C(\beta X)$-algebra with base map $\overline{\phi}$ and $\mathrm{Im}(\overline{\phi}) = \mathrm{cl}_{\beta X}(X_\phi)$.*

Now let A be a $C_0(X)$-algebra with base map ϕ and let $\overline{\phi} : \mathrm{Prim}(M(A)) \to \beta X$ be as in Proposition 2.2. For $x \in \beta X$, we define $H_x = \bigcap\{Q \in \mathrm{Prim}(M(A)) : \overline{\phi}(Q) = x\}$, a closed two-sided ideal of $M(A)$. Thus H_x is defined in relation to $(M(A), \beta X, \overline{\phi})$ in the same way that J_x (for $x \in X$) is defined in relation to (A, X, ϕ). It follows that for each $b \in M(A)$, the function $x \to \|b + H_x\|$ $(x \in \beta X)$ is upper semi-continuous.

The next proposition was proved in [7, Proposition 1.3].

Proposition 2.3. *Let A be a $C_0(X)$-algebra with base map ϕ, and set $X_\phi = \mathrm{Im}(\phi)$.*
(i) *For all $x \in X$, $J_x \subseteq H_x \subseteq \tilde{J}_x$ and $J_x = H_x \cap A$.*
(ii) *For all $x \in X$, H_x is strictly closed if and only if $H_x = \tilde{J}_x$.*
(iii) *For all $b \in M(A)$, $\|b\| = \sup\{\|b + \tilde{J}_x\| : x \in X_\phi\} = \sup\{\|b + H_x\| : x \in X_\phi\}$.*

We now turn to the subject of spectral synthesis and our first proposition seeks to justify the use of the name. Recall that in the the theory of commutative Banach algebras, spectral synthesis holds at a point x in the maximal ideal space provided that each element of the algebra whose Gelfand transform vanishes at x can be approximated in (the original) norm by elements whose transforms vanish in a neighbourhood of x. At this stage we need some more notation.

For a C*-algebra A, let $Z(A)$ denote the centre of A. Now let A be a $C_0(X)$-algebra with base map ϕ. For $b \in M(A)$, let $Z(b) = \{x \in X_\phi : b \in \tilde{J}_x\}$ and let $\mathrm{Int}\ Z(b)$ be the interior of $Z(b)$ relative to X_ϕ. Recall that the Dauns-Hofmann isomorphism $\theta_A : C^b(\mathrm{Prim}(A)) \to Z(M(A))$ has the property that $\theta_A(f)a + P = f(P)(a+P)$ for $f \in C^b(\mathrm{Prim}(A))$, $a \in A$, and $P \in \mathrm{Prim}(A)$ (equivalently, $\theta_A(f) + \tilde{P} = f(P)(1+\tilde{P})$).

Proposition 2.4. *Let A be a $C_0(X)$-algebra with base map ϕ and let $x \in X_\phi$. Set $H_x^{\mathrm{alg}} = \{b \in M(A) : x \in \mathrm{Int}\ Z(b)\}$. Then $H_x^{\mathrm{alg}} \subseteq H_x$ and H_x is the norm-closure of H_x^{alg}.*

Proof. Let $b \in H_x^{\mathrm{alg}}$. Then x lies in the interior U of $Z(b)$ in X_ϕ. There exists $f \in C^b(X_\phi)$ such that $f(x) = 0$ and $f(X_\phi \setminus U) = \{1\}$. Let $z = \theta_A(f \circ \phi) \in Z(M(A))$. Suppose that $Q \in \mathrm{Prim}(M(A))$ and $Q \supseteq H_x$. Let (P_α) be a net in $\mathrm{Prim}(A)$ such that $\tilde{P}_\alpha \to Q$. Since z is central,

$$\|z + Q\| = \lim \|z + \tilde{P}_\alpha\| = \lim |f(\phi(P_\alpha))| = |f(\overline{\phi}(Q))| = |f(x)| = 0.$$

Thus $z \in Q$ and hence $z \in H_x$. For $P \in \mathrm{Prim}(A)$,

$$zb + \tilde{P} = f(\phi(P))(b + \tilde{P}) = b + \tilde{P}$$

because $b \in \tilde{J}_{\phi(P)} \subseteq \tilde{P}$ whenever $f(\phi(P)) \neq 1$. Hence $zb = b$ and therefore $b \in H_x$.

For the second part of the proof, let $b \in H_x$ and $\epsilon > 0$. By upper semi-continuity, there is an open neighbourhood U of x in X such that $\|b + H_y\| < \epsilon$ for all $y \in U$. Let $V = U \cap X_\phi$. There exists a continuous function $f : X_\phi \to [0,1]$ such that $f(x) = 0$ and $f(X_\phi \setminus V) = \{1\}$. Define $g : [0,1] \to [0,1]$ by $g(t) = 0$ $(0 \le t \le 1/2)$ and $g(t) = 2t - 1$ $(1/2 < t \le 1)$. Let $h = g \circ f$. Then $h(X_\phi \setminus V) = \{1\}$ and there exists an open neighbourhood W of x in X_ϕ such that $h|_W = 0$. Let $z = \theta_A(h \circ \phi) \in Z(M(A))$.

Suppose that $y \in W$, $P \in \mathrm{Prim}(A)$ and $\phi(P) = y$. Then $zb + \tilde{P} = h(y)(b + \tilde{P}) = 0$. It follows that $zb \in \tilde{J}_y$ and hence $zb \in H_x^{\mathrm{alg}}$.

Finally, for $P \in \mathrm{Prim}(A)$,

$$\|(b - bz) + \tilde{P}\| = (1 - h(\phi(P)))\|b + \tilde{P}\| \le (1 - h(\phi(P)))\|b + H_{\phi(P)}\| < \epsilon$$

since $h(\phi(P)) = 1$ if $\phi(P) \notin U$. Hence $\|b - zb\| < \epsilon$. □

It follows from Proposition 2.3 and Proposition 2.4 that, for $x \in X_\phi$, the ideal H_x is strictly closed if and only if every element $b \in M(A)$ which vanishes at x can be approximated in norm by elements vanishing in a neighbourhood of x in X_ϕ. Thus we think of H_x being strictly closed as corresponding to 'spectral synthesis holding at x'.

As in [6], we define $\mu : C_0(X) \to Z(M(A))$ by $\mu(f) = \theta_A(f \circ \phi)$ $(f \in C_0(X))$ Now set $Z'(A) = \mu(C_0(X)) \cap A$ and note that $Z'(A) \subseteq Z(A)$. The next lemma was proved in [6, Lemma 2.1].

Lemma 2.5. *Let A be a $C_0(X)$-algebra with base map ϕ and let $x \in X_\phi$. Then $J_x \supseteq Z'(A)$ if and only if there exists $R \in \mathrm{Prim}(M(A))$ with $R \supseteq A$ such that $\overline{\phi}(R) = x$.*

We can now give the basic results on spectral synthesis for general $C_0(X)$-algebras.

Proposition 2.6. *Let A be a $C_0(X)$-algebra with base map ϕ and let $x \in X_\phi$.*
(i) *If $J_x \not\supseteq Z'(A)$ then H_x is strictly closed in $M(A)$.*
(ii) *If x is an isolated point in X_ϕ then H_x is strictly closed in $M(A)$.*
(iii) *If $J_x \supseteq Z'(A)$ and A/J_x is unital then H_x is not strictly closed in $M(A)$.*

Proof. (i) This was proved in [6, Proposition 2.2].

(ii) Since x is an isolated point in X_ϕ, $W = \phi^{-1}(x)$ is a clopen subset of $\mathrm{Prim}(A)$. Set $Y = \mathrm{Prim}(A) \setminus W$. Suppose that $R \in \mathrm{Prim}(M(A))$ and $R \not\supseteq \tilde{J}_x$. Then R is not in the closure of the set $\{\tilde{P} : P \in W\}$. Since the set $\{\tilde{P} : P \in \mathrm{Prim}(A)\}$ is dense in $\mathrm{Prim}(M(A))$, R must lie in the closure of the set $\{\tilde{P} : P \in Y\}$. Hence the continuity of $\overline{\phi}$ implies that $\overline{\phi}(R)$ lies in the closure of $\{\phi(P) : P \in Y\}$. Thus $\overline{\phi}(R) \in X \setminus \{x\}$ and so $R \not\supseteq H_x$.

(iii) Let $p \in A$ such that $p + J_x$ is the identity for A/J_x. Then $ap - a, pa - a \in J_x$ for all $a \in A$ and hence $1 - p \in \tilde{J}_x$. On the other hand, by Lemma 2.5 there exists $R \in \mathrm{Prim}(M(A))$ with $R \supseteq A$ such that $\overline{\phi}(R) = x$. Since $R \supseteq A$, $p \in R$, and hence $1 - p \notin R$. Thus $1 - p \notin H_x$. □

If $J_x \not\supseteq Z'(A)$ then A/J_x is unital [6, Proposition 2.2]. So the three cases of Proposition 2.6 cover all possibilities for x except when x is a non-isolated point of X_ϕ with A/J_x non-unital. This is the case of interest which will occupy us for the rest of the paper. We will use the following notation. Let $U_\phi = \{x \in X_\phi : J_x \not\supseteq Z'(A)\}$, an open subset of X_ϕ (see [6, Section 2]); and let $W_\phi = X_\phi \setminus U_\phi$. Let ∂U_ϕ denote the boundary of U_ϕ in X_ϕ.

Finally in this section, we consider spectral synthesis for a closed subset E of X_ϕ. Define $J_E = \bigcap\{P \in \mathrm{Prim}(A) : \phi(P) \in E\}$, $H_E = \bigcap\{Q \in \mathrm{Prim}(M(A)) : \overline{\phi}(P) \in E\}$ and $H_E^{\mathrm{alg}} = \{b \in M(A) : E \subseteq \mathrm{Int}\, Z(b)\}$. In the following analogue of Proposition 2.4, we restrict to the case where A is σ-unital in order to ensure that X_ϕ is normal.

Proposition 2.7. *Let A be a σ-unital $C_0(X)$-algebra with base map ϕ and let E be a closed subset of X_ϕ. Then $H_E^{\mathrm{alg}} \subseteq H_E$ and H_E is the norm-closure of H_x^{alg}.*

Proof. Let $b \in H_E^{\mathrm{alg}}$. Then E is contained in the interior U of $Z(b)$ in X_ϕ. Since X_ϕ is normal, there exists $f \in C^b(X_\phi)$ such that $f(E) = \{0\}$ and $f(X_\phi \setminus U) = \{1\}$. Let $z = \theta_A(f \circ \phi) \in Z(M(A))$. As in the proof of Proposition 2.4, $z \in H_E$ and $b = zb \in H_E$.

For the second part of the proof, let $b \in H_E$ and $\epsilon > 0$. By upper semi-continuity, there is an open neighbourhood U of E in X such that $\|b + H_y\| < \epsilon$ for all $y \in U$. Let $V = U \cap X_\phi$. Since X_ϕ is normal, there exists an open neighbourhood W of E in X_ϕ, with closure contained in V, and a continuous function $f : X_\phi \to [0, 1]$ such that $f(W) = \{0\}$ and $f(X_\phi \setminus V) = \{1\}$. Let $z = \theta_A(f \circ \phi) \in Z(M(A))$. As in the proof of Proposition 2.4, $zb \in H_x^{\mathrm{alg}}$ and $\|b - zb\| < \epsilon$. □

Let A be a σ-unital $C_0(X)$-algebra with base map ϕ and let E be a closed subset of X_ϕ. If $P \in \mathrm{Prim}(A)$ and $P \supseteq J_E$ then $\phi(P) \in E$ because ϕ is continuous and E is closed. Hence

$$J_E = \bigcap_{x \in E} J_x \subseteq \bigcap_{x \in E} H_x = H_E \subseteq \bigcap_{x \in E} \tilde{J}_x = \tilde{J}_E,$$

by Proposition 2.1(iv). It follows from Proposition 2.1(i) and Proposition 2.7 that H_E^{alg} is dense in $\bigcap_{x \in E} \tilde{J}_x$ (which may be thought of as spectral synthesis for E) if and only if H_E

is strictly closed in $M(A)$. We therefore define *spectral synthesis for E* to mean that H_E is strictly closed in $M(A)$. If E and F are closed subsets of X_ϕ then $H_{E\cup F} = H_E \cap H_F$. In contrast to the theory of commutative Banach algebras, it follows that if E and F have spectral synthesis then so does $E \cup F$.

We shall see in Proposition 2.9 that the question of spectral synthesis for a closed subset E of X_ϕ (i.e. whether H_E is strictly closed in $M(A)$) can be reduced to the question of spectral synthesis for a singleton, but at the expense of changing the base map ϕ. We will need the following standard topological lemma, where X/E is the quotient space of a topological space X obtained by identifying all of the points in a given subset E.

