EXTENDING A RESULT OF RYAN ON WEAKLY COMPACT OPERATORS

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ABSTRACT: An elegant result of Ryan, gives a characterisation of weakly compact operators from a Banach space A into $c_0(X)$, the space of null sequences in a Banach space X. It would be a useful tool if the analogue of Ryan's result were valid when $c_0(X)$ is replaced by $c(X)$, the space of convergent sequences in X. This seems plausible and has been assumed true by some authors. Unfortunately it is false in general; Ylinen has produced a counterexample. But when A is a C^*-algebra, or, more generally, the dual of A is weakly sequentially complete we show that the desired extension of Ryan's result does hold. The latter result turns out to be 'best possible'.

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INTRODUCTION

The origin of this paper stems from observing that some results on non-commutative, finitely additive vector measures (i.e. weakly compact operators from a C^*-algebra to a Banach space) do not depend on the domain being a C*-algebra but are essentially Banach space results.

Let A and X be Banach spaces and let $(T_n)(n = 1, 2...)$ be a sequence of weakly compact operators mapping A into X. For each $z \in A^{**}$ let $(T_n^{**}z)(n = 1, 2...)$ be a Cauchy sequence. Since, for each n, T_n is weakly compact, the range of T_n^{**} is in X. By the Uniform Boundedness Theorem there is a bounded operator $T^\# : A^{**} \longmapsto X$ such that $\lim_{n\to\infty} T_n^{**}z = T^\# z$ for each z in A^{**}. It would be natural to expect $T^\#$ to be weakly compact but, in general, this is false. This follows from the following example constructed by Ylinen [6].

In Proposition 2.1 [6], $A = l^1 = X$. For each n, $T_n : l^1 \longmapsto l^1$ is defined by $T_n(x_1, x_2, ..., x_k, ...) = (x_1, x_2, ..., x_n, 0, 0, ...)$. Then each T_n is weakly compact (because its range is finite dimensional). Ylinen proves that $(T_n^{**}z)(n = 1, 2...)$ converges for each z in the dual of l^∞ but the pointwise limit of the sequence of operators $(T_n)(n = 1, 2...)$ is not weakly compact.

However if A is a C^*-algebra then there does exist a weakly compact operator $T : A \longmapsto X$ such that $\lim_{n\to\infty} T_n^{**}z = T^{**}z$ for each z in A^{**}. This is an immediate consequence of Corollary 3.3 [1]. In this note we show that a positive result is also obtained if A^* is weakly complete. (We recall that the dual of a C^*-algebra is always weakly complete.) We shall also see that, in a sense made precise here, the latter result is 'best possible'.

Ryan [4] characterised weakly compact operators from a Banach space A into $c_0(X)$, the space of null sequences in a Banach space X; see Propostion 2.4 below. When $c_0(X)$ is replaced by $c(X)$, the space of convergent sequences in X, the natural extension of Ryan's characterisation does not hold, in general. But when X^* is weakly (sequentially) complete then we show, in Section 2, that Ryan's characterisation can be generalised successfully by applying the results we obtain in Section 1. This can then be applied to underpin some fundamental work on weak compactness and multi-linear operators on Banach spaces [3].

1. CONVERGENT SEQUENCES OF WEAKLY COMPACT OPERATORS

Let us recall that a Banach space Z is said to be *weakly complete* if, whenever $(z_n)(n = 1, 2...)$ is a sequence in Z such that $(\phi z_n)(n = 1, 2...)$ is a Cauchy sequence for every ϕ in Z^*, then there exists z in Z such that $\phi z_n \to \phi z$ for every ϕ in Z^*. Some authors use the term *weakly sequentially complete* for the same property.

