INFINITARY TABLEAU FOR SEMANTIC TRUTH

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ABSTRACT. We provide infinitary proof theories for three common semantic theories of truth: strong Kleene, van Fraassen supervaluation and Cantini supervaluation. The value of these systems is that they provide an easy method of proving simple facts about semantic theories. Moreover we shall show that they also give us a simpler understanding of the computational complexity of these definitions and provide a direct proof that the closure ordinal for Kripke's definition is ω_1^{CK}. This work can be understood as an effort to provide a proof-theoretic counterpart to Welch's game-theoretic [17].

1. INTRODUCTION

In [8], Kripke introduced fixed point theories of truth to the philosophy of language. The definitions of these truth predicates are conducted using transfinite recursion and the theory of inductive definitions. While such theories are well understood in mathematical logic [10], the resultant definitions are complicated both in terms of heuristics and computation. Proofs about membership in these fixed points are conducted informally in the metalanguage and are often contingent on a series of lemmas establishing various properties about the fixed point in question. This sort of reasoning is analogous to the way one may reason about a modal logic using its semantics. With modal logic, however, we usually also have a proof theory which, having established soundness and completeness, allows us to establish claims in a simple and transparent manner.

The aim of this paper is to provide a similarly simple and transparent means of verifying simple claims about the extension of the truth predicate. To do this we shall make use of infinitary tableau systems. We shall then establish that each of the systems provided is sound and complete with respect to their associated fixed points. The paper is broken into a section for each of the tableau systems developed. After this, we prove that each system gives a Π^1_1-complete set as its intended extension and provide a direct proof showing that the height of the fixed point is ω_1^{CK}.

1.1. **Semantic theories of truth.** In this section, we define each of the fixed point truth definitions used in this paper, although we shall assume some familiarity with the basic construction [8]. We restrict ourselves to providing a truth definition for the standard model

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of arithmetic, $\mathbb{N}$ and we assume that we are in the language $\mathcal{L}_T$ of arithmetic expanded with a predicate T intended to represent truth. The language of arithmetic will be denoted as $\mathcal{L}$. Let $Sent_{\mathcal{L}}$ and $Sent_{\mathcal{L}_T}$ denote the sentences of $\mathcal{L}$ and $\mathcal{L}_T$ respectively. We assume that we have a recursive bijection $\ulcorner \cdot \urcorner : Sent_{\mathcal{L}_T} \cong \omega$. We use φ, ψ, χ as variables for sentences from $Sent_{\mathcal{L}_T}$ in the metalanguage.

1.1.1. *Strong Kleene.* Let Φ stand for a pair of sets of sentences $\langle \Phi^+, \Phi^- \rangle$ which we shall call the *extension* and *anti-extension* respectively. These pairs will play the role of *guesses* or *approximations* of the truth predicate's intended meaning. We let $AAtom$ stand for a recursive predicate which is satisfied by arithmetic atomic sentences. Let $AArith$ be a recursive predicate which is satisfied by true arithmetic atomic sentences. We shall write $\psi_x(\underline{\mathrm{n}})$ to mean the sentence obtained by substituting the numeral $\underline{\mathrm{n}}$ (which represents $n \in \omega$) for the variable x in the formula ψ. In most cases, we shall suppress the $_x$ as this should result in no confusion.

Definition 1. We define the partial function $Val : (\mathcal{P}(Sent_{\mathcal{L}_T}) \times \mathcal{P}(Sent_{\mathcal{L}_T})) \times Sent_{\mathcal{L}_T} \rightharpoonup 2$ by recursion on the complexity of sentences as follows:[1]

$$
\begin{aligned}
Val_\Phi(\varphi) = 1 \quad \text{iff} \quad & (\varphi \in AAtom \wedge \varphi \in AArith) \vee \\
& (\varphi := T\ulcorner \psi \urcorner \wedge \psi \in \Phi^+) \vee \\
& (\varphi := (\neg \psi) \wedge Val_\Phi(\psi) = 0) \vee \\
& (\varphi := (\psi \wedge \chi) \wedge Val_\Phi(\psi) = 1 \wedge Val_\Phi(\chi) = 1) \vee \\
& (\varphi := (\forall x \psi) \wedge \forall n \in \omega \; Val_\Phi(\psi_x(\underline{\mathrm{n}})) = 1))
\end{aligned}
$$

$$
\begin{aligned}
Val_\Phi(\varphi) = 0 \quad \text{iff} \quad & (\varphi \in AAtom \wedge \varphi \notin AArith) \vee \\
& (\varphi := T\ulcorner \psi \urcorner \wedge \psi \in \Phi^-) \vee \\
& (\varphi := (\neg \psi) \wedge Val_\Phi(\psi) = 1) \vee \\
& (\varphi := (\psi \wedge \chi) \wedge (Val_\Phi(\psi) = 0 \vee Val_\Phi(\chi) = 0) \vee \\
& (\varphi := (\forall x \psi) \wedge \exists n \in \omega \; Val_\Phi(\psi_x(\underline{\mathrm{n}})) = 0))
\end{aligned}
$$

Informally speaking, this function takes a guess and a sentence and gives us the semantic value of that sentence according to that particular guess. Note that the function defined above is not total. For example, suppose $\Phi = \langle \emptyset, \emptyset \rangle$ and consider the sentence $T\ulcorner 0 = 1 \urcorner$. Since $0 = 1$ is in neither the extension nor the anti-extension, $Val_\Phi(T\ulcorner 0 = 1 \urcorner)$ is not defined.

We now define the so-called jump function which provides the inductive engine behind the definition.

[1]We shall be somewhat sloppy in the metalanguage. For example, we are using the symbol '$\wedge$' in both the object language and the metalanguage. In an effort to avoid confusion, we shall use parentheses for emphasis. Thus we write $\varphi := (\neg \psi)$ to indicate that the sentence denoted by φ is identical to that denoted by $\neg \psi$.

Definition 2. Let $j_{sK} : \mathcal{P}(Sent_{\mathcal{L}_T}) \times \mathcal{P}(Sent_{\mathcal{L}_T}) \to \mathcal{P}(Sent_{\mathcal{L}_T}) \times \mathcal{P}(Sent_{\mathcal{L}_T})$ be such that

$$j_{sK}(\Phi) = \langle \{\varphi \mid Val_\Phi(\varphi) = 1\}, \{\varphi \mid Val_\Phi(\varphi) = 0\} \rangle$$

Intuitively, j_{sK} takes one guess at the extension and anti-extension of the truth predicate and returns the set of sentences which would be evaluated as true according to that guess. For example, if $1 = 0$ was in Φ, then $T\ulcorner 1 = 0 \urcorner$ would be in $j_{sK}(\Phi)$. We shall say that a set Φ is *sound* if $\Phi \subseteq j_{sK}(\Phi)$.

Definition 3. We define the intended interpretation, Γ_{sK} for the truth predicate by transfinite recursion:

$$\begin{aligned}
\Gamma_0 &:= \langle \emptyset, \emptyset \rangle \\
\Gamma_{\alpha+1} &:= j_{sK}(\Gamma_\alpha) \\
\Gamma_\beta &:= \langle \bigcup_{\alpha<\beta} \Gamma^+_\alpha, \bigcup_{\alpha<\beta} \Gamma^-_\alpha \rangle \text{ for limit } \beta
\end{aligned}$$

Let $\Gamma_{sK} := \Gamma_\alpha$ for the least α such that $\Gamma_\alpha = \Gamma_{\alpha+1}$.

We are guaranteed that a fixed point will exist because it can be shown that the jump function is monotonic [3, 10]. Moreover, we should note that other, larger fixed points can be defined by starting with a different (although sound in the sense defined above) set at the base of the construction. The definition we offer here is known as the *minimal fixed point* and it is the only one that we shall be concerned with in this paper.

1.1.2. *Supervaluation.* We now define the supervaluational truth definitions which employ the van Fraassen scheme and Cantini schemes. We shall now only consider *classical* interpretations for the truth predicate for which each $\Phi = \langle \Phi^+, \Phi^- \rangle$ is such that $\Phi^+ \cup \Phi^- = Sent_{\mathcal{L}_T}$ and $\Phi^+ \cap \Phi^- = \emptyset$ [3]. For ease of notation, we now let Φ stand for subsets of $Sent_{\mathcal{L}_T}$, thus leaving out the now redundant anti-extension. We let $\dot\neg\Phi = \{(\neg\varphi) \mid \varphi \in \Phi\}$.

Definition 4. Ψ is a vF-expansion of Φ, abbreviated $\Psi \sqsupseteq_{vF} \Phi$ if $\Psi \supseteq \Phi$ and $\Psi \cap \dot\neg\Phi = \emptyset$.

We now define a new jump function j_{vF} which exploits the notion of an expansion. Informally speaking, we are admitting sentences into the extension of the truth predicate if every *safe* (in a way which will soon be described) expansion of the current guess agrees on that sentence. With the van Fraassen expansion, we ensure that we only consider alternative truth extensions that both expand upon what we already know but do not contradict anything we have already learnt. Once again, it takes one guess and returns another which is intended to be an improvement.

Definition 5. Let $j_{vF} : \mathcal{P}(Sent_{\mathcal{L}_T}) \to \mathcal{P}(Sent_{\mathcal{L}_T})$ be such that

$$j_{vF}(\Phi) = \{\varphi \mid \forall \Psi \sqsupseteq_{vF} \Phi \; Val_\Psi(\varphi) = 1\}.$$

Our intended extension of the truth predicate is then defined in much the same way as the strong Kleene definition.

Definition 6.

$$
\begin{aligned}
\Gamma_0 &:= \emptyset \\
\Gamma_{\alpha+1} &:= j_{vF}(\Gamma_\alpha) \\
\Gamma_\beta &:= \bigcup_{\alpha\in\beta} \Gamma_\alpha \text{ for limit } \beta
\end{aligned}
$$

Finally, let Γ_{vF} be Γ_α for the first α such that $\Gamma_{\alpha+1} = \Gamma_\alpha$.

Once again, the jump function is monotonic, so we are guaranteed that there is a fixed point. Γ_{vF} then forms the intended extension of the truth predicate in our standard model.

The Cantini evaluation scheme is defined in a similar fashion; the difference is that we use what we call a Ca-expansion as opposed to a vF-expansion. It is defined as follows:

Definition 7. Ψ is a Ca-expansion, abbreviated $\Psi \sqsupseteq_{Ca} \Phi$ of Φ if $\Psi \supseteq \Phi$ and for all φ it is not the case that $\varphi \in \Psi$ and $(\neg\varphi) \in \Psi$.

Informally speaking, we only consider expansions which agree with what we already know and in addition to this are consistent, in the sense of not containing any sentence and its negation. The rest of the definition is the same as for van Fraassen supervaluation, except we use Ca-expansions rather than vF-expansions. The monotonicity results still hold and we let Γ_{Ca} be the least fixed point of the resultant definition [4]. We observe that none of the definitions are equivalent. Indeed, we have

$$\Gamma_{sK} \subsetneq \Gamma_{vF} \subsetneq \Gamma_{Ca}.$$

To see that $\Gamma_{vF} \nsubseteq \Gamma_{sK}$, let λ be a liar sentence such that $\lambda \leftrightarrow \neg T\ulcorner\lambda\urcorner$. We observe that $\neg(T\ulcorner\lambda\urcorner \wedge \neg T\ulcorner\lambda\urcorner)$ is in Γ_{vF} but not in Γ_{sK}. The reason is that for any sentence (including λ) every classical interpretation will be such that λ is either in the extension or not. But on the other hand, there is no point in the strong Kleene induction at which $\neg(T\ulcorner\lambda\urcorner \wedge \neg T\ulcorner\lambda\urcorner)$ could get into the extension of the truth predicate Γ_{sK}.[2] To see that $\Gamma_{Ca} \nsubseteq \Gamma_{vF}$, consider the sentence $\psi := \neg(T\ulcorner\lambda\urcorner \wedge T\ulcorner\neg\lambda\urcorner)$. We have $\psi \in \Gamma_{Ca}$ but $\psi \notin \Gamma_{vF}$. We demonstrate this in Examples 21 and 27. We leave the inclusions as an exercise for the reader. The reasoning here is informal and too brief. There are a number of other claims that should be established before we reason so informally. This is part of the problem with reasoning directly with the semantic definition. We shall see that simple proofs of these claims can easily be established with the tableau systems offered below.

2. Strong Kleene

We are now ready to provide the framework for the tableau systems. We commence by providing an alternative, more fine-grained definition of the minimal strong Kleene fixed point. This definition can be found in [6] (page 202) and will be easier to work with. In

[2] We shall see this more clearly later, once the tableau system is in place.

contrast to the previous approach, we now treat the truth predicate more like one of the other logical connectives. We thus abandon the Val function and define a jump function h_{sK} which does all the work for all the connectives (including truth) at once.