Lemma 2.8. *Let X be a normal, Hausdorff space and let E be a non-empty closed subset. Set $Y = X/E$. Then Y is normal and Hausdorff and hence completely regular.*

Proof. Let $q : X \to Y$ be the quotient map. For $y \in Y$, $q^{-1}(y)$ is closed and so Y is a T_1 space. Let B and C be disjoint, non-empty closed subsets of Y. Then $G = q^{-1}(B)$ and $H = q^{-1}(C)$ are disjoint closed sets in the normal space X, and hence there exist disjoint open sets U and V such that $G \subseteq U$ and $H \subseteq V$. Without loss of generality, we may assume that $G \not\supseteq E$ and hence $G \cap E = \emptyset$. Thus, replacing U by $U \setminus E$, we may assume that U does not meet E. Hence U is saturated with respect to the equivalence relation corresponding to q and so $q(U)$ is an open neighbourhood of B and is disjoint from $q(V)$. If $H \supseteq E$ then V is saturated and so $q(V)$ is open, and if $H \not\supseteq E$ then $V \setminus E$ is a saturated open set containing H and so $q(V \setminus E)$ is an open neighbourhood of C. Hence Y is normal, and being T_1, it is also Hausdorff and completely regular. □

Note that if X is Hausdorff but non-normal then X has a closed set E such that X/E is not completely regular. To see this, let E and F be disjoint closed subsets of X which cannot be separated by disjoint open sets. Let $Y = X/E$ with quotient map q and set $q(E) = e$. Then e and the closed set $q(F)$ cannot be separated by disjoint open sets, so Y is not even regular.

Now let A be a $C_0(X)$-algebra with base map ϕ. Let E be a non-empty closed subset of X_ϕ. Set $Y = X_\phi/E$ and let $q : X_\phi \to Y$ be the quotient map. If A is σ-unital then Y is completely regular by Lemma 2.8 and so A is a $C(\beta Y)$-algebra with base map $\psi = q \circ \phi$. Let $e = q(E)$. The next proposition relates J_E and J_e, and H_E and H_e.

Note that if $f \in C^b(\mathrm{Prim}(A))$ then the element $\theta_A(f) \in Z(M(A))$ induces a function $\overline{f} \in C(\mathrm{Prim}(M(A))$ such that

$$\theta_A(f) + Q = \overline{f}(Q)(1 + Q) \qquad (Q \in \mathrm{Prim}(M(A))).$$

In particular, $\overline{f}(\tilde{P}) = f(P)$ for all $P \in \mathrm{Prim}(A)$.

Proposition 2.9. *Let A be a σ-unital $C_0(X)$-algebra with base map ϕ and let E be a non-empty closed subset of X_ϕ. Set $Y = X_\phi/E$ and let $q : X_\phi \to Y$ be the quotient map. Let $\psi = q \circ \phi$. Then A is a $C(\beta Y)$-algebra with base map ψ, and $J_e = J_E$ and $H_e = H_E$, where $\{e\} = q(E)$.*

Proof. For $P \in \mathrm{Prim}(A)$, $\phi(P) \in E$ if and only if $\psi(P) = e$ and therefore $J_E = J_e$.

For $P \in \mathrm{Prim}(A)$, $\overline{q \circ \phi}(\tilde{P}) = (q \circ \phi)(P) = \overline{q}(\overline{\phi}(\tilde{P}))$ and hence $\overline{q \circ \phi} = \overline{q} \circ \overline{\phi}$ by continuity. Thus if $Q \in \mathrm{Prim}(M(A))$ and $\overline{\phi}(Q) \in E$ then $\overline{\psi}(Q) = e$. Hence $H_E \supseteq H_e$. For the reverse inclusion, suppose that $Q \in \mathrm{Prim}(M(A))$ with $Q \notin W := \{R \in \mathrm{Prim}(M(A)) : R \supseteq H_E\}$. Let D be a compact neighbourhood of Q in $\mathrm{Prim}(M(A))$ with D disjoint from W. Then $F := \overline{\phi}(D)$ is a compact subset of βX_ϕ and hence $L := F \cap X_\phi$ is closed in X_ϕ and disjoint from E, and $\overline{\phi}(Q)$ lies in the closure of L in βX_ϕ. Since X_ϕ is normal, $\overline{\phi}(Q)$ does not lie in the closure of E in βX_ϕ and hence there is a continuous, real-valued function f on βX_ϕ such that $f(\overline{\phi}(Q)) = 1$ and $f(E) = \{0\}$. There is a well-defined function $g : Y \to \mathbf{R}$ such that $g \circ q = f|_{X_\phi}$. Let U be an open subset of $\mathbf{R}$. Then $q^{-1}(g^{-1}(U)) = f^{-1}(U) \cap X_\phi$, an open subset of X_ϕ. Thus $g^{-1}(U)$ is open in Y and so g is continuous. Let $\overline{g}$ be the extension of g to a continuous function on βY. Then $\overline{g} \circ \overline{q} = f$ by continuity and so $\overline{g} \circ \overline{q} \circ \overline{\phi}(Q) = 1$. Hence $\overline{\psi}(Q) = \overline{q} \circ \overline{\phi}(Q) \neq e$. Thus $Q \not\supseteq H_e$, and hence $H_E = H_e$. □

3. Global and local spectral synthesis

In this section we characterize 'global spectral synthesis' by showing that, for a σ-unital $C_0(X)$-algebra A, H_x is strictly closed for all $x \in X_\phi$ if and only if J_x is locally modular for all $x \in X_\phi$ and ϕ is a closed map relative to its image (Corollary 3.7). For separable A we can also characterize 'spectral synthesis at a point': if A is separable then for $x \in X_\phi$, H_x is strictly closed if and only if J_x is locally modular and ϕ is locally closed at x (Corollary 3.10).

We begin by analyzing the property of H_x being strictly closed into two separate sub-properties (Proposition 3.2). For this, we need the following lemma.

Lemma 3.1. *Let A be a σ-unital C^*-algebra and let Y and Z be disjoint closed subsets of* $\mathrm{Prim}(A)$. *Then the closures of $\tilde{Y}$ and $\tilde{Z}$ are disjoint in* $\mathrm{Prim}(M(A))$.

Proof. Set $J = \ker(Y \cup Z)$ and let $B = A/J$. Let $\pi : A \to B$ be the quotient map. Then the map $P \mapsto \pi(P)$ $(P \in Y \cup Z)$ carries $Y \cup Z$ homeomorphically onto $\mathrm{Prim}(B)$. Thus there exists $b \in Z(M(B))$ such that $b + \pi(P)^\sim = 1 + \pi(P)^\sim$ for $P \in Y$ and $b + \pi(P)^\sim = 0$ for $P \in Z$. By [30, Theorem 10] the canonical map $M(A) \to M(B)$ is surjective, and hence there exists $a \in M(A)$ such that $a + \tilde{P} = 1 + \tilde{P}$ for $P \in Y$ and $a + \tilde{P} = 0$ for $P \in Z$. Thus, by definition of the hull-kernel topology, $a + Q = 0$ for all Q in the closure of $\tilde{Z}$ in $\mathrm{Prim}(M(A))$, and by considering $(1 - a)$ we see that likewise $a + Q = 1 + Q$ for all Q in the closure of $\tilde{Y}$ in $\mathrm{Prim}(M(A))$. Hence $\tilde{Y}$ and $\tilde{Z}$ have disjoint closures in $\mathrm{Prim}(M(A))$. □

The first of the two sub-properties into which we analyze the property of strict closure is as follows. Recall that $H(x) = \phi^{-1}(x)$ $(x \in X_\phi)$. For $x \in X_\phi$ we say that the base map ϕ is *locally closed at* x [24, §13.XIV] if whenever Y is a closed subset of $\mathrm{Prim}(A)$ such that x lies in the closure of $\phi(Y)$ then $x \in \phi(Y)$, that is, $Y \cap H(x)$ is non-empty. For example, if x is an isolated point in X_ϕ (which implies that $H(x)$ is a clopen subset of $\mathrm{Prim}(A)$) then ϕ is trivially locally closed at x. Note, however, that $H(x)$ could be clopen, yet x non-isolated in X_ϕ. In this case, ϕ would not be locally closed at x, see Example 4.8(ii). We say that ϕ is *relatively closed* if $\phi(Y)$ is closed in X_ϕ for all closed subsets Y of $\mathrm{Prim}(A)$. Clearly ϕ is relatively closed if and only if ϕ is locally closed at each $x \in X_\phi$.

Proposition 3.2. *Let A be a $C_0(X)$-algebra with base map ϕ and let $x \in X_\phi$. Consider the following properties:*

(i) *H_x is strictly closed;*
(ii) *ϕ is locally closed at x;*

(iii) *for each* $b \in \tilde{J}_x$ *and* $\epsilon > 0$ *there is an open set* $V \subseteq \mathrm{Prim}(A)$ *with* $H(x) \subseteq V$ *such that* $\|b + \tilde{P}\| < \epsilon$ *for all* $P \in V$.
Then (i)⇒(iii) *and* (ii)+(iii)⇒(i). *If* A *is* σ*-unital then* (i)⇒(ii) *and hence* (i) *is equivalent to* (ii)+(iii).

Proof. (i)⇒(iii) Suppose that $H_x = \tilde{J}_x$ and let $b \in \tilde{J}_x$ and $\epsilon > 0$. Then by the upper semi-continuity of norm functions there is an neighbourhood U of x in X_ϕ such that $\|b + H_y\| < \epsilon$ for all $y \in U$. Set $V = \phi^{-1}(U)$. Then $H(x) \subseteq V$ and for all $P \in V$

$$\|b + \tilde{P}\| \leq \|b + \tilde{J}_{\phi(P)}\| \leq \|b + H_{\phi(P)}\| < \epsilon.$$

(ii)+(iii)⇒(i) Let $b \in \tilde{J}_x$ and let $\epsilon > 0$ be given. Then by assumption there is an open set $V \subseteq \mathrm{Prim}(A)$ with $H(x) \subseteq V$ such that $\|b + \tilde{P}\| < \epsilon$ for all $P \in V$. Set $Y = \mathrm{Prim}(A) \setminus V$. Then Y is closed and $Y \cap H(x)$ is empty. Since ϕ is locally closed at x, it follows that x does not belong to the closure of $\phi(Y)$ in X_ϕ. Hence there exists $g \in C^b(X_\phi)$ with $0 \leq g \leq 1$ such that $g(\phi(Y)) = \{1\}$ and $g(x) = 0$. Let $z = \theta_A(g \circ \phi) \in Z(M(A))$ so that $0 \leq z \leq 1$ and $z + \tilde{Q} = g(\phi(Q))(1 + \tilde{Q})$ for all $Q \in \mathrm{Prim}(A)$. Let $R \in \mathrm{Prim}(M(A))$ with $\overline{\phi}(R) = x$. There is a net (P_α) in $\mathrm{Prim}(A)$ such that $\tilde{P}_\alpha \to R$. Then $x = \overline{\phi}(R) = \lim \phi(P_\alpha)$ and so $g(\phi(P_\alpha)) \to g(x) = 0$. Thus $\|z + \tilde{P}_\alpha\| \to 0$ and so, since $z \in Z(M(A))$, $\|z + R\| = 0$. It follows that $z \in H_x$ and hence $zb \in H_x$. But

$$\|zb - b\| = \sup\{\|(zb - b) + \tilde{Q}\| : Q \in \mathrm{Prim}(A)\} = \sup\{\|(zb - b) + \tilde{Q}\| : Q \in V\} < \epsilon.$$

Since ϵ was arbitrary, $b \in H_x$ and hence $\tilde{J}_x = H_x$.

(i)⇒(ii) (assuming that A is σ-unital). Suppose that ϕ is not locally closed at x and let Y be a closed subset of $\mathrm{Prim}(A)$ such that x lies in the closure of $\phi(Y)$ but $H(x) \cap Y$ is empty. Let W be the closure of $\tilde{Y}$ in $\mathrm{Prim}(M(A))$. Then $\overline{\phi}(W)$ is a compact and hence closed subset of βX containing $\phi(Y)$. Hence there exists $R \in W$ such that $\overline{\phi}(R) = x$ and therefore $R \supseteq H_x$. By Proposition 2.1(iv), the closure of $\tilde{H}(x) = \{\tilde{P} : P \in \phi^{-1}(x)\}$ in $\mathrm{Prim}(M(A))$ is equal to $\mathrm{Prim}(M(A)/\tilde{J}_x)$ (where the latter is identified with the hull of $\tilde{J}_x$ in $\mathrm{Prim}(M(A))$). But, since A is σ-unital, W is disjoint from $\mathrm{Prim}(M(A)/\tilde{J}_x)$ by Lemma 3.1 Thus $R \not\supseteq \tilde{J}_x$ and so $\tilde{J}_x \neq H_x$. □

In particular, we notice that if A is σ-unital then a necessary condition for spectral synthesis at $x \in X_\phi$ is that ϕ should be locally closed at x.