THEOREM 1.1 *Let A be a Banach space such that A^* is weakly complete. Let X be a Banach space and let $(T_n)(n = 1, 2...)$ be a sequence of weakly compact operators from A into X. Let $(T_n^{**}z)(n = 1, 2...)$ be a Cauchy sequence for each z in A^{**}. Then there exists a weakly compact operator T such that $||(T^{**} - T_n^{**})z|| \to 0$ for each z in A^{**}.*

PROOF: Since T_n is weakly compact, T_n^{**} maps A^{**} into X. Let $T^{\#}z = lim T_n^{**}z$ for each z in A^{**}. Then, by the Uniform Boundedness Theorem, $T^{\#}$ is a bounded linear operator from A^{**} into X. Let T be the restriction of $T^{\#}$ to A.

Fix $\phi \in X^*$. Then, for each $z \in A^{**}$,

$$\lim_{n\to\infty} \ \langle T_n^{**}z, \phi\rangle \ = \left\langle T^{\#}z, \phi\right\rangle .$$

So $\lim_{n\to\infty} \langle z, T_n^*\phi\rangle = \left\langle T^{\#}z, \phi\right\rangle$. So $(T_n^*\phi)(n = 1, 2...)$ is a weakly Cauchy sequence in A^*. By the hypothesis that A^* is weakly complete, it follows that there exists a unique $\alpha \in A^*$ such that $<z, \alpha> = <T^{\#}z, \phi>$ for all z in A^{**}.

All that is now needed is to show that $T^{**} = T^{\#}$. Since this has been a source of error in the past we wish to avoid being too glib and so give a detailed elementary argument.

Let (z_t) be a net in A^{**} which converges to 0 in the $\sigma(A^{**}, A^*)$-topology. So $<z_t, \alpha> \to 0$. Thus $<T^{\#}z_t, \phi> \to 0$ for each ϕ in X^*. So $T^{\#}$ is a continuous map of A^{**}, equipped with the weak*-topology, to X equipped with the weak topology. Since the norm closed unit ball of A^{**} is weak* compact, the image of the unit ball of A^{**}under the map $T^{\#}$ is weakly compact. Hence $T^{\#}$, and its restriction to A, T, is weakly compact. Thus, by Lemma VI.2.3. and Theorem VI.4.2 of [2], T^{**} is weak* to weak continuous from A^{**} to X. By Goldstine's Theorem, see Theorem V.4.5 [2], the norm closed unit ball of A is weak*-dense in the norm closed unit ball of A^{**} . Hence $T^{\#} = T^{**}$.□

REMARK Let A be a C^*-algebra then its dual is the predual of a von Neumann algebra and so, by Corollary III. 5.2 [5], the dual of A is weakly complete. Hence Theorem 1.1 applies whenever A is a C^*-algebra.

It turns out that Theorem 1.1 is 'best possible'. To make this claim precise it is convenient to introduce the following definition:

DEFINITION 1.2 Let X be a Banach space. A Banach space A is said to have the *weak compactness stability property with respect to* X, if, given any sequence of weakly compact operators $(T_n)(n = 1, 2...)$, each mapping A into X, and with $(T_n^{**}z)(n = 1, 2...)$ a Cauchy sequence for each z in A^{**}, then there exists a weakly compact operator T such that $lim_{n\to\infty}T_n^{**}z = T^{**}z$ for each z in A^{**}.

PROPOSITION 1.3 *Let A be a Banach space with the weak compactness stability property with respect to some non-zero Banach space X. Then A^* is weakly complete.*

PROOF: Let $(\phi_n)(n = 1, 2...)$ be a weakly Cauchy sequence in A^*. Then, for each z in A^{**}, $\lim_{n\to\infty} < z, \phi_n >$ exists. By the Uniform Boundedness Theorem, there exists a bounded linear functional $\psi^{\#}$ on A^{**}such that $\psi^{\#}(z) = \lim_{n\to\infty} < z, \phi_n >$ for each z in A^{**}.

