MINIMAL ORDERINGS AND QUADRATIC FORMS ON A FREE MODULE OVER A SUPERTROPICAL SEMIRING

ZUR IZHAKIAN, MANFRED KNEBUSCH, AND LOUIS ROWEN

Abstract. This paper is a sequel to [6], in which we introduced quadratic forms on a module over a supertropical semiring R and analyzed the set of bilinear companions of a single quadratic form $V \to R$ in case the module V is free. Any (semi)module over a semiring gives rise to what we call its **minimal ordering**, which is a partial order iff the semiring is "upper bound." Any polynomial map q (or quadratic form) then induces a pre-order, which can be studied in terms of "q-minimal elements," which are elements a which cannot be written in the form $b+c$ where $b < a$ but $q(b) = q(a)$. We determine the q-minimal elements by examining their support.

But the class of *all* polynomial maps (in up to $\operatorname{rank}(V)$ variables) is itself a module over R, so the basic properties of the minimal ordering are applied to this R-module, or its submodule $\operatorname{Quad}(V)$ consisting of quadratic forms on V. This is a significant initial step in the classification of quadratic forms over semirings arising in tropical mathematics.

$\operatorname{Quad}(V)$ is the sum of two disjoint submodules $\operatorname{QL}(V)$ and $\operatorname{Rig}(V)$, consisting of the quasilinear and the rigid quadratic forms on V respectively (cf. [6]). Both $\operatorname{QL}(V)$ and $\operatorname{Rig}(V)$ are free with explicitly known bases, but $\operatorname{Quad}(V)$ itself is almost never free.

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1. Introduction

Let us set up some basic terminology (modules, quadratic forms, and supertropical algebra) in §1.1–§1.4, before describing what we are doing in this paper in §1.5.

1.1. A review of quadratic forms over semirings.

Let R be a semiring, here always assumed to be commutative and with 1. We review a few results from [6], [7].

Definition 1.1. *An R-**module** V is a semigroup with scalar multiplication $R \times V \to V$ satisfying the following axioms for all $r_i \in R$ and $v, w \in V$:*

(1) $r(v + w) = rv + rw$;

(2) $(r_1 + r_2)v = r_1v + r_2v$;

(3) $(r_1r_2)v = r_1(r_2v)$;

(4) $1v = v$;

(5) $r0_V = 0_V$;

(6) $0_Rv = 0_V$.

We write 0 for both 0_R and 0_V, hoping that the context is clear.

A **quadratic form** on an R-module V is a function $q : V \to R$ satisfying

$$q(ax) = a^2q(x) \tag{1.1}$$

for any $a \in R$, $x \in V$, such that there exists a symmetric bilinear form $b : V \times V \to R$ (not necessarily uniquely determined by q) with

$$q(x + y) = q(x) + q(y) + b(x, y) \tag{1.2}$$

for any $x, y \in V$. Every such bilinear form b is called a **companion** of q, and the pair (q, b) is called a **quadratic pair** on V.

A quadratic form q is called **quasilinear** if the bilinear form $b = 0$ is a companion of q, i.e., $q(x + y) = q(x) + q(y)$ for all $x, y \in V$. In the main case of this paper, that R is "supertropical" (cf, §1.4 below) these are the "diagonal" forms on V,

$$q\Big(\sum_i x_i\varepsilon_i\Big) = \sum_i q(\varepsilon_i)x_i^2, \tag{1.3}$$

due to the fact that then $(\lambda + \mu)^2 = \lambda^2 + \mu^2$ for all $\lambda, \mu \in R$, cf. [6, Proposition 0.5].

At the other end of the spectrum, q is called **rigid** if q has only one companion. This happens iff $q(\varepsilon_i) = 0$ for all vectors ε_i in the base $\{\varepsilon_i \mid i \in I\}$, cf. [6, Theorem 3.5].

Any quadratic form q on a free R-module can be written as a sum

$$q = q_{QL} + \rho, \tag{1.4}$$

where q_{QL} is a quasilinear (and uniquely determined by q) and ρ is rigid (but not unique), by [6, §4].

1.2. Ordered monoids versus semirings.

Since tropical geometry is based on valuations taking values in an ordered group (say $\mathbb{Q}$ or $\mathbb{R}$), we want to build up algebraic machinery from ordered groups and monoids. We start with a basic observation of Green on semilattices (sets with a "sup" function $\vee$).

Remark 1.2. *Any semilattice $(\mathcal{M}, \vee\ )$ gives rise to a semigroup, where we define $a + b$ to be $a \vee b$.*

Next we assume that our semilattice $\mathcal{M}$ acts on a monoid $(R, \cdot\,)$ [1], where the semilattice structure respects the monoid structure, in the following sense:

$$c(a \vee b) = ca \vee cb \tag{1.5}$$

for all elements $a, b \in \mathcal{M}$ and $c \in R$.

Lemma 1.3.

(i) $(\mathcal{M}, \vee\,)$ *gives rise to an additive monoid* $(M, +)$*, where we define* $a + b = a \vee b$*. When* $M = R$ *is a lattice-ordered monoid, this is a semiring.*

(ii) *Conversely, any additive monoid* M *gives rise to a transitive and reflexive binary relation, defined by* $b \leq a$ *if* $a = b + c$ *for some* $c \in M$*. When (i) holds, we have* $a = b + a$*.*

Proof. Distributivity follows from (1.5). □

As usual, $x < y$ means that $x \leq y$ and $x \neq y$.

