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## Traffic control and route choice; capacity maximization and stability

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### Abstract

This paper presents idealised natural general and special dynamical models of day-to-day re-routing and of day to day green-time response. Both green-time response models are based on the responsive control policy  $P_\theta$  introduced in Smith (1979a, b, c 1987). Several results are proved. For example, it is shown that, for any steady feasible demand within a flow model, if the general day to day re-routing model is combined with the general day to day green-time response model then under natural conditions any (flow, green-time) solution trajectory cannot leave the region of supply-feasible (flow, green-time) pairs and costs are bounded. Throughput is maximised in the following sense. Given any constant feasible demand; this demand is met as any routing / green-time trajectory evolves (following either the general or the special dynamical model). The paper then considers simple “pressure driven” responsive control policies, with explicit signal cycles of fixed positive duration. A possible approach to dynamic traffic control allowing for variable route choices is outlined. It is finally shown that modified Varaiya (2013) and Le at al (2013) pressure-driven responsive controls may not maximise network capacity, by considering a very simple one junction network. It is shown that (with each of these two modified policies) there is a steady demand within the capacity of the network for which there is no Wardrop equilibrium consistent with the policy. In contrast, responsive  $P_\theta$  on this simple network does maximise throughput at a quasi-dynamic user equilibrium consistent with  $P_\theta$ ; queues and delays remain bounded in natural dynamical evolutions in this case. It is to be expected that this  $P_\theta$  result may be extended to allow for certain time-varying demands on a much wider variety of networks; to show that this is indeed the case is a challenge for the future.

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### 1. Introduction

It is important

- (1) to use traffic signal control to make good use of the capacity of a given road network and
- (2) to do the best to ensure that the network, with the traffic control operating, is stable.

This paper considers both (1) and (2) within a day to day model with various responsive signal setting strategies.

First the paper considers, in sections 2-4, two simple dynamical [flow + green-time] models involving route flows and signal green-times in which the signal adjustments seek to ensure that consequent natural travellers' re-routing

decisions make the best utilisation of the capacity available on a given road network. In these two models (which we call “the general model” and “the special model”) signal setting changes actively encourage congestion-reducing route-swaps in the future; both models maximise network capacity under natural conditions. The capacity maximising effect arises because both models utilise the  $P_0$  signal control policy. This policy has been studied previously (see Smith (1979a, b, c, 1987, 2011), Smith et al (1987), Smith and Ghali (1990), Smith and van Vuren (1993), Smith and Mounce (2011), Smith (2015) and Liu and Smith (2015)). Both the performance and stability of the re-routeing – control interactions is considered. The dynamical models here, in sections 2-4, do not involve explicit queues.

Then, in sections 5-8, distributed traffic control / routeing models involving explicit queues are considered. Here the signal control policies include not only  $P_0$  but also naturally modified versions of control policies suggested by Varaiya (2013) and Le et al (2013); here called policy MV and policy ML. It is shown that neither MV nor ML are certain to maximise network capacity when travellers are free to choose their own routes, by displaying a network where neither of these control policies is consistent with equilibrium route choices by drivers: natural corresponding day to day models give rise to ever-increasing queues (if users continually swap to cheaper routes). It is shown that, on the other hand, the  $P_0$  policy does maximise the capacity of this network at a user-equilibrium routeing pattern, where all drivers are on cheapest routes: queues are then bounded in natural day to day models.

The policies suggested by Varaiya (2013) and Le et al (2013) are both motivated by the paper by Tassiulas and Ephremides (1992); on stability in the control of constrained queueing networks. This initial work was aimed at ensuring queueing stability (and maximum throughput) in multi-hop radio networks. In these networks only certain sets of nodes are allowed to transmit simultaneously due to power and interference limitations. These sets of nodes are rather like signal stages at junctions where only those links in a single stage are given green simultaneously.

### 1.1. Outline description and purpose of the day to day control/routeing models introduced here

The central control variables in severely congested networks are *green-times*; these are the proportions of time different links and stages are given green. We also (especially in sections 2-4) utilise red-times rather than green-times: using red-times gives an intuitive way of adding signal timings into traffic routeing models.

Sections 2-4 obtain capacity-maximization and stability results. In these sections the central formula for the “pressure” on a stage arises from the  $P_0$  signal setting policy. It is shown (in sections 2 – 4) that with this policy the routeing – control dynamical loop is both capacity maximising and stable for a general network without queues. Sections 5-8 show that other policies do not necessarily have this capacity-maximisation property.

Each dynamical [routeing and signal setting] model described in this paper may be regarded as a model of a system periodically updated by some new choices of route (by say car drivers) and some new choices of signal timings (by a signal engineer or by an automatic control system). To be specific in this paper we generally think of both the route choice and signal control dynamics as operating from day to day. (A more general context is possible: this is “epoch to epoch”. In this case both short-term within day route swapping and longer term week to week or month to month route swapping may be considered, very approximately along the lines described here.)

Route flow changes (in the general and the special model) are driven by the following principle. *On each day:*

*for each route with positive flow yesterday, some of that flow may swap today  
but only to a route which was less costly yesterday (and joins the same OD pair).* (1.1)

This is a natural if rather conservative behavioural assumption and depends on the definition of route cost. There is here no compulsion to swap route-flow; yesterday’s route flows are permitted to remain the same today. (Both the route-choice models and the above principle depends on the definition of route cost.) If no route flow changes consistent with (1.1) are possible then the [route flow, green-time] distribution is a Wardrop or a routeing equilibrium. (See Wardrop (1952).) Green-time changes (in both the general and special model presented here) are driven by the following principle: *On each day:*

*for each stage with positive green-time yesterday, some of that green-time may be swapped today  
but only to a stage which was under more pressure yesterday (and is at the same junction).* (1.2)

Again this is a natural if rather conservative responsive signal setting principle and depends on the definition of stage pressure. There is in this principle no compulsion to change signal timings; yesterday’s timings may remain today.

(The signal changing principle (1.2) depends on the definition of stage pressures.) If no change in green-time is possible consistent with (1.2) then the [route flow, green-time] distribution will be called a green-time equilibrium.

### 1.2. A brief context

The central work concerning traffic equilibria in capacitated networks (without traffic signals) is in Beckmann et al (1956). Allsop (1974), Gartner (1976) and Dickson (1981) were among the first to point to the need to combine models of route choice and traffic signal control; partly so that optimal controls taking account of routeing reactions might be found. This approach has been pursued by Meneguzzo (1997), Maher et al (2001) and many others.

Gartner et. al. (1975) considers a linear programming method for optimising signal timings assuming that routeing is fixed; Gartner (1983) designed the OPAC control system; Van Vuren and Van Vliet (1992) was an early study of route choice and signal control; Smith and van Vuren (1993) considered the equilibrium problem with responsive traffic control from a theoretical viewpoint. Hu and Mahmassani (1997), Liu et. al. (2006), Liu (2010) and Flötteröd and Liu (2014) have considered day to day evolution with reactive signal control using a micro-simulation model. Heydecker (2004), has considered modern objectives of traffic signal control; Aboudolas et al (2009) and Maher et al. (2013) consider different signal control optimisation methods without regarding route choices. Taale and van Zuylem (2001) provide an overview of the assignment / control problem and Schlaich and Haupt (2012) describe a large scale implementation of routeing and control within VISUM software with a view to determining suitable timings for a whole network. Shepherd (1992) gives a review of real-life traffic control systems.

Cantarella et al (1991), Cantarella (2010) and Cascetta et al (2006) consider the choice of optimal controls taking account of route choices; stability issues not dissimilar to those considered here arise.

Dynamical route-swap methods have been considered by Cascetta (1989), Bellei et al. (2005), Nie and Zhang (2005) and Nie (2010). Mounce (2006, 2009), Mounce and Carey (2011) and Mounce and Smith (2007) present route-swap results which are related to those presented here.

Bie and Lo (2010) have considered stability and attraction domains arising in route swap models and He et al (2010) have considered link-based models of route swapping.

Quasi-dynamic equilibrium networks, with explicit capacity constraints and explicit queues, have been studied by Bliemer et al (2012), Nesterov and de Palma (2003). These models have been combined with control by Thompson and Payne (1975), Smith (1987), Yang and Yagar (1995) and Yang (1996).

The day to day systems studied here are generalisations of day to day dynamical systems studied in Smith and van Vuren (1993). In that paper on day 1 the signals are held fixed and the route flows are equilibrated; on day 2 the flows are kept fixed and the signals are updated according to the policy being studied; on day 3 the signals are held fixed and the route flows are equilibrated; on day 4 the flows are kept fixed and the signals are updated according to the policy being studied; on day 5 the signals are held fixed and the route flows are equilibrated . . . . . Here in this paper (a) the adjustments of signals and flows is simultaneous (some adjustment of both may well occur every day) and also (b) each flow adjustment is not necessarily to an equilibrium (although that is not ruled out) and each control adjustment does not necessarily seek to satisfy the policy exactly (although that is not ruled out). Thus here we are looking at the disequilibrium day to day modelling of both routeing and green-time.

The two main contributions made in this paper are as follows.

(i) The paper shows that certain control adjustments (using the  $P_0$  policy) yield a stable dynamical system when these are combined with natural routeing adjustments; and that the dynamical systems which arise maximise throughput (with bounded costs) within a day to day system.

(ii) The paper shows that certain control policies which have been proposed recently may fail to maximise throughput (or network capacity); with these policies queues may be unbounded if users are assumed to vary their routes by continually switching to cheaper routes even though demand is within capacity.

### 1.3. The two dynamical route-flow swapping models considered in sections 2-4

The two main dynamical routeing models (the general model and the special model) in this paper arise by supposing that the same travellers traverse a fixed network day after day and that drivers may change their route from one day to the next. A general and a specific route swapping model are utilised in this paper; both are driven

by the principle given in (1.1). Plainly principle (1.1) depends on the definition of route cost. There is also to be a step length constraint within both the general and the specific route-flow swapping model. All the directions employed in the general route swapping model arise in Smith (1979a) and the single direction employed in the special route swapping model is derived from Smith (1984a).

#### 1.4. The two dynamical $P_0$ green-time or red-time swapping models considered in sections 2-4

In this paper the general and the special route-flow swapping models will be combined with corresponding general and special dynamical forms of the responsive  $P_0$  signal control green-time swapping policy. Both the general and the special dynamical  $P_0$  green-time swapping models satisfy principle (1.2).

Again there is no compulsion in this principle to swap green-time from one day to the next. Plainly the general green-time adjustment (1.2) depends on the definition of stage pressure. In sections 2-4 this pressure will be chosen to fit the  $P_0$  signal control policy, which has been specially designed to fit within route choice models *in strictly capacitated networks*; see, for example, Smith (1979a, b, c, 2010, 2011), Smith and Mounce (2011) and Smith et al. (2013). (A policy similar to the  $P_0$  policy is considered also by Bentley and Lambe (1980).)

$P_0$  signal control policies utilise the stage  $J$  pressure defined to be

$$\begin{aligned} & \text{the sum over all links } i \text{ in stage } J \text{ of the product} \\ & \{ \text{saturation flow of link } i \} \times \{ \text{bottleneck delay experienced at the exit of link } i \}. \end{aligned} \quad (1.3)$$

Then in this paper the general and specific  $P_0$  green-time swapping dynamical systems both satisfy (1.2) and a natural step length constraint.

In fact the paper initially utilises red-time rather than green-time. Given any stage  $J$  (this is a set of links given green simultaneously) *anti-stage*  $J$  is the set of all links at the same junction as stage  $J$  which are given red when stage  $J$  is green. Thus the red-time proportion allocated to anti-stage  $J$  equals the green-time proportion allocated to stage  $J$ . Then both of the dynamical  $P_0$  red-time systems may both be written in terms of the red-time cost  $RC_J$  of anti-stage  $J$ . This also is to be given by (1.3) but with “stage” replaced by “anti-stage”. Using anti-stages and anti-stage costs, the stage green-time swapping principle (1.2) becomes the following principle. *On each day:*

$$\begin{aligned} & \text{for each anti-stage with positive red-time yesterday, some of that red-time may be swapped today} \\ & \text{but only to an anti-stage which was less costly yesterday (and is at the same junction)}. \end{aligned} \quad (1.4)$$

#### 1.5. Stability and convergence results in sections 2-4

It is shown in this paper that, under natural conditions, the *general* combined (route-flow, anti-stage red-time) dynamical system directions (with a step length constraint) is stable in the sense that if the system is started at a feasible [route flow, anti-stage red-time] pair and follows the general dynamical system then

$$\begin{aligned} & \text{each possible solution trajectory never approaches the edge of the feasible region,} \\ & \text{and costs are bounded along any trajectory.} \end{aligned} \quad (1.5)$$

It will also be shown that the *particular* combined [route-flow, anti-stage red-time] dynamical system not only remains within the capacity of the network (with bounded costs) but also has a much more specific convergence property: the particular (route-flow, anti-stage red-time) dynamical system converges to a non-empty set of (route-flow, anti-stage red-time) equilibria consistent with  $P_0$ .

To state this property we need to define such consistent equilibria. First, a vector of route flows and red-times is a Wardrop equilibrium if no route-swapping is possible when principle (1.1) holds. Second, a vector of route flows and red-times is a  $P_0$ -equilibrium if no red-time swapping is possible when principle (1.4) holds.