2.1. **A finer-grained definition.** Our revised jump function is defined as follows:

Definition 8. Let $h_{sK} : \mathcal{P}(Sent_{\mathcal{L}_T}) \to \mathcal{P}(Sent_{\mathcal{L}_T})$ be such that $\chi \in h_{sK}(\Phi)$ iff:

- $\chi \in \Phi$;
- $\chi \in AAtom$ and $\chi \in AArith$;
- there is some φ such that $\ulcorner \chi \urcorner = \ulcorner \neg\varphi \urcorner$, $\varphi \in AAtom$ and $\varphi \notin AArith$;
- there is some φ such that $\ulcorner \chi \urcorner = \ulcorner \neg\neg\varphi \urcorner$ and $\varphi \in \Phi$;
- there are φ, ψ such that $\ulcorner \chi \urcorner = \ulcorner \varphi \wedge \psi \urcorner$ and both $\varphi \in \Phi$ and $\psi \in \Phi$;
- there are φ, ψ such that $\ulcorner \chi \urcorner = \ulcorner \neg(\varphi \wedge \psi) \urcorner$ and either $(\neg\varphi) \in \Phi$ or $(\neg\psi) \in \Phi$;
- there is some φ such that $\ulcorner \chi \urcorner = \ulcorner \forall x \varphi \urcorner$ and for all $n \in \omega$, $\varphi(\underline{\mathrm{n}}) \in \Phi$;
- there is some φ such that $\ulcorner \chi \urcorner = \ulcorner \neg\forall x \varphi \urcorner$ and there is some $n \in \omega$ such that $(\neg\varphi(\underline{\mathrm{n}})) \in \Phi$;
- there is some φ such that $\ulcorner \chi \urcorner = \ulcorner T \ulcorner \varphi \urcorner \urcorner$ and $\varphi \in \Phi$; or
- there is some φ such that $\ulcorner \chi \urcorner = \ulcorner \neg T \ulcorner \varphi \urcorner \urcorner$ and $(\neg\varphi) \in \Phi$.

We might see this definition as a generalisation of our ordinary satisfaction definition, in that it is expanded to incorporate a truth predicate. Then we define our intended interpretation of the truth predicate using an inductive definition. Also note that we have made do here with just an extension and not an extension/anti-extension pair. The negation clause ensures that this suffices with fixed points.

Definition 9. We define the intended extension Ξ_{sK} by transfinite recursion as follows:

$$\begin{aligned} \Xi_0 &:= \emptyset \\ \Xi_{\alpha+1} &:= h_{sK}(\Xi_\alpha) \\ \Xi_\beta &:= \bigcup_{\alpha \in \beta} \Xi_\alpha \text{ for limit } \beta. \end{aligned}$$

Let Ξ_{sK} be Ξ_α for the least α such that $\Xi_\alpha = \Xi_{\alpha+1}$.

We can verify, by inspection, that this jump function is monotonic and thus that we are justified in saying the recursion has a fixed point.

2.2. **Tableau ($\vdash_{sK}$).** We now define our first tableau system. Indeed, we can look to the definition of the h_{sK} to provide us with a guide as to how to do this. The tableau system is intended to capture Γ_{sK}, the extension of the truth predicate in the least fixed point of the three-valued strong Kleene evaluation scheme. The tableau is a tree consisting of sentences from $\mathcal{L}_T$ as nodes. The system we use is much the same as can be found in [15] or [13] except that we use an ω-rule to deal with the quantifiers. An example of a similar tableau system can be found in [16] although it is a trivial transformation of the more common Tait calculus which can be seen in [12].

2.2.1. *Starting the tableau.* To test whether some χ is in Γ, we start a tableau by writing $\neg\chi$. It will form the root of a tree which is constructed downwardly.

2.2.2. *Rules.* In completing the tableau, we must discharge our responsibilities to all of the sentences in the tableau. The rules below tells us what sentences must be added to the tableau in order to discharge those responsibilities.

$$(\wedge)\quad \begin{array}{c} \varphi \wedge \psi \\ | \\ \varphi \\ \psi \end{array} \qquad (\neg\wedge)\quad \begin{array}{c} \neg(\varphi \wedge \psi) \\ \diagup \quad \diagdown \\ \neg\varphi \quad \neg\psi \end{array}$$

$$(\neg\neg)\quad \begin{array}{c} \neg\neg\varphi \\ | \\ \varphi \end{array}$$

$$(\neg\forall)\quad \begin{array}{c} \neg\forall x\varphi \\ \diagup \quad | \quad \diagdown \\ \neg\varphi(0) \quad \neg\varphi(1) \quad \dots \end{array} \qquad (\forall)\quad \begin{array}{c} \forall x\varphi \\ | \\ \varphi(t) \end{array} \quad \text{where } t \text{ is an arbitrary term}$$

$$(T)\quad \begin{array}{c} T\ulcorner\varphi\urcorner \\ | \\ \varphi \end{array} \qquad (\neg T)\quad \begin{array}{c} \neg T\ulcorner\varphi\urcorner \\ | \\ \neg\varphi \end{array}$$

For the remainder of the paper, we shall assume that $\rightarrow$, $\wedge$, $\leftrightarrow$ and $\exists$ have been defined in the usual way; although we shall not mention them explicitly in definitions of proof systems or fixed points. Rules can be provided for them using [15, 16, 13].

Once we have discharged our responsibilities to a sentence, nothing more will be done with that sentence. Note that there will be no finite stage in the construction of a branch at which a $\forall x\varphi(x)$ will be fully discharged: an infinite branch is required for this. We shall say that a sentence is *discharged* if either every rule that can be applied to it has been applied or it is either an arithmetic atomic or negated arithmetic atomic.

We shall say that a branch of a tableau for $\neg\varphi$ is *properly formed for* $\neg\varphi$ if every sentence on the branch is either $\neg\varphi$ or the result of a rule having been applied to a sentence higher on the branch. A branch is *properly formed* if it is properly formed for some sentence ψ.

Proposition 10. *The set of (finite) branches which are properly formed by the rules is recursive.*

2.2.3. *Closing conditions.* A branch in a tableau closes if either: a false arithmetic *atomic* sentence occurs on the branch; or for some sentence $\varphi \in \mathcal{L}$ (i.e. not involving the truth predicate) both φ and $\neg\varphi$ occur on that branch.[3] Once the branch is closed no further sentences may be added to the branch. We say that a branch is open if it is fully discharged and not closed.

If all of the branches in the tableau for $\neg\varphi$ close, then φ is in the intended extension of the truth predicate which we abbreviate $\vdash_{sK} \varphi$.[4]

Fact 11. *For arithmetic sentences $\chi \in \mathcal{L}$ (i.e., not involving a truth predicate), $\vdash_{sK} \chi$ iff χ is true in the standard model of arithmetic (see* [12, 16]*).*

Given this fact, we shall augment our closing conditions for the tableau by adding that a branch also closes if a false arithmetic sentence occurs on that branch.

2.3. **Examples.**

Example 12. $\vdash_{sK} T\ulcorner 1 = 1 \urcorner$.

$$\neg T\ulcorner 1 = 1\urcorner$$
|
$$1 \neq 1$$
×

Let $app : \omega \to \omega$ be defined by primitive recursion in such a way that:

$$\begin{aligned} app(0) &:= \ulcorner 1 = 1 \urcorner \\ app(n+1) &:= \ulcorner T\ app(n) \urcorner \end{aligned}$$

Since app is recursive, there is an arithmetic formula representing it. Rather than using that formula we shall exploit Fact 11 and take it that the function does what it says it does. We shall assume that the recursive function is adequately represented by an arithmetic

[3]We note that if a sequent system were developed then this condition would correspond to a restricted version of the reflexivity rule $\varphi \vdash \varphi$.

[4]Observe that we can reformulate the tableau system as a game, similar to that of [17]. Player I, Abe, wants to show that χ is in the extension of the truth predicates. Player II, Elly, wants to show that it is not. Player Abe asks player Elly a series of questions which are designed to show that this could not occur.

Player Abe commences by asking player Elly if $\neg\chi$ is the case. At stage n of the game, Abe must query a sentence σ played previously by Elly. Elly must then respond by playing according to the table below:

If σ is of the form	then Elly must play
$\neg\neg\varphi$	φ
$\varphi \wedge \psi$	φ and ψ
$\neg(\varphi \wedge \psi)$	either $\neg\varphi$ or $\neg\psi$

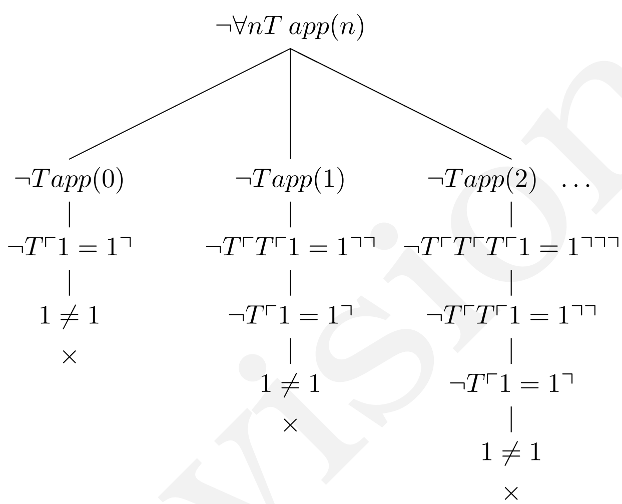
If σ is of the form	then Elly must play
$\forall x\varphi(x)$	$\varphi(n)$ for every term t
$\neg\forall x\varphi(x)$	$\neg\varphi(t)$ for some $n \in \omega$
$T\ulcorner\varphi\urcorner$	φ
$\neg T\ulcorner\varphi\urcorner$	$\neg\varphi$

For the case of $\forall x\varphi(x)$, Elly must find a way of ensuring that she plays $\varphi(t)$ for all terms by the end of the game. She obviously cannot do it in one move. Abe wins the game if Elly ever ends up either: having played both φ and $\neg\varphi$ for some arithmetic sentence; or a false arithmetic atomic sentence. Otherwise, Elly wins.

The tableau system can thus be understood as a means of tracking all of the different turns that the game could have taken when a formula is of the form $\neg(\varphi \wedge \psi)$ or $\neg\forall x\varphi(x)$.

formula and that our proof theory will only leave open branches in which the correct output occurs. Thus instead of including this working, we shall incorporate a rule which allows us to substitute the correct value of recursive functions. The following example illustrates this.

Example 13. $\vdash_{sK} \forall n\ T\ app(n)$.

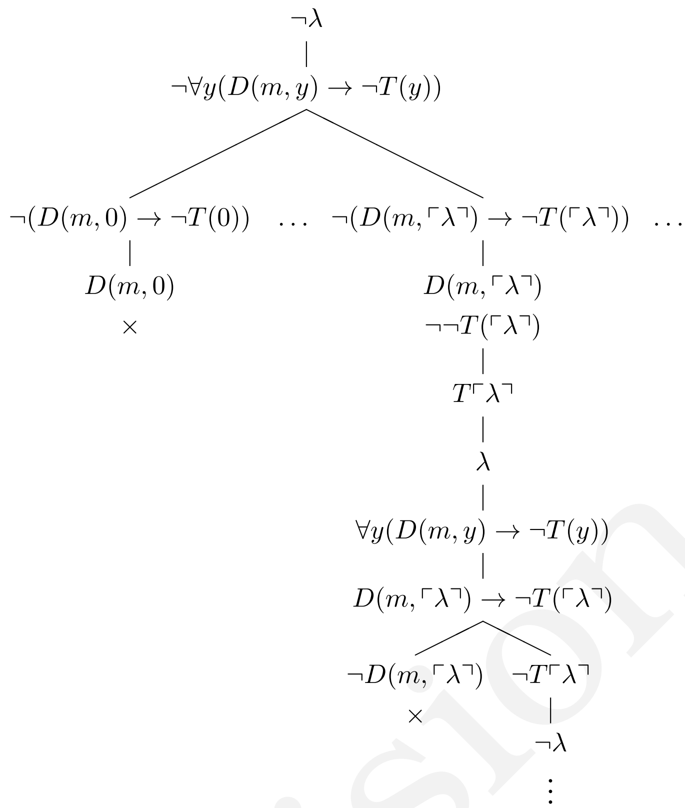


Let $D(x, y)$ be the recursive *diagonal* predicate, which says that x is the code of a formula with one free variable and y is the code of the sentence resulting from substituting the value x for the free variables in the formulae represented by x. For an example see [2] (page 222).

Let $\mu(x)$ be the formula $\forall y(D(x, y) \to \neg T(y))$. Let $m = \ulcorner\forall y(D(x, y) \to \neg T(y))\urcorner$. We define λ to be the formula $\mu(m)$, or in other words $\forall y(D(m, y) \to \neg T(y))$. Clearly $\lambda \leftrightarrow \neg T\ulcorner\lambda\urcorner$.

When using defined sentences on a branch we add a rule that permits us to place the definiendum at the end of the branch. Thus λ must be replaced by $\mu(m)$, which in turn must be replaced by $\forall y(D(x, m) \to \neg T(y))$, see [2].

Example 14. $\nvdash_{sK} \lambda$.



Note that at the first fork all of the other branches close because $D(m, y)$ is only true when $\ulcorner\lambda\urcorner$ is substituted for y. Similarly, when using the formula $\forall y...$ the only substitution worth making is $\ulcorner\lambda\urcorner$. This is because for any other n we will get a trivially open branch containing $\neg D(m, n)$. Thus, we see that the tableau will not close since we have just ended up back where we started and everything before this point has been discharged.

Observe that the right hand branch does not close despite having both λ and $\neg\lambda$ on it. The reason for this is that neither λ nor $\neg\lambda$ are in $\mathcal{L}$ the fragment of the language not involving the truth predicate. Thus the conditions for closure of the branch are not satisfied.

Moreover, we also note that our diagonal predicate did exactly what it ought to have. We saw that λ is equivalent to $\neg T\ulcorner\lambda\urcorner$ and that is where the branch leads us. So rather than write out the above in full, we shall be content with the following, much simpler, tableau:

$$\begin{array}{c}\neg\lambda \\ | \\ \neg\neg T\ulcorner\lambda\urcorner \\ | \\ T\ulcorner\lambda\urcorner \\ | \\ \lambda \\ | \\ \neg T\ulcorner\lambda\urcorner \\ | \\ \neg\lambda \\ \vdots\end{array}$$

in which we feel free to substitute the equivalent sentences resulting from the application of the diagonal lemma.