To understand the second sub-property (property (iii) of Proposition 3.2) into which we have analyzed the property of being strictly closed, we introduce the idea of local modularity. To define this it is helpful to have the following notation. For $x \in X_\phi$, let $\partial H(x)$ be the boundary and $U(x)$ the interior of $H(x)$ in $\mathrm{Prim}(A)$. We say that J_x is *locally modular* if for each $P \in \partial H(x)$ there exists a relatively open neighbourhood V of P in $\mathrm{Prim}(A) \setminus U(x)$ such that $A/\ker V$ is a unital C*-algebra.

For instance, if x is an isolated point of X_ϕ then $H(x)$ is clopen in $\mathrm{Prim}(A)$ and so $\partial H(x)$ is empty and hence J_x is vacuously locally modular. Secondly, if $J_x \not\supseteq Z'(A)$ (that is, $x \in U_\phi$) then by upper semi-continuity of norm functions and functional calculus we may find $z \in Z'(A)$ such that $z + J_y = 1_{A/J_y}$ for all y in an open neighbourhood V of x in X_ϕ (cf. the proof of [6, Proposition 2.3]). Hence $z + P = 1_{A/P}$ for all $P \in W = \phi^{-1}(V)$, and

$H(x) \subseteq W$, so again J_x is locally modular. On the other hand, if there exists $P \in \partial H(x)$ such that A/P is non-unital then clearly J_x is not locally modular.

The definition of local modularity is intrinsic to A, in that it does not mention $M(A)$, and it is a condition that should be possible to check in concrete cases. The following equivalent condition, however, seems easier to work with, although it does involve $M(A)$. To describe this, we need a slight variant of the definition of $\sim$. Recall from [33] that for $P, Q \in \mathrm{Prim}(A)$ we write $P \sim Q$ if P and Q cannot be separated by disjoint open subsets of $\mathrm{Prim}(A)$ (for a fuller discussion, see Section 5 below). For the multiplier algebra $M(A)$ of a $C_0(X)$-algebra A, we define $\sim_x$ as follows. For $x \in X_\phi$ and $Q, R \in \mathrm{Prim}(M(A)) \setminus \tilde{U}(x)$ we say that $Q \sim_x R$ if there is a net (P_α) in $\mathrm{Prim}(A) \setminus U(x)$ such that $(\tilde{P}_\alpha)$ converges to both Q and R. If $H(x)$ has empty interior then $\sim_x$ coincides with the relation $\sim$ on $\mathrm{Prim}(M(A))$ because the canonical image of $\mathrm{Prim}(A)$ is dense in $\mathrm{Prim}(M(A))$. Otherwise $Q \sim_x R \Rightarrow Q \sim R$, but the converse need not hold.

For the next lemma, we need the definition of a primal ideal and of the topology τ_s. An ideal J in a C*-algebra A is *primal* if whenever $I_1, \ldots, I_n$ is a finite collection of ideals of A with the product $I_1 \ldots I_n = \{0\}$ then $I_i \subseteq J$ for at least one $i \in \{1, \ldots, n\}$ [5]. Every primitive ideal is prime and hence primal. The set of proper primal ideals of A is denoted $\mathrm{Primal}'(A)$. The τ_s topology on $\mathrm{Primal}'(A)$ is defined to be the weakest topology for which all the norm functions $I \to \|a + I\|$ $(a \in A,\ I \in \mathrm{Primal}'(A))$ are continuous (see [17, Section II]). If A is unital then $\mathrm{Primal}'(A)$ is τ_s-compact [4, Proposition 4.1].

Lemma 3.3. *Let A be a $C_0(X)$-algebra with base map ϕ and let $x \in X_\phi$. Consider the following conditions:*

(i) *J_x is locally modular;*

(ii) *for all $P \in \partial H(x)$ and $R \in \mathrm{Prim}(M(A)/A)$, $\tilde{P} \not\sim_x R$.*

Then (i)$\Rightarrow$(ii)*, and* (i) *and* (ii) *are equivalent if A is σ-unital.*

Proof. Suppose first that (i) holds, and let $P \in \partial H(x)$. Let V be an open neighbourhood of P in $\mathrm{Prim}(A) \setminus U(x)$ such that $A/\ker V$ is unital. Let W be the closure of V in $\mathrm{Prim}(A)$, so that $\ker V = \ker W$. Write $J = \ker W$ (so that $W = \mathrm{Prim}(A/J)$) and recall that the quotient map $q_J : A \to A/J$ has a canonical extension $\tilde{q}_J : M(A) \to M(A/J) = A/J$. For each $b \in M(A)$ there exists $a \in A$ such that $b - a \in \ker \tilde{q}_J = \tilde{J}$, so $A + \tilde{J} = M(A)$. Thus if $R \in \mathrm{Prim}(M(A)/A)$ then $R \not\supseteq \tilde{J}$. It follows that $\tilde{W} = \mathrm{Prim}(M(A)/\tilde{J})$, a closed subset of $\mathrm{Prim}(M(A))$. Hence if (P_α) is a net in $\mathrm{Prim}(A) \setminus U(x)$ with $P_\alpha \to P$ then eventually $P_\alpha \in V$, so all the limits of the net $(\tilde{P}_\alpha)$ in $\mathrm{Prim}(M(A))$ lie in $\tilde{W}$. Thus $\tilde{P} \not\sim_x R$ for any $R \in \mathrm{Prim}(M(A)/A)$.

Conversely, suppose that (ii) holds and that A is σ-unital. Let u be a strictly positive element in A with $\|u\| = 1$. Let $Q \in \partial H(x)$ and suppose that $A + \tilde{Q} \neq M(A)$. Then there exists a maximal ideal M of $M(A)$ with $M \supseteq A + \tilde{Q}$, and hence $M \sim_x \tilde{Q}$ contradicting (ii). Thus $A + \tilde{Q} = M(A)$, so A/Q is unital and $\|(1 - u) + \tilde{Q}\| < 1$.

Let $P \in \partial H(x)$ and suppose, for a contradiction, that there is a net (P_α) in $\mathrm{Prim}(A) \setminus U(x)$ with $P_\alpha \to P$ and $\|(1 - u) + \tilde{P}_\alpha\| \to 1$. By the τ_s-compactness of $\mathrm{Primal}'(M(A))$, and by passing to a subnet if necessary, we may assume that there exists $J \in \mathrm{Primal}'(M(A))$ such that $\tilde{P}_\alpha \to J$ (τ_s). Hence $\|(1 - u) + J\| = 1$ and so there exists $R \in \mathrm{Prim}(M(A)/J)$ such that $\|(1 - u) + R\| = 1$ [12, 3.3.6]. Since $\tilde{P}_\alpha \to J$ (τ_s), $\tilde{P}_\alpha \to R$ and so $R \not\supseteq A$ by (ii). Then $R = \tilde{Q}$, where $Q := R \cap A \in \mathrm{Prim}(A)$, and so $P_\alpha \to Q$. Since ϕ is continuous, $\phi(Q) = \phi(P) = x$

and so $Q \in H(x)$. But $P_\alpha \in \mathrm{Prim}(A) \setminus U(x)$ for all α and so $Q \in \partial H(x)$, contradicting the fact that $\|(1-u)+\tilde{Q}\| = 1$. Thus no such net (P_α) exists and so there exists $\epsilon > 0$ and a neighbourhood V of P in $\mathrm{Prim}(A) \setminus U(x)$ such that $\|(1-u)+\tilde{Q}\| < 1-\epsilon$ for all $Q \in V$.

Now let f be a continuous function on $[0,1]$ with $f(0) = 0$, $f([\epsilon,1]) = \{1\}$, and $\|f\| = 1$, and set $v = f(u)$. Then $(1-v)+\tilde{Q} = 0$ for all $Q \in V$ and so $v + \ker V$ is the identity in $A/\ker V$. Hence J_x is locally modular, as required. □

The next proposition shows part of the connection between local modularity and property (iii) of Proposition 3.2.

Proposition 3.4. *Let A be a $C_0(X)$-algebra with base map ϕ and let $x \in X_\phi$. If J_x is locally modular then property (iii) of Proposition 3.2 holds, namely for each $b \in \tilde{J}_x$ and $\epsilon > 0$ there is an open set $V \subseteq \mathrm{Prim}(A)$ with $H(x) \subseteq V$ such that $\|b + \tilde{P}\| < \epsilon$ for all $P \in V$. Hence if J_x is locally modular and ϕ is locally closed at x then H_x is strictly closed.*

Proof. Suppose that J_x is locally modular and let $b \in \tilde{J}_x$ and $\epsilon > 0$. Let $P \in \partial H(x)$ and suppose for a contradiction that there is a net (P_α) in $\mathrm{Prim}(A) \setminus U(x)$ such that $P_\alpha \to P$ and $\|b + \tilde{P}_\alpha\| \geq \epsilon$. Then by compactness of the set $W = \{R \in \mathrm{Prim}(M(A)) : \|b + R\| \geq \epsilon\}$, and by passing to a subnet of (P_α) if necessary, there exists $R \in W$ such that $\tilde{P}_\alpha \to R$. Since J_x is locally modular, it follows from Lemma 3.3 that $R = \tilde{Q}$ for some $Q \in \mathrm{Prim}(A)$. Since $P_\alpha \to Q$, $\phi(Q) = \phi(P) = x$ and so $b \in \tilde{Q}$, contradicting the fact that $\|b + \tilde{Q}\| \geq \epsilon$. Therefore no such net (P_α) exists. Thus there is an open set V_P containing P such that $\|b + \tilde{Q}\| < \epsilon$ for all $Q \in V_P$. Taking $V = U(x) \cup \bigcup_{P \in \partial H(x)} V_P$ gives the required set V.

The final statement now follows from Proposition 3.2 ((ii)+(iii)⇒(i)). □

It will follow from Theorem 3.6 that the converse to Proposition 3.4 holds if A is separable. For a general σ-unital $C_0(X)$-algebra A, however, it is possible for property (iii) of Proposition 3.2 to hold for a particular $x \in X_\phi$ without J_x being locally modular, see Example 4.5. Nevertheless the relation between the two properties is very close, as we shall see.

To show this, we need the following theorem from [7, Theorem 2.5]. For an element a in a C*-algebra A, let $\mathrm{sp}(a)$ denote the spectrum of a; and for $a \geq 0$, let $\min \mathrm{sp}(a)$ be the smallest number in $\mathrm{sp}(a)$. The function g from the unit interval $[0,1]$ to the space $C[0,1]$ is as follows (where for $r \in [0,1]$, g_r is the continuous function on $[0,1]$ corresponding to r):

$$g_0(t) = 1 \text{ for all } t \in [0,1];$$

$$\text{for } 0 < r \leq 1/2, \quad g_r(t) = \begin{cases} 0 & (0 \leq t \leq r/2) \\ (2t/r) - 1 & (r/2 \leq t \leq r) \\ 1 & (r \leq t \leq 1); \end{cases}$$

$$g_r = g_{1/2} \text{ for } r \geq 1/2.$$

Theorem 3.5. *Let A be a σ-unital $C_0(X)$-algebra with base map ϕ and set $X_\phi = \mathrm{Im}(\phi)$. Let u be a strictly positive element in A with $\|u\| = 1$. Let $f \in C^b(X_\phi)$ with $0 \leq f \leq 1$, let U be the cozero set of f and let $V = \{x \in U : 2\min\mathrm{sp}(u + J_x) \leq f(x)\}$. Let $\mathrm{cl}(U)$ and $\mathrm{cl}(V)$ be the closures of U and V respectively in X_ϕ. Then there exists $b \in M(A)$ with $0 \leq b \leq 1$ such that*

(i) $b + \tilde{J}_x = g_{f(x)}(u + \tilde{J}_x) \qquad (x \in X_\phi)$;

(ii) $b \in A + H_x \subseteq A + \tilde{J}_x$ *for all* $x \in U$*;*
(iii) $1 - b \in \tilde{J}_x$ *for all* $x \in X_\phi \setminus U$ *and* $1 - b \in H_x$ *for all* $x \in X_\phi \setminus \mathrm{cl}(U)$*;*
(iv) $\|(1-b) + \tilde{J}_x\| = 1$ *for all* $x \in V$ *and* $\|(1-b) + H_x\| = 1$ *for all* $x \in \mathrm{cl}(V)$*.*
Furthermore,
(v) H_x *is not strictly closed in* $M(A)$ *for all* $x \in \mathrm{cl}(V) \setminus U$*.*

In the context of Theorem 3.5, note that if $x \in U$ and A/J_x is non-unital then $0 \in \mathrm{sp}(u + J_x)$ and hence $x \in V$.