The paper shows that under suitable conditions every solution of the specific dynamical system converges to a non-empty feasible set of [route flow, anti-stage red-time] equilibrium pairs; any such pair  $(\mathbf{X}, \mathbf{R})$  is simultaneously a Wardrop equilibrium and a  $P_0$ -equilibrium. (Such an  $[\mathbf{X}, \mathbf{R}]$  will be called a Wardrop -  $P_0$  equilibrium.)

The general stability property (1.5) implies that the dynamical  $P_0$  policy “maximises network capacity” in a very general way. This is because (1.5) says that if the steady demand is such that there is a feasible start point (that is: there is a feasible [route flow, anti-stage red-time] pair) then any solution trajectory of the dynamical system (1.1) + (1.4) (beginning at a feasible [flow, red-time] start point) never hits or even approaches the edge of the feasible region. (Thus the steady demand is fulfilled and travel costs remain bounded throughout any solution trajectory.)

Of course it would not be good if an adaptive control system, when interacting with reasonable routing changes, either (i) reduces network capacity (by forcing the system toward points which are not supply-feasible so that costs become very large or unbounded) or (ii) fails to have reasonable convergence properties; because such an adaptive control system when combined with reasonable routing dynamics may then on occasion create a costly system or an unpredictable system or both. The “general” results in sections 2-4 of this paper shows that with the  $P_0$  responsive signal control (i) cannot happen in the general model described here. The “special” (convergence) result shows that if  $P_0$  is utilized and the special dynamical system is followed then (ii) cannot happen either.

## 2. Some simple dynamical systems embracing route-flows and green-times (or red-times); and stability

### 2.1. Route-flow costs and stage red-time costs

Now consider a network with  $K_I$  OD pairs and each OD pair  $p$  is joined by  $N_p$  routes and also now there are to be  $K_2$  nodes and each node  $n$  has a signal with  $N_j$  stages. A route is a contiguous sequence of nodes and links without repetition; and a stage is a maximal set of approaches to a junction which may be shown green simultaneously. We suppose that if a particular lane is shown green then all movements along that lane are shown green and that if two lanes are shown green simultaneously then all movements from these two links are free to flow without interference (so for each stage no two movements given green simultaneously conflict).

In this paper we consider anti-stages and anti-stage red-times as well as stages and stage green-times. Suppose that stage  $J$  at node  $n$  is green for a proportion of time  $G_J$ . Let anti-stage  $J$  be the set of all those approaches or links terminating at node  $n$  which are *not* in stage  $J$ ; then all links in anti-stage  $J$  are shown red simultaneously when stage  $J$  is shown green and anti-stage  $J$  is red for a proportion of time  $R_J$  equal to  $G_J$ .

There is now a simple way of placing control within the route assignment model above. This is to think of the red time awarded to an anti-stage (and hence to the links in that anti-stage) as a different type of flow (called red-time) taking up some of the available capacity at the exits of those approaches in that anti-stage. Then the *aggregated* flow on link  $i$  will comprise the flow of real vehicles added to a suitable multiple of link  $i$  “red-time” (designed to take up the capacity which cannot be used while the signal is red for link  $i$ ).

Henceforth for each link  $i$  we let the new “total volume”  $v_i = x_i + s_i r_i$ ; where  $x_i$  represents the “real” vehicular flow on link  $i$  (in vehicles per second say) and  $r_i$  represents the proportion of time link  $i$  is red. The multiple  $s_i r_i$  (vehicles per second) is the capacity lost due to the proportion ( $r_i$ ) of red time, bearing in mind the saturation flow  $s_i$  (vehicles per second) at the link exit.

Then we suppose that the cost (in seconds) of traversing approach  $i$  equals

$$c_i(x_i) + f_i(x_i + s_i r_i).$$

Here  $c_i(x_i)$  now represents the cost (in seconds) of traversing the length of the link when the flow is  $x_i$  and  $f_i$  represents the bottleneck delay (in seconds) felt at the traffic signal when the flow is  $x_i$  and the red time is  $r_i$ . Both  $c_i(\cdot)$  and  $f_i(\cdot)$  are here to be non-decreasing non-negative functions. The slope of  $c_i$  may be shallow and the slope of  $f_i$  may be steep:  $f_i$  may even have a vertical asymptote at  $s_i$  and such an asymptote naturally represents the finite capacity of most links in real life, prohibiting flows which exceed this capacity.

A further natural “justification” of the form of the bottleneck delay formula above lies in looking at other delay formulae for traffic signalled approaches. The most famous such delay formula is that stated by Webster (1958). The second (unbounded) term of Webster’s two-term formula for the average delay experienced at a signalised exit of link  $i$  is  $Ax_i/[s_i g_i (s_i g_i - x_i)]$ . Now, writing this using the red-time proportion  $r_i$ :

$$Ax_i/[s_i g_i (s_i g_i - x_i)] = A/[s_i g_i - x_i] - A/(s_i g_i) = A/[s_i - (x_i + s_i r_i)] - A/[s_i - s_i r_i]$$

where  $A = 9/20$ ,  $g_i$  is the green-proportion awarded to link  $i$ ,  $s_i$  is the saturation flow at the link  $i$  exit and  $x_i$  is the average flow along  $i$ . So one natural steep cost function, with the form suggested above, is

$$b(x_i + s_i r_i) = A/[s_i - (x_i + s_i r_i)].$$

This particular function captures the unbounded part of the second term of Webster’s delay formula. It would be natural to extend the theory in sections 2-4 here to allow for the whole second term, including  $-A/[s_i - s_i r_i]$ .

2.2. The central control assumptions in sections 2-4

For definiteness and clearness we will now suppose that

(i)  $c_i(x_i)$  is non-negative, non-decreasing and continuous for all  $x_i$  such that  $0 \leq x_i \leq s_i$  and that (2.1a)

(ii)  $f_i(v_i)$  is non-negative, non-decreasing and continuous on  $[0, s_i)$  and tends to infinity as  $v_i$  tends to  $s_i$ . (2.1b)

It is natural to insist as we do here that both  $f_i$  and  $c_i$  are non-decreasing but this is not strictly necessary for all of the analysis below. These suppositions (2.1a, b) essentially follow Beckmann et al (1956) and ensure that the network is capacity constrained. (2.1a, b) also allow a very generous dynamical model of control and routing to be constructed (with very many solution trajectories) and thus enable a very general stability result to be proved. In essence we now have a *two commodity* link model where the two commodities are:

$x_i$  = vehicular flow on link  $i$  (vehicles per second) and

$r_i$  = red-time on link  $i$  (a proportion and dimensionless).

(We also have  $g_i$  = green-time on link  $i$  (again a proportion and dimensionless).)

For any two vectors  $\mathbf{x}, \mathbf{y}$  of the same length we define:

$$\mathbf{x} \bullet \mathbf{y} = [x_1, x_2, x_3, \dots, x_n] \bullet [y_1, y_2, y_3, \dots, y_n] = [x_1y_1, x_2y_2, x_3y_3, \dots, x_ny_n].$$

This is the Hadamard product of the vectors  $\mathbf{x}$  and  $\mathbf{y}$ .

Let the link-route incidence matrix be  $\mathbf{A}$  and the link anti-stage incidence matrix be  $\mathbf{B}$ , so that

$A_{ir} = 1$  if link  $i$  forms part of route  $r$  and  $= 0$  otherwise; and

$B_{iJ} = 1$  if link  $i$  forms part of anti-stage  $J$  and  $= 0$  otherwise.

Suppose that a fixed demand transportation network model with  $N$  routes and  $m$  links is given. Each link  $i$  has a link-exit-capacity or saturation flow  $s_i$  and two cost-flow functions satisfying (2.1) above, so the links are all capacitated. Using the Hadamard product defined above, we say that  $(\mathbf{X}, \mathbf{R})$  is supply-feasible if and only if

$$S = \{(\mathbf{X}, \mathbf{R}); \mathbf{AX} + \mathbf{s} \bullet (\mathbf{BR}) < \mathbf{s}\}; \tag{2.2}$$

and then to ensure supply-feasibility of any non-negative vector  $(\mathbf{X}, \mathbf{R})$  we suppose that  $(\mathbf{X}, \mathbf{R}) \in S$ . (Non-negativity will be ensured by making a separate assumption.)

2.3. The network and signal stages in sections 2-4

For each OD pair  $p$  the total of the flows  $X_r$  along all routes  $r$  joining OD pair  $p$  is  $\rho_p$  (fixed and non-negative). At each node  $n$  the total of the green-time proportions  $G_J$  allocated to the stages at that junction is 1 and so the total of the red time proportions  $R_J$  allocated to the anti-stages  $J$  at that junction  $n$  is also 1. Here we suppose zero lost times.

A set  $D$  of demand-feasible route-flow vectors  $\mathbf{X}$  is defined by:

$$D = \{\mathbf{X} \geq \mathbf{0}; \sum_{\{r:rjoinsp\}} X_r = \rho_p \text{ for all OD pairs } p\} \tag{2.3a}$$

where the  $\rho_p$  are given OD pair  $p$  demands and  $rjoinsp$  means that route  $r$  joins OD pair  $p$ . The set  $RD$  of feasible anti-stage red-time vectors  $\mathbf{R}$  is defined by:

$$RD = \{\mathbf{R} \geq \mathbf{0}; \sum_{(J,Jatn)} R_J = 1 \text{ for all junctions } n\} \tag{2.3b}$$

where  $Jatn$  means that antistage  $J$  is at node  $n$ . The [routing + red-time] dynamical systems in this paper are: at each OD pair some real vehicular flow may switch to cheaper routes as in section 2 and now also at each node some red-time may switch to “cheaper” anti-stages.

To determine the costs of routes and anti-stages (which then fix the permitted route-flow and red-time swap directions in the (routing, control) dynamical systems to be stated) relevant link costs are added.

For route  $r$  the relevant link costs are the *link flow-costs*  $c_i(x_i) + f_i(x_i + s_i r_i)$  and for stage  $J$  the relevant link costs are the *link red-time-costs*  $s_i f_i(x_i + s_i r_i)$ . The (flow-) cost  $C_r$  of traversing route  $r$  is then the sum over all links  $i$  in route  $r$  of the link flow-costs  $c_i(x_i) + f_i(x_i + s_i r_i)$  and the (anti-stage red-time) cost  $RC_J$  of anti-stage  $J$  is the sum over all links in anti-stage  $J$  of link (red-time)-costs  $s_i b_i(x_i + s_i r_i)$ . Thus

$$C_r = C_r(\mathbf{X}, \mathbf{R}) = \sum_{i \in R_r} [c_i(x_i) + f_i(x_i + s_i r_i)] \tag{2.4a}$$

$$RC_J = RC_J(\mathbf{X}, \mathbf{R}) = \sum_{i \in A_J} s_i f_i(x_i + s_i r_i). \quad (2.4b)$$

In (2.4b),  $A_J$  is the set of links  $i$  in anti-stage  $J$  and  $RC_J$  is the red-time cost felt by the anti-stage  $J$  red-time.

The vector  $\mathbf{x}$  of link flows and the vector  $\mathbf{r}$  of link red-times here are determined from  $\mathbf{X}$  and  $\mathbf{R}$  via:

$$\mathbf{x} = \mathbf{A}\mathbf{X} \text{ and } \mathbf{r} = \mathbf{B}\mathbf{R}. \quad (2.5)$$

Here the link red-time costs  $s_i f_i(x_i + s_i r_i)$  are those which define the  $P_0$  control policy (Smith (1979a, b, c)). Other control policies and so other “allowed” swap directions arise if this link red-time cost formula is changed. So, for example, the equi-saturation policy arises if we specify the link  $i$  red-time cost as the degree of saturation

$$x_i / g_i s_i = x_i / [(1-r_i)s_i] = x_i / [s_i - s_i r_i]. \quad (2.6)$$

It may be seen from the above allowed swapping directions that at a junction with two approaches the  $P_0$  policy (in choosing red times) may be thought of as seeking to ensure that

$$s_1 f_1(x_1 + s_1 r_1) = s_2 f_2(x_2 + s_2 r_2), \quad (2.7)$$

since this holds when equilibrium is reached in the sense that no red-time swapping between anti-stages occurs in the  $P_0$  case; similarly the equi-saturation policy may be thought of as seeking to ensure that

$$x_1 / g_1 s_1 = x_2 / g_2 s_2,$$

since this holds when equilibrium is reached and there is then no red-time swapping between anti-stages.

*A comment:* It is clear from the above equation (2.7) in the  $P_0$  case that if the saturation flow  $s_2$  is high then the  $P_0$  policy will (by a suitable choice of  $\mathbf{R}$  and so  $\mathbf{r}$ ) seek to ensure that the bottleneck delay  $f_2$  will tend to be small; encouraging the use of the approach with the higher saturation flow (even if the actual flow on that approach is small). The policy is designed to encourage re-routing toward higher capacity routes rather than rewarding travellers on existing routes. The results in this paper show that in a sense this is generally true. In contrast, standard traffic control policies such as the well-known equi-saturation policy tend to give greatest green-times to the currently most-used approaches and so may encourage increased usage of already highly used approaches.

The results in this paper show that, under natural conditions, the  $P_0$  policy maximises network throughput at a feasible equilibrium distribution of traffic flows. This tends to confirm that the policy encourages routeing shifts over time to more economical routeing patterns. (The capacity-maximisation proofs given here are natural developments of those in Smith (1979a, b, c) and Smith (1987); this last paper deals with a quasi-dynamic setting.)