2.4. **Equivalence.** We now show that each of the definitions of the minimal strong Kleene fixed point are equivalent. We assume that we have a uniform procedure for constructing tableau. This is just a way of ordering the moves so that we can associate with each sentence a particular tableau. For each φ we shall call this the *standard tableau for* φ.

Definition 15. Let the *tableau-rank* of a tableau, $\mathcal{T}$, be the least ordinal such that there is an order preserving map from $\mathcal{T}$ into the ordinals where $\mathcal{T}$ is considered to be a tree formed from finite sequences of formula. Let the *tableau-rank* of φ, abbreviated $\rho_{Tab}(\varphi)$, be the tableau-rank of the standard tableau for φ, if it is closed and ∞ otherwise. For $\varphi \in \Gamma^+_{sK} \cup \Gamma^-_{sK}$ ($\varphi \in \Xi_{sK}$), let the Γ-rank (Ξ-rank) of a sentence φ, abbreviated $\rho_\Gamma(\varphi)$ ($\rho_\Xi(\varphi)$) be the least α such that that $\varphi \in \Gamma^+_\alpha \cup \Gamma^-_\alpha$ ($\varphi \in \Xi_\alpha$) and ∞ if there is no such ordinal.

Theorem 16. *The following are equivalent:*

(1) $\vdash_{sK} \chi$*;*
(2) $\chi \in \Gamma^+_{sK}$*; and*
(3) $\chi \in \Xi_{sK}$*.*

Proof. (2.$\leftrightarrow$3.) is a folk result (see [6]).

(3.$\rightarrow$1.) By induction on the Ξ-rank. Suppose χ is a true arithmetic literal. Then $\vdash_{sK} \chi$, follows by definition. Suppose that for all $\beta \leq \alpha$ we have $\varphi \in \Xi_\beta \Rightarrow \vdash_{sK} \varphi$ and that $\chi \in \Xi_{\alpha+1}$. It suffices to show that $\vdash_{sK} \chi$. Suppose χ is of the form $\varphi \wedge \psi$. Then both $\varphi \in \Xi_\alpha$ and $\psi \in \Xi_\alpha$. By induction hypothesis, $\vdash_{sK} \varphi$ and $\vdash_{sK} \psi$. Let us take closed tableau $\mathcal{T}_\varphi$ and $\mathcal{T}_\psi$ respectively. Then the following tree closes:

$$\begin{array}{c}\neg\chi \\ \diagup\quad\diagdown \\ \mathcal{T}_\varphi \quad \mathcal{T}_\psi\end{array}$$

and thus $\vdash_{sK} \chi$. The other cases are similar. The limit case is trivial. (1.$\rightarrow$3.) By induction on the tableau-rank. □

3. van Fraassen Supervaluation

In the introduction, we observed that for the strong Kleene definition, logical truths like $\neg(T\ulcorner\lambda\urcorner \wedge \neg T\ulcorner\lambda\urcorner)$ failed to get into the extension of the truth predicate. In this section, we explore one method of recapturing this. To motivate the approach we attempt some diagnosis of this problem. With the previous definition we restricted our closure conditions in such a way that only arithmetic sentences could cause a branch to close. Thus, there is a sense in which our truth definition is generated upon a basis of arithmetic truths. The arithmetic basis can be observed more clearly in the finer-grained definition of Ξ_{sK}. Now if arithmetic is the only intended ground of our definition, then it seems as if the strong Kleene definition is the appropriate one to take up. However, we may also want to capture the logical truths. Informally speaking, our goal in this section is to provide a truth definition which takes as its basis both logical and arithmetic truths. We shall do this in two stages. First we shall make an alternative definition of the van Fraassen supervaluation truth extension. The goal of this is to make our informal motivation more transparent. Second, we provide an infinitary tableau for these definitions and show that all three are equivalent.

We first provide a high-level description of the process we shall use. Given that we want to capture logical truths, an obvious way of doing this is to use a conception of proof. We shall thus define an infinitary proof system to ensure that we capture logical truths as well as arithmetic ones. The proof notion defined will form the engine for induction steps in the truth definition. It will not be the main tableau system, which gives the intended extension of the truth predicate; rather it will be a *bridging* system which allows us to keep improving our guesses about truth. Intuitively speaking, the natural place to start is with the atomic arithmetic sentences. They form our pool of axioms and we start off by proving everything that we can from them. Now since we have proven these sentences, we can, indeed ought to, throw these into the extension of the truth predicate. Thus, we shall augment our axiom pool with these new sentences. We then prove everything we can from the new axiom set. We then repeat the process into the transfinite reaching a fixed point, which we shall call Ξ_{vF}.

3.1. Bridging tableau ($\Vdash_{vF}$).

We now define the conception of proof that will be used for the induction step. We shall regard it as a bridging system between Γ_{vF} and Ξ_{vF}. Intuitively, the tableau is designed to show that given $\Phi \subseteq \mathcal{L}_T$, $\chi \in \mathcal{L}_T$ is the case in the standard model of arithmetic expanded with a one place relation T constrained in such a way that the interpretation of T is both a superset of Φ and does not intersect $\dot\neg\Phi$. Or informally, given the truth-guess Φ, we ought to add χ to the extension of the truth predicate. We shall write this as $\Phi \Vdash_{vF} \chi$.

3.1.1. *Starting conditions.*

To attempt to show that $\Phi \Vdash_{vF} \chi$, we commence the tableau by placing $\neg\chi$ at the root.

3.1.2. *Rules.* We take the rules $(\wedge)$, $(\neg\wedge)$, $(\neg\neg)$, $(\forall)$ and $(\neg\forall)$. These are just the connective rules and quantifier rules from Section 2.2.2.

In place of the truth rules, we add the following *axiom rules*. Given a set of axioms Φ we may apply either of the rules below at any point in the construction of a branch.

$$(AxT) \qquad \begin{array}{c} | \\ T\ulcorner\varphi\urcorner \\ \text{where } \varphi \in \Phi \end{array} \qquad\qquad (Ax\neg T) \qquad \begin{array}{c} | \\ \neg T\ulcorner\varphi\urcorner \\ \text{where } (\neg\varphi) \in \Phi \end{array}$$

Informally speaking, the new truth rules allow us to pull sentences from our stock of axioms. We are allowing ourselves to add $T\ulcorner\varphi\urcorner$ if φ is an axiom; thus, cutting off any branches that attempt to run with its negation.

3.1.3. *Closing conditions .* A branch $\mathcal{B}$ closes if either:

- some formula φ and its negation $\neg\varphi$ occurs on $\mathcal{B}$; or
- a false arithmetic sentence occurs on $\mathcal{B}$.

With the proof system defined, we may now define our jump function.

Definition 17. Let $h_{vF} : \mathcal{P}(Sent) \to \mathcal{P}(Sent)$ be such that

$$\varphi \in h_{vF}(\Phi) \;\Leftrightarrow\; \Phi \Vdash_{vF} \varphi.$$

The jump function takes a set of sentences as axioms and puts everything that can be proven from those axioms Φ, into the extension of the truth predicate.

Definition 18. The intended extension Ξ_{vF} is defined by transfinite recursion as follows:

$$\begin{aligned} \Xi_0 &:= \emptyset \\ \Xi_{\alpha+1} &:= h_{vF}(\Xi_\alpha) \\ \Xi_\beta &:= \bigcup_{\alpha\in\beta} \Xi_\alpha \text{ for limit } \beta. \end{aligned}$$

Let Ξ_{vF} be Ξ_α for the least α such that $\Xi_\alpha = \Xi_{\alpha+1}$.

Clearly h_{vF} is monotonic, so we are justified in our assumption that Ξ_{vF} has a fixed point.

3.2. **Main tableau ($\vdash_{vF}$).** We now define an infinitary tableau for Ξ_{vF}, which we shall abbreviate as $\vdash_{vF}$. This tableau will give us the intended extension of the truth predicate.

3.2.1. *Starting the tableau.* The tableau for $\chi \in \mathcal{L}_T$ is commenced by placing $\neg\chi$ at the root of the tree.

3.2.2. *Rules .* We take the rules $(\wedge)$, $(\neg\wedge)$, $(\neg\neg)$, $(\forall)$ and $(\neg\forall)$ from Section 2.2.2.

In place of the truth rules we add the following *subtableau rule*:

(Sub) If at any point in the construction of a branch $\mathcal{B}$, a formula of the form $T\ulcorner\psi\urcorner$ or $\neg T\ulcorner\psi\urcorner$ occurs on $\mathcal{B}$, then a subtableau may be constructed which commences with ψ or $\neg\psi$ respectively. This subtableau is governed by the same rules as the main tableau.

3.2.3. *Closing conditions.* A branch $\mathcal{B}$ is deemed to close if either:

- for some sentence $\varphi \in \mathcal{L}_T$ (i.e. including truth), both φ and $\neg\varphi$ occur on $\mathcal{B}$;
- a false arithmetic sentence occurs on $\mathcal{B}$; or
- a subtableau for one of the sentences on $\mathcal{B}$ is closed, where a subtableau is closed if all of its branches are closed.

If all the branches of a tableau for χ close, then that tableau is closed. We abbreviate this as $\vdash_{vF} \chi$. Observe that a proof in this system may involve a nested tree of subtableau.

3.3. **Examples.**

Example 19. $\vdash_{vF} \lambda \vee \neg\lambda$.

$$\begin{array}{c} \neg(\lambda \vee \neg\lambda) \\ | \\ \neg\lambda \\ \neg\neg\lambda \\ \times \end{array}$$

We shall indicate a subtableau by placing a $\Rightarrow$ between the original tableau and its subtableau. This works well enough for one or two tableau but will quickly become too complex with more subtableau. In these cases, a more elaborate book-keeping system may be adopted which connects the subtableau to their origin point on an earlier tableau. We shall not need this here.

Example 20. $\nvdash_{vF} \lambda$.

$$\begin{array}{ccccc} \neg\lambda & \Rightarrow & \lambda & \Rightarrow & \neg\lambda \\ | & & | & & | \\ \neg\neg T\ulcorner\lambda\urcorner & & \neg T\ulcorner\lambda\urcorner & & \vdots \\ | & & & & \\ T\ulcorner\lambda\urcorner & & & & \end{array}$$

Example 21. $\nvdash \neg(T\ulcorner\lambda\urcorner \wedge T\ulcorner\neg\lambda\urcorner)$.

$$
\begin{array}{ccccccccccc}
\cdots \Leftarrow & \neg\lambda & \Leftarrow & \neg\neg(T\ulcorner\lambda\urcorner \wedge T\ulcorner\neg\lambda\urcorner) & \Rightarrow & \lambda & \Rightarrow & \neg\lambda & \Rightarrow & \lambda & \Rightarrow \cdots \\
 & | & & | & & | & & | & & | & \\
 & \neg\neg T\ulcorner\lambda\urcorner & & T\ulcorner\lambda\urcorner \wedge T\ulcorner\neg\lambda\urcorner & & \neg T\ulcorner\lambda\urcorner & & \neg\neg T\ulcorner\lambda\urcorner & & \neg T\ulcorner\lambda\urcorner & \\
 & | & & | & & & & | & & & \\
 & T\ulcorner\lambda\urcorner & & T\ulcorner\lambda\urcorner & & & & T\ulcorner\lambda\urcorner & & & \\
 & & & T\ulcorner\neg\lambda\urcorner & & & & & & &
\end{array}
$$

3.4. **Equivalence.** We now prove that each of the definitions of the minimal van Fraassen fixed point are equivalent. First we state and prove the crucial lemma, which will allow us to link the original definition Γ_{vF} from the introduction to Ξ_{vF}. The following definition is useful.

Definition 22. The *positive complexity* of a sentence $\varphi \in Sent_{\mathcal{L}_T}$, abbreviated $\kappa(\varphi)$ is defined by recursion as follows:

- $\kappa(\psi \wedge \chi) = max(\kappa(\psi), \kappa(\chi)) + 1$
- $\kappa(\neg(\psi \wedge \chi)) = max(\kappa(\psi), \kappa(\chi)) + 1$
- $\kappa(\neg\neg\psi) = \kappa(\psi) + 1$
- $\kappa(\forall x\psi) = \kappa(\psi(\underline{\mathrm{n}})) + 1$ for some/any $n \in \omega$
- $\kappa(\neg\forall x\psi) = \kappa(\neg\psi(\underline{\mathrm{n}})) + 1$ for some/any $n \in \omega$

Lemma 23. *$\forall\Psi \sqsupseteq_{vF} \Phi$, $Val_\Psi(\chi) = 1$ iff $\Phi \Vdash_{vF} \chi$.*

Proof. ($\to$) Suppose $\Phi \nVdash_{vF} \chi$. Then in any bridging tableau for χ and Φ there will be an open branch $\mathcal{B}$. It will suffice to show that there is some $\Psi \sqsupseteq_{vF} \Phi$ such that $Val_\Psi(\chi) = 0$.