The semiring structure opens the way to use basic tools of linear algebra and geometry (matrices, polynomials), and more sophisticated ones such as quadratic forms to handle questions about angles and trigonometry. Unfortunately the ensuing theory is considerably more intricate, which is what led us in the first place to an in-depth study starting with [3, 5, 6, 7], some of which turns out to be somewhat technical.

Our first task is to determine when our relation $\leq$ is antisymmetric and thus a partial order, in terms of the intrinsic structure of the semiring. We recall a definition from [4, Definition 11.5].

Definition 1.4. *We say that an additive monoid* V *is* ***u.b.*** *(for "upper bound") if* $a+b+c = a$ *always implies* $a + b = a$*, and we say that* V ***lacks zero sums*** *if* V *has the weaker property that* $a + b = 0$ *implies* $a = b = 0$*.*

A semiring R *is* ***u.b.*** *(respectively* ***lacks zero sums****) if its underlying additive monoid has this property.*

For example, the max-plus algebra is an u.b. semifield. Any polynomial semiring over a u.b. semiring is u.b.. A semiring whose underlying additive semigroup lacks zero sums is called an **antiring** in [1, 9], and **zerosumfree** in [2].

In [1, 9] various properties of antirings were developed that tie in with tropical linear algebra. (For example, the only invertible matrices over antirings are generalized permutation matrices). The point of this definition is seen in the next observation (see also [4, Remark 11.6]:

Proposition 1.5. *Assume that* V *is an additive monoid. Then the binary relation given by*

$$a \geq b \quad \Leftrightarrow \quad a = b + c \quad \text{for some } c \in V$$

is a partial order iff V *is u.b..*

Proof. ($\Rightarrow$): Suppose $a + b + c = a$. Then $a \geq a + b$, and obviously $a + b \geq a$, so $a + b = a$.

($\Leftarrow$): Suppose $a \geq b$ and $b \geq a$. Then $a = b + c$ and $b = a + d$, for suitable c, d, so $a = (a + d) + c$, implying $a = a + d = b$. □

In this paper we always assume that R is an u.b semiring and V an u.b. module.

[1] A monoid is a semigroup that has a neutral element.

Definition 1.6. *The partial order of Proposition 1.5 is called the* ***minimal order*** *on V, respectively R.*

We want to understand this minimal order as best as we can, and how it impacts on quadratic forms.

The word "minimal order" is justified by the following fact.

Proposition 1.7. *Assume that V is an additive monoid which admits a partial ordering α, such that $0 \leq_\alpha x$ for all $x \in V$. Then V is u.b. and α refines the minimal order of V.*

Proof. If $x + y + z = x$, we obtain from $0 \leq_\alpha y, z$ that $x \leq_\alpha x + y \leq_\alpha x + y + z = x$, and so $x = x + y$, which proves that V is u.b. If $x \leq y$, then $y = x + z$ for some z, and so we obtain form $0 \leq_\alpha z$ that $x \leq_\alpha x + z = y$. $\square$

The word "u.b." alludes to the property that for any $x, y \in V$ we have an upper bound u encoded in the additive structure of V, namely $u = x + y$. Lemma 1.3 deals with the special case that $x + y$ is the least upper bound (=maximum) of x and y, which may be false in general. In this case the criterion for the minimal order to be total, i.e., any two elements a, b are comparable, ensues the condition that $a + b \in \{a, b\}$, which sometimes is called **bipotence**.

1.3. **Polynomials and polynomial functions.** The next step is to bring in polynomial functions on modules, to provide another approach to quadratic forms. For convenience, we assume that V is a free module with a base $\{\varepsilon_i \mid i \in I\}$. Occasionally I is taken infinite, but always is an ordered set. Usually I is assumed to be finite, of order n, called the **rank** of V.

We assume that a semiring R lacks zero sums and also is closed under multiplication. (In other words, $R \setminus \{0\}$ is a semiring without 0.) This is a natural condition in tropical mathematics, and then every free module V of rank n has a unique base $\{\varepsilon_1, \dots, \varepsilon_n\}$ (up to permutation and scalar multiple), cf. [1] and [7, Theorem 1.2]. Hence a polynomial $q(\lambda_1, \dots, \lambda_n) \in R[\lambda_1, \dots, \lambda_n]$ defines a "polynomial function" on R^n by

$$\sum_{i=1}^{n} x_i \varepsilon_i \longmapsto q(x_1, \dots, x_n).$$

When q is homogeneous of degree 2, it is easy to see that this is a quadratic form. Namely, if $q(\lambda_1, \dots, \lambda_n) = \sum_{i=1}^{n} \alpha_i \lambda_i^2 + \sum_{i<j} \beta_{ij} \lambda_i \lambda_j$, $x = \sum_{i=1}^{n} x_i \varepsilon_i$, $y = \sum_{i=1}^{n} y_i \varepsilon_i$, we define the quadratic form

$$q(x) = \sum_i \alpha_i x_i^2 + \sum_{i<j} \beta_{ij} x_i x_j, \tag{1.6}$$

and bilinear companion

$$b(x, y) = \sum_i 2\alpha_i x_i y_i + \sum_{i<j} \beta_{ij}(x_i y_j + x_j y_i).$$

Thus, each homogeneous polynomial function of degree 2 is a quadratic form q with $q(\varepsilon_i) = \alpha_i$ and $b(\varepsilon_i, \varepsilon_j) = \beta_{i,j}$, and we will present some of our results more generally for polynomial functions, although our focus is on quadratic forms. (To distinguish the two concepts, one could speak of a polynomial quadratic form versus a functional quadratic form.) It turns out that every functional quadratic form is obtained in this way, cf. [5, §1].