### 3. Two simple capacity-maximising results

#### 3.1. Route-flow costs and anti-stage red-time costs

We now utilize the 2-commodity link cost-flow function

$$(c_i(x_i) + f_i(x_i + s_i r_i), s_i f_i(x_i + s_i r_i)) \quad (3.1)$$

essentially just introduced above. The first component gives the link flow-cost felt by “real” vehicle link flow and the second component gives the link red-time cost felt by the link red-time. This link 2-vector (3.1) will in what follows give rise to all permitted route flow swaps and stage red-time swaps; since by summing it specifies the flow costs of all routes and the red-time costs of all anti-stages. The cost of a route is obtained by adding relevant link flow-costs (those flow costs corresponding to all links in the route); and the red-time cost of an anti-stage is obtained by adding relevant link red-time costs (those red-time costs corresponding to all links in the anti-stage). These summations are given in (2.4a) and (2.4b).

#### 3.2. [Routeing + $P_0$ control] assignment intervals: a general route-flow swap and red-time swap dynamical system

The permitted flow and red-time swaps will depend on the specifications of the route costs and the anti-stage red-time costs. These costs are given in (2.4a) and (2.4b) in terms of link flows and link red-times.

In this section we restrict the permissible swaps as follows, as indicated in section 2. Suppose that

$$[\mathbf{X}, \mathbf{R}] + t [\Delta\mathbf{X}, \Delta\mathbf{R}]$$

is both demand and supply feasible, and so belongs to  $(D \times RD) \cap S$ , for all  $t$  such that  $0 \leq t < 1$ . Consider moving

from  $[\mathbf{X}, \mathbf{R}]$  to  $[\mathbf{X}, \mathbf{R}] + [\Delta\mathbf{X}, \Delta\mathbf{R}]$   
 along the straight line path  
 $\{[\mathbf{X}, \mathbf{R}] + t[\Delta\mathbf{X}, \Delta\mathbf{R}]; 0 \leq t \leq 1\}$  (3.2)  
 by steadily increasing  $t$  from 0 to 1. Any such path will be called an interval.

3.3. Definition of a routing- $P_0$  assignment interval

We shall call this straight line path or interval in (3.2) a *routing /  $P_0$ -control assignment interval* if

- (i)  $[\mathbf{X}, \mathbf{R}] + t[\Delta\mathbf{X}, \Delta\mathbf{R}]$  is demand-feasible (or belongs to  $D \times RD$ ) for all  $t$  satisfying  $0 \leq t \leq 1$ ;
- (ii)  $[\mathbf{X}, \mathbf{R}] + t[\Delta\mathbf{X}, \Delta\mathbf{R}]$  is supply-feasible (or belongs to  $S$ ) for all  $t$  satisfying  $0 \leq t < 1$ ; and also
- (iii)  $-(\mathbf{C}, \mathbf{RC})([\mathbf{X}, \mathbf{R}] + t[\Delta\mathbf{X}, \Delta\mathbf{R}]) \cdot [\Delta\mathbf{X}, \Delta\mathbf{R}] \geq 0$  for all  $0 \leq t < 1$ .

A routing /  $P_0$ -control assignment interval is thus a straight line path (3.2) which is demand and supply feasible at each point apart (possibly) from  $[\mathbf{X}, \mathbf{R}] + [\Delta\mathbf{X}, \Delta\mathbf{R}]$  corresponding to  $t = 1$ . Also, the direction  $[\Delta\mathbf{X}, \Delta\mathbf{R}]$  of the straight line path must have a non-negative dot product with

$$-(\mathbf{C}, \mathbf{RC})([\mathbf{X}, \mathbf{R}] + t[\Delta\mathbf{X}, \Delta\mathbf{R}])$$

for all  $0 \leq t < 1$ . (At the final point  $([\mathbf{X}, \mathbf{R}] + [\Delta\mathbf{X}, \Delta\mathbf{R}])$ ,  $(\mathbf{C}, \mathbf{RC})$  may not be defined as this final point may not be supply-feasible.) Rather as before  $-(\mathbf{C}, \mathbf{RC})([\mathbf{X}, \mathbf{R}] + t[\Delta\mathbf{X}, \Delta\mathbf{R}])$  may be thought of as a force pushing

$$([\mathbf{X}, \mathbf{R}] + t[\Delta\mathbf{X}, \Delta\mathbf{R}])$$

in the direction  $[\Delta\mathbf{X}, \Delta\mathbf{R}]$ . For routing /  $P_0$  control assignment intervals this push is never negative.

The ideas here are developed from Smith (1979a); the key paper on dynamical systems such as those described just above was written by Smale (1976).

At any  $[\mathbf{X}, \mathbf{R}]$ , any direction arising from (1.1) and (1.4) gives rise to a routing- $P_0$  assignment interval, provided the step length constraint to be introduced holds.

3.4. A general stability result involving linear route flow swaps and anti-stage red-time swaps

With the above specification of an allowable path, or a *routing/ $P_0$ -control assignment interval*, in section (3.2):

if  $\{[\mathbf{X}, \mathbf{R}] + t[\Delta\mathbf{X}, \Delta\mathbf{R}]; 0 \leq t \leq 1\}$  is a routing -  $P_0$  control assignment interval then  $[\mathbf{X}, \mathbf{R}] + [\Delta\mathbf{X}, \Delta\mathbf{R}] \in S$ .

This means that even with our *very wide* collection of admissible route flow and stage red-time swaps (giving rise to all possible routing/ $P_0$ -control assignment intervals),

$$\text{a routing/ $P_0$ -control assignment interval does not leave } S. \tag{3.3}$$

It then follows that along any routing/ $P_0$ -control assignment interval travel costs are bounded. The proof of this reasonably general result is given in appendix A below. The above result (3.3) shows that stage red-time adjustments following a dynamic form of policy  $P_0$ , when combined with the generous re-routing rules in section 2 above creates a stable routing/ $P_0$ -control system in as much as *there is no routing/ $P_0$ -control assignment interval which leaves the set  $(D \times RD) \cap S$* . It follows that along any routing/ $P_0$ -control assignment interval travel costs are bounded. It is easy to check, by giving an example, *that no similar result is possible for the equi-saturation policy*. See for example Smith (1979c).

3.5. A stronger stability result using a slightly stronger assumption.

A slightly stronger condition than (2.1b) above is:

$f_i(v_i)$  is non-negative, non-decreasing and continuous on  $[0, s_i)$  and

$$\int_0^{s_i} b_i(v_i) dv_i = \infty. \tag{3.4}$$

This condition is plainly somewhat stronger than (2.1b). Assuming that (3.4) and (2.1a) hold we may utilize Lyapunov arguments like those in Smith and Mounce (2010). For any  $(x_i, r_i)$  such that

$$x_i + s_i r_i < s_i,$$

consider the standard Beckmann et al (1956) objective function

$$Z(\mathbf{x}) = \sum_i \int_0^{x_i} c_i(u) du \tag{3.5}$$

and also the red-time-modified Beckmann objective function

$$W(\mathbf{x}, \mathbf{r}) = \sum_i \int_0^{x_i + s_i r_i} b_i(u) du. \tag{3.6}$$

Then let

$$V(\mathbf{x}, \mathbf{r}) = V_{Beckmann}(\mathbf{x}, \mathbf{r}) = Z(\mathbf{x}) + W(\mathbf{x}, \mathbf{r}). \tag{3.7}$$

It follows that:

$$\partial V / \partial x_i = c_i(x_i) + f_i(x_i + s_i r_i) \text{ and } \partial V / \partial r_i = s_i f_i(x_i + s_i r_i)$$

and so

$$\text{grad } V(\mathbf{x}, \mathbf{r}) = [\mathbf{c}(\mathbf{x}) + \mathbf{f}(\mathbf{x}, \mathbf{r}), \mathbf{s} \bullet \mathbf{f}(\mathbf{x}, \mathbf{r})].$$

(This uses the Hadamard product defined above in section 2.2.) It now further follows from (iii) in section 3.3 that  $V$  cannot increase at any point along any assignment / control interval; so the values taken by  $V$  along this interval cannot exceed the value of  $V = V(\mathbf{x}^0, \mathbf{r}^0)$  at the start of the interval.

Now (3.4) implies that  $V(\mathbf{x}, \mathbf{r}) = Z(\mathbf{x}) + W(\mathbf{x}, \mathbf{r})$  tends to infinity as  $(\mathbf{x}, \mathbf{r})$  approaches the boundary of  $S$  where  $x_i + s_i r_i = s_i$  for at least one link. It follows that no assignment / control interval (along which  $V$  does not increase) can even get close to the unfeasible boundary of  $S$ , because if it did the corresponding  $V$  values would exceed  $V(\mathbf{x}^0, \mathbf{r}^0)$ .

More can now be said: under condition (3.4) no *sequence* of assignment intervals can approach the boundary of  $S$  since such a sequence beginning at say  $(\mathbf{x}^0, \mathbf{r}^0)$  must remain within

$$\{(\mathbf{x}, \mathbf{r}) \in (D \times RD) \cap S; V(\mathbf{x}, \mathbf{r}) \leq V(\mathbf{x}^0, \mathbf{r}^0)\}$$

and this set is a positive distance from the boundary of  $S$ . It follows that in this case (where 3.4 holds) travel costs are bounded along *any sequence* of assignment- $P_0$  control intervals.

#### 4. Outline of a simple global convergence result as route flows and stage red-times follow a single trajectory

Here we consider certain sequences of particular routing /  $P_0$ -control assignment intervals. For any such sequence we demonstrate convergence to the set of those (route flow vector, anti-stage red-time vector) or  $(\mathbf{X}, \mathbf{R})$  pairs which are Wardrop -  $P_0$  consistent equilibria. Thus the sequence not only stays clear of the boundary of the feasible set but also converges to a non-empty set of Wardrop -  $P_0$  consistent equilibria.

##### 4.1. The modified proportional switch route-flow and stage red-time adjustment process (MPAP)

Let us suppose that a fixed demand model is given. There are to be  $K$  OD pairs, each OD pair  $p$  is joined by  $N_p$  routes, and for each  $p$  the total flow for OD pair  $p$  is  $\rho_p$  (fixed and non-negative). There are also a number of junctions and at each junction there are a number of stages and anti-stages. Each route  $r$  has an associated flow variable  $X_r$  and each anti-stage  $s$  has an associated red-time variable  $R_s$ .

For route-flow,  $X$ , subscripts,  $r \sim s$  means that route  $r$  and route  $s$  join the same OD pair and are different. For any route-flow subscripts  $r, s$  we define (the route-flow swap from route  $r$  to route  $s$  vector)  $\Delta_{rs}$  as follows:

$$\Delta_{rsr} = -1 \text{ and } \Delta_{rss} = +1 \text{ if } r \sim s; \text{ and } \Delta_{rsq} = 0 \text{ in all other cases.}$$

For red-time,  $R$ , subscripts,  $r \sim s$  means that stage  $r$  and stage  $s$  are at the same junction and are different. For any red-time subscripts  $r, s$  we define (the red-time swap from stage  $r$  to stage  $s$  vector)  $R\Delta_{rs}$  as follows:

$$R\Delta_{rsr} = -1 \text{ and } R\Delta_{rss} = +1 \text{ if } r \sim s; \text{ and } R\Delta_{rsq} = 0 \text{ in all other cases.}$$

We define a direction satisfying principles (1.1) and (1.4) at every feasible  $(\mathbf{X}, \mathbf{R})$ . This is to be  $\mathbf{U}(\mathbf{X}, \mathbf{R})$  where:

$$\mathbf{U}(\mathbf{X}, \mathbf{R}) = \sum_{\{(r,s); r \sim s\}} k(\mathbf{X}, \mathbf{R}) X_r \phi[C_r(\mathbf{X}, \mathbf{R}) - C_s(\mathbf{X}, \mathbf{R})] \Delta_{rs} + \sum_{\{(r,s); r \sim s\}} k(\mathbf{X}, \mathbf{R}) R_r \phi[RC_r(\mathbf{X}, \mathbf{R}) - RC_s(\mathbf{X}, \mathbf{R})] (R\Delta)_{rs}. \tag{4.1}$$

Here  $k(\mathbf{X}, \mathbf{R})$  is a scalar and  $k(\mathbf{X}, \mathbf{R})$  is to be a continuous function of  $(\mathbf{X}, \mathbf{R})$ .  $\Delta_{rs}$  is the “swap flow from route  $r$  to route  $s$  vector” and  $(R\Delta)_{rs}$  is the “swap red-time from anti-stage  $r$  to anti-stage  $s$ ” vector defined above. We insist that the function  $\varphi$  is smooth, non-negative, non-decreasing; also we insist that

$$\varphi(x) = 0 \text{ if } x \leq 0; \varphi(x) > 0 \text{ if } x > 0 \text{ and } \varphi(x) \text{ tends to } 1 \text{ as } x \text{ tends to } +\infty.$$

Additionally, the factor  $k(\mathbf{X}, \mathbf{R})$  in (4.1) is to be chosen so that for each  $(\mathbf{X}, \mathbf{R})$  which is both supply and demand feasible (see 2.2, 2.3a, 2.3b),

$(\mathbf{X}, \mathbf{R}) + \mathbf{U}(\mathbf{X}, \mathbf{R})$  is also demand and supply feasible; and

$[(\mathbf{X}, \mathbf{R}), (\mathbf{X}, \mathbf{R}) + \mathbf{U}(\mathbf{X}, \mathbf{R})]$  is an assignment- $P_0$  control interval.