Theorem 3.6. *Let* A *be a* σ*-unital* $C_0(X)$*-algebra with base map* ϕ*. Let* $x \in X_\phi$ *and let* Z *be a zero set of* X_ϕ *with* $x \in Z$*. Suppose that* J_x *is not locally modular. Then there exists* $y \in Z$ *for which property (iii) of Proposition 3.2 fails, that is, for which there exists* $c \in \tilde{J}_y$ *and* $\epsilon > 0$ *such that there is no open set* $V \subseteq \mathrm{Prim}(A)$ *with* $H(y) \subseteq V$ *and* $\|c + \tilde{P}\| < \epsilon$ *for all* $P \in V$*. In particular,* H_y *is not strictly closed.*

Proof. Since J_x is not locally modular, it follows from Lemma 3.3 that there exist $P \in \partial H(x)$ and $R \in \mathrm{Prim}(M(A)/A)$ such that $\tilde{P} \sim_x R$. Let D be a compact neighbourhood of P in $\mathrm{Prim}(A)$. Let $u \in A$ be a strictly positive element with $\|u\| = 1$. Since $u \in A$, it follows that $\|(1-u) + R\| = 1$. Hence by [12, 3.3.2], for each n the set

$$U_n = \{Q \in \mathrm{Prim}(M(A)) : \|(1-u) + Q\| > 1 - 1/2^{n+2}\}$$

is an open neighbourhood of R in $\mathrm{Prim}(M(A))$. Let $h : X_\phi \to [0,1]$ be a continuous function such that $Z(h) = Z$ and, for $n \geq 1$, let $V_n = \{Q \in \mathrm{Prim}(A) : h(\phi(Q)) < 1/n\}$. Then V_n is an open neighbourhood of P in $\mathrm{Prim}(A)$. Since $\tilde{P} \sim_x R$, it follows that

$$U_1 \cap \tilde{V}_1 \cap (\mathrm{Int}\, D)^\sim \cap (\mathrm{Prim}(A) \setminus U(x))^\sim$$

is a non-empty, relatively open, subset of $(\mathrm{Prim}(A) \setminus U(x))^\sim$.

Since $\mathrm{Prim}(A) \setminus H(x)$ is dense in $\mathrm{Prim}(A) \setminus U(x)$, we may choose $Q_1 \in \mathrm{Prim}(A) \setminus H(x)$ with $\tilde{Q}_1 \in U_1 \cap \tilde{V}_1 \cap (\mathrm{Int}\, D)^\sim$. Set $x_1 = \phi(Q_1)$. Then $x_1 \neq x$ and so there exists a continuous function $f_1 : X_\phi \to [0, 1/4]$ with $f_1(x_1) = 1/4$ and $f_1(x) = 0$. For $n \geq 2$, we will inductively define points $x_n \in X_\phi$ and continuous functions $f_n : X_\phi \to [0, 1/2^{n+1}]$ with $f_n(x_n) = 1/2^{n+1}$ and $f_n(x) = 0 = f_n(x_m)$ for $1 \leq m \leq n-1$. Note that x_1 and f_1 satisfy these conditions.

Suppose that $n \geq 2$ and that $x_1, \ldots, x_{n-1}$ and $f_1, \ldots, f_{n-1}$ satisfy the required conditions. Let $W_n = \{Q \in \mathrm{Prim}(A) : \sum_{i=1}^{n-1} f_i(\phi(Q)) < 1/2^{n+1}\}$. Then W_n is an open neighbourhood of P and hence, since $\tilde{P} \sim_x R$, it follows that

$$U_n \cap \tilde{V}_n \cap (\mathrm{Int}\, D)^\sim \cap \tilde{W}_n \cap (\mathrm{Prim}(A) \setminus U(x))^\sim$$

is a non-empty, relatively open, subset of $(\mathrm{Prim}(A) \setminus U(x))^\sim$. Thus we may choose $Q_n \in \mathrm{Prim}(A) \setminus H(x)$ with $\tilde{Q}_n \in U_n \cap \tilde{V}_n \cap (\mathrm{Int}\, D)^\sim \cap \tilde{W}_n$. Set $x_n = \phi(Q_n)$. Then $x_n \neq x$ and $x_n \neq x_m$ for $1 \leq m \leq n-1$ because

$$\sum_{i=1}^{n-1} f_i(x_m) \geq f_m(x_m) = \frac{1}{2^{m+1}} > \frac{1}{2^{n+1}} > \sum_{i=1}^{n-1} f_i(x_n).$$

Thus there exists a continuous function $f_n : X_\phi \to [0, 1/2^{n+1}]$ with $f_n(x_n) = 1/2^{n+1}$ and $f_n(x) = 0 = f_n(x_m)$ for $1 \leq m \leq n-1$.

Set $f = \sum_{n=1}^{\infty} f_n$. Then f is continuous and, for each $n \geq 1$,

$$\frac{1}{2^{n+1}} = f_n(x_n) \leq f(x_n) = \sum_{i=1}^{n-1} f_i(x_n) + f_n(x_n) < \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}} = \frac{1}{2^n}.$$

Thus $f(x_n) \to 0$ as $n \to \infty$. Furthermore, since $\|(1-u) + \tilde{Q}_n\| > 1 - 1/2^{n+2}$, it follows that

$$\min \operatorname{sp}(u + \tilde{Q}_n) < \frac{1}{2^{n+2}} \leq \frac{f(x_n)}{2}$$

and hence $0 \in \operatorname{sp}(g_{f(x_n)}(u + \tilde{Q}_n))$ for $n \geq 1$.

Since $Q_n \in D$ for all n, the compactness of D implies that there exists $Q \in D$ and a subnet (Q_{n_α}) of (Q_n) such that $Q_{n_\alpha} \to Q$. Set $y = \phi(Q)$. Since $Q_n \in V_n$ $(n \geq 1)$, $h(x_n) \to 0$ as $n \to \infty$. It follows that $h(y) = 0$ and hence $y \in Z$. Furthermore, $f(x_{n_\alpha}) \to f(y)$ and so $f(y) = 0$. Let b be an element of $M(A)$ corresponding to f as in Theorem 3.5. Then $1 - b \in \tilde{J}_y$ by Theorem 3.5(iii). Since $\phi(Q_n) = x_n$ $(n \geq 1)$, it follows that $\tilde{Q}_n \supseteq \tilde{J}_{x_n}$ and hence that $b + \tilde{Q}_n = g_{f(x_n)}(u + \tilde{Q}_n)$ by Theorem 3.5(i). Then

$$1 \geq \|(1-b) + \tilde{Q}_n\| \geq \|(1 + \tilde{Q}_n) - g_{f(x_n)}(u + \tilde{Q}_n)\| = 1 - \min \operatorname{sp}(g_{f(x_n)}(u + \tilde{Q}_n)) = 1.$$

Hence $\|(1-b) + \tilde{Q}_{n_\alpha}\| = 1$ for all α. Suppose that V is an open subset of $\operatorname{Prim}(A)$ such that $H(y) \subseteq V$. Then $Q \in V$ and so eventually $Q_{n_\alpha} \in V$. Thus property (iii) of Proposition 3.2 fails at y for $c = 1 - b \in \tilde{J}_y$ and $\epsilon = 1/2$. Hence H_y is not strictly closed by Proposition 3.2. □

Regarding the hypotheses of Theorem 3.6, we note that the complete regularity of X_ϕ implies that every neighbourhood of x contains a zero set containing x. In the proof of Theorem 3.6, we could have checked that $y \in \operatorname{cl}(V) \setminus U$ (in the terminology of Theorem 3.5) and used Theorem 3.5(v) to deduce that H_y is not strictly closed. We have preferred, however, to obtain the stronger result that property (iii) of Proposition 3.2 fails at y.

Armed with Theorem 3.6, we can now prove some of the main results of the paper.

Corollary 3.7 (global spectral synthesis). *Let A be a $C_0(X)$-algebra with base map ϕ. Consider the following two properties.*

(i) *H_x is strictly closed for all $x \in X_\phi$.*

(ii) *J_x is locally modular and ϕ is locally closed at x for all $x \in X_\phi$.*

Then (ii)⇒(i), *and* (i) *and* (ii) *are equivalent if A is σ-unital.*

Proof. (ii)⇒(i). This follows from Proposition 3.4.

(i)⇒(ii) (assuming that A is σ-unital). This follows from (i)⇒((ii)+(iii)) of Proposition 3.2 together with Theorem 3.6 (taking $Z = X_\phi$, for example). □

Corollary 3.8 (spectral synthesis at a point). *Let A be a σ-unital $C_0(X)$-algebra with base map ϕ and let x be a G_δ-point in X_ϕ. Then H_x is strictly closed if and only if J_x is locally modular and ϕ is locally closed at x.*

Proof. This follows from Proposition 3.4 and from Proposition 3.2 and Theorem 3.6 taking the zero set Z in Theorem 3.6 to be $Z = \{x\}$. □

Recall that a completely regular (Hausdorff) space X is *perfectly normal* if every closed subset of X is the zero set of a continuous real-valued function on X. Every metric space is perfectly normal.

Lemma 3.9. *Let A be a separable $C_0(X)$-algebra with base map ϕ. Then X_ϕ is perfectly normal.*

Proof. Let V be an open subset of X_ϕ. Set $Y = \phi^{-1}(V)$ and let J be the ideal of A such that $\mathrm{Prim}(J)$ is canonically homeomorphic to Y. Then J is separable. Let u be a strictly positive element in J. For each $n \in \mathbf{N}$, set $Y_n = \{P \in \mathrm{Prim}(A) : \|u + P\| \geq 1/n\}$. Then Y_n is compact and so $\phi(Y_n)$ is a compact (hence closed) subset of X_ϕ. Since $V = \bigcup_{n=1}^{\infty} \phi(Y_n)$, we see that V is an F_σ subset of X_ϕ. But X_ϕ is normal, and in a normal space an open F_σ subset is a cozero set. Thus V is a cozero set. □

Corollary 3.10. *Let A be a separable $C_0(X)$-algebra with base map ϕ and let $x \in X_\phi$. Then H_x is strictly closed if and only if J_x is locally modular and ϕ is locally closed at x.*

Proof. This follows as for Corollary 3.8, noting that $\{x\}$ is a zero set of X_ϕ for each $x \in X_\phi$ by Lemma 3.9. □

4. Pointwise spectral synthesis and P-points

Recall from Section 2 that if A is a $C_0(X)$-algebra with base map ϕ then $U_\phi = \{x \in X_\phi : J_x \not\supseteq Z'(A)\}$, an open subset of X_ϕ, ∂U_ϕ denotes the boundary of U_ϕ in X_ϕ and $W_\phi = X_\phi \backslash U_\phi$. Here $Z'(A) = \mu(C_0(X)) \cap A$. We saw in Proposition 2.6 that if $x \in U_\phi$ or if x is an isolated point of X_ϕ then H_x is strictly closed, while if A/J_x is unital but $x \notin U_\phi$ then H_x is not strictly closed. We remarked that the remaining points to consider are those for which A/J_x is non-unital and $x \in W_\phi$ and is non-isolated in X_ϕ. From Section 3 we now have a complete characterization of global spectral synthesis for σ-unital A, and a complete characterization of pointwise spectral synthesis for separable A, but only a near-characterization of pointwise spectral synthesis for the more general case when A is σ-unital.

In this section we approach the σ-unital case from another angle. We characterize the points x in the interior of W_ϕ in X_ϕ for which H_x is closed (these turn out to be precisely the P-points) and make partial progress for the difficult case of points $x \in \partial U_\phi$ (cf. for example [6, Proposition 3.5]). We also give a complete characterization of pointwise spectral synthesis for the important case when the base map ϕ is open (Theorem 4.6). We begin with the following lemma.