We write $\mathrm{Pol}(V, R)$ for the set of polynomial functions on V. We also define $\mathrm{Fun}(V, R)$ to be the set of all functions from V to R. Viewing a polynomial as a function enables us to embed $\mathrm{Pol}(V, R)$ into $\mathrm{Fun}(V, R)$.

On the R-module $\mathrm{Fun}(V, R)$ (actually an R-algebra), the minimal order of R induces the **"functional" partial order**, given by

$$f \le g \quad \Leftrightarrow \quad f(v) \le g(v) \text{ for all } v \in V.$$

This order coincides with the minimal order on $\mathrm{Fun}(V, R)$ since clearly $f \le g$ iff $f + h = g$ for some h. But when we restrict this order to $\mathrm{Pol}(V, R)$ it may be finer than the minimal ordering on $\mathrm{Pol}(V, R)$. We pursue this aspect below in §1.5.

An easy exercise reveals that every polynomial function q is monotone, in the sense that $y \le x$ implies $q(y) \le q(x)$ (as always, in the minimal orderings), cf. Example 4.2 below. We need this here for q a quadratic form, where it is obvious from the definition (1.1): If $x = y + z$ then $q(x) = q(y) + q(z) + b(x, y) \ge q(y)$. Thus q induces a partial pre-order on V (given by $x \ge_q y$ if $q(x) \ge q(y)$), which is coarser then the minimal order. We call it the q-**preorder**.

1.4. **The supertropical connection.** We recall ([6, Definition 0.3] and [4, §3]), that a semiring R is called **supertropical** if $e := 1_R + 1_R$ is an idempotent (i.e., $e = 1_R + 1_R = 1_R + 1_R + 1_R + 1_R = e + e$), and the following axioms hold for all $x, y \in R$:

$$\text{If } ex \neq ey, \ \text{ then } x + y \in \{x, y\}, \tag{1.7}$$

$$\text{If } ex = ey, \ \text{ then } x + y = ey. \tag{1.8}$$

Then the ideal eR of R is a semiring with unit element e, which is **bipotent**, i.e., for any $u, v \in eR$ the sum $u+v$ is either u or v. It follows that eR carries a total ordering, compatible with addition and multiplication, which is given by

$$u \le v \Leftrightarrow u + v = v. \tag{1.9}$$

(Note that this is the minimal ordering on eR.)

The addition in a supertropical semiring is determined by the map $x \mapsto ex$ and the total ordering on eR, as follows: If $x, y \in R$, then

$$x + y = \begin{cases} y & \text{if } ex < ey, \\ x & \text{if } ex > ey, \\ ey & \text{if } ex = ey. \end{cases} \tag{1.10}$$

In particular, taking $y = 0$ in (1.10) or in (1.8),

$$ex = 0 \Rightarrow x = 0. \tag{1.11}$$

It follows from (1.9) that every supertropical semiring is u.b.

For the convenience of the reader, we set more terminology.

Notation 1.8. *Let* $\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

If R is a supertropical semiring, then

- $\mathcal{T}(R) := R \setminus eR$, *called the set of* ***tangible*** *elements* $\neq 0$.
- $\mathcal{G}(R) := eR \setminus \{0\}$, *called the set of* ***ghost*** *elements* $\neq 0$.
- ν_R *denotes the ghost map* $R \to eR$, $a \mapsto ea$.

When there is no ambiguity, we write $\mathcal{T}$, $\mathcal{G}$, ν instead of $\mathcal{T}(R)$, $\mathcal{G}(R)$, ν_R.
We also use the following notation. If $a, b \in R$, then $a \leq_\nu b$ means that $ea \leq eb$, $a \cong_\nu b$ ("ν-equivalent") means that $ea = eb$, while $a <_\nu b$ means that $ea < eb$.

We do not assume that the restriction of the ghost map ν_R to $\mathcal{T}$ is necessarily 1:1. Up to §5 we assume that the set $\mathcal{T}$ **is closed under multiplication**, and so is a multiplicative monoid. The zero of R is regarded both as tangible and ghost. The semiring R itself is called **tangible** if R is generated by $\mathcal{T}(R)$ as a semiring. Clearly, this happens iff $e\mathcal{T}(R) = \mathcal{G}(R)$. If $\mathcal{T}(R) \neq \emptyset$, then the set

$$R' := \mathcal{T}(R) \cup e\mathcal{T}(R) \cup \{0\}$$

is the largest subsemiring of R which is tangible supertropical. (We have discarded the "superfluous" ghost elements.)

We also assume up to §5 that $\mathcal{G}$ **is closed under multiplication** (equivalently, R has no zero divisors, cf. (1.10)), and that the monoid $\mathcal{G}$ **is cancellative** ($ac = bc \Rightarrow a = b$). Clearly this holds when $\mathcal{G}$ is a group, and all the more, when R is a **supersemifield**[2], i.e., when both $\mathcal{G}$ and $\mathcal{T}$ are groups, which is the case in most tropical applications.