Under reasonable conditions such a function  $k$  exists. It follows immediately that, for any feasible  $(\mathbf{X}, \mathbf{R})$ ,

$$V[(\mathbf{X}, \mathbf{R}) + \mathbf{U}(\mathbf{X}, \mathbf{R})] < V(\mathbf{X}, \mathbf{R}).$$

Now consider the dynamical system:

$$(\mathbf{X}, \mathbf{R})(0) = (\mathbf{X}, \mathbf{R})^0 \text{ and } (\mathbf{X}, \mathbf{R})(t+1) = (\mathbf{X}, \mathbf{R})(t) + \mathbf{U}(\mathbf{X}(t), \mathbf{R}(t)) \text{ for } t = 0, 1, 2, 3, \dots \tag{4.2}$$

where  $(\mathbf{X}, \mathbf{R})^0$  is a given feasible starting [route flow vector, anti-stage red-time vector]; this starting point is to be both demand and supply feasible. That is  $(\mathbf{X}, \mathbf{R})(0) = (\mathbf{X}, \mathbf{R})^0$  belongs to  $(D \times RD) \cap S$  where  $D$  and  $RD$  are given by 2.3a and 2.3b and  $S$  is given by (2.2).

Now  $\mathbf{U}(\mathbf{X}, \mathbf{R})$  (in (4.2)) is a continuous bounded function of (current) flows, red-times, route costs and anti-stage costs so, for any given continuous cost flow function

$$[\mathbf{C}, \mathbf{RC}] = [\mathbf{C}(\mathbf{X}, \mathbf{R}), \mathbf{RC}(\mathbf{X}, \mathbf{R})]$$

defined on  $(D \times F) \cap S$ ,  $\mathbf{U}(\mathbf{X}, \mathbf{R})$  becomes a continuous function of  $(\mathbf{X}, \mathbf{R})$  also defined on  $(D \times F) \cap S$ .

The dynamical system (4.2) gives rise to a sequence of assignment  $P_0$ -control intervals and  $V$  declines to zero along this sequence. This is proved below.

#### 4.2. Convergence of MPAP with $P_0$ signal adjustments

Suppose that  $(D \times RD) \cap S$  is non-empty and that  $(\mathbf{X}, \mathbf{R}) \in (D \times RD) \cap S$ . If  $(\mathbf{X}, \mathbf{R})$  is not a Wardrop- $P_0$  equilibrium then the direction  $\mathbf{U}(\mathbf{X}, \mathbf{R})$  is not zero and  $(\mathbf{X}, \mathbf{R}) + \mathbf{U}(\mathbf{X}, \mathbf{R}) \in (D \times RD) \cap S$  by our construction (4.2).

It now follows that the set of Wardrop- $P_0$  equilibria is nonempty and that any solution  $\{(\mathbf{X}, \mathbf{R})(t)\}$  of the dynamical system (4.2) converges to the non-empty set of Wardrop- $P_0$  equilibria. (Or: the distance between  $(\mathbf{X}, \mathbf{R})(t)$  and the set of Wardrop- $P_0$  equilibria tends to zero.) This proof is given in detail in appendix B below.

### 5. Stable responsive traffic control policies with explicit queues and cycle times

In the previous section it is supposed that for each link there is a “real” delay formula  $f_i$  and that the bottleneck delay felt at the link  $i$  exit equals  $f_i(x_i + s_i r_i)$ , where  $s_i$  is the saturation flow at the link exit,  $x_i$  is the flow out of the link exit and  $r_i$  is the proportion of time that the link  $i$  exit is red. This is a major supposition which may not hold. To exploit the analysis in sections 2-4 it is thus natural to consider what happens if the cost of travel along link  $i$  has a different form.

In a quasi-dynamic network link  $i$  a reasonable delay formula is  $Q_i / s_i g_i$ ; where  $Q_i$  is the number of vehicles in a vertical queue at the link  $i$  exit and  $g_i$  is the proportion of time that the link  $i$  exit is green. To make the previous analysis work in this case suppose given a quasi-dynamic network and a non-decreasing unbounded function  $f_i$  for each link  $i$ . Then (given these  $f_i$ ), suppose given a bottleneck delay  $b_i$ , flow  $x_i$  and “red-time proportion”  $r_i$  satisfying

$$b_i = f_i(x_i + s_i r_i).$$

A little thought shows that typically (in a quasi-dynamic setting) the  $r_i$  here cannot in fact be the real red-time: however  $r_i$  can be no greater than the true or real red-time. So here it is necessary to add in a “slack” red-time  $r_i^s$  which “corrects” each  $r_i$ ; so that the “real” red-time is  $r_i + r_i^s = r_i^*$ , (say). The red-times  $r_i^*$ , now comprise a feasible set of “real” link red-times. (Both  $r_i$  and  $r_i^s$  need to be determined as controls.) Now we select the  $r_i$  to fit the given artificial delay formulae  $f_i$  and then  $r_i^s$  are determined to ensure that the total red-time is feasible, within the quasi-dynamic model. (For now let us suppose that the link-antistage matrix  $\mathbf{B}$  is invertible; this might be relaxed.)

Suppose that for each link  $i$  at a certain time  $[x_i, b_i, r_i, r_i^s]$  are known and satisfy:

$$b_i = f_i(x_i + s_i r_i) \text{ and } x_i + s_i r_i + s_i r_i^s = s_i. \tag{5.1}$$

Let  $\mathbf{X}$ ,  $\mathbf{R}$ ,  $\mathbf{R}^s$  and  $\mathbf{R}^*$  satisfy  $\mathbf{x} = \mathbf{A}\mathbf{X}$ ,  $\mathbf{r} = \mathbf{B}\mathbf{R}$ ,  $\mathbf{r}^s = \mathbf{B}\mathbf{R}^s$  and  $\mathbf{r}^* = \mathbf{B}\mathbf{R}^*$ . To exploit the results in sections 2-4, we consider route-flow swaps and red-time swaps  $[\Delta\mathbf{X}, \Delta\mathbf{R}^*]$  such that  $[\mathbf{X}, \mathbf{R}^*] + t[\Delta\mathbf{X}, \Delta\mathbf{R}^*]$  is both demand and supply feasible, and so belongs to  $(D \times RD) \cap S$ , for all  $t$  such that  $0 \leq t < 1$ . Consider moving

from  $[\mathbf{X}, \mathbf{R}]$  to  $[\mathbf{X}, \mathbf{R}^*] + [\Delta\mathbf{X}, \Delta\mathbf{R}^*]$

along the straight line path

$$\{[\mathbf{X}, \mathbf{R}^*] + t[\Delta\mathbf{X}, \Delta\mathbf{R}^*]; 0 \leq t \leq 1\} \tag{5.2}$$

by steadily increasing  $t$  from 0 to 1. Such a [route-flow, red-time] path will be called an interval. As  $t$  increases from 0 to 1 the separate component red times  $\mathbf{R}(t)$  and  $\mathbf{R}^s(t)$  must evolve in such a way that (5.1) holds at each  $t$  in  $[0, 1)$ . Red times  $\mathbf{R}(t)$  and  $\mathbf{R}^s(t)$  evolve along a curve which is not typically a straight line path because (5.1) always holds.

Imitating section 4, we here call a straight line path in (5.2) a *routeing / P<sub>0</sub>-control assignment interval* if

- (i)  $[\mathbf{X}, \mathbf{R}] + t[\Delta\mathbf{X}, \Delta\mathbf{R}^*]$  is demand-feasible (or belongs to  $D \times RD$ ) for all  $t$  satisfying  $0 \leq t \leq 1$ ;
- (ii)  $[\mathbf{X}, \mathbf{R}] + t[\Delta\mathbf{X}, \Delta\mathbf{R}^*]$  is supply-feasible (or belongs to  $S$ ) for all  $t$  satisfying  $0 \leq t < 1$ ; and also
- (iii)  $-(\mathbf{C}, \mathbf{RC})([\mathbf{X}, \mathbf{R}](t)) \cdot [\Delta\mathbf{X}(t), \Delta\mathbf{R}(t)] \geq 0$  for all  $0 \leq t < 1$ .

Here, at each  $t$  in  $[0, 1)$  the direction  $[\Delta\mathbf{X}(t), \Delta\mathbf{R}(t)]$  arising from the straight line path (5.2) must have a non-negative dot product with  $-(\mathbf{C}, \mathbf{RC})([\mathbf{X}(t), \mathbf{R}(t)])$  for all  $0 \leq t < 1$ . (At the final point,  $[\mathbf{X}(1), \mathbf{R}(1)]$ ,  $(\mathbf{C}, \mathbf{RC})$  may not be defined as this final point may not be supply-feasible.) As before  $-(\mathbf{C}, \mathbf{RC})([\mathbf{X}(t), \mathbf{R}(t)])$  may be thought of as a force pushing  $[\mathbf{X}(t), \mathbf{R}(t)]$  in the direction  $[\Delta\mathbf{X}(t), \Delta\mathbf{R}(t)]$ . For routeing / P<sub>0</sub> control assignment intervals this push is never negative.

With this new specification of a routeing / P<sub>0</sub> control assignment interval the results established in section 4 also hold in this quasi-dynamic context. In this new setting, it is still true that no routeing / P<sub>0</sub> control assignment interval can leave  $S$ , since  $V$  declines along any such interval and  $V(\mathbf{x}, \mathbf{r})$  tends to infinity as  $(\mathbf{x}, \mathbf{r})$  tends to the boundary of  $S$ .

### 5.1. Pressure-driven responsive control policies

$RC_r(\mathbf{X}, \mathbf{R})$  given in (2.4b) is the *anti-stage r cost* but we may also use for each *stage r* the stage pressures  $PRESS_r(\mathbf{X}, \mathbf{R})$ . These are to be felt by *stage r green-time*. Anti-stage dynamics above may then be written instead as pressure driven green-time dynamics; and this is done in the rest of the paper.

Many real life traffic control systems have green-times which are pressure-driven"; green-time proportions are continually swapped toward those signal stages under greatest pressure and away from stages under the least pressure. A point at which the green-time proportions do not move is a point at which all the stages which receive some green-time are under equal pressure; and so it is impossible then to move green-time toward any more pressurised stage. The simplest of these pressure driven real life responsive traffic control systems have fixed cycle times and the proportions of green-time awarded to the stages during each cycle are determined at the start of that cycle. We consider such simple systems in this section. These systems are more realistic than those described previously in this paper in part because signal cycles and queues are explicitly represented.

Here we suppose that each stage pressure is constructed from "pressures" on the links comprising that stage. Examples of such link pressures from the previous sections are (i)  $x_i/s_i g_i$  and (ii)  $s_i b_i$ . A stage pressure equal to the maximum of the relevant link pressures  $x_i/s_i g_i$  gives rise to the equisaturation policy and a stage pressure obtained by adding the relevant  $s_i b_i$  (where  $b_i$  is in the discussion above given by a function  $f_i$ ) gives rise to the P<sub>0</sub> policy.

We now have explicit cycle times and queues. Also here  $b_i$  may be the measured bottleneck delay felt on exiting link  $i$ , and is not necessarily given by a function of flow and green-time (or red-time). At each junction the signal cycles are all to be the following time intervals of duration  $\tau$  seconds:

$$[0, \tau], [\tau, 2\tau], [2\tau, 3\tau], \dots, [(t-1)\tau, t\tau], \dots$$

The cycle  $[(t-1)\tau, t\tau]$  will be called the  $t^{\text{th}}$  cycle. Other notation here is as follows:

- $s_i$  = the saturation flow at the link  $i$  exit (veh/sec; for all  $i$ );
- $C_r$  = the free-flow cost / time of travel via route  $r$  (seconds; for all  $r$ );
- $x_i(t)$  = the average outflow from link  $i$  during the  $t^{\text{th}}$  cycle (veh/sec; for all  $i, t$ );
- $b_i(t)$  = the average bottleneck delay experienced on exiting from link  $i$  exit during the  $t^{\text{th}}$  cycle (secs, for all  $i, t$ );
- $Q_i(t)$  = the average number of vehicles queueing on link  $i$  during the  $t^{\text{th}}$  cycle (vehicles; for all  $i, t$ );
- $G_k(t)$  = the proportion of time that stage  $k$  is green during the  $t^{\text{th}}$  cycle (for all  $k, t$ ); and
- $g_i(t)$  = the proportion of time that link  $i$  exit is green during the  $t^{\text{th}}$  cycle (for all  $i, t$ ).

In this section the responsive control will have the following form. At each junction at the end of cycle  $t$  the stage green-times are changed (for implementation during the following cycle, cycle  $t+1$ ) only by green-time swaps from one stage to another stage under more pressure, and by sums of such swaps. In order to do this, at the end of cycle  $t$ ,  $\mathbf{x}(t)$ ,  $\mathbf{Q}(t)$ ,  $\mathbf{G}(t)$ ,  $\mathbf{b}(t)$  are all supposed known and for each stage  $j$  the pressure  $PRESS_j(\mathbf{x}(t), \mathbf{Q}(t), \mathbf{G}(t), \mathbf{b}(t))$  at time  $t\tau$  is then determined. The stage green time vector  $\mathbf{G}(t+1)$  to be utilised in cycle  $t+1$  is then in turn determined (for implementation during the next cycle (cycle  $t+1$ ) by adding to  $\mathbf{G}(t)$  a sum of elementary swaps; each elementary swap must obey the following rule.