To construct the model, we take the domain ω and let the terms denote their corresponding numbers. The arithmetic part of the signature is interpreted in the usual way. We then let Ψ be the set of φ such that either: $T\ulcorner\varphi\urcorner$ occurs on $\mathcal{B}$; or φ is of the form $(\neg\psi)$ and $\neg T\ulcorner\psi\urcorner$ occurs on $\mathcal{B}$. Since $\mathcal{B}$ is open, every instance of (AxT) and ($Ax\neg T$) must be used, so we have $T\ulcorner\zeta\urcorner$ and $\neg T\ulcorner\delta\urcorner$ on $\mathcal{B}$ for all $\zeta, (\neg\delta) \in \Phi$. Thus $\Psi \sqsupseteq_{vF} \Phi$.

The following claim suffices.

Claim. If σ is on $\mathcal{B}$, then $Val_\Psi(\sigma) = 1$.

Proof. By induction on the positive complexity of sentences. For illustration, we do some of the cases.

If σ is an arithmetic literal, then σ cannot be false in the standard model, thus $Val_\Psi(\sigma) = 1$.

If σ is of the form $\varphi \wedge \psi$, then both φ and ψ are on $\mathcal{B}$ via the tableau rule ($\wedge$). Then by induction hypothesis, $Val_\Psi(\varphi) = 1$ and $Val_\Psi(\psi) = 1$; and thus, $Val_\Psi(\varphi \wedge \psi) = 1$.

If σ if of the form $\neg(\varphi \wedge \psi)$, then either $(\neg\varphi)$ or $(\neg\psi)$ is on $\mathcal{B}$. Assume the first case. Then by induction hypothesis, $Val_\Psi(\neg\varphi) = 1$ and $Val_\Psi(\neg(\varphi \wedge \psi)) = 1$; similarly if $(\neg\psi)$ is on $\mathcal{B}$. The quantifier cases are similar.

If σ is of the form $T\ulcorner\psi\urcorner$, then by definition, $\psi \in \Psi$ and thus $Val_{\Psi}(T\ulcorner\psi\urcorner) = 1$. If σ is of the form $\neg T\ulcorner\psi\urcorner$, the tableau rules ensure that $T\ulcorner\psi\urcorner$ does not occur on $\mathcal{B}$. Thus $\psi \notin \Psi$ and $Val_{\Psi}(\neg T\ulcorner\psi\urcorner) = 1$. □

($\leftarrow$) Suppose there is some $\Psi \sqsupseteq_{vF} \Phi$ such that $Val_{\Psi}(\chi) = 0$. It will suffice to show that there is a completed open branch in the tableau for χ. We shall do this in two stages. First we shall construct a tableau $\mathcal{T}$ that makes no use of the (AxT) and $(Ax\neg T)$ rules. Then we shall show how to find a particular open branch $\mathcal{B}$ in $\mathcal{T}$ which could not be closed using applications of (AxT) or $(Ax\neg T)$.

We use Ψ as guide for the construction of an open branch $\mathcal{B}$ such that for every φ on $\mathcal{B}$, $Val_{\Psi}(\varphi) = 1$. The proof proceeds by induction on the length of branches in the tableau. We look at one case for illustration. Suppose we come to a sentence of the form $\neg(\varphi \wedge \psi)$. Then by induction hypothesis, we have $Val_{\Psi}(\neg(\varphi \wedge \psi)) = 1$. The tableau rules dictate that there will be branches which fork into a $\neg\varphi$ and a $\neg\psi$ path. By the Val definition, $Val_{\Psi}(\neg\varphi) = 1$ or $Val_{\Psi}(\neg\psi) = 1$; and thus, the branch can be continued by selecting one in which the formula is evaluated to 1. Clearly, this will give us an open branch $\mathcal{B}$.

Now suppose for *reductio*, that some application of the (AxT) or $(Ax\neg T)$ could close $\mathcal{B}$. We first observe that for all sentences φ on $\mathcal{B}$, we have $Val_{\Psi}(\varphi) = 1$ by construction. Thus for all sentences of the form $T\ulcorner\varphi\urcorner$ or $\neg T\ulcorner\psi\urcorner$ that occur on $\mathcal{B}$, we have $\varphi, (\neg\psi) \in \Psi$. But then since $\Psi \sqsupseteq_{vF} \Phi$, it is not possible to find a sentence in Φ which could close $\mathcal{B}$. □

Proposition 24. *If* $B \subseteq \Gamma$ *and* $B \Vdash_{vF} \chi$*, then* $\Gamma \Vdash_{vF} \chi$*.*

In order to prove the equivalence between the three van Fraassen supervaluation definitions, we shall need to place a rank on a tree of tableau which form a proof of some φ. We do this by considering an alternative method of notating the tableau. The change is merely for bookkeeping purposes and is not particularly visible in the actual proof.

Instead of starting a subtableau on some branch $\mathcal{B}$, we shall continue along the same branch but tag the sentences that would have occurred on the subtableau with a flag (some $n \in \omega$) which distinguishes them from other sentences on that branch. To deal with embedded subtableau we shall flag sentences with a sequence of natural numbers, $\langle n_1, ..., n_m\rangle$. Thus, we do not just place sentences φ on a branch but rather pairs of the form $\varphi, \langle n_1, ..., n_m\rangle$. The truth rules thus become:

$$\begin{array}{cc} T\ulcorner\psi\urcorner, \langle n_1, ..., n_m\rangle & \neg T\ulcorner\psi\urcorner, \langle n_1, ..., n_m\rangle \\ | & | \\ \psi, \langle n_1, ..., n_m, k\rangle & \neg\psi, \langle n_1, ..., n_m, k\rangle \end{array}$$

where k is is the first $k \in \omega$ such that $\varphi, \langle n_1, ..., n_m, k\rangle$ does not occur on the branch for any φ. We shall call the tuple $\langle n_1, ..., n_m, k\rangle$ a *flag*. The rules for closure need to be slightly amended such that a branch $\mathcal{B}$ closes when a sentence φ and $\neg\varphi$ occur on $\mathcal{B}$ and both φ and $\neg\varphi$ have the same flag. Thus if we have φ, n and $\neg\varphi, n$ on branch $\mathcal{B}$, then $\mathcal{B}$ is closed. It is clear that the proof systems are equivalent. We shall call a tableau constructed in this way a *flagged* tableau.

Definition 25. We let the vF-rank of φ, abbreviated $\rho_{vF}(\varphi)$ be the least ordinal α such that there is an order preserving map from the flagged tableau commencing with φ into α, if $\vdash_{vF} \varphi$; and ∞ otherwise.

Using this notation, we may also extract from some tableau $\mathcal{T}$ a subtableau indexed by $\langle n_1, ..., n_m \rangle$ by removing all the sentence-indexes tagged by sequences which do not commence with $\langle n_1, ..., n_m \rangle$.

We now prove the main theorem. The basic strategy is to use the bridging tableau system as a means of setting up the induction step for the main tableau. Essentially, we are going to use the sentences from the axiom pool of a bridging tableau to house the sentences we could have already proved using a subtableau. Thus, the bridging tableau will form the lever for the induction argument.

Theorem 26. *The following are equivalent:*

(1) $\varphi \in \Gamma_{vF}$;
(2) $\varphi \in \Xi_{vF}$;
(3) $\vdash_{vF} \varphi$.

Proof. (3.→1.) By induction on the $\vdash_{vF}$-rank of the tableau. Suppose for all $\beta < \alpha$ we have

$$\rho_{vF}(\neg\psi) = \beta \Rightarrow \psi \in \Gamma_{vF} \text{ and } \rho_{vF}(\psi) = \beta \Rightarrow (\neg\psi) \in \Gamma_{vF}$$

Now suppose $\rho_{vF}(\neg\varphi) = \alpha$; i.e., $\vdash_{vF} \varphi$ and the closed tableau verifying this has vF-rank α. Let

$$B^+ = \{\neg\psi \mid T\ulcorner\psi\urcorner \text{ occurs in the tableau for } \neg\varphi \text{ and a closed subtableau results from } \psi\}$$
$$B^- = \{\psi \mid \neg T\ulcorner\psi\urcorner \text{ occurs in the tableau for } \neg\varphi \text{ and a closed subtableau results from } \neg\psi\}$$

Then clearly, $B^+ \cup B^- \Vdash_{vF} \varphi$. Suppose $(\neg\psi) \in B^+$. Then since this means there is a closed subtableau commencing with ψ, there is a $\beta < \alpha$ such that $\rho_{vF}(\psi) = \beta$. Thus, by induction we have $(\neg\psi) \in \Gamma_{vF}$. Similarly, if $\psi \in B^-$, we have $\psi \in \Gamma_{vF}$. Thus $B^+ \cup B^- \subseteq \Gamma_{vF}$ and by Proposition 24, we have $\Gamma_{vF} \Vdash_{vF} \varphi$; and by Lemma 23, we have $\varphi \in \Gamma_{vF}$ since Γ_{vF} is a fixed point.

Now suppose $\rho_{vF}(\varphi) = \alpha$; i.e., $\vdash_{vF} \neg\varphi$ and the closed tableau verifying this has vF-rank $\alpha+1$. The proof is similar to the previous case in that we show $B^+ \cap B^- \subseteq \Gamma_{vF}$, except B^+ and B^- are defined relative to a main tableau commencing with $\neg\neg\varphi$ rather than φ. We note that for $(\neg\psi) \in B^+$, the closed subtableau commencing with ψ must be such that $\rho_{vF}(\psi) = \beta < \alpha$ since the main tableau with vF-rank $\alpha+1$ commenced with $\neg\neg\varphi$ and so the first rule applied must have been $(\neg\neg)$. Similarly for $\psi \in B^-$.

(1.→3.) By induction on the Γ-rank of sentences. Suppose that for all $\beta \leq \alpha$ we have $\psi \in \Gamma_\beta \Rightarrow \vdash_{vF} \psi$ and that $\varphi \in \Gamma_{\alpha+1}$. It suffices to show that $\vdash_{vF} \varphi$. By hypothesis, we know that for all $\Psi \sqsupseteq_{vF} \Gamma_\alpha$, $Val_\Psi(\varphi) = 1$. Thus by Lemma 23, $\Gamma_\alpha \Vdash \varphi$. Let us call this tableau $\mathcal{T}_{\Vdash}$.

Now suppose we start the vF-tableau for φ and construct it the same way as $\mathcal{T}_{\Vdash}$ except that applications of (AxT) and $(Ax\neg T)$ from $\mathcal{T}_{\Vdash}$ are not performed. Call this tableau $\mathcal{T}_{vF}$. Now suppose for that some branch $\mathcal{B}_{vF}$ is open in $\mathcal{T}_{vF}$. We claim that wherever this occurs it could have been closed via a subtableau. Thus $\vdash_{vF} \varphi$.

The only way that $\mathcal{B}_{vF}$ could have become closed while its counterpart $\mathcal{B}_{\Vdash}$ in $\mathcal{T}_{\Vdash}$ remained open is if for some ψ both $T\ulcorner\psi\urcorner$ and $\neg T\ulcorner\psi\urcorner$ occur on some branch $\mathcal{B}_{\Vdash}$ but only one of them occurs on $\mathcal{B}_{vF}$. But by definition of $\mathcal{T}_{\Vdash}$, we see that either $\psi \in \Gamma_\alpha$ or $(\neg\psi) \in \Gamma_\alpha$; i.e., exactly one of them must have resulted from either the (AxT) or $(Ax\neg T)$ rules. Assuming the former case, we then have $\vdash_{vF} \psi$ by the induction hypothesis and thus a subtableau commencing would ψ close. The latter case is similar.

(2.↔1.) This follows straightforwardly from Lemma 23. □

4. Cantini Supervaluation

We now discuss a second form of supervaluation as developed by Cantini in [4]. The usual way of defining it was discussed in the introduction, but like van Fraassen supervaluation, it can also be defined using a proof theoretic device. In this section, we shall construct a bridging tableau system like the one from the beginning of the previous section. Then we construct a tableau for the minimal fixed point and finally, we show that all three definitions are equivalent.

4.1. **Bridging tableau ($\Vdash_{Ca}$).**

4.1.1. *Starting the tableau.* To attempt to show that $\Phi \Vdash_{Ca} \chi$, we commence tableau by placing $\neg\chi$ at the root of the tree.

4.1.2. *Rules.* We take the rules $(\wedge)$, $(\neg\wedge)$, $(\neg\neg)$, $(\forall)$, $(\neg\forall)$, (AxT) and $(Ax\neg T)$.[5]

4.1.3. *Closing conditions.* A branch $\mathcal{B}$ is deemed to close if either:

- some sentence $\varphi \in Sent_{\mathcal{L}_T}$ (i.e. including truth), both φ and $\neg\varphi$ occur on $\mathcal{B}$;
- some sentence $\varphi \in Sent_{\mathcal{L}_T}$, both $T\ulcorner\varphi\urcorner$ and $T\ulcorner\neg\varphi\urcorner$; or
- a false arithmetic sentence occurs on $\mathcal{B}$.

The key difference from the van Fraassen system is the second condition, which demands that we close a branch if it exhibits a truth extension that is, in some sense, inconsistent.

If all the branches of the tableau are closed, then that tableau is closed. We abbreviate this as $\Phi \Vdash_{Ca} \chi$.

[5]Recall the rules (AxT) and $(Ax\neg T)$ allow us to exploit the set of sentences Φ.

4.2. **Main tableau ($\vdash_{Ca}$).**

4.2.1. *Starting condition.* We commence a tableau for χ and Φ by placing the sentence $\neg\chi$ at the root of the tree.

4.2.2. *Rules.* We take the rules $(\wedge)$, $(\neg\wedge)$, $(\neg\neg)$, $(\forall)$, $(\neg\forall)$ and (Sub).