Remark 1.9. *If $a \cong_\nu b$ with $b \in \mathcal{G}$, then $b = ea$. Trivially $a \leq b$ implies $ac \leq bc$ for all $c \in R$. We conclude that if $a, b, c \in \mathcal{G}$ and $a < b$, then $ac < bc$. Note also that, if $a, b \in R$ with $a < b$ are given and $ac = bc$ for some $c \neq 0$, then $b = ea$.*

Given a quadratic space (V, q), we say that an element $x \in V$ is g**-anisotropic** if $0 \neq q(x) \in \mathcal{T}(R)$. Otherwise $q(x) \in \mathcal{G}(R) \cup \{0\}$, and we say $x \in V$ is g**-isotropic** ("g" alludes to "ghost"). If $q(x) = 0$ we call x **isotropic**.

1.5. **Goals of this paper.** In the present paper, a sequel of [6], we continue the study of quadratic forms and pairs on R-modules with R a supertropical semiring, often more specifically a supersemifield. Our approach is in terms of the q-preorder. The first question to ask for what elements is $q(x) = q(y)$?

Definition 1.10. *An element $x \in V$ is q-**minimal**, if $x' <_q x$ for every element $x' < x$ (with respect to the minimal ordering of V and R).*

In Theorems 2.4 and 5.2 we prove that every element x dominates a q-minimal element of the same q-value. Towards this end, we build an (algorithmic) process of reducing an arbitrary element.

In Sections §4.1 and §4.2, we obtain a detailed description of all minimal vectors and certain relations between them in the case that R is tangible supertropical with $\mathcal{G}(R)$ a cancellative monoid under multiplication (in particular, if R is a tangible supersemifield). Specifically, in the quasi-linear case, it is easy to check that the q-minimal g-anisotropic elements x are just the multiples of the g-anisotropic base elements. For general q, by Corollary 4.4, all q-minimal q-anisotropic elements have support of order at most 2.

If V is a free module with base $\{\varepsilon_i \mid i \in I\}$, and q is quadratic form on V, then by Proposition 4.3, every q-minimal vector $x \in V$ is contained in a smallest submodule $V_J = \sum_{i \in J} Rv_i$ of V with $|J| \leq 4$. In Theorem 4.6 we easily find all q-minimal vectors for $|J| \leq 2$ (vectors of "small support").

Then in Theorem 4.10 we characterize all q-minimal vectors for subsets J of I with $|J| \geq 3$ (vectors of "large support"). Continuing, we prove in Theorems 4.11 and 4.13 that for $|J| = 3$

[2] called a "supertropical semifield in [5, 6]. We avoid this term here, since supersemifields are not semifields in the technical sense, where elements $\neq 0$ are invertible.

or $|J| = 4$ a q-minimal vector x is the maximum $y \vee z$ of a pair of q-minimal vectors y and z of small support which is uniquely determined by x, except in one case, where y and z can be freely chosen in a triplet y_1, y_2, y_3 of q-minimal vectors of small support, uniquely determined by x. Conversely, we find out which sups $y \vee z$ of q-minimal vectors y, z with small support are again q-minimal.

One can use the same ideas to compare different forms as functions. Recall that the functional order on $\mathrm{Fun}(V, R)$ is given by:

$$f \geq g \quad \text{iff} \quad f(v) \geq g(v) \text{ for each } v \in V.$$

This restricts to a partial order[3] on $\mathrm{Pol}(V, R)$, which thereby becomes a module under scalar multiplication and pointwise addition. But the set of quadratic forms on $\leq \mathrm{rank}(V)$ variables is itself a module over R, so the basic properties of the minimal order can be applied to the R-module $\mathrm{Quad}(V)$ consisting of *all* quadratic forms on V. This leads to a subtle distinction. Suppose $q \leq q'$ are quadratic forms, so that $q' = q + h$. When can we take h also to be a quadratic form? In this case, we write $q \preceq q'$, and one of our major objectives (to be considered below) is to determine when quadratic forms satisfy $q \preceq q'$.

In case R is a nontrivial tangible supersemifield, we can determine all pairs (q, q') in $\mathrm{Quad}(V)$ for which $q \leq q'$ implies $q \preceq q'$, by using the results in [6, §7] on the companions of a given quadratic form [loc. cit, Proposition 7.9, Theorems 7.11 and 7.12]. Perhaps surprisingly, it is rare that $q \leq q'$ without $q \preceq q'$. One step in this direction is to be given in Theorem 9.10, which draws on the "companion table" $(C_{i,j}(q))$ studied in [6, §6].

Let us note in passing that this kind of problem disappears for the classical theory over rings instead of semirings. If, say, R is an ordered field, and we consider positive semidefinite forms on an R-module V, then the relation $q \leq q'$ implies that $q' = q + q_1$ with q_1 again positive semidefinite, namely $q_1 = q' - q$.