*Elementary (pairwise) green-time swap rule.* For each pair of stages at a single junction, say stage  $j$  and stage  $k$ :

$$\text{If } PRESS_j(\mathbf{x}(t), \mathbf{Q}(t), \mathbf{G}(t), \mathbf{b}(t)) > PRESS_k(\mathbf{x}(t), \mathbf{Q}(t), \mathbf{G}(t), \mathbf{b}(t)) \text{ then some green-time is swapped from stage } k \text{ to stage } j \text{ and no green-time is swapped from stage } j \text{ to stage } k \tag{5.1}$$

The elementary swap rule (5.1) is applied to each pair of stages at time  $t\tau$ ; then the whole green-time vector change determined at time  $t\tau$  must be a feasible sum  $\mathbf{S}$  of such pair-wise elementary swaps which each follow (5.1). The updated stage green-time vector  $\mathbf{G}(t+1) = \mathbf{G}(t) + \mathbf{S}$  then determines the green-times to be implemented in cycle  $t+1$ .

Rule (5.1) depends on the functions  $PRESS_k$  and so the choice of these functions is critical. Also (5.1) allows a large family of policies even if the functions  $PRESS_j$  are given. Control policies suggested by Smith (1979a, b, c), Wongpiromsarn et al (2012), Varaiya (2013), Le et al (2013) and Gregoire et. al (2014) all belong to one of these families. For example the  $P_0$  signal control policy belongs to this family if

$$PRESS_k(\mathbf{x}, \mathbf{Q}, \mathbf{G}, \mathbf{b}) = \sum_{\text{link } i \text{ belongs to stage } k} s_i b_i \tag{5.2}$$

or 
$$PRESS_k(\mathbf{x}, \mathbf{Q}, \mathbf{G}, \mathbf{b}) = \sum_{\text{link } i \text{ belongs to stage } k} Q_i / g_i. \tag{5.3}$$

### 5.2. Green-time equilibrium and some reasonable stage pressures

Here we call the state vector  $(\mathbf{x}(t), \mathbf{Q}(t), \mathbf{G}(t), \mathbf{b}(t))$  a green-time equilibrium if for each pair of stages  $j, k$  at the same junction less pressurised stages receive no green-time or:

$$PRESS_j(\mathbf{x}(t), \mathbf{Q}(t), \mathbf{G}(t), \mathbf{b}(t)) > PRESS_k(\mathbf{x}(t), \mathbf{Q}(t), \mathbf{G}(t), \mathbf{b}(t)) \text{ implies that } G_j(t) = 0.$$

In this case the signal green-time is not changed. It is natural to expect that under certain conditions a responsive control policy should have a green-time equilibrium. Further it is natural also to expect that there should under reasonable conditions be a sequence  $\{(\mathbf{x}(t), \mathbf{Q}(t), \mathbf{G}(t), \mathbf{b}(t))\}$  which is both a routeing equilibrium (where more costly routes are not used) and also a green-time equilibrium at each cycle  $t$ .

Table 1 gives a list of some reasonable stage pressure formulae and selected relevant papers. Each sum is over all links  $i$  in stage  $k$  and  $p_{ij}$  is the proportion of traffic leaving link  $i$  to enter the downstream link  $j$ . The idea of using backpressure (in telecommunication networks) seems to have first arisen in Tassiulas and Ephremides (1992).

Table 1. Some stage pressure formulae and references where corresponding control policies are considered. The sign “(BP)” means that the relevant stage  $k$  pressure formula involves backpressure and the sign “(NBP)” means that the relevant stage pressure does not involve backpressure. .

| Pressure number | Stage $k$ Pressure Formulae                        | Some corresponding references                                |
|-----------------|--|--|
| 1               | $\sum s_i [Q_i - \sum_j p_{ij} Q_j]$               | Tassiulas and Ephremides (1992), Wongpiromsarn et al (2012), |
| 2               | $\sum w_i [Q_i - \sum_j p_{ij} Q_j]$               | Gregoire et al (2014), Varaiya (2013a, b) (BP)               |
| 3               | $\{exp \sum s_i [Q_i - \sum_j p_{ij} Q_j]\} / G_k$ | Le et al (2014) (BP)   |
| 4               | $(\sum Q_i) / G_k$                                 | Le et al (2014) (NBP)  |
| 5               | $\sum (Q_i / g_i); \sum (s_i b_i)$                 | Smith (1979a, b, c, 1987) (NBP)                              |
| 6               | $\sum [Q_i - \sum_j p_{ij} Q_j] / g_i$             | This paper (BP)  |
| 7               | $\sum s_i [b_i - \sum_j p_{ij} b_j]$               | This paper (BP)  |

The argument in Le et al (2013) appears to apply to show that this new  $P_0$  backpressure policy 6 stabilises queue lengths if route choices (and so the  $p_{ij}$ ) are fixed.

5.3. An outline of an extension of assignment – control formulations in sections 1-4 into a dynamic regime using  $P_0$

To move the steady state theory in sections 1-4 toward a dynamical theory it is natural to consider the following dynamic (time-slice) variation of the standard Beckmann et al (1956) objective function:

$$V(\mathbf{x}, \mathbf{r}) = \sum_i \sum_{t=1}^N \int_0^{x_i(t)} c_i(u) du + \sum_i \sum_{t=1}^N \int_0^{x_i(t)+s_i r_i(t)} f_i(u) du.$$

Here there are  $N$  time slices corresponding to  $t = 1, 2, 3, \dots, N$ . Then as in section 4 above, for each  $(i, t)$ :

$$\partial V / \partial x_i(t) = c_i(x_i(t)) + f_i(x_i(t) + s_i r_i(t)) \text{ and } \partial V / \partial r_i(t) = s_i f_i(x_i(t))$$

and so  $\text{grad } V(\mathbf{x}, \mathbf{r}) = [\mathbf{c}(\mathbf{x}) + \mathbf{f}(\mathbf{x}, \mathbf{r}), \mathbf{s} \cdot \mathbf{f}(\mathbf{x}, \mathbf{r})]$ . This is similar to sections 2-4. To carry through the theory in sections 2-4 in this dynamic context it is necessary to impose conservation and FIFO constraints. This is not done here.

6. A modified Varaiya max pressure policy which is not capacity maximising when route choice is allowed for

The policies and models considered in 6, 7 and 8 below are smooth versions of certain policies considered in section 5 above; just think of the cycle time being very small, vehicles being very short and the lost time being zero.

There has recently been a sharp increase in interest in local distributed traffic signal control policies which are queue stabilising; see for example Varaiya (2013) and Le et al (2013). In almost all of this work the interaction between these policies and routeing decisions by travellers is not allowed for; and thus merits attention. In the special network in figure 1 below we consider naturally modified versions of the Varaiya Max-pressure control policy (called MV here) and the Le et al control policy (called policy ML here). We show that in this network neither policy MV nor policy ML makes the best of available network capacity when selfish route choices are allowed for; each policy is sometimes not consistent with routeing equilibrium and may lead to unbounded queues.

6.1. A network on which neither the MV control policy nor the ML control policy maximizes network capacity

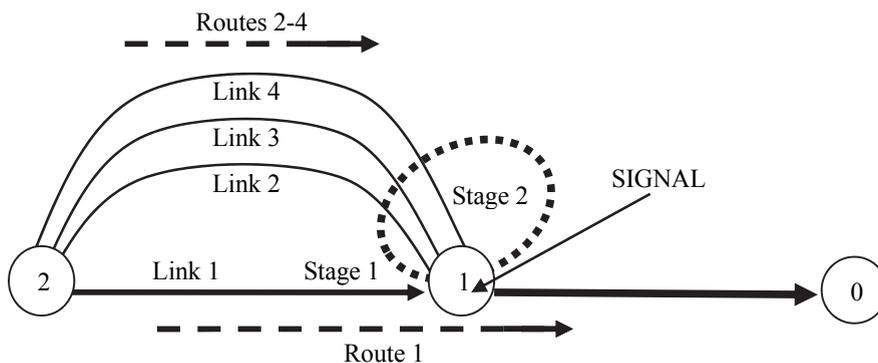


Figure 1. A four route signalised network; links 2-4 all have exit saturation flow = 1 veh/sec; link 1 has exit saturation flow 2 veh/sec. Stage 1 contains link 1; stage 2 contains links 2-4.

Consider the network in figure 1. Let:

- $s_i$  = the saturation flow at the link  $i$  exit (v/sec, for  $i = 1, 2, 3, 4$ );
- $C_i$  = the freeflow cost/time of travel via routes  $i = 1, 2, 3, 4$  (seconds; constant);
- $X_i$  = the flow on route  $i$  (veh/sec, for  $i = 1, 2, 3, 4$ );
- $b_i$  = the bottleneck delay at the link  $i$  exit (seconds, for  $i = 1, 2, 3, 4$ );
- $Q_i$  = the queue volume on link  $i$  (vehicles, for  $i = 1, 2, 3$  and 4);
- $G_i$  = the proportion of time that stage  $i$  is green ( $i = 1, 2$ );
- $g_1$  = the proportion of time that link 1 exit is green (equal to  $G_1$ );
- $g_2$  = the proportion of time that link 2 exit is green (equal to  $G_2$ );

$g_3$  = the proportion of time that link 3 exit is green (equal to  $G_2$ ); and  
 $g_4$  = the proportion of time that link 4 exit is green (equal to  $G_2$ ).

Suppose that

$$s_1 = 2 \text{ and } s_2 = s_3 = s_4 = 1$$

so that the greatest flow is possible when the green-time proportion awarded to stage 2 is as large as possible. Suppose also that the free-flow travel times  $C_i$  along the four routes satisfy

$$C_1 < C_2 < C_3 < C_4.$$

Of course we impose the natural constraints:

$$G_1 + G_2 = 1, G_1 \geq 0 \text{ and } G_2 \geq 0.$$

But we also here impose a further constraint on  $\mathbf{G} = (G_1, G_2)$ :

$$G_2 \leq 4/7 \text{ (or } G_1 \geq 3/7).$$

This green-time constraint is needed to make the counterexample here work in as simple a way as possible. However this constraint on  $\mathbf{G}$  may be very natural in practice. There are two possible scenarios which might require such a constraint in practice when signal control is being utilised to help with traffic management. First: the green-time constraint  $G_1 \geq 3/7$  and  $G_2 \leq 4/7$  may be thought of as protecting the environment along the three routes 2, 3 and 4, supposing that these three routes pass through sensitive areas, by disallowing large values of  $G_2$ . Second: suppose that link 1 is the main commercial street of a thriving town and the routes 2, 3 and 4 represent different “bypasses”. Then it would be natural to always encourage at least a minimum flow through the town main street for commercial purposes, and  $G_1 \geq 3/7$  might be thought of as doing this.

We suppose in this example that  $T$  is feasible, bearing in mind the saturation flows and the green-time constraint. Clearly (given that  $G_1 \geq 3/7$  and links 2 + 3 + 4 have together a greater saturation flow than link 1) choosing  $\mathbf{G} = [3/7, 4/7]$  maximises the green-time allocated to stage 2 and also the possible OD flow through this network. So the maximum feasible value of  $T$  = the maximum possible throughput =  $2.(3/7) + 3.(4/7) = 18/7$ .

So we also suppose here (for the purposes of this counterexample) that the feasible rigid demand  $T$  (veh/sec.) for travel from the origin to the destination satisfies

$$2 = 14/7 < T < 18/7;$$

so that  $T$  is feasible (because  $T < 18/7$ ) but route 1 alone has insufficient capacity for  $T$  even if link 1 were given green all the time: because  $T > 2$  the saturation flow of link 1. So if all of the given demand flow  $T$  (veh/sec) does reach the destination then some of that flow must use at least one of the routes 2-4.

Consider a fixed stage green-time vector  $\mathbf{G}$  and a corresponding quasi-dynamic equilibrium  $(\mathbf{x}, \mathbf{Q}, \mathbf{b}, \mathbf{G})$ . (This is a 4-vector of route flows, a 4-vector of queue volumes, a 4-vector of bottleneck delays and  $\mathbf{G}$ , in which flows are all on cheapest routes, unsaturated link exits have zero queue and  $b_i = Q_i/g_i s_i$ .) We assume that link 5 has a saturation flow  $> 3$  and so is wide enough to accommodate all possible outflows from links 1-4. We consider these 4-vectors.

We also suppose vertical queueing; so that the cost of traversing route  $i$  is  $C_i + b_i$ . In this paper we further assume, for each  $i = 1, 2, 3$  and 4, that the queue volume  $Q_i$ , the bottleneck delay  $b_i$  and the link green-time  $g_i$  are related by:

$$b_i = Q_i/g_i s_i. \quad (6.1)$$

(Of course  $g_1 = G_1, g_2 = G_2, g_3 = G_2$  and  $g_4 = G_2$ .) This formula (6.1) may be motivated in a dynamic context by assuming that the green-times are slowly varying (in which case (6.1) becomes approximately true); here however we are assuming that  $(\mathbf{x}, \mathbf{Q}, \mathbf{b}, \mathbf{G})$  is a quasi-dynamic equilibrium, so that  $(\mathbf{x}, \mathbf{Q}, \mathbf{b}, \mathbf{G})$  is a constant vector (not varying with time) and in this case (6.1) becomes accurately true.