4.2.3. *Closing conditions.* A branch $\mathcal{B}$ in a tableau $\mathcal{T}$ for some χ is closed if either:

- for some sentence $\varphi \in Sent_{\mathcal{L}_T}$ and its negation $\neg\varphi$ occurs on $\mathcal{B}$;
- for some sentence $\varphi \in Sent_{\mathcal{L}_T}$ both $T\ulcorner\varphi\urcorner$ and $T\ulcorner\neg\varphi\urcorner$ occur on $\mathcal{B}$;
- a false arithmetic sentence occurs on $\mathcal{B}$; or
- a truth-subtableau for $\mathcal{B}$ closes.

If all the branches of the tableau close, then the tableau is closed and we write $\Phi \vdash_{Ca} \chi$.[6]

4.3. **Examples.**

Example 27. $\vdash_{Ca} \neg(T\ulcorner\lambda\urcorner \wedge T\ulcorner\neg\lambda\urcorner)$.

$$\begin{array}{c}
\neg\neg(T\ulcorner\lambda\urcorner \wedge T\ulcorner\neg\lambda\urcorner) \\
| \\
(T\ulcorner\lambda\urcorner \wedge T\ulcorner\neg\lambda\urcorner) \\
| \\
T\ulcorner\lambda\urcorner \\
T\ulcorner\neg\lambda\urcorner \\
\times
\end{array}$$

Example 28. $\nvdash_{Ca} \neg(T\ulcorner T\ulcorner\lambda\urcorner\urcorner \wedge T\ulcorner\neg\lambda\urcorner)$.

$$\begin{array}{ccccccccccc}
\ldots\ \lambda & \Leftarrow & \neg\lambda & \Leftarrow & \neg\neg(T\ulcorner T\ulcorner\lambda\urcorner\urcorner \wedge T\ulcorner\neg\lambda\urcorner) & \Rightarrow & T\ulcorner\lambda\urcorner & \Rightarrow & \lambda & \Rightarrow & \neg\lambda\ \ldots \\
 & & | & & | & & & & | & & \\
 & & \neg\neg T\ulcorner\lambda\urcorner & & T\ulcorner T\ulcorner\lambda\urcorner\urcorner \wedge T\ulcorner\neg\lambda\urcorner & & & & \neg T\ulcorner\lambda\urcorner & & \\
 & & | & & | & & & & & & \\
 & & T\ulcorner\lambda\urcorner & & T\ulcorner T\ulcorner\lambda\urcorner\urcorner & & & & & & \\
 & & & & T\ulcorner\neg\lambda\urcorner & & & & & &
\end{array}$$

[6]For readers familiar with Cantini's [4], similarities may be discerned between Cantini's infinitary Tait calculus, Definition 4.1, and the tableau system discussed above. Cantini's AX.3 does much the same work as the second of our closure conditions above. Moreover, Cantini's (T) and $(\neg T)$ rules play a similar role to the subtableau used in the system above. The fact that we are using a subtableau roughly corresponds to Cantini's demand that proofs of sentences of the form $T\ulcorner\varphi\urcorner$ and $\neg T\ulcorner\varphi\urcorner$ must be established without recourse to auxiliary assumptions.

4.4. **Equivalence.** We now establish that the proof theory is sound and complete with regard to fixed point definition provided in Section 1. First we establish that our bridging tableau links with the jump function used in the introduction.

Lemma 29. *For all* $\Psi \sqsupseteq_{Ca} \Phi$, $Val_\Psi(\varphi) = 1$ *iff* $\Phi \Vdash_{Ca} \varphi$.

Proof. ($\rightarrow$) Similar to 23. Suppose $\Phi \nVdash_{Ca} \varphi$ and use an open branch $\mathcal{B}$ to show that there is some $\Psi \sqsupseteq_{Ca} \Phi$ such that $Val_\Psi(\varphi) = 0$. The only difference is that Ψ must also be consistent. Suppose it was not. Then for some ψ both $T\ulcorner\psi\urcorner$ and $T\ulcorner\neg\psi\urcorner$ must occur on $\mathcal{B}$ contradicting the fact that it is open. ($\leftarrow$) Similar to Lemma 23. □

Theorem 30. $\varphi \in \Gamma_{Ca}$ *iff* $\vdash_{Ca} \varphi$.

Proof. ($\leftarrow$) By induction on $\vdash_{Ca}$-rank. Suppose that for all $\beta < \alpha$ we have $\vdash^\beta_{Ca} \psi \Rightarrow \psi \in \Gamma_{Ca}$ and that $\vdash^\alpha_{Ca} \varphi$. It suffice to show that $\varphi \in \Gamma_{Ca}$. Similar to Theorem 26, except here we exploit Lemma 29.

($\rightarrow$) Similar to Theorem 26. □

4.5. **Other truth definitions.**

4.5.1. *Strengthening consistency.* The notion of consistency used for Cantini supervaluation is exceedingly weak. To see this, consider the sentence $\varphi := \lambda \wedge \lambda$. This is obviously logically equivalent to λ. However, while $\neg T\ulcorner\lambda\urcorner \vee \neg T\ulcorner\lambda\urcorner \in \Gamma_{Ca}$, it is not the case that $\neg T\ulcorner\varphi\urcorner \vee \neg T\ulcorner\neg\lambda\urcorner$ is in Γ_{Ca}. This is because the notion of a Ca-expansion does not recognise logical equivalence. It simply rules out those expansions which contain some sentence φ and its negation $\neg\varphi$: it only respects the syntactic form. Thus, it seems like it would be more interesting to consider expansions which are genuinely logically consistent.

Definition 31. Ψ is a Con-expansion of Φ, abbreviated $\Psi \sqsupseteq_{Con} \Phi$, if

- $\Psi \supseteq \Phi$; and
- Ψ is logically consistent.

The rest of the definition is then carried out in the usual way and we denote the resultant truth extension Γ_{Con}.

To provide a tableau system for this, we replace the new rule added to the Cantini truth definition with the following. At any point in the construction of a branch $\mathcal{B}$, a subtableau may be formed by taking a collection B of sentences of the form $T\ulcorner\varphi\urcorner$ and completing an ordinary first order logic tableau commencing with B. Such a tableau system can be found in [15] or [13]. If this tableau closes, then $\mathcal{B}$ is deemed to have closed.

Finally, we might also consider expansions that are not only consistent but complete.[7]

[7]We think of this as an epistemicist approach to truth on the following basis. We start by taking it that there is some fixed extension of the truth predicate which is both consistent and complete. We take it that while the *real* extension is, so to speak, out there, there is no way for us to come to know it. So the holder of such a theory is an epistemicist in the sense that they believe that the truth predicate, metaphysically speaking, has a fixed extension;

Definition 32. Ψ is an Ep-expansion of Φ, abbreviated $\Psi \sqsupseteq_{Ep} \Phi$ if

- $\Psi \supseteq \Phi$; and
- for all ψ, $\psi \in \Phi$ iff $(\neg\psi) \notin \Phi$.

We modify the tableau system by allowing us to pull sets of negated truth sentences from the tree and start subtableau using them.

5. Π^1_1 sets and Complexity

In this section we use the tableau system to give simple and transparent proofs:

- that the definitions provided above are Π^1_1-complete; and
- that the height of Kripke's fixed point is ω_1^{CK} (the supremum of the recursive ordinals).

The first result is not new, having first been claimed by Kripke and presented by Burgess in [3]. However, the manner in which the result is established is informative because it clearly illustrates that there is a sense in which our tableau definition *just is* one of the canonical representations of a Π^1_1-complete set. Moreover the result established below can be easily generalised to apply to each of the other tableau systems proposed in this paper.[8] The second result is a well-known folk theorem from Spector, however, the usual proofs in the literature require a significant detour through generalised recursion theory, admissible set theory or proof theory: see [14] page 78; [1] pages 173 and 210; and [12] page 94.[9] We provide a direct proof for the strong Kleene fixed point which, while somewhat technical, is self-contained. While the proof of complexity is relatively straightforward, it is anticipated that the reader may want to do some ancillary scribbling for the calculation of the fixed point.

The following two theorems may help motivate the importance of Π^1_1-sets from the point of view of descriptive set theory and generalised recursion theory.

Theorem 33. *(Gödel) For all recursively enumerable subsets $A \subseteq \omega$, there is a Σ_1 formula φ of set theory such that*

$$n \in A \Leftrightarrow L_\omega \models \varphi[n].$$

This is just a restatement of the result that recursively enumerable functions can be represented in the language of arithmetic by formulae that use a single unbounded existential quantifier (see, for example, [2] page 206).

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but we are not in an epistemic position to grasp it. However, even with only these constraints we can still say a great deal about what is true. By supervaluating over all of the extensions which are complete and consistent we are assured of obtaining sentences which are in the real extension.

[8]The easiest way to make this generalisation is by avoiding the subtableau and adopting the *flagged tableau* discussed above Definition 25.

[9]The techniques of the next section can be used to obtain more general results overlapping these areas of mathematical logic. If the reader is interested in further developing these skills, Barwise's framework of admissible set theory is probably the most versatile. However, a background in the constructible hierarchy [5] and descriptive set theory [11] is also helpful.

Theorem 34. *(Spector-Gandy) For all Π^1_1 subsets $A \subseteq \omega$ there is a Σ_1 formula φ of set theory such that*

$$n \in A \Leftrightarrow L_{\omega_1^{CK}} \models \varphi[n]$$

where ω_1^{CK} is the supremum of the recursive well-orderings.

Thus there is a sense a Π^1_1 set plays much the same role as a recursively enumerable set: it is a generalisation of that concept. $L_{\omega_1^{CK}}$ is the smallest chunk of set theory in which we can execute our infinitary proofs. Similarly L_ω is the smallest chunk of set theory in which our ordinary finitary proofs can be executed.

We shall use α, β for functions from ω to ω. Let $\omega^{<\omega}$ denote the set of finite sequences of natural numbers.[10] Let $\langle \cdot \rangle : \omega^{<\omega} \cong \omega$ be a recursive bijection coding finite sequences of natural numbers using the naturals. We write $\langle 1, 2, 3 \rangle = 5$ if 5 is the code number of the sequence consisting of $1, 2$ and 3. Write $n(i)$ for the i^{th} element of the sequence coded by n. Let $\bar{\cdot} : \omega \cong \omega^{<\omega}$ be the inverse of $\langle \cdot \rangle$. Let $lh : \omega \to \omega$ be a function taking the code of a sequence to its length; i.e., for $n \in \omega$, $lh(n)$ is the length of $\bar{n}$. Let $\cdot ^\frown \cdot : \omega \times \omega \cong \omega$ be a function which takes the codes of two sequences and returns the code of the first concatenated with the second.[11] For $\alpha \in \omega^\omega$, we shall write $\alpha|_n$ to mean the restriction of $\alpha \in \omega^\omega$ to its first n values; and for $n \in \omega$ we shall write $\bar{n}|_m$ to denote the restriction of the finite sequence $\bar{n} \in \omega^{<\omega}$ to its first m values assuming it has that many.

A tree S on ω^k is a set of k-tuples of finite sequences of natural numbers (i.e. $S \subseteq (\omega^{<\omega})^k$) such that:

- for all $(\bar{n}_1, ..., \bar{n}_k) \in S$, $lh(n_i) = lh(n_1)$ for $1 \leq i \leq k$; and
- if $(\bar{n}_1, ..., \bar{n}_k) \in S$ and $i < lh(n_1)$, then $(\bar{n}_1|_i, ..., \bar{n}_k|_i) \in S$.

We shall say that the function α *represents* the tree $S \subseteq (\omega^{<\omega})^k$ if

$$\forall n_1, ..., n_k((\bar{n}_1, ..., \bar{n}_k) \in S \leftrightarrow \alpha(\langle n_1, ..., n_k \rangle) = 0).$$

A tree S is *recursively enumerable* if it is represented by a partial recursive function. We shall now consider a particular type of tree on ω^2. Suppose $S \subseteq (\omega^{<\omega})^2$ is a tree such that for all $(\bar{n}, \bar{m}) \in S$ we have $n(i) = n(1)$ for all $i \leq lh(n)$. So we always have a constant function in the first component. This means we lose no information if we take S and form the set

$$S' = \{(k, \bar{m}) \mid \exists i \leq lh(m)(k = n(i) \wedge (\bar{n}, \bar{m}) \in S\} \cup \{(k, \bar{\langle\rangle})\}.$$

We call such a set a p-tree. Given a recursive p-tree S we let $S_n = \{\bar{m} \mid (n, \bar{m}) \in S\}$.

[10]Note that in this paper we also use α, β to represent ordinals, but we shall take care to ensure that this causes no confusion.

[11]We note that the usual practice when defining length and concatenation functions for sequences would be to define them on the sequences themselves rather than their codes [11]. However, given our need to articulate these facts in the language of arithmetic, this has the effect of making the syntax unpleasantly cumbersome.

Definition 35. (See [9] page 35) $A \subseteq \omega$ is Π^1_1 if there is a recursively enumerable p-tree S such that $n \in A$ iff

$$\forall \alpha \exists m \ (n, \alpha|_m) \notin S.$$

Remark 36. In other words $n \in A$ iff the tree S_n on ω is well founded. Thus to check whether $n \in A$, we need to show that there are no infinite paths through the tree S_n. We observe that the set of indices for well-founded recursive trees is a canonical example of a Π^1_1-complete set. It is also worth noting that we could have demanded that the p-tree S was recursive (and not merely recursively enumerable), but the condition above is more convenient for our purposes.