While the focus of [6] is mainly on a single quadratic form, we now are led to study the set $\mathrm{Quad}(V)$ of *all* quadratic forms on V. $\mathrm{Quad}(V)$ is not a free module, except in the trivial case when $n = 1$, but it does contain the free submodules $\mathrm{QL}(V)$ and $\mathrm{Rig}(V)$, consisting of the quasilinear and the rigid forms respectively, and their bases are easily described respectively, in Proposition 7.2, as

$$\mathfrak{D}_0 := \{d_i \mid i \in I\} \quad \text{and} \quad \mathfrak{H}_0 := \{h_{ij} \mid i < j\},$$

with

$$d_i(x) = x_i^2, \qquad h_{ij}(x) = x_i x_j$$

for $x = \sum_{i \in I} x_i \varepsilon_i$. $\mathrm{Quad}(V)$ is the sum of the submodules $\mathrm{QL}(V)$ and $\mathrm{Rig}(V)$, and so $\mathfrak{D}_0 \cup \mathfrak{H}_0$ generates $\mathrm{Quad}(V)$.

In §7 and §8 we prove, under mild Archimedean-type conditions on R (cf. Definition 7.4), that both $\mathfrak{D}_0$ and $\mathfrak{H}_0$ are uniquely determined (projectively) by the R-module structure of $\mathrm{Quad}(V)$, and so $\mathrm{QL}(V)$ and $\mathrm{Rig}(V)$ are encoded in the R-linear structure of $\mathrm{Quad}(V)$. Of course $\mathrm{QL}(V) \cap \mathrm{Rig}(V) = \{0\}$.

In Theorem 9.3 we show that both orderings $\leq$ and $\preceq$ coincide on $\mathrm{QL}(V)$ and also on $\mathrm{Rig}(V)$. This gives us the possibility, pursued further in §9, of describing the more difficult minimal ordering in terms of the function ordering and the quasilinear-rigid decomposition of the quadratic form on V (the main theme of [6]), cf. Corollary 9.4. For example, Theorem 9.8 gives a criterion for $q \preceq q'$ in terms of their restrictions to free modules of rank 2. This kind of analysis reaches its technical culmination in Theorem 9.10, which surprisingly shows in

[3] We always assume that R is u.b.

Corollary 9.11 that $\leq$ and $\preceq$ coincide for any free module V over a tangible supertropical semifield with densely ordered set $\mathcal{G}$.

2. Preliminary observations

Assume from now on that V is a free R-module with base $(\varepsilon_i \mid i \in I)$. We call the elements of V "vectors". If x, y are vectors in V with coordinates $(x_i \mid i \in I)$, $(y_i \mid i \in I)$, i.e.,

$$x = \sum_{i \in I} x_i \varepsilon_i \quad y = \sum_{i \in I} y_i \varepsilon_i,$$

where $x_i \neq 0$ or $y_i \neq 0$ for only finitely many $i \in I$, then clearly

$$x \leq_V y \quad \Leftrightarrow \quad x_i \leq_R y_i, \ \forall i \in I. \tag{2.1}$$

2.1. **The support.** We define the **support** of an element $x = \sum\limits_{i \in I} x_i \varepsilon_i$ of V to be

$$\operatorname{supp}(x) := \{ i \in I \mid x_i \neq 0 \} \tag{2.2}$$

and the **tangible support** of x to be

$$\operatorname{supp}_{\tan}(x) := \{ i \in I \mid x_i \in \mathcal{T} \}. \tag{2.3}$$

Notice that both $\operatorname{supp}(x)$ and $\operatorname{supp}_{\tan}(x)$ are essentially independent of the choice of the base $(\varepsilon_i \mid i \in I)$, since up to permutation every other base of V arises by multiplying the ε_i by units of R [6, Theorem 0.9].

Remarks 2.1. $\operatorname{supp}(x)$ *is empty iff* $x = 0$*, and* $y \leq x$ *implies* $\operatorname{supp}(y) \subseteq \operatorname{supp}(x)$*. Clearly* $\operatorname{supp}(x) = \operatorname{supp}(ex)$*. Also* $\operatorname{supp}(x + y) = \operatorname{supp}(x) \cup \operatorname{supp}(y)$ *for any* $x, y \in V$*.*

When computing $q(x)$ *for a vector* $x = \sum_{i \in I} x_i \varepsilon_i$*, we only need to consider indices* i *in* $\operatorname{supp}(x)$*, and so we quickly may reduce to the case that* I *is finite, say* $I = \{1, \dots, n\}$*. Then we write* $x = \sum_{i=1}^n x_i \varepsilon_i$*.*

Definition 2.2. *The component* $x_j \varepsilon_j$ *of* x *is* q-***essential*** *if, taking* $x' = \sum_{i \neq j} x_i \varepsilon_i$*, we have* $q(x') < q(x)$*.*

The ***index*** $\operatorname{ind}(x)$ *is the number of summands in the right side of* (1.6) *which are* ν*-equivalent to* $q(x)$*.*

Remark 2.3. *If* x *is* q*-minimal, then each component is* q*-essential.*

In the next theorem we exploit that the ordering on $\mathcal{G}$ is strongly consistent with multiplication (cf. Remarks 1.9). Later on, in Theorem 5.2, we give a more detailed result, but the proof is much longer and far more technical.