Since  $(\mathbf{x}, \mathbf{Q}, \mathbf{b}, \mathbf{G})$  is a quasi-dynamic equilibrium, all flow must be on cheapest/quickest routes and as shown above some flow must use at least one of the routes 2 – 4. Since these routes have a free-flow travel time which is greater than that of route 1, the bottleneck delay on route 1 (and possibly on other routes too) must equilibrate the network. Since  $C_1 < C_2 < C_3 < C_4$  these equilibrating bottleneck delays must thus satisfy:

$$b_4 \leq b_3 \leq b_2 < b_1. \quad (6.2)$$

(At a quasi-dynamic equilibrium the bottleneck delays on shorter routes must compensate exactly for the longer free-flow travel time of the longest utilised route.)

We will suppose that, in addition to the above conditions which include the quasi-dynamic user-equilibrium condition, the green times given to stages 1 and 2 are to satisfy a more general dynamic version of Varaiya’s control policy (2013); within this continuous model. We here also assume that link 5 has a very high capacity and that there is zero queue on link 5 so that the backpressure term in the Varaiya policy is zero.

### 6.2. A more general dynamic version of the Varaiya Max Pressure signal control Policy

Suppose for the moment now that time is slotted and that all data (on flows, queues and green-times) is available at the time when the “current” time-slot starts. To determine the signal green-times in the “current” time-slot the Varaiya Max Pressure signal control policy on this network utilises Varaiya stage pressures defined as follows:

$$VP_1 = \text{the Varaiya pressure on stage 1} = s_1 Q_1 \text{ and} \quad (6.3a)$$

$$VP_2 = \text{the Varaiya pressure on stage 2} = s_2 Q_2 + s_3 Q_3 + s_4 Q_4. \quad (6.3b)$$

The Varaiya Max Pressure policy then gives all green-time in the “current” short time-slot to the stage with the greatest pressure at the start time of the current time-slot. If the two stage pressures are both maximal (and so equal) then resolve the tie arbitrarily.

We modify this policy to the following smoother policy (where we allow all possible proportions of green-time within a time slot instead of 0 or 1). Suppose given the current queues  $Q_1, Q_2, Q_3, Q_4$ ; these are the queues at the start of the *current* time slot. Then the current stage pressures  $VP_1, VP_2$  are given by (6.3a) and (6.3b). Suppose given also the previous stage green-time proportions  $G_1, G_2$ ; these are the green-time proportions which were implemented in the *previous* time slot.

Then the modified Varaiya control policy MV allocates current green-times (to be implemented in the current time slot) according to the following principle:

*MV: given stage green-times in the previous time-slot, each stage green-time can only be reduced in the current time slot by swapping some of the previous stage green-time onto currently more pressurised stages.*

The policy of always swapping all green to the most pressurised stage satisfies principle MV, so Varaiya’s original policy satisfies MV. (However green-time allocations arising from principle MV swaps are not necessarily all-or-nothing; so it is more likely that there is quasi-dynamic equilibrium consistent with MV than with the original Varaiya policy.)

*Definition of an MV green-time equilibrium.* If in a time slot no green-time changes are possible under the defining principle of MV given above, then the distribution of queues and green-times is said to be an MV green-time equilibrium. An MV equilibrium is thus a triple  $(\mathbf{X}, \mathbf{Q}, \mathbf{b}, \mathbf{G})$  such that:

*less pressurised stages have no green-time*

where the pressures are given by  $VP_1$  and  $VP_2$ .

### 6.3. Policy MV is inconsistent with quasi-dynamic user equilibrium on some networks.

Under the conditions specified in section 6.2 above, assume now that we are at a quasi-dynamic equilibrium consistent with MV equilibrium. This is to be

- (a) a routeing equilibrium (where all used routes have the same travel time),
- (b) an MV equilibrium (as defined above) and
- (c) a queueing equilibrium (where queues are constant and occur only on saturated links).

(Such an equilibrium is a quasi-dynamic equilibrium [(a) and (c)] consistent with the MV equilibrium condition (b).) At such a consistent equilibrium there is no incentive for route flows, green-times or queue lengths to change.

Here (in our continuous context) we now show that (6.4) is impossible on this network shown in figure 1; even though the demand  $T$  (which satisfies  $2 = 14/7 < T < 18/7$ ) is within the network capacity.

So assume that  $2 = 14/7 < T < 18/7$  and that (6.4) holds at  $(\mathbf{X}, \mathbf{Q}, \mathbf{b}, \mathbf{G})$ . As we are at a user equilibrium and (6.1) holds it follows that (6.2) also holds and so

$$Q_4 / s_4 g_4 \leq Q_3 / s_3 g_3 \leq Q_2 / s_2 g_2 < Q_1 / s_1 g_1.$$

Then, using the given saturation flows and the stage green-times,

$$Q_4 / G_2 \leq Q_3 / G_2 \leq Q_2 / G_2 < Q_1 / 2G_1.$$

So

$$Q_4 \leq Q_3 \leq Q_2 < (Q_1 / 2)(G_2 / G_1)$$

This line above yields the following three inequalities:

$$Q_4 < (Q_1 / 2)(G_2 / G_1);$$

$$Q_3 < (Q_1 / 2)(G_2 / G_1); \text{ and}$$

$$Q_2 < (Q_1 / 2)(G_2 / G_1).$$

Adding the three inequalities:

$$Q_4 + Q_3 + Q_2 < 3(Q_1/2)(G_2/G_1).$$

Thus, since  $s_1 = 2, s_2 = s_3 = s_4 = 1$  and  $G_2 / G_1 \leq 4/3$  (this is the green-time constraint we are imposing),

$$s_4Q_4 + s_3Q_3 + s_2Q_2 < (3/2)(s_1Q_1/2)(G_2/G_1) = (3/4)(s_1Q_1)(G_2/G_1) \leq (3/4)(s_1Q_1)(4/3) = s_1Q_1.$$

It follows that at any user equilibrium and at any green-time vector  $\mathbf{G}$  (satisfying the green-time constraint):

$$\text{the Varaiya stage 2 pressure} = s_2Q_2 + s_3Q_3 + s_4Q_4 < s_1Q_1 = \text{the Varaiya stage 1 pressure.}$$

It now follows that, at an MV equilibrium green-time, the stage 2 green-time = 0 and the stage 1 green-time = 1. But  $T > 14/7 = 2$  (the saturation flow of link 1); and so the inflow  $T$  exceeds the maximum possible outflow of link 2 and the queue on link 1 cannot be constant.

Thus the any feasible demand  $T$  satisfying  $2 = 14/7 < T < 18/7$  cannot be satisfied at a quasi-dynamic equilibrium when the MV policy is followed. (A slow dynamical model will have unbounded queues.)

## 7. A modified Le et al signal control policy which is not capacity maximising when route choice is allowed for

### 7.1. The Le et. al. signal control policy may not be consistent with quasi-dynamic user equilibrium.

A similar analysis to that given above may be applied to a signal control policy designed by Le et al (2013), with no modification; still using the network in figure 1. In this section the definitions and constraints in section (6.1) all hold including the added green-time constraint  $G_1 \geq 3/7$ . One difference now is that time slots are replaced by “proper” signal cycles.

Stage pressures are also defined very differently by Le et al (2013) who start with stage weights as follows:

$$\text{Le et al stage 2 weight} = \exp(s_2Q_2 + s_3Q_3 + s_4Q_4) \text{ and}$$

$$\text{Le et al stage 1 weight} = \exp(s_1Q_1).$$

Then, given the values of these weights in a current cycle, Le et al (2013) suggest making the stage green-times during the next cycle proportional to these weights; or

$$G_1 = \exp(s_1Q_1) / [\exp(s_2Q_2 + s_3Q_3 + s_4Q_4) + \exp(s_1Q_1)]$$

and

$$G_2 = \exp(s_2Q_2 + s_3Q_3 + s_4Q_4) / [\exp(s_2Q_2 + s_3Q_3 + s_4Q_4) + \exp(s_1Q_1)]$$

Such a green-time vector  $\mathbf{G}$  equalises the two Le et al stage pressures given below:

$$LP_1 = \text{Le et al stage 1 pressure} = \exp(s_1Q_1)/G_1;$$

$$LP_2 = \text{Le et al stage 2 pressure} = \exp(s_2Q_2 + s_3Q_3 + s_4Q_4)/G_2.$$

Here we show that for a range of feasible demands  $T$  this policy is inconsistent with quasi-dynamic equilibrium on the network in figure 1, by using arguments very similar to those given above in the Varaiya case.

Suppose we are at a quasi-dynamic user equilibrium so that (6.1) and (6.2) both hold. Then as shown above in the Varaiya case (using (6.1), (6.2) and the green-time constraint  $G_1 \geq 3/7$ ):

$$s_4Q_4 + s_3Q_3 + s_2Q_2 < s_1Q_1.$$

It follows immediately that (at a quasi-dynamic user equilibrium)

$$\exp(s_4Q_4 + s_3Q_3 + s_2Q_2) < \exp(s_1Q_1)$$

and so if  $\mathbf{G}$  is to satisfy the Le et al policy then  $G_2 < G_1$  and so  $G_2 < 1/2$  and  $G_1 > 1/2$ .

Now, in this Le et. al. case,  $T$  is restricted by this additional green-time restriction ( $G_2 < 1/2$  and  $G_1 > 1/2$ ) which has arisen from the Le et al control policy combined with quasi-dynamic user equilibration. Any feasible  $T$  must therefore, at a quasi-dynamic equilibrium, satisfy:

$$T < 1/2 \cdot 3 + 1/2 \cdot 2 = 5/2.$$

Thus if  $5/2 = 35/14 < T < 36/14$  then there is no quasi-equilibrium which is also consistent with the Le et al policy.

### 7.2. A slow quasi-dynamic signal control adjustment with unbounded queues

Consider a natural slow dynamic (with green-times adjusting, according to stage pressures, only very slowly as in the modified Varaiya case described above, and flows adjusting to maintain the quasi-dynamic equilibrium state) starting at any quasi-dynamic equilibrium with  $35/14 < T < 36/14$ . Suppose that there are substantial positive initial queues and an initial stage green-time vector  $\mathbf{G}^0$  consistent with  $T$  and so satisfying  $G_2^0 > 1/2$ . Then the natural

dynamic would see  $G_2$  slowly decrease to  $< 1/2$  and  $G_1$  slowly increase to  $> 1/2$ , with green-time swapping slowly from the less pressurised stage 2 to the more pressurised stage 1, causing ever increasing queues as time passes.

## 8. The $P_0$ signal control policy is capacity maximising and queue-stabilising when route choice is allowed for in the network of figure 1.

### 8.1. $P_0$ is consistent with quasi-dynamic user equilibrium on this network

Now we apply a similar analysis to that given above but using the signal control policy  $P_0$ , still using the network in figure 1. The big difference is that signal control policy  $P_0$  is here shown to be consistent with quasi-dynamic user equilibrium. We follow Smith (1979a, b, c, 1987): for this two-stage network the two  $P_0$  pressures are given below:

$$P_0 \text{ stage 1 pressure} = s_1 b_1;$$

$$P_0 \text{ stage 2 pressure} = s_2 b_2 + s_3 b_3 + s_4 b_4.$$

The  $P_0$  policy is satisfied exactly if the two pressures above are equal. The most general formulation of  $P_0$  is the following dynamic formulation which allows for the case where the signals are not exactly satisfying the policy. Let  $\mathbf{G}$  be the signal green-time in the last stage. Then

$$\text{if } P_0 \text{ stage 1 pressure} = s_1 b_1 < s_2 b_2 + s_3 b_3 + s_4 b_4 = P_0 \text{ stage 2 pressure,}$$

increase  $G_2$  and decrease  $G_1$  in the current signal stage and

$$\text{if } P_0 \text{ stage 1 pressure} = s_1 b_1 > s_2 b_2 + s_3 b_3 + s_4 b_4 = P_0 \text{ stage 2 pressure,}$$

increase  $G_1$  and decrease  $G_2$  in the current signal stage.

Here we show that for any feasible demand this policy is consistent with quasi-dynamic equilibrium by constructing a quasi-dynamic equilibrium consistent with  $P_0$ .