Lemma 4.1. *Let A be a σ-unital $C_0(X)$-algebra with base map ϕ and let $x \in W_\phi$. Let $u \in A$ be strictly positive with $\|u\| \leq 1$. Then there is a net (x_α) in X_ϕ with $x_\alpha \to x$ and $\|(1-u) + \tilde{J}_{x_\alpha}\| \to 1$.*

Proof. By Lemma 2.5 there exists $R \in \mathrm{Prim}(M(A)/A)$ with $R \supseteq H_x$. Let (P_α) be a net in $\mathrm{Prim}(A)$ such that $\tilde{P}_\alpha \to R$. Then $\overline{\phi}(\tilde{P}_\alpha) \to \overline{\phi}(R) = x$. Hence $x_\alpha := \phi(P_\alpha) \to x$. On the other hand, since $R \supseteq A$,

$$\begin{aligned} 1 = \|(1-u) + R\| &\leq \liminf \|(1-u) + \tilde{P}_\alpha\| \\ &\leq \liminf \|(1-u) + \tilde{J}_{x_\alpha}\| \\ &\leq \limsup \|(1-u) + \tilde{J}_{x_\alpha}\| \leq 1. \end{aligned}$$

Hence $\|(1-u) + \tilde{J}_{x_\alpha}\| \to 1$. □

Next we recall the definition of a P-point. Let X be a completely regular (Hausdorff) space. A point $x \in X$ is a *P-point* if every continuous real-valued function vanishing at x

vanishes in a neighbourhood of x [18, 4L]. Equivalently, x is a P-point if x does not lie in the boundary of any cozero set. If the space X is perfectly normal then every singleton is a zero set and so a P-point is necessarily an isolated point. A space in which every point is a P-point is a *P-space*.

We are now ready for the first main result of this section.

Theorem 4.2. *Let A be a σ-unital $C_0(X)$-algebra with base map ϕ. If $x \in X_\phi$ is a P-point in X_ϕ then H_x is strictly closed. Conversely, if $x \in W_\phi$ and H_x is strictly closed then x is a P-point in W_ϕ.*

Proof. Let $x \in X_\phi$ and suppose that H_x is not strictly closed. We show that x is not a P-point in X_ϕ. By [21, Theorem 10.1.7], there exists $b \in \tilde{J}_x \setminus H_x$ with $\|b\| = \|b + H_x\| = 1$. Let u be a strictly positive element of A with $\|u\| = 1$, and recall that for $y \in X_\phi$, $b \in \tilde{J}_y$ if and only if $bu \in \tilde{J}_y$ (cf. [6, Section 2]). For each $n \geq 1$, set $W_n = \{y \in X_\phi : \|bu + \tilde{J}_y\| \geq 1/n\}$. By [7, Lemma 4.2], for every neighbourhood W of x in $\mathrm{cl}_{\beta X} X_\phi$,

$$||b + H_x|| \leq \sup\{||b + \tilde{J}_y|| : y \in X_\phi \cap W\}.$$

Hence for every neighbourhood V of x in X_ϕ there exists $y \in V$ such that $bu \notin \tilde{J}_y$. Thus x lies in the closure of $\bigcup_{n=1}^\infty W_n$. Since

$$W_n = \{y \in X_\phi : ||bu + J_y|| \geq 1/n\} = \phi(\{P \in \mathrm{Prim}(A) : ||bu + P|| \geq 1/n\}),$$

W_n is a compact subset of X_ϕ. But $x \notin W_n$ and hence there exists a continuous function $f_n : X_\phi \to [0, 1]$ such that $f_n(x) = 0$ and $f_n(W_n) = \{1\}$. Set $f = \sum_{n=1}^\infty f_n/2^n$. Then $f(x) = 0$, but x lies in the closure of the cozero set of f. Hence x is not a P-point.

Conversely, suppose that $x \in W_\phi$ is not a P-point in W_ϕ. Let f be a continuous function on W_ϕ with $f(x) = 0$ such that x lies in the closure of the cozero set of f. Replacing f by $\min\{|f|, 1\}$, we may assume that $0 \leq f \leq 1$. Since X_ϕ is normal and W_ϕ is closed in X_ϕ, we may extend f to a continuous function $\overline{f}$ on X_ϕ with $0 \leq \overline{f} \leq 1$. Let $y \in W_\phi \cap \mathrm{coz}(\overline{f})$. By Lemma 4.1 there is a net (y_α) in X_ϕ with $y_\alpha \to y$ and $\|(1-u) + \tilde{J}_{y_\alpha}\| \to 1$. Hence eventually $2 \min \mathrm{sp}(u + J_{y_\alpha}) = 2(1 - \|(1-u) + \tilde{J}_{y_\alpha}\|) \leq \overline{f}(y_\alpha)$, since $\overline{f}(y_\alpha) \to \overline{f}(y) > 0$. It follows that the set V of Theorem 3.5 (associated with the cozero set $\mathrm{coz}(\overline{f})$) has closure $\mathrm{cl}(V)$ containing $W_\phi \cap \mathrm{coz}(\overline{f})$. Hence $x \in \mathrm{cl}(V)$ since x lies in the the closure of $W_\phi \cap \mathrm{coz}(\overline{f}) = \mathrm{coz}(f)$. Thus $\tilde{J}_x \neq H_x$ by Theorem 3.5(v). □

Theorem 4.2 has some useful consequences. Let Int W_ϕ denote the interior of W_ϕ relative to X_ϕ.

Corollary 4.3. *Let A be a $C_0(X)$-algebra with base map ϕ and let $x \in \mathrm{Int}\ W_\phi$. If A is σ-unital then H_x is strictly closed if and only if x is a P-point in X_ϕ. If A is separable then H_x is strictly closed if and only if x is an isolated point in X_ϕ.*

Proof. Suppose that A is σ-unital and that x is not a P-point in X_ϕ. Then there exists $f \in C^b(X_\phi)$ such that $f(x) = 0$ and x lies in the closure of $\mathrm{coz}(f)$. Let (x_α) be a net in $\mathrm{coz}(f)$ such that $x_\alpha \to x$, and set $\overline{f} = f|_{W_\phi}$. Then $\overline{f}$ is continuous and $\overline{f}(x) = 0$ but eventually $x_\alpha \in \mathrm{Int}\ W_\phi$ and so x lies in the closure of the cozero set of $\overline{f}$. Thus x is not a P-point in W_ϕ and so H_x is not strictly closed by Theorem 4.2. If A is separable then $\{x\}$ is a zero set in X_ϕ by Lemma 3.9 and hence x is P-point in X_ϕ if and only if it is isolated in X_ϕ. □

Corollary 4.4. *Let A be a $C_0(X)$-algebra with base map ϕ and suppose that $Z'(A) = \{0\}$.*

(i) *Let $x \in X_\phi$. If A is σ-unital then H_x is strictly closed if and only if x is a P-point in X_ϕ. If A is separable then H_x is strictly closed if and only if x is an isolated point in X_ϕ.*

(ii) *If A is σ-unital then H_x is strictly closed for all $x \in X_\phi$ if and only if X_ϕ is discrete.*

Proof. Since $Z'(A) = \{0\}$, $U_\phi = \emptyset$ and $W_\phi = X_\phi$. Thus part (i) follows from Corollary 4.3. Part (ii) now follows because the space X_ϕ is σ-compact, and a σ-compact P-space is discrete (see, for example, the proof of [7, Lemma 4.4]). □

If A is a stable $C_0(X)$-algebra then $Z(A) = \{0\}$ and hence $Z'(A) = \{0\}$. Thus Corollary 4.4 extends [7, Theorem 4.5].

It follows from Theorem 4.2 and Proposition 3.2 that if A is a σ-unital $C_0(X)$-algebra and $x \in X_\phi$ is a P-point in X_ϕ then ϕ is locally closed at x and that property (iii) of Proposition 3.2 holds. On the other hand, J_x need not be locally modular as the following example shows.

Example 4.5. Let X be a compact Hausdorff space with a non-isolated P-point x (e.g. take X to be $\omega_1 + 1$, where ω_1 is the first uncountable ordinal, with the usual topology) and let $A = C(X) \otimes K(H)$ (where $K(H)$ is the algebra of compact operators on a separable, infinite-dimensional Hilbert space H). There is a homeomorphism $\phi : \mathrm{Prim}(A) \to X$ such that

$$\phi(\{f \in C_0(X) : f(y) = 0\} \otimes K(H)) = y \qquad (y \in X).$$

Then J_x is not locally modular because $\partial H(x)$ is non-empty but A/P is non-unital for all $P \in \mathrm{Prim}(A)$. Nevertheless, H_x is strictly closed by Theorem 4.2.

In Theorem 4.2 we saw a characterization, for A σ-unital, of when H_x is strictly closed for $x \in \mathrm{Int}\, W_\phi$ and we know that H_x is always strictly closed if $x \in U_\phi$ (Proposition 2.6(i)). The remaining points to consider are those in ∂U_ϕ. For general σ-unital $C_0(X)$-algebras we are not able to characterize the points $x \in \partial U_\phi$ for which H_x is strictly closed (though we have seen a necessary condition in Theorem 4.2), but if we make the further assumption that the base map ϕ is open then we can show that there are no such points.

Theorem 4.6. *Let A be a continuous, σ-unital $C_0(X)$-algebra with base map ϕ and let $x \in X_\phi$. Then the following are equivalent:*

(i) *H_x is strictly closed;*

(ii) *either $x \in U_\phi$, or $x \in \mathrm{Int}\, W_\phi$ and x is a P-point in X_ϕ.*

Proof. (ii)⇒(i) If $x \in U_\phi$, or if $x \in \mathrm{Int}\, W_\phi$ with x a P-point in X_ϕ, then H_x is strictly closed by Proposition 2.6(i) and Corollary 4.3.

(i)⇒(ii) Corollary 4.3 shows that if $x \in \mathrm{Int}\, W_\phi$ with H_x strictly closed then x must be a P-point in X_ϕ. It is enough, therefore, to show that if $x \in \partial U_\phi$ then H_x is not strictly closed. If A/J_x is unital then this follows from Proposition 2.6(iii). Hence we may assume that A/J_x is non-unital. Let u be a strictly positive element of A with $\|u\| = 1$. Then $\|(1-u) + \tilde{J}_x\| = 1$. For each $n \geq 1$, there exists $P_n \in \mathrm{Prim}(A/J_x)$ such that $\|(1-u) + \tilde{P}_n\| > 1 - 1/2^{n+1}$. Hence the set

$$V_n = \{Q \in \mathrm{Prim}(A) : \|(1-u) + \tilde{Q}\| > 1 - 1/2^{n+1}\}$$

is an open neighbourhood of P_n and so, since ϕ is open, the set $\phi(V_n)$ is an open neighbourhood of $\phi(P_n) = x$ in X_ϕ. Thus

$$X_n := X_\phi \setminus \phi(V_n) = \{y \in X_\phi : \|(1-u) + \tilde{J}_y\| \leq 1 - 1/2^{n+1}\}$$

is closed in X_ϕ and $x \notin X_n$. If $y \in U_\phi$ then A/J_y is unital [6, Proposition 2.2] and therefore $\|(1-u) + \tilde{J}_y\| < 1$. Hence $\bigcup_{n=1}^\infty X_n \supseteq U_\phi$ and it follows that x lies in the closure of $(\bigcup_{n=2}^\infty X_n) \setminus X_1$.

Since $x \notin X_n$ there exists $f_n \in C^b(X_\phi)$ with $0 \le f_n \le 1/2^n$ such that $f_n(x) = 0$ and $f_n(X_n) = \{1/2^n\}$. Set $f = \sum_{n=1}^\infty f_n$. Then $f \in C^b(X_\phi)$ with $0 \le f \le 1$ and $f(x) = 0$. Let W be the cozero set of f and let $V = \{y \in W : 2 \min \mathrm{sp}(u + J_y) \le f(y)\}$. Suppose that $y \in X_\phi$ with $y \in X_{n+1} \setminus X_n$ for some $n \ge 1$. Then $\|(1-u) + \tilde{J}_y\| > 1 - 1/2^{n+1}$ and so $\min \mathrm{sp}(u + J_y) < 1/2^{n+1}$. Since $y \in X_m$ for all $m \ge n+1$, $f(y) \ge 1/2^n$ and so $f(y) > 2 \min \mathrm{sp}(u + J_y)$. Hence $y \in V$, and thus $V \supseteq (\bigcup_{n=2}^\infty X_n) \setminus X_1$. Hence $x \in \mathrm{cl}(V) \setminus W$ and so it follows from Theorem 3.5(v) that $\tilde{J}_x \ne H_x$, as required. □

Corollary 4.7. *Let A be a continuous, σ-unital $C_0(X)$-algebra with base map ϕ. Then the following are equivalent:*

(i) *for all $x \in X_\phi$, H_x is strictly closed;*

(ii) *U_ϕ and W_ϕ are clopen in X_ϕ, and W_ϕ is discrete.*

Proof. (ii)⇒(i). This follows immediately from Theorem 4.6.