Theorem 2.4. *Any vector* $x = \sum_{i \in I} x_i \varepsilon_i$ *can be reduced to a* q*-minimal vector* x' *of same* q*-value by means of an algorithm given in the proof, whereby first* x *is replaced by its* q*-essential part and then some of the coefficients may be converted from ghosts to tangibles having the same* ν*-value.*

Proof. First, remove all inessential components. If one could lower some coefficient x_i say to x_i' without lowering $q(x)$, then x_i' must be inessential, and we can remove it. Thus, we first check whether we can remove some component of x without lowering $q(x)$. Of course we are lowering $\operatorname{ind}(x)$ at each stage, so we continue until $\operatorname{ind}(x)$ cannot be lowered further. Then we cannot lower any x_i any further, so the only way we can lower x without affecting $q(x)$ is by replacing some ghost coefficient by a tangible one, i.e., writing some $x_j = e x_j'$ for x_j'

tangible. (Here there may be several, perhaps infinitely many choices of x'_j, although $\nu(x_j)$ is given.) In other words, take $x' = x_j\varepsilon_j + \sum_{i\neq j} x_i\varepsilon_i$. But $\mathrm{supp}_{\tan}(x') \supset \mathrm{supp}_{\tan}(x)$ has been increased, and we can only do this a finite number of times until we reach $\mathrm{supp}(x)$, at which point x' must be q-minimal. □

3. The minimal ordering on supertropical semirings and modules

In this short section we provide background about the minimal order of a free module V over a supertropical semiring R and its relevance for quadratic forms on V. Except in Proposition 3.6 below our standard assumption, that both $\mathcal{T}$ and $\mathcal{G}$ are closed under multiplication and $\mathcal{G}$ is cancellative, is not needed. R could be any supertropical semiring.

Notation 3.1. *When no other modules come into play, we usually write $x \le y$ instead of $x \le_V y$. But notice that if W is a submodule of V, it may happen for $x, y \in W$ that $x \le_V y$ but not $x \le_W y$, since we could have $y = x + z$ for $z \in V \setminus W$.*

As usual, $x < y$ means that $x \le y$ and $x \ne y$.

In particular, R itself carries the minimal ordering $\le_R$, which already showed up in [4, Proposition 11.8] and [6, §5]. Again, we usually write $\lambda \le \mu$ instead of $\lambda \le_R \mu$.

Scalar multiplication is compatible with these orderings on R and V :

$$\lambda \le \mu,\ x \le y \quad \Rightarrow \quad \lambda x \le \mu y \tag{3.1}$$

for all $\lambda, \mu \in R,\ x, y \in V$.

The minimal ordering of R has the following detailed description in terms of the ν-dominance relation and the sets eR and $\mathcal{T} = R \setminus (eR)$.

Proposition 3.2.

a) *Assume that $x \in eR$. Then, using the ν-notation,*

$$x < y \quad \Leftrightarrow \quad x <_\nu y, \tag{3.2}$$

$$y < x \quad \Leftrightarrow \quad \text{either } y <_\nu x \text{ or } y \in \mathcal{T} \text{ and } y \cong_\nu x. \tag{3.3}$$

b) *Assume that $x \in \mathcal{T}$, $y \in R$. Then*

$$x < y \quad \Leftrightarrow \quad \text{either } x <_\nu y \text{ or } x \cong_\nu y \text{ and } y \in eR, \tag{3.4}$$

$$y < x \quad \Leftrightarrow \quad y <_\nu x. \tag{3.5}$$

Thus x and y are incomparable iff $x, y \in \mathcal{T}$ and $x \ne y$, but $x \cong_\nu y$.

c) *For any $x, y \in R$ there exists the maximum $x \vee y$ and $e(x \vee y) = (ex) \vee (ey)$. If x and y are incomparable, then*

$$x \vee y = ex = ey.$$

Proof. All of this can be read off from the description (1.10) of the sum $x + y$ of $x, y \in R$ in terms of the ν-dominance relation, recalled from [8, §2]. (We note that the general assumption in [8], that the monoid $(eR, \cdot\,)$ is cancellative, is irrelevant since products xy are not involved here.) □

It follows from Proposition 3.2.c that, for x, y in a free module V with base $(\varepsilon_i \mid i \in I)$, there exists the maximum $x \vee y$, namely, if $x = \sum_{i\in I} x_i\varepsilon_i$, $y = \sum_{i\in I} y_i\varepsilon_i$, then

$$x \vee y \quad = \quad \sum_{i\in I} (x_i \vee y_i)\varepsilon_i. \tag{3.6}$$

Furthermore

$$e(x \vee y) = (ex) \vee (ey) = ex + ey, \tag{3.7}$$

In general, for any $\lambda \in R$

$$(\lambda x) \vee (\lambda y) \leq \lambda(x \vee y), \tag{3.8}$$

as follows from (3.1), but here we have equality when R is a supersemifield.

Remark 3.3. *As before, let V be a module over a supertropical semiring R. If (q, b) is a quadratic pair on V, then for all $x, y, z, w \in V$ the following hold:*

$$x \leq_V z \quad \Rightarrow \quad q(x) \leq_R q(z), \tag{3.9}$$

$$x \leq_V z,\ y \leq_V w \quad \Rightarrow \quad b(x, y) \leq_R b(z, w), \tag{3.10}$$

$$b(x, y) \leq_R q(x + y). \tag{3.11}$$

This is evident from the definition of quadratic pairs, cf. (1.2).

On both R and V we have a natural equivalence relation compatible with the minimal order.

Definition 3.4. *Assume that X is any partial ordered set. We say that two elements x, y of X are* ***order-associated*** *and write $x \approx y$, if either $x = y$ or for any $z \in X$ the following holds:*

$$z < x \Leftrightarrow z < y, \qquad x < z \Leftrightarrow y < z.$$

We now look on the meaning of this when $X = R$ or $X = V$ as above, equipped with the minimal order.