The delay formula  $b_i = Q_i/g_i s_i$  is natural for quasi-dynamic networks and we suppose that this holds here at a quasi-dynamic equilibrium. This formula allows stage pressures to be written in terms of either  $b_i$  or  $Q_i$  (at a quasi-dynamic equilibrium). Using this delay formula, these stage pressures may be written:

$$P_0 \text{ stage 1 pressure} = s_1 b_1 = Q_1/g_1;$$

$$P_0 \text{ stage 2 pressure} = s_2 b_2 + s_3 b_3 + s_4 b_4 = Q_2/g_2 + Q_3/g_3 + Q_4/g_4.$$

The following specification gives a quasi-dynamic equilibrium  $(\mathbf{X}, \mathbf{Q}, \mathbf{b}, \mathbf{G})$  consistent with  $P_0$  for any  $T$  such that  $2 < T < 3$  (this is the relevant range of  $T$  for this network):

$$X_1 = 2(3 - T) > 0; X_2 = T - 2 > 0; X_3 = T - 2 > 0; X_4 = T - 2 > 0; G_1 = 3 - T > 0; G_2 = T - 2 > 0;$$

so that

$$X_1 + X_2 + X_3 + X_4 = 2(3 - T) + (T - 2) + (T - 2) + (T - 2) = T \text{ and } G_1 + G_2 = 3 - T + T - 2 = 1,$$

and  $(\mathbf{X}, \mathbf{G})$  is feasible. Also we put

$$b_1 = C_2 + C_3 + C_4 - 3C_1;$$

$$b_2 = C_3 + C_4 - 2C_1;$$

$$b_3 = C_2 + C_4 - 2C_1; \text{ and}$$

$$b_4 = C_2 + C_3 - 2C_1;$$

so that  $C_1 + b_1 = C_2 + C_3 + C_4 - 2C_1 = C_2 + b_2 = C_3 + b_3 = C_4 + b_4$  and the four route costs are equal. Thus  $\mathbf{b}$  ensures that the route flows are equilibrated. Also, using the equation  $b_i = Q_i/g_i s_i$  or  $Q_i = s_i g_i b_i$ , it is clear that for each  $i$ ,  $Q_i > 0$  and  $X_i = s_i g_i$  so that (the bottleneck queues are all non-zero and) the links are exactly saturated, and hence the queues are equilibrated. Thus  $(\mathbf{X}, \mathbf{Q}, \mathbf{b}, \mathbf{G})$  is a quasi-dynamic equilibrium. Finally,

$$s_1 b_1 = 2[C_2 + C_3 + C_4 - 3C_1] = 1[C_3 + C_4 - 2C_1] + 1[C_2 + C_4 - 2C_1] + 1[C_2 + C_3 - 2C_1] = s_2 b_2 + s_3 b_3 + s_4 b_4$$

so that policy  $P_0$  holds. Therefore this  $(\mathbf{X}, \mathbf{Q}, \mathbf{b}, \mathbf{G})$  is a quasi-dynamic equilibrium consistent with control policy  $P_0$ .

### 8.2. A slow quasi-dynamic signal control adjustment with bounded queues

Consider a natural slow dynamic (with green-times adjusting, according to stage pressures, only very slowly) starting at any quasi-dynamic equilibrium  $(\mathbf{X}, \mathbf{Q}, \mathbf{b}, \mathbf{G})$  with  $35/14 < T < 36/14$  and maintaining that quasi-dynamic state. Suppose that there are substantial positive initial queues and an initial stage green-time vector  $\mathbf{G}^0$  consistent with  $T$  and so with  $G_2^0 > 1/2$ . Then  $G_2$  would slowly converge to

$$T - 2 > 35/14 - 2 = 7/14 = 1/2$$

and  $G_l$  would slowly converge to

$$3 - T < 3 - 35/14 = (42-35)/14 = 7/14 = 1/2.$$

(In this case queues also converge to the above equilibrium queues as time passes.)

It appears possible that the above  $P_0$  analysis may also apply in certain cases when the demand varies with time and is such that there are time varying green-times and time varying route flows which ensure that all link-exit flows are within capacity; possibly following the approach outlined in section 5.4.

### 9. Conclusion

This paper has presented several idealised natural general and special dynamical models of day-to-day re-routing and of day to day green-time response. Both green-time response models have been based on the responsive control policy  $P_0$  introduced in Smith (1979a, b, c 1987). Several results have been proved. For example, it has been shown that, for any steady feasible demand within a special flow model, if the general day to day re-routing model is combined with the general day to day green-time response model then under natural conditions any (flow, green-time) solution trajectory cannot leave the region of supply-feasible (flow, green-time) pairs and costs are bounded. It has been shown that throughput is maximised in the following sense. Given any constant feasible demand; this demand is met as any routing / green-time trajectory evolves (following either the general or the special dynamical model). The paper has then considered simple “pressure driven” responsive control policies, with explicit signal cycles of fixed duration. A possible approach to responsive control within a within-day dynamic network, allowing for variable route choices has been very briefly outlined. It has finally been shown that modified Varaiya (2013) and Le et al (2013) pressure-driven responsive controls may not maximise network capacity when route choices are variable, by considering a very simple one junction network. There are many opportunities for further work in the directions discussed in this paper. For example it would be interesting to understand whether  $P_0$  (perhaps suitably modified) maximizes the capacity of within-day and day-to-day dynamic networks.

### Appendix A. Showing that there is no assignment – $P_0$ control interval which leaves $S$

Here we show that with the above general specification of an allowable path, or a routing/ $P_0$ -control assignment interval, in section 3.2,

$$a \text{ routing} / P_0 \text{ control-assignment interval does not leave } S. \tag{A.1}$$

(There are different ways of writing the following argument down. Below is chosen so as to be as simple and clear as possible. The argument follows that in section 2 above.)

To prove that (A.1) (or (3.3)) holds, suppose that there is a routing /  $P_0$ -control assignment interval which leaves  $S$ . Let this be  $\{[\mathbf{X}, \mathbf{R}] + t [\Delta\mathbf{X}, \Delta\mathbf{R}]; 0 \leq t \leq 1\}$ . Then (i), (ii) and (iii) hold in section 3.3 (where a “routing /  $P_0$ -control assignment interval” is defined and also

$$[\mathbf{X}, \mathbf{R}] + [\Delta\mathbf{X}, \Delta\mathbf{R}] \notin S.$$

It follows that, for this interval, as  $t$  increases  $[\mathbf{X}, \mathbf{R}] + t [\Delta\mathbf{X}, \Delta\mathbf{R}]$  swaps red-time and route flow in such a way that  $[\mathbf{X}, \mathbf{R}] + t [\Delta\mathbf{X}, \Delta\mathbf{R}] \in S$  when  $0 \leq t < 1$  but  $[\mathbf{X}, \mathbf{R}] + [\Delta\mathbf{X}, \Delta\mathbf{R}] \notin S$  (when  $t = 1$ ). This means that at least one link exit becomes exactly saturated just as  $t$  reaches 1. Here link  $i$  is said to be exactly saturated if

$$x_i + s_i r_i = s_i.$$

Let SAT denote the (non-empty) set of all those  $i$  such that link  $i$  becomes exactly saturated at  $t = 1$ . Since all links are unsaturated if  $t < 1$  (this is because  $[\mathbf{X}, \mathbf{R}] + t [\Delta\mathbf{X}, \Delta\mathbf{R}] \in (D \times RD) \cap S$  for all  $0 \leq t < 1$ ), each link  $i$  with  $i$  in SAT must have a steadily increasing  $(x_i + s_i r_i)$  as  $t$  increases. In fact  $(x_i + s_i r_i)$  must increase linearly with  $t$  (and must equal  $s_i$  just as  $t$  reaches 1).

Let  $\mathbf{x} = \mathbf{A}\mathbf{X}, \mathbf{r} = \mathbf{B}\mathbf{R}, [\Delta\mathbf{x}, \Delta\mathbf{r}] = [\mathbf{A}\Delta\mathbf{X}, \mathbf{B}\Delta\mathbf{R}]$  and also let

$$[\mathbf{X}, \mathbf{R}](t) = [\mathbf{X}, \mathbf{R}] + t [\Delta\mathbf{X}, \Delta\mathbf{R}] \text{ and } [\mathbf{x}, \mathbf{r}](t) = [\mathbf{x}, \mathbf{r}] + t [\Delta\mathbf{x}, \Delta\mathbf{r}] \text{ for } 0 \leq t \leq 1.$$

Now, if  $i \in \text{SAT}$ , as  $t$  increases to 1,  $(x_i + s_i r_i)$  must increase toward  $s_i$  and

$$\Delta(x_i + s_i r_i) > 0$$

and also (using (2.1b) in section 2)

$$b_i(x_i + s_i r_i) \text{ must increase toward infinity.}$$

It follows that both components of the 2 commodity link cost vector

$$[c_i(x_i) + b_i(x_i + s_i r_i), s_i b_i(x_i + s_i r_i)]$$

must increase toward infinity as  $t$  approaches 1. (This uses (2.1a) and (2.1b).

On the other hand, for all  $i \notin \text{SAT}$ , as  $t$  increases toward 1, link  $i$  remains unsaturated in

$$[\mathbf{X}, \mathbf{R}](t) = [\mathbf{X}, \mathbf{R}] + t [\Delta\mathbf{X}, \Delta\mathbf{R}] \text{ or in } [\mathbf{x}, \mathbf{r}](t) = [\mathbf{x}, \mathbf{r}] + t [\Delta\mathbf{x}, \Delta\mathbf{r}].$$

Hence (for all  $i \notin \text{SAT}$ )  $[x_i + s_i r_i](t)$  is bounded away from  $s_i$ , both components of the two-vector

$$[c_i(x_i) + b_i(x_i + s_i r_i), s_i b_i(x_i + s_i r_i)]$$

are bounded for all  $t$  such that  $0 \leq t \leq 1$  and the sum

$$\sum_{i \notin \text{SAT}} [c_i(x_i) + b_i(x_i + s_i r_i), s_i b_i(x_i + s_i r_i)](t) \cdot [\Delta x_i, \Delta r_i]$$

is bounded below (by  $B$  say) for all  $t$  such that  $0 \leq t < 1$ . (The dot products above may have either sign.) Hence:

$$-(\mathbf{C}, \mathbf{RC})([\mathbf{X}, \mathbf{R}] + t [\Delta\mathbf{X}, \Delta\mathbf{R}]) \cdot [\Delta\mathbf{X}, \Delta\mathbf{R}] = -(\mathbf{c} + \mathbf{b}, \mathbf{s} \bullet \mathbf{b})([\mathbf{x}, \mathbf{r}] + t [\Delta\mathbf{x}, \Delta\mathbf{r}]) \cdot [\Delta\mathbf{x}, \Delta\mathbf{r}]$$

$$= -(\mathbf{c} + \mathbf{b}, \mathbf{s} \bullet \mathbf{b})([\mathbf{x}, \mathbf{r}](t)) \cdot [\Delta\mathbf{x}, \Delta\mathbf{r}]$$

$$= - \sum_i (c_i + b_i, s_i b_i)([x_i, r_i](t)) \cdot [\Delta x_i, \Delta r_i] = - \sum_i [c_i(x_i) + b_i(x_i + s_i r_i), s_i b_i(x_i + s_i r_i)](t) \cdot [\Delta x_i, \Delta r_i]$$

$$= - \sum_{i \in \text{SAT}} [c_i(x_i) + b_i(x_i + s_i r_i), s_i b_i(x_i + s_i r_i)](t) \cdot [\Delta x_i, \Delta r_i] - \sum_{i \notin \text{SAT}} [c_i(x_i) + b_i(x_i + s_i r_i), s_i b_i(x_i + s_i r_i)](t) \cdot [\Delta x_i, \Delta r_i]$$

$$< - \sum_{i \in \text{SAT}} [c_i(x_i) + b_i(x_i + s_i r_i), s_i b_i(x_i + s_i r_i)](t) \cdot [\Delta x_i, \Delta r_i] + B$$

$$\text{(since } - \sum_{i \notin \text{SAT}} [c_i(x_i) + b_i(x_i + s_i r_i), s_i b_i(x_i + s_i r_i)](t) \cdot [\Delta x_i, \Delta r_i] \text{ is bounded above by } B \text{ as } t \text{ varies)}$$

$$\leq - \sum_{i \in \text{SAT}} [b_i(x_i + s_i r_i), s_i b_i(x_i + s_i r_i)](t) \cdot [\Delta x_i, \Delta r_i] + B'$$

(since  $c_i(x_i(t)) \cdot \Delta x_i$  is bounded for all  $t \leq 1$  and all links  $i$ )

$$= - \sum_{i \in \text{SAT}} [\Delta x_i b_i(x_i + s_i r_i) + \Delta r_i s_i b_i(x_i + s_i r_i)](t) + B' = - \sum_{i \in \text{SAT}} [\Delta x_i + s_i \Delta r_i] b_i(x_i + s_i r_i)(t) + B'$$

and this tends to  $-\infty$  as  $t$  tends to 1 since, for  $i \in \text{SAT}$ ,  $(\Delta x_i + s_i \Delta r_i) > 0$ ,

$$(x_i + s_i r_i)(t) \text{ tends to } s_i \text{ as } t \text{ tends to 1 and}$$

$$b_i(x_i + s_i r_i)(t) \text{ tends to } +\infty \text{ as } t \text{ tends to 1.}$$

Thus, under the current conditions,

$$-(\mathbf{C}, \mathbf{RC})([\mathbf{X}, \mathbf{R}] + t [\Delta\mathbf{X}, \Delta\mathbf{R}]) \cdot [\Delta\mathbf{X}, \Delta\mathbf{R}] = -(\mathbf{C}, \mathbf{RC})([\mathbf{X}, \mathbf{R}] + t [\Delta\mathbf{X}, \Delta\mathbf{R}]) \cdot [\Delta\mathbf{X}, \Delta\mathbf{R}]$$

tends to  $-\infty$  as  $t$  tends to 1. It follows immediately that

$$-(\mathbf{C}, \mathbf{RC})([\mathbf{X}, \mathbf{R}] + t [\Delta\mathbf{X}, \Delta\mathbf{R}]) \cdot [\Delta\mathbf{X}, \Delta\mathbf{R}] < 0$$

for all  $t$  sufficiently close to 1. This implies that

$$-(\mathbf{C}, \mathbf{RC})([\mathbf{X}, \mathbf{R}] + t [\Delta\mathbf{X}, \Delta\mathbf{R}]) \cdot [\Delta\mathbf{X}, \Delta\mathbf{R}] \geq 0 \text{ for all } 0 \leq t < 1$$

cannot hold (under the current conditions) and so  $\{[\mathbf{X}, \mathbf{R}] + t [\Delta\mathbf{X}, \Delta\mathbf{R}]; 0 \leq t \leq 1\}$  is *not* a routing /  $P_0$ -control assignment interval as we have defined it.