(i)⇒(ii). Theorem 4.6 implies that ∂U_ϕ is empty and hence X_ϕ has the required decomposition into clopen sets U_ϕ and W_ϕ. Since X_ϕ is σ-compact, W_ϕ must be σ-compact as well. But by Theorem 4.6, W_ϕ is a P-space, and a σ-compact P-space is discrete, see [7, Lemma 4.4]. □

It is interesting to compare Corollary 4.7 with [6, Theorem 3.8] which characterizes, for a continuous σ-unital $C_0(X)$-algebra A, when $M(A)$ is a continuous $C(\beta X)$-algebra. The conditions in Corollary 4.7 are markedly stronger than those in [6, Theorem 3.8], notably the requirement that W_ϕ be discrete as against being a basically disconnected space. On the other hand, if A is separable then there is a much closer fit with [6, Corollary 3.9]: indeed, if A is continuous and separable and $X_\phi = X$ then H_x is strictly closed for all $x \in X_\phi$ if and only if $M(A)$ is continuous for $\overline{\mu}$.

We conclude this section with a couple of elementary abelian examples, part of whose significance will appear in the next two sections.

Example 4.8. (i) *An abelian $C_0(X)$-algebra A with $x \in W_\phi$ such that H_x is strictly closed in $M(A)$.* Let $Y = \{(x,y) \in \mathbf{R}^2 : y \ge 0\}$ be the upper half-plane, and let $L = \{(x,y) \in Y : y = 0\}$ be the x-axis. Let Y/L be the quotient space (which is completely regular by Lemma 2.8). Set $A = C_0(Y)$. Then we may identify $\mathrm{Prim}(A)$ with Y in the usual way and define $\phi : \mathrm{Prim}(A) \to \beta(Y/L)$ by $\phi((x,y)) = [(x,y)] \in Y/L$. Thus $X_\phi = Y/L$. Then $J_{[(0,0)]}$ is locally modular (since every point in L has a compact neighbourhood in Y) and ϕ is locally closed at $[(0,0)]$ (since Y is normal). However, $[(0,0)] \notin U_\phi$ because if $f \in C(\beta Y/L)$ and $f \circ \phi \in A = C_0(Y)$ then $f([(0,0)]) = 0$.

(ii) *An abelian $C_0(X)$-algebra A with $x \in X_\phi$ such that H_x is not strictly closed in $M(A)$.* Let $A = C_0([0,1) \cup [2,3])$ and set $X = [0,1]$. Then we may identify $\mathrm{Prim}(A)$ with $[0,1) \cup [2,3]$. Let $\phi : \mathrm{Prim}(A) \to X$ be given by $\phi(x) = x$ $(0 \le x < 1)$ and $\phi(x) = 1$ $(2 \le x \le 3)$. Then J_x is locally modular for all $x \in X$, but ϕ is not locally closed at $x = 1$. Hence H_1 is not strictly closed in $M(A)$.

5. $C_0(X)$-algebras where the base map ϕ is the complete regularization map

In this section we investigate $C_0(X)$-algebras A where the base map ϕ is the complete regularization map on Prim(A) [18, Theorem 3.9]. Thus X may be taken to be the complete regularization space (in cases where this is locally compact) or its Stone-Čech compactification. Restricting ϕ in this way places a considerable constraint on its behaviour, as we shall see. This case is of special interest for two reasons: firstly, the complete regularization map interacts with the topology on Prim(A) in a way that is lacking with more general continuous maps, and secondly, every continuous map from Prim(A) to a locally compact Hausdorff space factors through the complete regularization map.

Under the hypotheses that ϕ is the complete regularization map and that $\mathrm{Orc}(A) < \infty$ (a technical assumption which is usually satisfied), we show that if A is σ-unital and J_x is locally modular then ϕ is locally closed at x and H_x is strictly closed (Theorem 5.3). Thus the 'locally closed at x' part of the hypothesis in the final sentence of Proposition 3.4 is automatically satisfied in this case (contrast with Example 4.8(ii)).

We begin by explaining the notation Orc(A) [33]. Recall that for a C*-algebra A and for $P, Q \in \mathrm{Prim}(A)$ we write $P \sim Q$ if P and Q cannot be separated by disjoint open sets in Prim(A). The relation $\sim$ on Prim(A) induces a graph structure on Prim(A) whereby P and Q are adjacent if $P \sim Q$. The distance $d_A(P, Q)$ between P and Q is then defined as the length of the shortest path from P to Q (and is ∞ if no such path exists). The diameter of a $\sim$-component of Prim(A) is the supremum of the distances between primitive ideals in the component (with the convention that a singleton component, such as when Prim(A) is Hausdorff, has diameter 1). The *connecting order*, Orc(A), is the supremum of the diameters of $\sim$-components of Prim(A). Clearly Orc(A) is an integer between 1 and ∞, and all possibilities occur, including ∞ [33, Example 2.8] (see also Example 6.4(ii) below). The smaller that Orc(A) is, the nearer Prim(A) is to being Hausdorff, with the case $\mathrm{Orc}(A) = 1$ corresponding to $\sim$ being an equivalence relation on Prim(A). For a subset $Y \subseteq \mathrm{Prim}(A)$ and for $n \geq 0$, let $Y^n = \{P \in \mathrm{Prim}(A) : \exists\, Q \in Y,\ d_A(P, Q) \leq n\}$.

We also need the following topological lemma characterizing separation by open sets. We say that a topological space is *locally compact* if every point has a neighbourhood base of compact sets.

Lemma 5.1. *Let X be a locally compact topological space and let Y and Z be subsets of X which are Lindelof in the relative topology. Then the following are equivalent:*
(i) *The closure of Y^1 does not meet Z and the closure of Z^1 does not meet Y.*
(ii) *There exist disjoint open subsets U and V of X with $Y \subseteq U$ and $Z \subseteq V$.*

Proof. Suppose first that (ii) holds. Then $X \setminus V$ is disjoint from Z and is a closed set containing the neighbourhood U of Y and hence containing Y^1. Similarly $X \setminus U$ is disjoint from Y and is a closed set containing Z^1. Hence (i) holds.

Conversely, suppose that (i) holds. Let $x \in Y$. Since the closure of Z^1 does not meet Y, x has an open neighbourhood disjoint from Z^1. Hence, by the local compactness of X, x has a compact neighbourhood $U_x \subseteq X \setminus Z^1$. Then U_x^1 is closed because U_x is compact, and U_x^1 does not meet Z because U_x does not meet Z^1. Similarly for each $x \in Z$ there exists a neighbourhood V_x of x such that the closure of V_x does not meet Y.

Since Y and Z are Lindelof, we may obtain a countable collection, say $U_1, U_2, \ldots$, of the sets Int U_x such that $\bigcup_{i=1}^{\infty} U_i$ covers Y, and likewise a countable collection $V_1, V_2, \ldots$

of the sets Int V_x such that $\bigcup_{i=1}^{\infty} V_i$ covers Z. For each $i \geq 1$, set $U_i' = U_i \setminus \left(\bigcup_{j=1}^{i} \overline{V}_j\right)$ and $V_i' = V_i \setminus \left(\bigcup_{j=1}^{i} \overline{U}_j\right)$ (where $\overline{V}_i$ denotes the closure of V_i, etc). Set $U = \bigcup_{i=1}^{\infty} U_i'$ and $V = \bigcup_{i=1}^{\infty} V_i'$. Then it is easily checked that $Y \subseteq U$ and $Z \subseteq V$. If $x \in U \cap V$ then there exist U_i' and V_j' such that $x \in U_i' \cap V_j'$. Without loss of generality we may suppose that $i \geq j$. But then $V_j' \subseteq V_j$ which is disjoint from U_i', a contradiction. Hence U and V are disjoint, and thus (ii) holds. □

Now let A be a σ-unital $C_0(X)$-algebra. Let E be a non-empty closed subset of X_ϕ, set $Y = X_\phi/E$ and let $q : X_\phi \to Y$ be the quotient map. Set $\psi = q \circ \phi$ and $\{e\} = q(E)$. We saw in Proposition 2.9 that the question of spectral synthesis for the set E can be reduced to that of the point e, and for this reason we have previously confined ourselves to considering spectral synthesis at points. If we restrict ϕ to be the complete regularization map, however, then we can no longer make this reduction (because the reduction changes the base map), and we will therefore have to work with closed sets in this section.

With this in mind, and with the notation above, we say that ϕ is *locally closed at E* if ψ is locally closed at e, and that J_E is *locally modular* if J_e is locally modular. Elementary topological arguments show that ϕ is locally closed at E if and only if whenever Y is a closed subset of Prim(A) such that $\phi(Y) \cap E = \emptyset$ (i.e. $Y \cap \phi^{-1}(E) = \emptyset$) then $\overline{\phi(Y)} \cap E = \emptyset$. Note that $H(e) = (q \circ \phi)^{-1}(e) = \phi^{-1}(E)$, the hull of J_E in Prim(A), and recall that $U(e)$ is the interior of $H(e)$ in Prim(A). For $Q, R \in \mathrm{Prim}(M(A)) \setminus \tilde{U}(e)$, recall that we write $Q \sim_e R$ if there is a net (P_α) in $\mathrm{Prim}(A) \setminus U(e)$ such that $\tilde{P}_\alpha \to Q, R$.

Proposition 5.2. *Let A be a σ-unital $C_0(X)$-algebra with base map ϕ and let E be a non-empty closed subset of X_ϕ. Consider the following conditions*
(i) *H_E is strictly closed.*
(ii) *J_E is locally modular and ϕ is locally closed at E.*
Then (ii)⇒(i)*, and* (i) *and* (ii) *are equivalent if A is separable.*

Proof. With the notation above, we have $J_e = J_E$ and $H_e = H_E$ by Proposition 2.9. Hence (ii)⇒(i) follows immediately from Proposition 3.4.

Conversely, suppose that A is separable and that (i) holds. Then H_e is strictly closed by Proposition 2.9 and it follows from Corollary 3.10 that $J_e = J_E$ is locally modular and $q \circ \phi$ is locally closed at e. Hence ϕ is locally closed at E by definition. □

We are now ready for the main theorem of the section.

Theorem 5.3. *Let A be a σ-unital $C_0(X)$-algebra and suppose that ϕ is the complete regularization map for* Prim(A) *and that* Orc(A) $< \infty$. *Let E be a non-empty closed subset of X_ϕ and suppose that J_E is locally modular. Then ϕ is locally closed at E and H_E is strictly closed.*

Proof. As before, let $q : X_\phi \to X_\phi/E$ be the quotient map and $\{e\} = q(E)$. By Proposition 5.2, it suffices to show that ϕ is locally closed at E. Let V be a proper open subset of Prim(A) with $V \supseteq H(e)$. It suffices to produce a continuous function g on Prim(A) with $g(\mathrm{Prim}(A) \setminus V) = \{1\}$ and $g(H(e)) = \{0\}$ (for then g induces a continuous function on $\phi(\mathrm{Prim}(A))$ separating $\phi(\mathrm{Prim}(A) \setminus V)$ and E, so that $\overline{\phi(\mathrm{Prim}(A) \setminus V)} \cap E = \emptyset$).

Set $Y = \{Q \in \mathrm{Prim}(A) \setminus U(e) : \exists R \in \mathrm{Prim}(M(A)/A) \text{ with } \tilde{Q} \sim_e R\}$. Then $Y \cap H(e)$ is empty by Lemma 3.3. We claim that Y is a closed subset of Prim(A). To see this, let

(Q_α) be a net in Y with $Q_\alpha \to Q \in \mathrm{Prim}(A)$. Then $Q \notin U(e)$ since $\mathrm{Prim}(A) \setminus U(e)$ is closed. By definition of Y, there is a net (R_α) in $\mathrm{Prim}(M(A)/A)$ with $\tilde{Q}_\alpha \sim_e R_\alpha$ for each α. By the compactness of $\mathrm{Prim}(M(A)/A)$, there is a subnet (R_β) of (R_α) with $R_\beta \to R \in \mathrm{Prim}(M(A)/A)$. Then $\tilde{Q} \sim_e R$ and so $Q \in Y$, as required.

Next we note that Y^1 is closed in $\mathrm{Prim}(A)$. To see this, let (Q_α) be a net in Y^1 with $Q_\alpha \to Q \in \mathrm{Prim}(A)$. Then $Q \notin U(e)$, for otherwise eventually $Q_\alpha \in U(e) \subseteq H(e)$, which is impossible since $H(e)$ is a $\sim$-saturated set disjoint from Y. Let (P_α) be a net in Y such that $P_\alpha \sim Q_\alpha$ for each α. Since $P_\alpha, Q_\alpha \notin H(e)$, $\tilde{P}_\alpha \sim_e \tilde{Q}_\alpha$. By the compactness of $\mathrm{Prim}(M(A))$ there exists $R \in \mathrm{Prim}(M(A))$ and a subnet (P_β) of (P_α) such that $\tilde{P}_\beta \to R$. Hence $R \sim_e \tilde{Q}$. If $R = \tilde{S}$ for some $S \in Y$ then $Q \in Y^1$ as required. Otherwise $R \in \mathrm{Prim}(M(A)/A)$ and hence $Q \in Y \subseteq Y^1$. Thus Y^1 is closed.