Proposition 3.5.

(i) *Let $a, b \in R$. Then $a \approx b$ iff $ea = eb$ and the elements a, b are of the same type (tangible, ghost, zero).*

(ii) *Let $x = \sum_{i \in I} x_i \varepsilon_i$, $y = \sum_{i \in I} y_i \varepsilon_i$ be vectors in V. Then $x \approx y$ iff $x_i \approx y_i$ for every $i \in I$.*

Proof. i) follows form the description of the minimal order on R in Proposition 3.2, and then ii) is clear from (2.1) above. □

These equivalence relations on V and R are respected by any quadratic form $q : V \to R$.

Proposition 3.6. *If $x, y \in V$ and $x \approx y$, then $q(x) \approx q(y)$.*

Proof. We write, as in (1.6)

$$q(x) = \sum_i \alpha_i x_i^2 + \sum_{i<j} \beta_{ij} x_i x_j, \tag{$*$}$$

$$q(y) = \sum_i \alpha_i y_i^2 + \sum_{i<j} \beta_{ij} y_i y_j. \tag{$**$}$$

There is nothing to prove if $x = y$. Assume that $x \neq y$. It follows from $x \approx y$ that $ex = ey$, and then

$$eq(x) = q(ex) = q(ey) = eq(y).$$

This implies that $q(x) = 0$ iff $q(y) = 0$. If $q(x) \neq 0$, then $q(x) \in \mathcal{T}$ iff in the sum $(*)$ there is only one ν-dominate summand, and this is tangible, as follows from (1.10). But then the same holds in the sum $(**)$, due to our standard assumption that both $\mathcal{T}$ and $\mathcal{G}$ are closed under multiplication. Thus $q(x) \in \mathcal{T}$ implies $q(y) \in \mathcal{T}$. By symmetry $q(y) \in \mathcal{T}$ implies $q(x) \in \mathcal{T}$. □

4. All q-minimal vectors have support ≤ 4

In the next few sections, through §5, we study the precise nature of the q-minimal vectors x in the minimal ordering of a free module V over a supertropical semiring R. Recall from §2.1 that the support $\operatorname{supp}(x)$ essentially does not depend on the chosen base $\{\varepsilon_i \mid i \in I\}$ of V since any other base of V (as a set) is obtained by multiplying the ε_i by units of R.

We start by showing that $|\operatorname{supp}(x)| \le 4$, and then explicitly compute the various possibilities.

Definition 4.1.

a) *We call a map $\phi : V \to W$ between R-modules V, W* ***monotonic*** *if for any $x, y \in V$*

$$y \le x \quad \Rightarrow \quad \phi(y) \le \phi(x).$$

b) *Given a monotonic map $\phi : V \to W$, we call a vector $x \in V$ ϕ-**minimal**, if there **does not** exist a vector $x' < x$ in V with $\phi(x') = \phi(x)$.*

Examples 4.2.

i) *For any $n \in \mathbb{N}$ and $c \in R$, the map $R \to R$ given by $x \mapsto cx^n$, is monotonic. More generally, every monomial map $R^n \to R$,*

$$(x_1, \dots, x_n) \mapsto c x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \qquad (\alpha_i \in \mathbb{N}_0),$$

is monotonic, and hence every polynomial map $f : R^n \to R$ is monotonic.

ii) *Every quadratic form $q : V \to R$ on an R-module V is monotonic, cf. Remark 3.3.*

We note the trivial fact that an isotropic vector $x \in V \setminus \{0\}$ is never q-minimal, since then $0 < x$, but $q(x) = 0 = q(0)$.

Given a quadratic form $q : V \to R$, we turn to the problem of determining the q-minimal vectors in V in case the R-module V is free. The following distinction of the vectors in V will be useful here and elsewhere. Assume that V is free with base $(\varepsilon_i \mid i \in I)$.

Proposition 4.3. *Let $x \in V \setminus \{0\}$ be q-minimal. Then $|\operatorname{supp}(x)| \le 2$ if $q(x) \in \mathcal{T}$, and $|\operatorname{supp}(x)| \le 4$ if $q(x) \in \mathcal{G}$.*

Proof. We have a finite nonempty subset $J = \operatorname{supp}(x)$ of I, such that $x = \sum_{i \in J} x_i \varepsilon_i$, all $x_i \ne 0$. We choose a companion b of q. Then

$$q(x) = \sum_{i \in J} x_i^2 q(\varepsilon_i) + \sum_{\substack{i<j \\ i,j \in J}} x_i x_j b(\varepsilon_i, \varepsilon_j). \tag{$*$}$$

and $q(x) \ne 0$, since x is q-minimal and $x \ne 0$.