We have established a contradiction and therefore the supposition which led to that contradiction cannot in fact hold. Our supposition was that “there is a routing /  $P_0$ -control assignment interval which leaves  $S$ .” So this cannot hold and no routing /  $P_0$ -control assignment interval can leave  $S$ ; which is the result we are seeking to prove.  $\square$

The above result shows that stage red-time adjustments following a very general dynamic form of policy  $P_0$ , when combined with the generous re-routing rules in section 2 above creates a stable routing/ $P_0$ -control system in as much as there is no routing/ $P_0$ -control assignment interval which leaves the set  $(D \times RD) \cap S$ . It follows that along

any routing/ $P_0$ -control assignment interval travel costs are bounded. It is easy to check, by giving an example, *that no similar result is possible for the equi-saturation policy*. See for example Smith (1979c).

**Appendix B. Proving convergence to equilibrium of MPAP with  $P_0$  signal adjustments following (4.2)**

This appendix gives the detailed proof of the assertion in section 4 above that dynamical system (4.2) converges to a non-empty set of consistent routing /  $P_0$  green-time equilibria.

Suppose that  $(D \times RD) \cap S$  is non-empty and that  $(\mathbf{X}, \mathbf{R}) \in (D \times RD) \cap S$ . If  $(\mathbf{X}, \mathbf{R})$  is not a Wardrop- $P_0$  equilibrium then the direction  $\mathbf{U}(\mathbf{X}, \mathbf{R})$  is not zero and  $(\mathbf{X}, \mathbf{R}) + \mathbf{U}(\mathbf{X}, \mathbf{R}) \in (D \times RD) \cap S$  by our construction (4.2).

In this section we prove that the set of Wardrop- $P_0$  equilibria is nonempty and that any solution  $\{(\mathbf{X}, \mathbf{R})(t)\}$  of the dynamical system (4.2) converges to the non-empty set of Wardrop- $P_0$  equilibria. (Or that the distance between  $\{(\mathbf{X}, \mathbf{R})(t)\}$  and the set of Wardrop- $P_0$  equilibria tends to zero.)

To do this we suppose that  $\{(\mathbf{X}, \mathbf{R})(t)\}$  is any solution of (4.2). We (i) show that the set  $L$  of all limit points of the sequence  $\{(\mathbf{X}, \mathbf{R})(t)\}$  is non-empty. This implies that  $\{(\mathbf{X}, \mathbf{R})(t)\}$  converges to this non-empty set  $L$  of limit points. We then (ii) and (iii) show that each member of  $L$  is feasible and a routing /  $P_0$  green-time equilibrium so that the set  $L$  of limit points is contained in the set of Wardrop- $P_0$  equilibria. It then follows that the sequence  $\{(\mathbf{X}, \mathbf{R})(t)\}$ , in converging to the set of limit points  $L$ , must also converge to the set of routing /  $P_0$  routing equilibria, because this set contains  $L$ .

(i) *Showing that  $L$  is non-empty.*  $\{(\mathbf{X}, \mathbf{R})(t)\}$  following (4.2) is an infinite sequence of points in  $(D \times RD) \cap S$  and so in  $(D \times RD) \cap (\text{cl}S)$  where  $\text{cl}S$  stands for the closure of  $S$ . Now  $(D \times RD) \cap (\text{cl}S)$  is a closed bounded subset of Euclidean space and so is compact; this implies that each sequence of points in  $(D \times RD) \cap (\text{cl}S)$  has at least one limit point. Thus  $L$  is non-empty.

(ii) *Showing that each member of  $L$  is in  $(D \times RD) \cap S$ .* Let  $(\mathbf{X}, \mathbf{R})^*$  be a point of  $L$ . Then at least along a subsequence  $\{(\mathbf{X}, \mathbf{R})(t)\}$  converges to  $(\mathbf{X}, \mathbf{R})^*$  and so along this subsequence  $\{V(\mathbf{X}(t), \mathbf{R}(t))\}$  of  $V$ -values is strictly decreasing and so cannot tend to  $+\infty$ . This implies that  $(\mathbf{X}, \mathbf{R})^*$  cannot belong to  $\text{bdry}S$ , the boundary of  $S$ ; because  $V(\mathbf{X}, \mathbf{R})$  tends to infinity as  $(\mathbf{X}, \mathbf{R})$  tends to  $\text{bdry}S$ .

(iii) *Showing that each member of  $L$  is a Wardrop –  $P_0$  consistent equilibrium (in  $(D \times RD) \cap S$ ).* We have shown above that all limit points of our sequence lie in  $(D \times RD) \cap S$ . So now we just need to show that if  $(\mathbf{X}_0, \mathbf{R}_0)$  belongs to  $(D \times RD) \cap S$  and  $(\mathbf{X}_0, \mathbf{R}_0)$  is not a Wardrop –  $P_0$  equilibrium then  $(\mathbf{X}_0, \mathbf{R}_0)$  cannot belong to  $L$ . So suppose that  $(\mathbf{X}_0, \mathbf{R}_0)$  belongs to  $(D \times RD) \cap S$  and also that  $(\mathbf{X}_0, \mathbf{R}_0)$  is not a Wardrop –  $P_0$  equilibrium.

We need to show that our sequence  $\{\mathbf{X}(t), \mathbf{R}(t)\}$  cannot have  $(\mathbf{X}_0, \mathbf{R}_0)$  as a limit point; so let us suppose that the non-Wardrop –  $P_0$  equilibrium  $(\mathbf{X}_0, \mathbf{R}_0)$  in  $(D \times RD) \cap S$  is a limit point of  $\{\mathbf{X}(t), \mathbf{R}(t)\}$ .

We will show that this leads to a contradiction. *This will demonstrate that such a non-equilibrium point cannot be a limit point of  $\{\mathbf{X}(t), \mathbf{R}(t)\}$  and so all limit points are Wardrop- $P_0$  equilibria.*

Our assumption is that the sequence  $\{\mathbf{X}(t), \mathbf{R}(t)\}$  does have the non-equilibrium  $(\mathbf{X}_0, \mathbf{R}_0) \in (D \times RD) \cap S$  as a limit point. It follows at once (since  $V$  is continuous) that  $\{V(\mathbf{X}(t), \mathbf{R}(t))\}$  has  $V(\mathbf{X}_0, \mathbf{R}_0)$  as a limit point.

By the definition of the direction  $\mathbf{U}$  in (4.2) and continuity of  $k$  and  $\mathbf{C}$ , each assignment interval generating the sequence  $\{\mathbf{X}(t), \mathbf{R}(t)\}$  begins with a subinterval along which  $V$  strictly decreases. So there are two constants  $h, a > 0$  (possibly small; probably  $h < 1$ ); so small that

$$- [\mathbf{C}, \mathbf{RC}]((\mathbf{X}_0, \mathbf{R}_0) + \theta h \mathbf{U}((\mathbf{X}_0, \mathbf{R}_0))) \cdot \mathbf{U}((\mathbf{X}_0, \mathbf{R}_0)) > a \text{ for all } 0 \leq \theta \leq 1.$$

Then, integrating with respect to  $\theta$  over  $[0, 1]$ ,

$$\{V[(\mathbf{X}_0, \mathbf{R}_0) + h \mathbf{U}((\mathbf{X}_0, \mathbf{R}_0))] - V[(\mathbf{X}_0, \mathbf{R}_0)]\} < - ah$$

and so,

$$\{V[(\mathbf{X}_0, \mathbf{R}_0) + \mathbf{U}((\mathbf{X}_0, \mathbf{R}_0))] - V[(\mathbf{X}_0, \mathbf{R}_0)]\} < - ah. \tag{B.1}$$

This is because (since  $[(\mathbf{X}_0, \mathbf{R}_0), (\mathbf{X}_0, \mathbf{R}_0) + \mathbf{U}(\mathbf{X}_0, \mathbf{R}_0)]$  is an assignment interval)

$$- [\mathbf{C}, \mathbf{RC}]((\mathbf{X}_0, \mathbf{R}_0) + \theta \mathbf{U}(\mathbf{X}_0, \mathbf{R}_0)) \cdot \mathbf{U}((\mathbf{X}_0, \mathbf{R}_0)) \geq \theta \text{ for all } 0 \leq \theta \leq 1$$

and so  $V((\mathbf{X}_0, \mathbf{R}_0) + \theta \mathbf{U}(\mathbf{X}_0, \mathbf{R}_0))$  is non-decreasing throughout  $0 \leq \theta \leq 1$ . (Note that, by the definition of the direction  $\mathbf{U}$  and continuity, each assignment interval begins with a subinterval along which  $V$  strictly decreases.)

We are supposing that  $\{(\mathbf{X}(t), \mathbf{R}(t))\}$  has  $(\mathbf{X}_0, \mathbf{R}_0)$  as a limit point, at least along a subsequence SUB of 1, 2, 3, . . .

$(\mathbf{X}(t), \mathbf{R}(t))$  converges to  $(\mathbf{X}_0, \mathbf{R}_0)$ ;

$[(\mathbf{X}(t), \mathbf{R}(t)) + \mathbf{U}(\mathbf{X}(t), \mathbf{R}(t))]$  converges to  $[(\mathbf{X}_0, \mathbf{R}_0) + \mathbf{U}(\mathbf{X}_0, \mathbf{R}_0)]$ ;

and by continuity of  $V$ :

$V(\mathbf{X}(t), \mathbf{R}(t))$  converges to  $V(\mathbf{X}_0, \mathbf{R}_0)$ ; and

$V[(\mathbf{X}(t), \mathbf{R}(t)) + \mathbf{U}(\mathbf{X}(t), \mathbf{R}(t))]$  converges to  $V[(\mathbf{X}_0, \mathbf{R}_0) + \mathbf{U}(\mathbf{X}_0, \mathbf{R}_0)]$ .

Therefore, at least along the subsequence SUB,

$V[(\mathbf{X}(t), \mathbf{R}(t)) + \mathbf{U}(\mathbf{X}(t), \mathbf{R}(t))] - V(\mathbf{X}(t), \mathbf{R}(t))$  converges to  $V[(\mathbf{X}_0, \mathbf{R}_0) + \mathbf{U}(\mathbf{X}_0, \mathbf{R}_0)] - V(\mathbf{X}_0, \mathbf{R}_0)$ .

Hence (using (B.1)) there is a  $t_0$  belonging to SUB such that whenever  $t > t_0$  and belongs to SUB:

$V(\mathbf{X}(t+1), \mathbf{R}(t+1)) - V(\mathbf{X}(t), \mathbf{R}(t)) = V[(\mathbf{X}(t), \mathbf{R}(t)) + \mathbf{U}(\mathbf{X}(t), \mathbf{R}(t))] - V(\mathbf{X}(t), \mathbf{R}(t)) < -ah/2 < 0$ .

Moreover there must then also be  $t_1 > t_0$  such that whenever  $t \geq t_1$  and belongs to SUB:

$V(\mathbf{X}(t), \mathbf{R}(t)) - V(\mathbf{X}_0, \mathbf{R}_0) < ah/4$ .

Thus for all  $t \geq t_1 > t_0$ , belonging to SUB:

$V(\mathbf{X}(t+1), \mathbf{R}(t+1)) = V[(\mathbf{X}(t), \mathbf{R}(t)) + \mathbf{U}(\mathbf{X}(t), \mathbf{R}(t))] < V(\mathbf{X}(t), \mathbf{R}(t)) - ah/2 < V(\mathbf{X}_0, \mathbf{R}_0) + ah/4 - ah/2 < V(\mathbf{X}_0, \mathbf{R}_0)$ .

Now  $V(\mathbf{X}(t), \mathbf{R}(t))$  is in any case strictly decreasing as  $t$  increases. So the above inequality ensures that:

$V(\mathbf{X}(t), \mathbf{R}(t)) < V(\mathbf{X}_0, \mathbf{R}_0)$  for all  $t \geq t_1$ .

Again since  $V(\mathbf{X}(t), \mathbf{R}(t))$  is strictly decreasing as  $t$  increases, it now follows that

$V(\mathbf{X}(t), \mathbf{R}(t))$  cannot converge to  $V(\mathbf{X}_0, \mathbf{R}_0)$  as  $t$  tends to infinity in SUB.

Now  $V$  is continuous and it then follows that  $\{\mathbf{X}(t), \mathbf{R}(t)\}$  cannot converge to  $(\mathbf{X}_0, \mathbf{R}_0)$  as  $t$  tends to infinity in SUB.

But we are assuming that  $(\mathbf{X}(t), \mathbf{R}(t))$  does converge to  $(\mathbf{X}_0, \mathbf{R}_0)$  for  $t$  in SUB. So our assumption has led to a contradiction and therefore this assumption cannot in fact hold. Thus  $\{\mathbf{X}(t), \mathbf{R}(t)\}$  cannot converge to the non- $P_0$ -equilibrium  $(\mathbf{X}_0, \mathbf{R}_0)$  as  $t$  tends to infinity in SUB. It follows that the only limit points of the sequence  $\{(\mathbf{X}(t), \mathbf{R}(t))\}$  must be Wardrop- $P_0$  equilibria; and hence that  $\{(\mathbf{X}(t), \mathbf{R}(t))\}$  converges to the set of Wardrop- $P_0$  equilibria.  $\square$

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