Since A is σ-unital, $\mathrm{Prim}(A)$ is σ-compact and hence every closed subset of $\mathrm{Prim}(A)$ is a Lindelof space. Thus, since $H(e)$ is closed and $\sim$-saturated, we may apply Lemma 5.1 to $H(e)$ and Y to obtain disjoint open subsets V' and V'' of $\mathrm{Prim}(A)$ with $H(e) \subseteq V'$ and $Y \subseteq V''$. By intersecting V' with the open set V of the first paragraph, we may assume that $V' \subseteq V$. Set $Z = \mathrm{Prim}(A) \setminus V'$. Then $Z \supseteq V'' \supseteq Y$ and the boundary of Z in $\mathrm{Prim}(A)$ does not meet V''. Let $k = \mathrm{Orc}(A)$. Then the same argument that showed that Y^1 is closed also shows, inductively, that $Z^1, \ldots, Z^k$ are closed. Note that if Q belongs to the boundary of Z^k in $\mathrm{Prim}(A)$ then $Q \notin V''$ and so $Q \notin Y$. Thus there does not exist $R \in \mathrm{Prim}(M(A)/A)$ with $\tilde{Q} \sim_e R$.

Note that the disjoint sets Z^k and $H(e)$ are $\sim$-saturated and hence so is the set $F := \mathrm{Prim}(A) \setminus (Z^k \cup H(e))$. We now define an equivalence relation $\diamond$ on Prim(A) as follows. The $\diamond$-equivalence classes are: Z^k, $H(e)$, and the $\sim$-components of F. Set $W = \mathrm{Prim}(A)/\diamond$, equipped with the quotient topology, and let $p : \mathrm{Prim}(A) \to W$ be the quotient map. We show that p is a closed map. Let T be a closed subset of $\mathrm{Prim}(A)$ and set $T' = p^{-1}(p(T))$. Let (Q_α) be a net in T' with a limit $Q \in \mathrm{Prim}(A)$. We must show that $Q \in T'$. For each Q_α there exists $R_\alpha \in T$ such that $Q_\alpha \diamond R_\alpha$, and by the compactness of $\mathrm{Prim}(M(A))$ and by passing to subnets of (Q_α) and (R_α), if necessary, we may assume that there exists $R \in \mathrm{Prim}(M(A))$ with $\tilde{R}_\alpha \to R$.

We consider various cases. If (Q_α) is frequently in Z^k then $Q \in Z^k$, since Z^k is closed. Hence $Q \diamond Q_\alpha$ for $Q_\alpha \in Z^k$ and so $Q \in T'$ since T' is $\diamond$-saturated. A similar argument shows that $Q \in T'$ if (Q_α) is frequently in $H(e)$. Hence we may restrict attention to the case when $Q_\alpha \in F$ for all α. This implies that $d_A(Q_\alpha, R_\alpha) \leq k$ and hence that for each α there is a walk $Q_\alpha \sim Q^1_\alpha \sim \ldots \sim Q^k_\alpha = R_\alpha$ (possibly with repetitions) of length k between Q_α and R_α. Hence $\tilde{Q}_\alpha \sim_e \tilde{Q}^1_\alpha \sim_e \ldots \sim_e \tilde{Q}^k_\alpha = \tilde{R}_\alpha$. Using the compactness of $\mathrm{Prim}(M(A)) \setminus \tilde{U}(e)$, and passing to successive subnets, we obtain a walk $\tilde{Q} \sim_e Q^1 \sim_e \ldots \sim_e Q^k = R$ of length k in $\mathrm{Prim}(M(A)) \setminus \tilde{U}(e)$ such that $\tilde{Q}, Q^1, \ldots, Q^k$ all lie in $(\tilde{F})^-$, the closure of $\tilde{F}$ in $\mathrm{Prim}(M(A))$.

Suppose that $P \in F^-$ (the closure of F in $\mathrm{Prim}(A)$) and that $\tilde{P} \sim_e P'$ for some $P' \in (\tilde{F})^-$. Then $P \notin U(e)$ and $P \notin Y$ and hence, by the definition of Y, $P' = \tilde{S}$ for some $S \in \mathrm{Prim}(A)$. Furthermore, $S \in F^-$ because $\tilde{S} \in (\tilde{F})^-$. Since $Q \in F^-$, it follows by induction that $Q^i = \tilde{S}_i$ $(1 \leq i \leq k)$ for some $S_i \in \mathrm{Prim}(A)$, and hence that $Q \diamond S_k$. But $S_k \in T$, since T is closed in $\mathrm{Prim}(A)$, and hence $Q \in T'$ as required. Thus we have shown that p is a closed map.

Now let C and D be non-empty, disjoint closed subsets of W. Then $C' := p^{-1}(C)$ and $D' := p^{-1}(D)$ are disjoint closed $\sim$-saturated subsets of $\mathrm{Prim}(A)$. Thus C' and D' are

Lindelof and so Lemma 5.1 implies the existence of disjoint open sets E' and F' containing C' and D' respectively. We now use a standard characterization (see e.g. [24, §13.XIV Theorem 3]): a quotient map $r : M \to N$ is closed if and only if whenever $d \in N$ and O is an open set containing $r^{-1}(d)$ then there exists a saturated open set H such that $r^{-1}(d) \subseteq H \subseteq O$ (where H is saturated if $H = r^{-1}(r(H))$). Applying this characterization in the present case to each of the points of C and D relative to E' and F' we obtain $\diamond$-saturated open sets E'' and F'' such that $C' \subseteq E'' \subseteq E'$ and $D' \subseteq F'' \subseteq F'$. Hence $p(E'')$ and $p(F'')$ are disjoint open sets of W containing C and D respectively. Thus W is normal.

It follows that there is a positive continuous function f on W with $\|f\| = 1$ such that $f(p(Z^k)) = \{1\}$ and $f(p(H(e))) = \{0\}$. Then $g = f \circ p$ is a continuous function on $\mathrm{Prim}(A)$ with $g(Z^k) = \{1\}$ and $g(H(e)) = \{0\}$. Since $\mathrm{Prim}(A) \setminus V \subseteq Z^k$, g has the property required at the start of the proof. □

Now let A be a $C_0(X)$-algebra with base map ϕ. Then ϕ factors as $\phi = \psi \circ \phi_A$ where ϕ_A is the complete regularization map and ψ is continuous. The advantage of the following result over Proposition 5.2 is that ψ is a map between completely regular spaces and should therefore be simpler to analyze.

Let X_{ϕ_A} denote the image of $\mathrm{Prim}(A)$ under the complete regularization map ϕ_A. Then ψ is a map from $X_{\phi_A} \to X_\phi$. By analogy with our earlier definition, we say that ψ is locally closed at a non-empty subset $E \subseteq X_\phi$ if whenever W is a closed subset of X_{ϕ_A} with $\psi(W) \cap E = \emptyset$ (i.e. $W \cap \psi^{-1}(E) = \emptyset$) then $\overline{\psi(W)} \cap E = \emptyset$.

Corollary 5.4. *Let A be a σ-unital $C_0(X)$-algebra with base map ϕ and suppose that* $\mathrm{Orc}(A) < \infty$*. Write $\phi = \psi \circ \phi_A$ where ϕ_A is the complete regularization map for* $\mathrm{Prim}(A)$*. Let E be a non-empty closed subset of X_ϕ. Consider the following conditions:*

(i) *H_E is strictly closed;*

(ii) *J_E is locally modular and $\psi : X_{\phi_A} \to X_\phi$ is locally closed at E.*

Then (ii)⇒(i), *and* (i) *and* (ii) *are equivalent if A is separable.*

Proof. Set $H(E) = \phi^{-1}(E)$. First, suppose that (ii) holds and let Y be a closed subset of $\mathrm{Prim}(A)$ with $Y \cap H(E)$ empty. Let W be the closure of $\phi_A(Y)$ in X_{ϕ_A}. Then W does not meet $\phi_A(H(E)) = \psi^{-1}(E)$ by Theorem 5.3. Hence $\psi(W) \cap E = \emptyset$ and so the closure of $\psi(W)$ in X_ϕ does not meet E, since ψ is locally closed at E. But $\phi(Y) \subseteq \psi(W)$ and hence $\overline{\phi(Y)} \cap E = \emptyset$. Thus ϕ is locally closed at E and so J_E is strictly closed by Proposition 5.2.

Conversely, suppose that (i) holds and that A is separable. Then J_E is locally modular and ϕ is locally closed at E by Proposition 5.2. Let W be a closed set in X_{ϕ_A} such that $W \cap \psi^{-1}(E)$ is empty. Set $Y = \phi_A^{-1}(W)$. Then Y is closed in $\mathrm{Prim}(A)$ and $Y \cap H(E)$ is empty. Hence the closure of $\phi(Y)$ does not meet E. But $\phi(Y) = \psi(W)$ and hence $\overline{\psi(W)} \cap E = \emptyset$. Thus ψ is locally closed at E. □

In particular, if A in Corollary 5.4 is separable and ψ is a closed map (for example, the identity map when $\phi = \phi_A$) then H_E is strictly closed if and only if J_E is locally modular.

6. Locally modular ideals

In this final section we look at locally modular ideals in the case when ϕ is the complete regularization map and $\mathrm{Orc}(A) < \infty$. We saw immediately after the definition of local modularity that there are two 'easy' ways for J_x to be locally modular: if $x \in U_\phi$ or if x is an isolated point in X_ϕ. Example 4.8 gave two examples where J_x is locally modular with $x \in \partial U_\phi$, and in the first of these $H(x)$ has empty interior in $\mathrm{Prim}(A)$. We will show that such behaviour cannot occur when ϕ is the complete regularization map and $\mathrm{Orc}(A) < \infty$. In this case, if J_x is locally modular then either $x \in U_\phi$ or $H(x)$ has non-empty interior (Corollary 6.3).

Recall that for a C*-algebra A, we say that $P, Q \in \mathrm{Prim}(A)$ belong to the same *Glimm class* if $f(P) = f(Q)$ for all continuous, bounded, real-valued functions f on $\mathrm{Prim}(A)$ (equivalently, $\phi_A(P) = \phi_A(Q)$, where ϕ_A is the complete regularization map on $\mathrm{Prim}(A)$). The algebra $A + Z(M(A))$ in the next result was introduced by Dixmier [11].

Lemma 6.1. *Let A be a C*-algebra with* $\mathrm{Orc}(A) < \infty$ *and let $C = A + Z(M(A))$. Then* $\mathrm{Orc}(C) \leq 2\ \mathrm{Orc}(A) + 2$.

Proof. First note that A is an essential ideal in C, so that $\mathrm{Prim}(A)$ is (homeomorphic to) a dense open subset of $\mathrm{Prim}(C)$. Suppose that Q_1, Q_2 are distinct elements of $\mathrm{Prim}(C) \setminus \mathrm{Prim}(A)$. Then $Q_i = M_i + A$ where M_i is a maximal ideal of $Z(M(A))$ containing $A \cap Z$ $(i = 1, 2)$. It follows that $Z(M(A))$ separates Q_1 and Q_2. Thus each Glimm class in $\mathrm{Prim}(C)$ contains at most one element of $\mathrm{Prim}(C) \setminus \mathrm{Prim}(A)$. Hence if $P_1 \sim P_2 \sim \ldots \sim P_n$ is a path in $\mathrm{Prim}(C)$ then at most one element from $\mathrm{Prim}(C) \setminus \mathrm{Prim}(A)$ can occur among the P_i's. It follows that $d_C(P_1, P_n) \leq 2\ \mathrm{Orc}(A) + 2$ as required. □

The next theorem is a general result giving a dichotomy for $\sim$-components in $\mathrm{Prim}(A)$ for any C^*-algebra A for which $\mathrm{Orc}(A) < \infty$.

Theorem 6.2. *Let A be a $C_0(X)$-algebra and suppose that ϕ is the complete regularization map for* $\mathrm{Prim}(A)$ *and that* $\mathrm{Orc}(A) < \infty$. *Let T be a $\sim$-component of* $\mathrm{Prim}(A)$, *so that $T \subseteq H(x)$ for some $x \in X_\phi$. Then either*

(i) $J_x \not\supseteq Z(A)$ *and* $T = H(x)$; *or*

(ii) $J_x \supseteq Z(A)$ *and there exist* $P \in T$ *and* $R \in \mathrm{Prim}(M(A))$ *with* $R \supseteq A$ *such that* $\tilde{P} \sim R$.