Definition 37. $A \subseteq \omega$ is Π^1_1-*complete* if A is Π^1_1 and A is Π^1_1-*hard* where that means that for any Π^1_1, $B \subseteq \omega$, there is a recursive function f such that

$$n \in B \leftrightarrow f(n) \in A.$$

There is a sense in which a Π^1_1-complete set is a universal machine for all of the Π^1_1 sets. For any Π^1_1 set there is a simple means of figuring out its contents using a Π^1_1-complete set. The proof of the theorem below illustrates this. This result can be generalised to apply to any tableau system possessing a sufficiently similar truth rule, which is most of them.

5.1. Γ_{sK} is a Π^1_1-complete set.

Theorem 38. $\{\ulcorner \varphi \urcorner \mid \ \vdash_{sK} \varphi\} = A$ *is* Π^1_1-*complete.*

Remark 39. Showing that A is Π^1_1 is quite straightforward. In order to verify Π^1_1-hardness, our strategy will be to use our infinitary tableau to track infinite potential paths through the trees which represent Π^1_1 sets. We shall use the diagonal lemma to construct a sentence such that the tableau theory forces us to play out all of the possible paths through the tree. Thus if the tableau closes, then none of the paths could have been infinite and the tree is well founded. The resultant tableau is, in some sense, the *same* as the recursive tree; and once this is seen, the result is obvious. The work of the proof is just the task of making the correspondence explicit.

Proof. (of Theorem 38) It should be clear that a tableau for φ can be construed as a recursively enumerable tree $S_{\ulcorner \varphi \urcorner}$.[12] Moreover, with a slight tweak, $\varphi's$ being in A is reducible to the well-foundedness of $S_{\ulcorner \neg\varphi \urcorner}$.[13] Thus, A is Π^1_1.

We now verify that A is Π^1_1-hard. Take an arbitrary Π^1_1 set, B. By Definition 35, there is some recursively enumerable p-tree S such that

$$n \in B \leftrightarrow S_n \text{ is well founded.}$$

[12] See the proof of Lemma 48 for more detail.

[13] The tweak required is to ensure that an infinite branch ensues when a branch is not closed. For example, the branch consisting of just $0 = 0$ is open but not infinite. To remedy this, we just conjoin the sentence $\forall x \ x = x$ to the sentence at the top of the tableau. This new sentence has no effect on the outcome of the tableau but will ensure that an open branch is infinite.

By the diagonal lemma,[14] let $\psi(n,m)$ be such that

$$\psi(n,m) \leftrightarrow ((n,\bar{m}) \in S \to \forall i\, T\ulcorner\psi(n, m^\frown\langle i\rangle)\urcorner).$$

Let $f : \omega \to \omega$ be the recursive function such that

$$n \mapsto \ulcorner\psi(n,\langle\rangle)\urcorner.$$

It suffices to show that

$$f(n) \in A \;\Leftrightarrow \vdash_{sK} \psi(n,\langle\rangle) \;\Leftrightarrow\; S_n \text{ is well founded.}$$

The first ($\Leftrightarrow$) follows by definition, so we concentrate on the second.

($\Rightarrow$) Suppose S_n is not well founded. Then there is some α such that for all m, $(n,\alpha|_m) \in S$. Fix such an α. It will suffice to show that there is an open branch $\mathcal{B}$ in the tableau commencing with $\neg\psi(n,\langle\rangle)$. We construct $\mathcal{B}$ by recursion and show by induction that:

- for all $m \in \omega$, $\mathcal{B}_m$ is not closed; and
- that $\mathcal{B} = \bigcup_m \mathcal{B}_m$ is fully discharged.

This then gives us an open branch which suffices for the proof.

Let $\mathcal{B}_0$ be the branch (indicated in bold) of the tableau commencing as follows:

$$\boldsymbol{\neg\psi(n,\langle\rangle)}$$
$$|$$
$$\boldsymbol{\neg((n,\bar{\langle\rangle}) \in S \to \forall i T\ulcorner\psi(n,\langle i\rangle)\urcorner)}$$
$$|$$
$$\boldsymbol{(n,\bar{\langle\rangle}) \in S}$$
$$\boldsymbol{\neg\forall i(T\ulcorner\psi(n,\langle i\rangle)\urcorner)}$$

$$\neg T\ulcorner\psi(n,\langle 0\rangle)\urcorner \qquad \neg T\ulcorner\psi(n,\langle 1\rangle)\urcorner \quad \dots \quad \boldsymbol{\neg T\ulcorner\psi(n,\langle\alpha|_1\rangle)\urcorner} \;\dots$$

The tree branches infinitely here, so we must eventually come to the first initial segment of α: $\alpha|_1$. Now suppose we are given $\mathcal{B}_m$. $\mathcal{B}_{m+1}$ is then obtained by extending the branch along the bold part of the following tableau.

---

[14] We are being a little loose with the notation here. First, we shall be somewhat relaxed about the use of numerals. Thus, we shall mostly just write n instead of the more correct $\underline{\mathrm{n}}$. Second, we note that it is essential that n, m and i are free in the diagonal sentence above. We also ought to define a recursive function $\ulcorner\psi(\cdot,\cdot^\frown\langle\cdot\rangle)\urcorner : \omega^3 \to \omega$ which takes m, n and i and returns the code number of the formula $\psi(\underline{\mathrm{m}},\underline{\mathrm{n}}^\frown\langle\underline{\mathrm{i}}\rangle)$ where the numerals for m, n and i have been simultaneously substituted into the three distinct variable places in $\psi(\cdot,\cdot^\frown\langle\cdot\rangle)$ respectively. We may then represent such a function using the language of arithmetic. The reader will see that this is a simple, albeit tedious, task. Finally, $(n,\bar{m}) \in S$ is not part of the object language. However, since S is recursively enumerable and m represents the sequence $\bar{m}$, there is no harm in this shorthand.

$$\boldsymbol{\neg\psi(n, \langle\alpha|_m\rangle)}$$

$$\boldsymbol{\neg((n, \alpha|_m) \in S \to \forall i T\ulcorner\psi(n, \langle\alpha|_m\rangle^\frown\langle i\rangle)\urcorner)}$$

$$\boldsymbol{(n, \alpha|_m) \in S}$$

$$\boldsymbol{\neg\forall i(T\ulcorner\psi(n, \langle\alpha|_m\rangle^\frown i)\urcorner)}$$

$$\neg T\ulcorner\psi(n, \langle\alpha|_m\rangle^\frown\langle 0\rangle)\urcorner \qquad \neg T\ulcorner\psi(n, \langle\alpha|_m\rangle^\frown\langle 1\rangle)\urcorner \quad \ldots \quad \boldsymbol{\neg T\ulcorner\psi(n, \langle\alpha|_{m+1}\rangle)\urcorner} \quad \ldots$$

Again, at every level, we must eventually come to the appropriate initial segment of α: $\alpha|_{m+1}$. In this manner, we use α as a guide for the construction of the infinite open branch.

By construction, we see that for all m every sentence on $\mathcal{B}_m$ is discharged on the branch $\mathcal{B}_{m+1}$. The only truth free sentences on $\mathcal{B}$ are of the form $(n, \alpha|_m) \in S$ for some m, and each of these are true by our assumption that α is an infinite path through S; thus $\mathcal{B}$ is not closed. Thus $\mathcal{B} = \bigcup_m \mathcal{B}_m$ is open.

($\Leftarrow$) Suppose $\nvdash_{sK} \psi(n, \langle\rangle)$. Then for any tableau commencing with $\neg\psi(n, \langle\rangle)$ there will be an open (and infinite) branch. It will suffice to find an α such that for all m, $(n, \alpha|_m) \in S$. We use an open branch $\mathcal{B}$ of a tableau $\mathcal{T}$ to do this. We recursively define a particular tableau $\mathcal{T}$ as follows:

Let $\mathcal{T}_0$ be the tableau commencing with $\neg\psi(n, \langle\rangle)$ and proceeding as follows:

$$\neg\psi(n, \langle\rangle)$$

$$\neg((n, \bar{\langle\rangle}) \in S \to \forall i T\ulcorner\psi(n, \langle i\rangle)\urcorner)$$

$$(n, \bar{\langle\rangle}) \in S$$

$$\neg\forall i(T\ulcorner\psi(n, \langle i\rangle)\urcorner)$$

$$\neg T\ulcorner\psi(n, \langle 0\rangle)\urcorner \quad \neg T\ulcorner\psi(n, \langle 1\rangle)\urcorner \quad \ldots$$

At stage m we have a tableau $\mathcal{T}_m$ such that the final point on each branch is a sentence of the form $\neg T\ulcorner\psi(n, j)\urcorner$ where $\bar{j}$ is a sequence of length $m + 1$. We extend each of these branches in the following manner:

$$\neg\psi(n, j)$$
$$|$$
$$\neg((n, \bar{j}) \in S \to \forall i T^{\ulcorner}\psi(n, j^\frown\langle i\rangle)^{\urcorner})$$
$$|$$
$$(n, \bar{j}) \in S$$
$$\neg\forall i(T^{\ulcorner}\psi(n, j^\frown\langle i\rangle)^{\urcorner})$$
$$\neg T^{\ulcorner}\psi(n, j^\frown\langle 0\rangle)^{\urcorner} \quad \neg T^{\ulcorner}\psi(n, j^\frown\langle 1\rangle)^{\urcorner} \quad \ldots$$

Call the result $\mathcal{T}_{m+1}$. Let $\mathcal{T}$ be the limit of the $\mathcal{T}_m$'s. Now fix an (infinite) open branch $\mathcal{B}$ from $\mathcal{T}$. We use the sentences on $\mathcal{B}$ of the form $(n, j) \in S$ to define the appropriate α. Formally, we let $\alpha \in \omega^\omega$ be such that

$$\alpha = \{(i, k) \mid \exists \bar{m} \in \omega^{<\omega} \text{ `}(n, \bar{m}) \in S\text{' occurs on } \mathcal{B} \text{ and } \exists i < lh(m)\ m(i) = k\}.$$

Clearly $\alpha \in \omega^\omega$ and α is a path through the tree S_n. Thus S_n is not well-founded. □

We now use the tableau from the proof above to make a definition that will be useful in the next section.

Definition 40. Let $S \subseteq \omega^{<\omega}$ be an arithmetic tree. Let $S^\dagger = \{(0, \langle n_1, ..., n_k\rangle) \in \omega \times \omega^{<\omega} \mid \langle n_1, ..., n_k\rangle \in S\}$. Using the diagonal lemma in the same way as we did in the proof above, let $\psi_S(n, m)$ be such that

$$\psi_S(n, m) \leftrightarrow ((n, \bar{m}) \in S^\dagger \to \forall i\, T^{\ulcorner}\psi_S(n, \bar{m}^\frown\langle i\rangle)^{\urcorner})).$$

Let the *canonical tableau for* S be the tableau commencing with $\neg\psi_S(0, \langle\rangle)$ and constructed exactly as in the second half of the proof of Theorem 38.

Note that the "0" plays no real role. It is just a place filler that allows us to re-use much the same tableau as in Theorem 38. Also note that below, we shall mostly be concerned with trees $S \subseteq \omega^{<\omega}$ that are recursive.

Proposition 41. *S is well-founded iff $\vdash_{sK} \psi_S(0, \langle\rangle)$ (i.e., the canonical tableau for S is closed).*

5.2. **Fixed point height.** We now prove that the fixed point height for the strong Kleene truth definition is ω_1^{CK}. Our strategy is to confirm that the ranks of the tableau proofs have a supremum of ω_1^{CK} and use that fact to show that the closure ordinal for Kripke's original definition is also ω_1^{CK} (i.e., Definition 3). There is a sense in which we almost have the result in our grasp from the beginning. We note the following fact can be easily found or established [7, 14, 9].

Fact 42. *The supremum of the tree-ranks of trees on $\omega^{<\omega}$ of complexity between Δ_1^0 (recursive) and Σ_1^1 is ω_1^{CK}: the supremum of the recursive ordinals.*

It should be pretty clear (although we'll discuss it further) that each of our tableaux is essentially a recursive tree. Thus, the tableau-ranks of our tableau will be bounded by

ω_1^{CK}. Moreover, using our canonical tableau from the proof of Theorem 38, it is easy to see that for every recursive tree there will be a tableau with much the same rank - in fact, a little greater. Thus, the recursive ordinals are exhausted and we get a lower bound of ω_1^{CK} too.

This sketch does not, however, give us our target result. We want to calculate the closure ordinal of Kripke's original definition from [8]. Nonetheless, this sketch does tell us that ω_1^{CK} is the natural conjecture and should guide our intuitions through the proof. In making all this precise, we shall establish a number of other comparative results. To get a clearer idea of the strategy, the reader may prefer to work backwards from the main result, Corollary 55.

In this section, we shall make use of a restricted form of our tableau in which we remove the closure condition which states that a branch containing a sentence $\varphi \in \mathcal{L}$ and its negation is closed. This has no effect on the completeness of the system. We shall also assume that we have a uniform recursive procedure for constructing tableau. This is just a way of ordering the moves so that we can associate with each sentence a particular tableau. For each φ we shall call this the *standard tableau* for φ.

Given that the stages of Kripke's inductive definition measure, loosely speaking, the number of truth predicates that are prefixed to a sentence, it will be useful to make a similar measure of truth-rank for the tableau system.