Since X is a non-zero Banach space it contains a non zero element x_0. For each n, let $T_n : A \longmapsto X$ be defined by

$$T_n(a) = \langle a, \phi_n \rangle\, x_0.$$

Then T_n has one dimensional range and so is (weakly) compact. Furthermore $T_n^{**}(z) = \langle z, \phi_n \rangle\, x_0$ for each z in A^{**}. It now follows from the weak compactness stability property for X that there exists a weakly compact operator T mapping A into X, such that

$$T^{**}(z) = \lim_{n\to\infty} T_n(z) = \lim_{n\to\infty} \langle z, \phi_n \rangle\, x_0 = \psi^{\#}(z)x_0 \text{ for each } z \text{ in } A^{**}.$$

Since T is weakly compact then, as remarked in the proof of Theorem 1.1, T^{**} is weak* to weak continuous as a map from A^{**} to X. Thus $\psi^{\#}$ is a weak* continuous linear functional on A^{**}. So, by Theorem V.3.9 [2], $\psi^{\#}$ may be identified with an element of A^*. Hence $(\phi_n)(n = 1, 2...)$ is weakly convergent. Thus A^* is weakly complete. □

COROLLARY 1.4 *Let A be a Banach space. Then the following conditions are equivalent:*
(i) A^* is weakly complete.
(ii) A has the weak compactness stability property with respect to some Banach space of non-zero dimension.
(iii) A has the weak compactness stability property with respect to every Banach space X.

PROOF: By Theorem 1.1, (i) implies (iii). Trivially (iii) implies (ii). By Proposition1.3 (ii) implies (i). □

2. EXTENDING RYAN'S LEMMA

For any Banach space X, let $c(X)$ be the Banach space of all (norm) convergent sequences in X, equipped with the supremum norm. Those elements of $c(X)$ which are sequences in X converging (in norm) to 0, form a closed subspace which is denoted by $c_0(X)$.

For each positive integer n, let T_n be a bounded linear operator from a Banach space A into a Banach space X. Let $\lim T_n a$ exist for each a in A. Then $(T_n a)(n = 1, 2...)$is a vector in $c(X)$. Let T_∞ be the linear map from A into X defined by $T_\infty a = \lim T_n a$ for each a in A. We use $\mathbf{T}$ to denote the operator from A to $c(X)$ associated with the sequence $(T_n)(n = 1, 2...)$ and defined by $\mathbf{T}(a) = (T_n a)(n = 1, 2...)$. By applying the Uniform Boundedness Theorem we see that T_∞ and $\mathbf{T}$ are both bounded linear operators. Conversely, every bounded operator from A into $c(X)$ arises in this way from a sequence of operators from A into X.

Let us recall [4] that, for $1 \leq p < \infty$, and X an arbitrary Banach space, $l^p(X)$ is the Banach space whose points are the sequences $\mathbf{x} = (x_n)\,(n = 1, 2...)$ in X for which $\sum_1^\infty ||x_n||^p < \infty$.The norm of $\mathbf{x}$ is defined to be $(\sum_1^\infty ||x_n||^p)^{1/p}$. Also $l^\infty(X)$ is defined to be the Banach space whose points are all bounded sequences in X and where the norm of $\mathbf{x} = (x_n)\,(n = 1, 2...)$ is defined to be $\sup\{||x_n|| : 1 \leq n\}$.

Given $\boldsymbol{\phi} = (\phi_0, \phi_1, ...)$ in $l^1(X^*)$ and $\mathbf{x} = (x_n)\,(n = 1, 2...)$ in $c(X)$, let $L_{\boldsymbol{\phi}}(\mathbf{x}) = \phi_0(\lim x_n) + \sum_{n=1}^\infty \langle x_n, \phi_n\rangle$. Then straightforward calculations show that $L_{\boldsymbol{\phi}}$ is a bounded linear functional on $c(X)$ and its norm is $\sum_{n=0}^\infty ||\phi_n||$. Furthermore the map $\boldsymbol{\phi} \longmapsto L_{\boldsymbol{\phi}}$ can be shown to be a surjective isometry of $l^1(X^*)$ onto $c(X)^*$. It follows from the remarks in [4] that the dual of $l^1(X^*)$ can be identified in a natural way with $l^\infty(X^{**})$. Thus $c(X)^{**}$ can be identified with $l^\infty(X^{**})$. Let $\natural$ be the canonical embedding of X into X^{**}. Then a sequence $(x_n)(n = 1, 2...)$ in $c(X)$ is mapped to $(\lim \natural x_n, \natural x_1, \natural x_2, ...)$ in $l^\infty(X^{**})$.