Proof. Suppose that $P \in T$ and $R \in \mathrm{Prim}(M(A))$ with $R \supseteq A$ and $\tilde{P} \sim R$. Then $\overline{\phi}(R) = \overline{\phi}(\tilde{P}) = \phi(P) = x$ and it follows from Lemma 2.5 that $J_x \supseteq Z(A)$. We must show, therefore, that if there do not exist $P \in T$ and $R \in \mathrm{Prim}(M(A))$ with $R \supseteq A$ such that $\tilde{P} \sim R$ then alternative (i) holds.

Suppose, then, that for all $P \in T$ and $R \in \mathrm{Prim}(M(A))$ with $R \supseteq A$, $\tilde{P} \not\sim R$. Set $k = \mathrm{Orc}(A)$, and let $Q \in T$. Note that $\tilde{T} = \{R \in \mathrm{Prim}(M(A)) : d_{M(A)}(\tilde{Q}, R) \leq k\}$, by the supposition that for all $P \in T$ and $R \in \mathrm{Prim}(M(A))$ with $R \supseteq A$, $\tilde{P} \not\sim R$. Hence $\tilde{T}$ is a closed (and thus compact) subset of $\mathrm{Prim}(M(A))$ by [33, Corollary 2.3] applied k times. Set $L = \ker \tilde{T}$, a closed ideal of $M(A)$. If $A + L$ were a proper ideal of $M(A)$ there would exist $R \in \mathrm{Prim}(M(A))$ such that $R \supseteq A + L$. Hence $R \supseteq L$ and so $R \in \tilde{T}$ since $\tilde{T}$ is a closed subset of $\mathrm{Prim}(M(A))$, but also $R \supseteq A$. This is a contradiction, and hence $A + L = M(A)$.

Now set $C = A + Z(M(A))$, and for each $P \in \mathrm{Prim}(A)$ let P' be the unique primitive ideal in C such that $P' \cap A = P$.

Let $T' := \{P' : P \in T\}$. We claim that $K \not\sim P'$ whenever $P \in T$ and $K \in \mathrm{Prim}(C)$ with $K \supseteq A$. Supposing otherwise, there exist $P \in T$, $K \in \mathrm{Prim}(C)$ with $K \supseteq A$, and a net (P_α) in $\mathrm{Prim}(A)$ such that $P'_\alpha \to P'$ and $P'_\alpha \to K$. Since $K \supseteq A$ and $C \subseteq M(A) = A + L$, $K \not\supseteq L \cap C$. Hence there exists $c \in L \cap C$ such that $\|c + K\| = 1$. By lower semi-continuity, $\|c + P'_\alpha\| \geq 1/2$ eventually. Hence $\|c + \tilde{P}_\alpha\| \geq 1/2$ eventually (because both $\|c + P'_\alpha\|$ and

$\|c + \tilde{P}_\alpha\|$ are equal to $\|\overline{\pi}_\alpha(c)\|$ where π_α is an irreducible representation of A with kernel P_α and $\overline{\pi}_\alpha$ is its unique ultra-weakly continuous extension to A^{**}). By the compactness of the set $\{S \in \mathrm{Prim}(M(A)) : \|c + S\| \geq 1/2\}$, we may assume, without loss of generality, that $\tilde{P}_\alpha \to R$ in $\mathrm{Prim}(M(A))$, where $\|c + R\| \geq 1/2$.

Since $P'_\alpha \to P'$ we have $P_\alpha \to P$ and $\tilde{P}_\alpha \to \tilde{P}$. Thus $\tilde{P} \sim R$. By the supposition of the second paragraph, it must be that $R = \tilde{S}$ for some $S \in \mathrm{Prim}(A)$. Hence $S \in T$. Thus we have $c \in L \subseteq \tilde{S} = R$, contradicting the fact that $\|c+R\| \geq 1/2$. It follows, then, that $K \not\sim P'$ whenever $P \in T$ and $K \in \mathrm{Prim}(C)$ with $K \supseteq A$, and hence that T' is a $\sim$-component in $\mathrm{Prim}(C)$.

By Lemma 6.1, $\mathrm{Orc}(C) \leq 2\ \mathrm{Orc}(A) + 2 < \infty$, and since $\mathrm{Prim}(C)$ is compact it follows that T' is a Glimm class in $\mathrm{Prim}(C)$ [33, Corollary 2.7]. It follows at once that T is a Glimm class in $\mathrm{Prim}(A)$. Thus $T = H(x)$. Now, let $P \in T$, and let $\phi_C : \mathrm{Prim}(C) \to Y$ be the complete regularization map for $\mathrm{Prim}(C)$, where Y is the space of Glimm ideals of C with the induced completely regular topology. Then the set $W = \phi_C(\mathrm{Prim}(C) \setminus \mathrm{Prim}(A))$ is compact and hence closed in Y, and does not contain the point $\phi_C(P)$. Thus there exists a continuous function f on Y taking the value 1 on $\phi_C(P')$ and 0 on W. Since f vanishes on W, it follows that the central element z of C induced by f actually belongs to A. But $z \notin P'$, and $P' \supseteq P \supseteq J_x$, and thus $z \notin J_x$. Hence $J_x \not\supseteq Z(A)$. □

Corollary 6.3. *Let A be a $C_0(X)$-algebra and suppose that ϕ is the complete regularization map for* $\mathrm{Prim}(A)$ *and that* $\mathrm{Orc}(A) < \infty$. *Let $x \in X_\phi$ with J_x locally modular. Then either $x \in U_\phi$ or $H(x)$ has non-empty interior.*

Proof. Suppose that $x \notin U_\phi$ and that $H(x)$ has empty interior. Let T be a $\sim$-component of $H(x)$. By Theorem 6.2 there exists $P \in T$ and $R \in \mathrm{Prim}(M(A)/A)$ such that $\tilde{P} \sim R$. Since $H(x)$ has empty interior, $P \in \partial H(x)$ and $\tilde{P} \sim_x R$. By Lemma 3.3, J_x is not locally modular. □

We conclude with two examples of $x \in \partial U_\phi$ with J_x locally modular. The first has $\mathrm{Orc}(A) < \infty$ and $H(x)$ with non-empty interior. The second has $\mathrm{Orc}(A) = \infty$ and $H(x)$ with empty interior, showing that the condition $\mathrm{Orc}(A) < \infty$ in Corollary 6.3 is not redundant.

Example 6.4. (i) *A $C_0(X)$-algebra with $z \in \partial U_\phi$ such that J_z is locally modular, ϕ is locally closed at z, and $H(z)$ has non-empty interior.*

As in Example 4.8(i), let $Y = \{(x,y) \in \mathbf{R}^2 : y \geq 0\}$ be the upper half-plane, and let $L = \{(x,y) \in Y : y = 0\}$ be the x-axis. Set $B = C_0(Y)$ and $C = C_0(L)$, and let $\pi : B \to C$ be the surjective $*$-homomorphism given by $\pi(b) = b|_L$ $(b \in B)$. Let H be a separable, infinite-dimensional Hilbert space, $B(H)$ the algebra of bounded operators on H, and $K(H)$ the algebra of compact operators on H. Let $\rho : C \to B(H)$ be a *-monomorphism such that $\rho(C) \cap K(H) = \{0\}$. Set $D = \rho(C) + K(H)$, a C*-subalgebra of $B(H)$. Note that each element $d \in D$ can be uniquely expressed in the form $d = g + T$ where $g \in \rho(C)$ and $T \in K(H)$.

Set $A = \{(b,d) \in B \oplus D : \rho(\pi(b)) = g\}$. Then A is separable. For $(x,y) \in Y$, let $\theta_{x,y}$ be the character on A given by $\theta_{x,y}(b,d) = b((x,y))$. Set $G = \{(b,d) \in A : \pi(b) = 0,\ T = 0\}$. Since any irreducible representation of A extends to an irreducible representation of $B \oplus D$ (on a possibly larger Hilbert space), $\mathrm{Prim}(A) = \{\ker\theta_{x,y} : (x,y) \in Y\} \cup \{G\}$. Note that $G \subseteq \ker\theta_{x,0}$ for all $x \in R$. It follows that a subset W of $\mathrm{Prim}(A)$ is closed if and only if (i)

$\{(x,y) \in Y : \ker\theta_{x,y} \in W\} \cap Y$ is closed in Y (with the usual topology), and (ii) if $G \in W$ then $\ker\theta_{x,0} \in W$ for all $x \in \mathbf{R}$. In particular $\{G\}$ is an open subset of $\mathrm{Prim}(A)$.

Set $X_\phi = Y/L$ and let $q : Y \to X_\phi$ be the quotient map. Set $X = \beta X_\phi$. Define $\phi : \mathrm{Prim}(A) \to X_\phi \subseteq X$ by $\phi(\theta_{x,y}) = q(x,y)$ $((x,y) \in Y)$ and $\phi(G) = q(0,0)$. Then ϕ is the complete regularization map for $\mathrm{Prim}(A)$ and $\phi(G)$ is non-isolated in X_ϕ. For $(x,y) \in Y \setminus L$, $J_{q(x,y)} = \ker\theta_{x,y}$ while $J_{q(0,0)} = G$. Each point of Y has a compact neighbourhood in Y and hence J_x is locally modular for each $x \in X_\phi$, although A/G is non-unital. Since Y is normal, the map ϕ is easily seen to be relatively closed and hence H_x is strictly closed for each $x \in X_\phi$ by Proposition 3.4. Taking $z = q(0,0) = q(G)$, however, we have that

$$H(q(0,0)) = \{\ker\theta_{x,0} : x \in \mathbf{R}\} \cup \{G\}$$

and this has non-empty interior $\{G\}$.

(ii) *A $C_0(X)$-algebra with $x \in \partial U_\phi$ such that J_x is locally modular, ϕ is locally closed at x, and $H(x)$ has empty interior.*

Let A be the C*-algebra defined as follows (see [33, Example 2.8]). Let B be the C*-algebra consisting of all continuous functions from the interval $[0,1]$ into the 2×2 complex matrices. Let $B(1)$ be the C*-subalgebra of B consisting of those functions $f \in B$ satisfying $f(2^{-n}) = \mathrm{diag}(\lambda_{2n-1}(f), \lambda_{2n}(f))$, $(n \geq 1)$, and $f(0) = \mathrm{diag}(\lambda(f), \lambda(f))$, for some complex numbers $\lambda(f)$, $\lambda_n(f)$ $(n \geq 1)$. Let $A = \{f \in B(1) : \lambda_{2n}(f) = \lambda_{2n+1}(f) \quad (1 \leq n < \infty) \text{ and } \lambda(f) = 0\}$. Then A is separable.

For $y \in (0,1] \setminus \{2^{-n} : n \geq 1\}$, set $P_y = \{f \in A : f(y) = (0)\}$. Then

$$\mathrm{Prim}(A) = \{P_y : y \in (0,1] \setminus \{2^{-n} : n \geq 1\}\} \cup \{\ker(\lambda_i) : i = 1,3,5,\ldots\}.$$

Set $X_\phi = (0,1]/*$ where for $r,s \in (0,1]$, $r * s$ if $r = s$ or if $r,s \in \{2^{-n} : n \geq 1\}$ and let $q : (0,1] \to X_\phi$ denote the quotient map. Set $X = \beta X_\phi$ and $\infty = q(1/2)$. Define $\phi : \mathrm{Prim}(A) \to X$ by $\phi(P_y) = y$ $(y \in (0,1] \setminus \{2^{-n} : n \geq 1\})$ and $\phi(\lambda_i) = \infty$ $(i = 1,3,5,\ldots)$. Then ϕ is the complete regularization map for $\mathrm{Prim}(A)$.

If $x \in X_\phi \setminus \{\infty\}$ then $J_x \not\supseteq Z(A) = Z'(A)$ and hence $x \in U_\phi$ and J_x is locally modular. It is easy to see directly that J_∞ is also locally modular. But A/J_∞ is non-unital, since $\mathrm{Prim}(A/J_\infty) = \{\ker(\lambda_i) : i = 1,3,5,\ldots\}$ is non-compact, and hence $J_\infty \supseteq Z(A)$. Thus $U_\phi = X_\phi \setminus \{\infty\}$ and $W_\phi = \{\infty\}$. We show that ϕ is a relatively closed map. Let Y be a closed subset of $\mathrm{Prim}(A)$ and set $Y' = \phi^{-1}(\phi(Y))$. Then $Y' = Y$ if $Y \cap \{\ker(\lambda_i) : i = 1,3,5,\ldots\}$ is empty, and $Y' = Y \cup \{\ker(\lambda_i) : i = 1,3,5,\ldots\}$ otherwise. In either case Y' is closed, and hence ϕ is relatively closed. It follows, therefore, from Proposition 3.4 that H_x is strictly closed for every $x \in X_\phi$.

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