Definition 43. Let $\mathcal{T}$ be a strong Kleene tableau and let $\vec{\varphi}, \vec{\psi}$ be finite sequences of sentences which are (not necessarily proper) initial segments of branches in $\mathcal{T}$. Let

$$\vec{\varphi} \prec_T^{\mathcal{T}} \vec{\psi} \quad \leftrightarrow \quad \vec{\varphi} \text{ is a proper initial segment of } \vec{\psi} \text{ and}$$
$$\text{a truth rule (i.e., } (T) \text{ or } (\neg T)) \text{ is applied in the part of } \vec{\psi} \text{ extending } \vec{\varphi}$$

We first define the *truth-rank* of some finite branch $\vec{\varphi}$ in tableau $\mathcal{T}$ by recursion with a function $\rho_T^{\mathcal{T}} : (Sent_{\mathcal{L}_T})^{<\omega} \to On$ such that:

$$\rho_T^{\mathcal{T}}(\vec{\varphi}) = sup\{\rho_T^{\mathcal{T}}(\vec{\psi}) + 1 \mid \vec{\varphi} \prec_T^{\mathcal{T}} \vec{\psi}\}.$$

The *truth-rank* of a tableau $\mathcal{T}$ is $\rho_T^{\mathcal{T}}(\langle\varphi\rangle)$ where φ is at the top of the tableau. We let the *truth-rank* of a sentence φ, $\rho_T(\varphi)$, be the truth-rank of the standard tableau commencing with φ if such exists; otherwise ∞.

Remark 44. Observe that the truth-rank of a tableau is a coarser grained measure than its tableau-rank; i.e., the tableau-rank of a tableau will be greater than or equal to its truth-rank.

Proposition 45. *For all* $\varphi \in Sent_{\mathcal{L}_T}$, $\rho_T(\varphi) \leq \rho_{Tab}(\varphi)$.[15]

[15]For a couple of examples, we note that: $\rho_T(T^\ulcorner 0 \neq 0^\urcorner) = 1$ while $\rho_{Tab}(T^\ulcorner 0 \neq 0^\urcorner) = 2$; and $\rho_T(\forall n Tapp(n)) = \rho_{Tab}(\forall n Tapp(n)) = \omega$. The discrepancy in the finite cases is caused by a slightly eccentric difference between our

We now show that the truth-ranks of tableau exhaust the recursive ordinals and are bounded by their supremum. We let the *tree-rank* of a tree $S \subseteq \omega^{<\omega}$ be defined as follows. Let $\rho^S_{Tree} : \omega^{<\omega} \to On$ be defined by recursion for $s \in S$ such that

$$\rho^S_{Tree}(s) = sup\{\rho^S_{Tree}(t) + 1 \mid t \text{ extends } s \text{ in } S\}.$$

We let $\rho_{Tree}(S) = \rho^S_{Tree}(\langle\rangle)$.

Remark 46. Note that we now have five different ranking functions in play:

(1) ρ_Γ (Kripke's original - see Definition 15);
(2) ρ_Ξ (for the finer grained jump - see Definition 15);
(3) ρ_{Tab} (for tableau - see Definition 15);
(4) ρ_T (for calculating truth usage in a tableau - see Definition 43); and
(5) ρ_{Tree} (for trees on $\omega^{<\omega}$ - see remarks immediately above).

Lemma 47. *For any well-founded recursive tree $S \subseteq \omega^{<\omega}$ of tree-rank α, there is a sentence $\varphi \in Sent_{\mathcal{L}_T}$ such that:*

$$\rho_T(\varphi) = \rho_{Tree}(S).$$

Proof. Let $S \subseteq \omega^{<\omega}$be a well-founded recursive tree. Then it can be seen from the proof of Theorem 38, that the canonical tableau $\mathcal{T}$ for S commencing with $\neg\psi_S(0, \langle\rangle)$ is such that

$$\rho_T(\neg\psi_S(0, \langle\rangle)) = \rho_{Tree}(S).$$

We leave the proof for the reader noting that it is an induction on the tree-ranks of recursive trees. □

Lemma 48. *For all $\varphi \in Sent_{\mathcal{L}_T}$ the standard strong Kleene tableau $\mathcal{T}_\varphi$ for φ is isomorphic to a recursively enumerable tree $S \subseteq \omega^{<\omega}$: i.e., there is a structure preserving bijection between $\mathcal{T}_\varphi$ and S as trees.*

Proof. $\mathcal{T}_\varphi$ may be represented by a set of sequences of sentences closed under initial segments; i.e., $\mathcal{T}_\varphi$ is a tree on $(Sent_{\mathcal{L}_T})^{<\omega}$. By using our coding function $\ulcorner\cdot\urcorner : Sent_{\mathcal{L}_T} \cong \omega$, we may transform $\mathcal{T}_\varphi$ into a tree $S \subseteq \omega^{<\omega}$. Moreover, if we consider a sequence $s \in \omega^{<\omega}$ we see (by appeal to the Church-Turing thesis) that a Turing machine could be devised which verified whether $s \in S$; thus, S is recursively enumerable. □

Theorem 49. *The supremum of the truth ranks of $\varphi \in Sent_{\mathcal{L}_T}$ is ω_1^{CK}.*

Proof. (Lower bound) By Fact 42 and Lemma 47, we see that the recursive ordinals are exhausted by the truth-ranks of sentences $\varphi \in Sent_{\mathcal{L}_T}$. (Upper bound) By Lemma 48, we

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definition of ρ_{Tab} and ρ_T. An equivalent definition of ρ_{Tab} can be provided which has much the same form as the definition of ρ_T above. When defining $\prec^{\mathcal{T}}_{Tab}$ we ask for mere proper extensions, rather than also demanding that a truth-rule has been applied. We then define $\rho^{\mathcal{T}}_{Tab}$ in the same way. However, when we come to define the tableau-rank of the tableau, we take the rank of a point immediately above the root of the tableau: $\rho^{\mathcal{T}}_{Tab}(\langle\rangle)$. This is required for the equivalence, but not convenient below. We leave establishing their equivalence as an exercise for the reader who may wish to consult [9] see for any sentence $\varphi \in Sent_{\mathcal{L}_T}$, we may find a recursively enumerable tree S which is isomorphic as a tree to the tableau $\mathcal{T}_\varphi$ for φ ; thus, $\rho_{Tree}(S) = \rho_{Tab}(\varphi)$. Moreover, it is clear by Proposition 45 that $\rho_{Tab}(\varphi) \geq \rho_T(\varphi)$. Thus, for every sentence $\varphi \in Sent_{\mathcal{L}_T}$ there is a recursively enumerable tree S such that $\rho_{Tree}(S) \geq \rho_T(\varphi)$. Fact 42 then tells us that ω_1^{CK} is an upper bound on the truth-rank of sentences. □

We shall exploit this fact about truth-ranks to fix the closure ordinal for Kripke's inductive definition. We now show that there is a sense in which the T-rank of a sentence is always greater than its Γ-rank. We shall exploit this to put an upper bound on the Γ-ranks of sentences.

Lemma 50. *For all* $\varphi \in \Gamma^+_{sK}$, $\rho_\Gamma(\varphi) \leq \rho_{Tab}(\neg\varphi) + 1$.[16]

Proof. We show that for all $\varphi \in \Gamma^+_{sK}$:

(1) $\rho_{Tab}(\neg\varphi) + 1 \geq \rho_\Xi(\varphi)$; and
(2) $\rho_\Xi(\varphi) \geq \rho_\Gamma(\varphi)$.

(1.) By induction on tableau rank. We suppose that for all $\beta < \alpha$ if $\rho_{Tab}(\neg\psi) = \beta$, then $\rho_\Xi(\psi) \leq \rho_{Tab}(\neg\psi) + 1$. Suppose that $\rho_{Tab}(\neg\varphi) = \alpha$. We show that for all the ways of forming such a closed tableau, $\rho_\Xi(\varphi) \leq \rho_{Tab}(\neg\varphi) + 1$.

Suppose φ is a true arithmetic atomic sentence. Then $\rho_{Tab}(\neg\varphi) = 1$ and $\rho_\Xi(\varphi) = 1$. Thus $\rho_{Tab}(\neg\varphi) + 1 \geq \rho_\Xi(\varphi)$.

Suppose $\varphi := \neg\neg\psi$. Then $\rho_{Tab}(\neg\varphi) = \rho_{Tab}(\neg\neg\neg\psi) = \rho_{Tab}(\neg\psi) + 1$, and by the induction hypothesis we have $\rho_\Xi(\psi) \leq \rho_{Tab}(\neg\psi) + 1$. Thus

$$\begin{aligned} \rho_{Tab}(\neg\neg\neg\psi) + 1 &= (\rho_{Tab}(\neg\psi) + 1) + 1 \\ &\geq \rho_\Xi(\psi) + 1 \\ &= \rho_\Xi(\neg\neg\psi). \end{aligned}$$

Suppose $\varphi := \psi \wedge \chi$. By the induction hypothesis we have $\rho_\Xi(\psi) \leq \rho_{Tab}(\neg\psi) + 1$ and $\rho_\Xi(\chi) \leq \rho_{Tab}(\neg\chi) + 1$. Then we see that,

$$\begin{aligned} \rho_{Tab}(\neg(\psi \wedge \chi)) + 1 &= sup\{\rho_{Tab}(\neg\psi) + 1, \rho_{Tab}(\neg\chi) + 1\} + 1 \\ &\geq sup\{\rho_\Xi(\psi), \rho_\Xi(\chi)\} + 1 \\ &= \rho_\Xi(\psi \wedge \chi). \end{aligned}$$

Suppose $\varphi := \neg(\psi \wedge \chi)$. Then since the tableau commencing with $\neg\neg(\psi \wedge \chi)$ is closed, it can be seen that for some $\zeta \in \{\psi, \chi\}$, $\rho_{Tab}(\neg\neg(\psi\wedge\chi)) \geq \rho_{Tab}(\neg\neg\zeta)+1$. It is here that we require the

[16] The $+1$ on the right hand side is caused by a discrepancy between the treatment of limit ordinals in ranking tableau and running Kripke's inductive definition: no sentence has Γ-rank β for β a limit ordinal.

restriction on the closure conditions in the standard tableau.[17] Without loss of generality, suppose ψ is such a ζ. Then using the induction hypothesis, we see that

$$\begin{aligned} \rho_{Tab}(\neg\neg(\psi \wedge \chi)) + 1 &\geq (\rho_{Tab}(\neg\neg\psi) + 1) + 1 \\ &\geq \rho_{\Xi}(\neg\psi) + 1 \\ &\geq \rho_{\Xi}(\neg(\psi \wedge \chi)). \end{aligned}$$

Suppose $\varphi := \forall x \psi(x)$. Then using the induction hypothesis we see that

$$\begin{aligned} \rho_{Tab}(\neg\forall x\psi(x)) + 1 &= sup\{\rho_{Tab}(\neg\psi(\underline{\mathrm{n}})) + 1 \mid n \in \omega\} + 1 \\ &\geq sup\{\rho_{\Xi}(\neg\psi(\underline{\mathrm{n}})) \mid n \in \omega\} + 1 \\ &= \rho_{\Xi}(\forall x\psi(x) \mid n \in \omega\}. \end{aligned}$$

Suppose $\varphi := \neg\forall x\psi(x)$. Then since the tableau commencing with $\neg\neg\forall x\psi(x)$ is closed, it can be seen that there is some $n \in \omega$ such that $\rho_{Tab}(\neg\neg\forall x\psi(x)) \geq \rho_{Tab}(\neg\neg\psi(\underline{\mathrm{n}})) + 1$. Fix such an n. Then using the induction hypothesis we have:

$$\begin{aligned} \rho_{Tab}(\neg\neg\forall x\psi(x)) + 1 &\geq (\rho_{Tab}(\neg\neg\psi(\underline{\mathrm{n}})) + 1) + 1 \\ &\geq \rho_{\Xi}(\neg\psi(\underline{\mathrm{n}})) + 1 \\ &\geq \rho_{\Xi}(\neg\forall x\psi(x)). \end{aligned}$$

Suppose $\varphi := T\ulcorner\psi\urcorner$. Then

$$\begin{aligned} \rho_{Tab}(\neg T\ulcorner\psi\urcorner) + 1 &= (\rho_{Tab}(\neg\psi) + 1) + 1 \\ &\geq \rho_{\Xi}(\psi) + 1 \\ &= \rho_{\Xi}(T\ulcorner\psi\urcorner). \end{aligned}$$

Suppose $\varphi := \neg T\ulcorner\psi\urcorner$. Then

$$\begin{aligned} \rho_{Tab}(\neg\neg T\ulcorner\psi\urcorner) + 1 &= (\rho_{Tab}(\neg\neg\psi) + 1) + 1 \\ &\geq \rho_{\Xi}(\neg\psi) + 1 \\ &= \rho_{\Xi}(\neg T\ulcorner\psi\urcorner). \end{aligned}$$

(2.) By induction on Ξ-rank. We suppose that for all $\beta \leq \alpha$ if $\psi \in \Xi_{\beta+1} \backslash \Xi_{\beta}$, then $\psi \in \Gamma^{+}_{\beta+1}$. We then suppose that $\varphi \in \Xi_{\alpha+1} \backslash \Xi_{\alpha}$. We must show that $\varphi \in \Gamma^{+}_{\alpha+1}$. It will suffice to consider the ways that φ could have entered $\Xi_{\alpha+1}$.

[17] To see why this is the case, let φ be the sentence $0 \neq 1 \wedge 0 = 1$ and consider the tableaux for $\varphi \wedge \neg\varphi$ which respectively do and do not include the inconsistency closure rule. With the rule, the former tableau has rank 3; but without that closure rule, the other tableau has rank 5 (assuming the obvious ordering of moves). However, the tableau for the first conjunct φ has rank 3; thus, with the inconsistency closure condition the claim above is violated. We leave it to the reader to satisfy themselves that the standard tableau - which omits this closure condition - satisfies the claim. Similar remarks apply to the $\neg\forall$ case.