LEMMA 2.1 *Let* $\mathbf{T}$ *be a bounded operator from a Banach space* A *into* $c\,(X)$ *and let* $T_n(n = 1, 2...)$ *and* T_∞ *be the operators from* A *into* X *associated with* $\mathbf{T}$ *as above. Fix* L *in* $c(X)^*$. *Then let* $\boldsymbol{\phi} = (\phi_0, \phi_1, ...)$ *be the corresponding element of* $l^1(X^*)$. *Then, for each* $z \in A^{**}$, $\langle \mathbf{T}^{**}z, L\rangle = \langle T_\infty^{**}z, \phi_0\rangle + \sum_{n=1}^\infty \langle T_n^{**}z, \phi_n\rangle$.

PROOF: For each $a \in A$,

$$\langle \mathbf{T}a, L\rangle = \langle T_\infty a, \phi_0\rangle + \sum_{n=1}^\infty \langle T_n a, \phi_n\rangle.$$

Now let z be in the unit ball of A^{**}. Then, by Goldstine's Theorem (see above) there is a net (a_t) in the unit ball of A which converges weak* to z. Then $\mathbf{T}a_t \to \mathbf{T}^{**}z$ in the weak* topology of $c(X)^{**}$. So $\langle \mathbf{T}a_t, L\rangle \to \langle \mathbf{T}^{**}z, L\rangle$ .Similarly, for each N,

$$\langle T_\infty a_t, \phi_0\rangle + \sum_{n=1}^{N} \langle T_n a_t, \phi_n\rangle \to \langle T_\infty^{**} z, \phi_0\rangle + \sum_{n=1}^{N} \langle T_n^{**} z, \phi_n\rangle.$$

Choose $\varepsilon > 0$. Choose N large enough to ensure that $||\mathbf{T}|| \sum_{n=N+1}^{\infty} ||\phi_n|| \leq \varepsilon$. Then for any w in the unit ball of A^{**},

$|| \sum_{n=N+1}^{\infty} \langle T_n^{**} w, \phi_n\rangle || \leq ||\mathbf{T}|| \sum_{n=N+1}^{\infty} ||\phi_n|| \leq \varepsilon$. From this it follows by routine arguments that $\langle \mathbf{T}^{**} z, L\rangle = \langle T_\infty^{**} z, \phi_0\rangle + \sum_{n=1}^{\infty} \langle T_n^{**} z, \phi_n\rangle$. $\square$

We have seen that $c(X)^{**}$can be identified with $l^\infty(X^{**})$. When this identification is made appropriately, we have:

COROLLARY 2.2 *For each z in A^{**}*we have

$$\mathbf{T}^{**}(z) = (T_\infty^{**} z, T_1^{**} z, T_2^{**} z, ..., T_n^{**} z, ...)\ .$$

The following lemma is, in essence, proved by Ylinen [6]. For the convenience of the reader, we give a brief proof here as an application of Corollary 2.2.

LEMMA 2.3 *Let A and X be Banach spaces and let $\mathbf{T}$ be a weakly compact operator from A into $c(X)$. Let $(T_n)(n = 1, 2...)$ be the sequence of operators from A into X such that $\mathbf{T}(a) = (T_n a)(n = 1, 2...)$for each a* in A. *Then each T_n is weakly compact. Also T_∞, the pointwise limit of $(T_n)(n = 1, 2...)$, is weakly compact. Furthermore* $\lim T_n^{**}(z) = T_\infty^{**} z$ *for each z* in A^{**}.

PROOF: We recall that the product of a bounded operator and a weakly compact operator is weakly compact. Let π_n be the canonical projection of $c(X)$ onto the n^{th}coordinate. Then $T_n = \pi_n \mathbf{T}$. Hence T_n is weakly compact. Let π_∞ be the operator which maps $(a_1, a_2, ...)$ in $c(X)$ to $\lim a_n$. Then $T_\infty = \pi_\infty \mathbf{T}$ and so is also weakly compact.

Since $\mathbf{T}$ is weakly compact, $\mathbf{T}^{**}$maps into (the canonical image of) $c(X)$. So$(T_\infty^{**} z, T_1^{**} z, T_2^{**} z, ..., T_n^{**} z, ...)$ is a convergent sequence in X with limit $T_\infty^{**} z$. $\square$

PROPOSITION 2.4 (Ryan [4]) *Let A and X be Banach spaces. Let $(T_n)(n = 1, 2...)$ be a sequence of bounded operators from A into X. Let $||T_n z|| \to 0$ for each z in A. Then $\mathbf{T}$ is a weakly compact operator from A into $c_0(X)$ if, and only if, each T_n is weakly compact and $||T_n^{**} z|| \to 0$ for each z in A^{**}. When $\mathbf{T}$ is weakly compact, $\mathbf{T}^{**}(z) = (T_n^{**}(z))(n = 1, 2...)$ for each z in A^{**}.*

Proposition 2.4 is a special case of the following result of Ylinen [6]:

PROPOSITION 2.5 *Let A and X be Banach spaces and let $\mathbf{T}$ be a bounded operator from A into $c(X)$. Let $(T_n)(n = 1, 2...)$ be the sequence of operators from A into X such that $\mathbf{T}(a) = (T_n a)(n = 1, 2...)$ for each a in A. Then $\mathbf{T}$ is weakly compact if and only if the following conditions are satisfied:*

(i) *For each n, T_n is weakly compact.*

(ii) *For each z in A^{**}, $\lim T_n^{**}(z)$ exists.*

(iii) *The operator $T_\infty : A \longmapsto X$ is weakly compact, where $T_\infty(a) = \lim T_n(a)$ for each a in A.*

PROOF: By Lemma 2.3, when $\mathbf{T}$ is weakly compact the three conditions are satisfied.

Now suppose that the conditions are satisfied. So, for each z in A^{**}, (iii) implies that $T_\infty^{**}z$ is in X and (i) implies that $T_n^{**}z$ is in X for each n. Hence by (ii), $(T_\infty^{**}z, T_1^{**}z, T_2^{**}z, ..., T_n^{**}z, ...)$ is in $c(X)$. Hence, by Corollary 2.2, $\mathbf{T}^{**}$maps A^{**} into $c(X)$. So $\mathbf{T}$ is weakly compact. □

THEOREM 2.6 *Let X be any Banach space. Let A be a Banach space whose dual space, A^*, is weakly complete. Let $(T_n)(n = 1, 2...)$ be a sequence of weakly compact operators from A into X such that $(T_n^{**}(z))(n = 1, 2...)$ is a Cauchy sequence for each z in A^{**}. Then $\mathbf{T}$ is a weakly compact operator from A into $c(X)$.*

PROOF: Because the dual of A is weakly complete, Theorem 1.1 implies the existence of a weakly compact operator $T_\infty : A \longmapsto X$ such that $T_n^{**}(z) \to T_\infty^{**}(z)$ for each z in A^{**}. So conditions (i), (ii) and (iii) of Proposition 2.5 are satisfied. □

REMARK: If $A^*$is not weakly complete then it follows from Proposition 1.3 that we can find a sequence of weakly compact operators, $(T_n)(n = 1, 2...)$, each mapping A into c, such that $(T_n^{**}(z))(n = 1, 2...)$ is a convergent sequence for each z in A^{**} but T_∞ is not weakly compact. (Where $T_\infty(a) = \lim T_n(a)$ for each a in A.) It then follows from Proposition 2.5 that $\mathbf{T}$ is not weakly compact. So the hypothesis that $A^*$is weakly complete is essential for the validity of Theorem 2.6.

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