($\neg\neg$) Suppose that $\varphi := \neg\neg\psi$. Then we can see that it must be the case that $\psi \in \Xi_{\gamma+1}\backslash\Xi_\gamma$ where $\alpha = \gamma + 1$.[18] Then by induction hypothesis, we see that $\psi \in \Gamma^+_{\gamma+1}$. Thus $Val_{\Gamma_\gamma}(\psi) = Val_{\Gamma_\gamma}(\neg\neg\psi) = 1$; and so $(\neg\neg\psi) \in \Gamma^+_{\gamma+1} \subseteq \Gamma^+_{\alpha+1}$.

The other *logical* cases are similar, so we conclude by looking at the truth cases.

(T) Suppose $\varphi := T\ulcorner\psi\urcorner$. Then it must be the case that $\psi \in \Xi_{\gamma+1}\backslash\Xi_\gamma$ where $\alpha = \gamma + 1$. Then by induction hypothesis, we see that $\psi \in \Gamma^+_{\gamma+1}$. Thus, $Val_{\Gamma_{\gamma+1}}(T\ulcorner\psi\urcorner) = 1$; and so $(T\ulcorner\psi\urcorner) \in \Gamma^+_{\gamma+2} = \Gamma^+_{\alpha+1}$.

($\neg T$) Suppose $\varphi := \neg T\ulcorner\psi\urcorner$. Then it must be the case that $(\neg\psi) \in \Xi_{\gamma+1}\backslash\Xi_\gamma$ where $\alpha = \gamma + 1$. By induction, we see that $(\neg\psi) \in \Gamma^+_{\gamma+1}$ and so it can be seen that $\psi \in \Gamma^-_{\gamma+1}$. Thus, $Val_{\Gamma_{\gamma+1}}(T\ulcorner\psi\urcorner) = 0$, which means that $Val_{\Gamma_{\gamma+1}}(\neg T\ulcorner\psi\urcorner) = 1$; and so, $(\neg T\ulcorner\psi\urcorner) \in \Gamma^+_{\gamma+2} = \Gamma^+_{\alpha+1}$.

□

This gives us an upper bound on the strong Kleene fixed point.

Corollary 51. *The height of the strong Kleene fixed point is* $\leq \omega_1^{CK}$.

Proof. By Lemma 50, we see that the Γ-rank of any sentence $\varphi \in \Gamma^+_{SK}$ is less than its tableau-rank $+1$. Since Lemma 48 and Fact 42 tell us that the tableau ranks are bounded by a limit ordinal ω_1^{CK}, the result follows. □

Finally, we need to show that the recursive ordinals are exhausted by the Γ-ranks of sentences. To do this we introduce a bridging tableau for the strong Kleene system. We introduce this for similar reasons as we did in the equivalence proofs for the supervaluation systems. The bridging tableau allows us to cordon off that part of an ordinary tableau which takes us up to the first application of the truth rules. With this in hand, we have something much closer to Kripke's original jump function and a useful tool for the execution of the inductive proofs that follow. The system is very similar to the van Fraassen bridging tableau except that we introduce a new predicate U to the language.

5.2.1. *Starting conditions* ($\Vdash_{sK}$). To attempt to show that $\Phi \Vdash_{sK} \chi$, we commence the tableau by placing $\neg\chi$ at the root.

5.2.2. *Rules* ($\Vdash_{sK}$). We take the rules ($\wedge$), ($\neg\wedge$), ($\neg\neg$), ($\forall$) and ($\neg\forall$). These are just the connective rules and quantifier rules from Section 2.2.2.

In place of the truth rules, we add the following *axiom rules*. Given a set of axioms Φ we may apply either of the rules below at any point in the construction of a branch.

$$(AxU) \quad \begin{array}{c} | \\ U\ulcorner\varphi\urcorner \\ \text{where } \varphi \in \Phi^+ \end{array} \qquad\qquad (Ax\neg U) \quad \begin{array}{c} | \\ \neg U\ulcorner\varphi\urcorner \\ \text{where } \varphi \in \Phi^- \end{array}$$

[18] Note that α could not be a limit ordinal since nothing new is added in the limit stages of the Ξ construction.

Remark 52. We might think of this new predicate U as the official truth, while T is merely provisional. This allows us to avoid problems with pathological sentences like the liar sentence.

5.2.3. *Closing conditions* ($\Vdash_{sK}$) *.* A branch $\mathcal{B}$ closes if either:

- for some sentence φ, $\neg T\ulcorner\varphi\urcorner$ and its negation $U\ulcorner\varphi\urcorner$ occurs on $\mathcal{B}$;
- for some sentence φ, $T\ulcorner\varphi\urcorner$ and its negation $\neg U\ulcorner\varphi\urcorner$ occurs on $\mathcal{B}$;
- for some sentence φ, $U\ulcorner\varphi\urcorner$ and its negation $\neg U\ulcorner\varphi\urcorner$ occurs on $\mathcal{B}$; or
- a false arithmetic sentence occurs on $\mathcal{B}$.

If all of the branches in the tableau commencing with $\neg\psi$ and axioms $\langle\Phi^+,\Phi^-\rangle$ close, then the tableau is closed and we write $\langle\Phi^+,\Phi^-\rangle \Vdash_{sK} \psi$.

Lemma 53. $\langle\Phi^+,\Phi^-\rangle \Vdash_{sK} \psi$ *iff* $Val_{\langle\Phi^+,\Phi^-\rangle}(\psi) = 1$.

Remark. The proof is very similar to that of Lemma 23, although care needs to be taken to deal with the cases when the valuation function is undefined.

Lemma 54. *For any recursive tree* $S \subseteq \omega^{<\omega}$ *there is a sentence* $\varphi \in Sent_{\mathcal{L}_T}$ *whose* Γ*-rank is greater than or equal to the tree-rank of* S*.*

Proof. Let $S^\dagger = \{(0,\bar{m}) \mid \bar{m} \in S\}$ and $\psi_S(n,m)$ be as described in Definition 40; and let $\mathcal{T}$ be the canonical tableau commencing with $\neg\psi_S(0,\langle\rangle)$. We first claim that the $\psi_S(0,\langle\rangle)$ has Γ-rank greater than or equal to the truth rank of $\mathcal{T}$.

Before we commence the main business of the proof, we make a couple of useful observations about the canonical tableau. It should be clear that every sentence that appears in the tableau appears only once. Thus, we may abuse our rank notation and consider the truth-rank of sentences in $\mathcal{T}$ rather than the truth-ranks of finite sequences of sentences. We shall thus write $\rho_T^{\mathcal{T}}(\psi_S)$ to mean the truth-rank of ψ in the tableau $\mathcal{T}$.

We also note that the canonical tableau $\mathcal{T}_1$ for $\neg\psi_S(0,\langle m_1,...,m_p,n_1,...,n_k\rangle)$ is exactly the same as that part of the canonical tableau $\mathcal{T}_2$ for

$$\neg\psi_S(0,\langle m_1,...,m_p\rangle)$$

which proceeds from the point $\neg\psi_S(0,\langle m_1,...,m_p,n_1,...,n_k\rangle)$ occurring in $\mathcal{T}_2$. We assume that $\neg\psi_S(0,\langle m_1,...,m_p,n_1,...,n_k\rangle)$ occurs in $\mathcal{T}_2$; i.e., the branch has not closed before it gets the chance to be added. Moreover, it can easily be seen that

$$\neg\psi_S(0,\langle m_1,...,m_p,n_1,...,n_k\rangle)$$

has the same truth-rank regardless of which tableau we calculate it in; i.e.,

$$\rho_T^{\mathcal{T}_1}(\neg\psi_S(0,\langle m_1,...,m_p,n_1,...,n_k\rangle)) = \rho_T^{\mathcal{T}_2}(\neg\psi_S(0,\langle m_1,...,m_p,n_1,...,n_k\rangle)).$$

We can thus ignore the tableau superscripts and just write $\rho_T(\neg\psi_S(0,\langle m_1,...,m_p,n_1,...,n_k\rangle))$ for sentences of the form $\neg\psi_S(0,\langle m_1,...,m_p,n_1,...,n_k\rangle)$ to mean $\rho_T^{\mathcal{T}}(\neg\psi_S(0,\langle m_1,...,m_p,n_1,...,n_k\rangle))$ for some canonical tableau $\mathcal{T}$ in which $\neg\psi_S(0,\langle m_1,...,m_p,n_1,...,n_k\rangle)$ occurs.

We now proceed to establish the lemma by induction on truth-rank. Suppose that for all sequences $(n_1, ..., n_k) \in \omega^{<\omega}$ and for all $\beta < \alpha$, if

$$\rho_T(\neg\psi_S(0, \langle n_1, ..., n_k\rangle)) = \beta,$$

then

$$\rho_T(\neg\psi_S(0, \langle n_1, ..., n_k\rangle)) \leq \rho_\Gamma(\psi_S(0, \langle n_1, ..., n_k\rangle)).$$

Now suppose that $\rho_T(\neg\psi_S(0, \langle m_1, ..., m_l\rangle)) = \alpha$. Let $\mathcal{T}^\dagger$ be the closed canonical tableau commencing with $\neg\psi_S(0, \langle m_1, ..., m_l\rangle)$ with truth rank α. We establish that the induction hypothesis also holds for $\neg\psi_S(0, \langle m_1, ..., m_l\rangle)$. Let

$$D^+ = \{\psi_S(0, \langle m_1, ..., m_l, i\rangle) \mid i \in \omega\}.$$

Then clearly, D^+ is the $\subseteq$-minimal set of sentences such that $\langle D^+, \emptyset\rangle \Vdash_{sK} \psi_S(0, \langle m_1, .., m_l\rangle)$. Then for all $\delta \in D^+$ we have $\rho_T(\neg\delta) = \beta < \alpha$ for some β; and by induction, we have $\rho_T(\neg\delta) \leq \rho_\Gamma(\delta)$. Let $\gamma = sup(\rho_\Gamma\text{“}D^+) = sup\{\rho_\Gamma(\psi_S(0, \langle m_1, ..., m_l, i\rangle)) \mid i \in \omega\}$. Then

$$\begin{aligned} \rho_T(\neg\psi_S(0, \langle m_1, ..., m_l\rangle)) &= sup(\{\rho_T(\neg\psi_S(0, \langle m_1, ..., m_l, i\rangle)) + 1 \mid i \in \omega\}) \\ &\leq sup(\{\rho_\Gamma(\psi_S(0, \langle m_1, ..., m_l, i\rangle)) + 1 \mid i \in \omega\}) \\ &\leq sup(\{\rho_\Gamma(\psi_S(0, \langle m_1, ..., m_l, i\rangle)) \mid i \in \omega\}) + 1 \\ &= \gamma + 1. \end{aligned}$$

It will suffice to show that $\rho_\Gamma(\psi_S(0, \langle m_1, ..., m_l\rangle)) = \gamma+1$. It can be seen that $D^+ \subseteq \Gamma^+_\gamma$ but $D^+ \not\subseteq \Gamma^+_\xi$ for any $\xi < \gamma$. From the first of these facts we see that $\langle\Gamma^+_\gamma, \Gamma^-_\gamma\rangle \Vdash_{sK} \psi_S(0, \langle m_1, ..., m_l\rangle)$; and by Lemma 53, $Val_{\Gamma_\gamma}(\psi_S(0, \langle m_1, ..., m_l\rangle)) = 1$; so $\psi_S(0, \langle m_1, ..., m_l\rangle) \in \Gamma^+_{\gamma+1}$. We then claim that

$$\psi_S(0, \langle m_1, ..., m_l\rangle) \notin \Gamma^+_\gamma.$$

Suppose not. Then for some $\zeta < \gamma$, $\psi_S(0, \langle m_1, ..., m_l\rangle) \in \Gamma^+_{\zeta+1}$. Then by Lemma 53, we see that $\langle\Gamma^+_\zeta, \Gamma^-_\zeta\rangle \Vdash_{sK} \psi_S(0, \langle m_1, ..., m_l\rangle)$; but then $D^+ \subseteq \Gamma^+_\xi \not\subseteq \Gamma^+_\gamma$ contradicting the minimality of γ.

We have now shown that the Γ-rank of $\psi_S(0, \langle\rangle)$ is greater than or equal to the truth rank of $\neg\psi_S(0, \langle\rangle)$. But from the proof of Lemma 47 it can be seen that the truth rank of $\neg\psi_S(0, \langle\rangle)$ is equal to the tree-rank of S. □

Corollary 55. *The height of the strong Kleene fixed point is* ω_1^{CK}.

Proof. (Upper bound) By Corollary 51. (Lower bound) By Lemma 54, we see that the Γ-ranks of sentences from $Sent_{\mathcal{L}_T}$ exhaust the recursive ordinals. □

6. Conclusion

We have provided simple infinitary tableau systems for the minimal fixed points based on the strong Kleene, van Fraassen supervaluation and Cantini supervaluation schemes. Moreover, we have indicated how modifications may be made so that other semantic truth definitions may also be given proof systems. We have used this approach to provide a simple proof of the complexity of these definitions and a direct proof that that closure ordinal of Kripke's strong Kleene definition is ω_1^{CK}. In the future, it is hoped that these techniques may be useful in the provision of consistency proofs for axiomatic theories and logics of truth and as a guide for the development of new axiomatic theories.

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