If $q(x) \in \mathcal{T}$, the sum on the right side of $(*)$ contains a unique ν-dominant term. If this term is $x_k^2 q(\varepsilon_k)$, then $x_k \varepsilon_k \le x$ and $q(x_k \varepsilon_k) = q(x)$; hence $x = x_k \varepsilon_k$ and $J = \{k\}$. If the ν-dominant term is $x_k x_\ell b(\varepsilon_k, \varepsilon_\ell)$, then $x_k \varepsilon_k + x_\ell \varepsilon_\ell \le x$, and again both vectors have the same q-values, and hence $x = x_k \varepsilon_k + x_\ell \varepsilon_k$, and $J = \{k, \ell\}$. Indeed, then

$$q(x) = x_k x_\ell b(\varepsilon_k, \varepsilon_\ell) \le q(x_k \varepsilon_k + x_\ell \varepsilon_\ell) \le q(x).$$

If $q(x) \in \mathcal{G}$, then on the right of $(*)$ there exists either a ν-dominant term, which is ghost, or there exist two ν-dominant terms which are tangible. In the first case, we see as above that $|J| \le 2$, and in the second that $|J| \le 4$. □

Corollary 4.4. *Assume in Proposition 4.3 that q also is quasilinear. Then $|\operatorname{supp}(x)| = 1$ if $q(x) \in \mathcal{T}$, and $|\operatorname{supp}(x)| \leq 2$ if $q(x) \in \mathcal{G}$.*

Proof. We choose the companion $b = 0$. Now, in the above arguments no ν-dominant terms $x_k x_\ell b(\varepsilon_k, \varepsilon_\ell)$ show up. □

Recall that for vectors x', x in V with $x' \leq x$ the support of x' is contained in the support of x. Thus in searching for q-minimal vectors in V it is no loss of generality to assume that $|I| \leq 4$. If q is quasilinear we may even assume that $|I| \leq 2$.

4.1. **q-minimal vectors with small support.** We deal now with the case that $|I| \leq 2$, postponing the cases $|I| = 3$ and $|I| = 4$ to the next subsection. In all the following **we assume that** $\mathcal{G} = e\mathcal{T}$, i.e., the supertropical semiring R is tangible.

Proposition 4.5.

a) *Assume that V is free with a single base vector ε_1. When $q(\varepsilon_1) \in \mathcal{T}$, all vectors in V are q-minimal. If $q(\varepsilon_1) \in \mathcal{G}$, a vector $\lambda\varepsilon_1$ is q-minimal iff $\lambda \in \mathcal{T}$.*

b) *Assume that V is free with base $(\varepsilon_1, \varepsilon_2)$, and that q is quasilinear. Let $\alpha_1 := q(\varepsilon_1)$, $\alpha_2 := q(\varepsilon_2)$. A vector $x = \lambda\varepsilon_1 + \mu\varepsilon_2$ with $\lambda, \mu \neq 0$ is q-minimal iff $\lambda, \mu, \alpha_1, \alpha_2 \in \mathcal{T}$ and $\lambda^2\alpha_1 \cong_\nu \mu^2\alpha_2$. (Thus every q-minimal vector with $|\operatorname{supp}(x)| = 2$ is g-isotropic.)*

Proof. a): Let $\alpha_1 := q(\varepsilon_1)$ and $x := \lambda\varepsilon_1 \in V$. We have $q(x) = \lambda^2\alpha_1$. Assume first that $\alpha_1 \in \mathcal{T}$. If $x' = \lambda'\varepsilon_1$ is a second vector, then $x' < x$ iff $\lambda' < \lambda$, iff $\lambda'^2\alpha_1 < \lambda^2\alpha_1$. Thus x is q-minimal. Assume now that $\alpha_1 \in \mathcal{G}$. If $\lambda \in \mathcal{G}$, there exists $\lambda' \in \mathcal{T}$ with $\lambda' \cong_\nu \lambda$, and then $\lambda' < \lambda$. For $x' = \lambda'\varepsilon_1$ we have $x' < x$, but $q(x') = \lambda'^2\alpha_1 = \lambda^2\alpha_1 = q(x)$. Thus x is not q-minimal. If $\lambda \in \mathcal{T}$ and $\lambda' < \lambda$ then $\lambda' <_\nu \lambda$ (cf. (3.5)); hence

$$q(x') = \lambda'^2\alpha_1 <_\nu \lambda^2\alpha_1 = q(x).$$

Thus x is q-minimal.

b): We have $q(x) = \lambda^2\alpha_1 + \mu^2\alpha_2$. If $q(x) = 0$, then x is not q-minimal (cf. Example 4.2.ii).

Assume now that $q(x) \neq 0$. If $\lambda^2\alpha_1 <_\nu \mu^2\alpha_2$ then $q(x) = \mu^2\alpha_2 = q(\mu\varepsilon_2)$, and x is not q-minimal, and likewise if $\lambda^2\alpha_1 >_\nu \mu^2\alpha_2$. Assume henceforth that $\lambda^2\alpha_1 \cong_\nu \mu^2\alpha_2$. Then $q(x) \in \mathcal{G}$ and $\alpha_1 \neq 0$, $\alpha_2 \neq 0$. If $\lambda^2\alpha_1$ or $\mu^2\alpha_2$ is ghost, then $q(x) = q(\lambda\varepsilon_1)$, resp. $q(x) = q(\mu\varepsilon_2)$, and thus x is not q-minimal.

We are left with the case that both $\lambda^2\alpha_1$, $\mu^2\alpha_2$ are tangible, i.e., $\lambda, \mu, \alpha_1, \alpha_2 \in \mathcal{T}$.

If $x' < x$, then either $x' \leq \lambda'\varepsilon_1 + \mu\varepsilon_2$ or $x' \leq \lambda\varepsilon_1 + \mu'\varepsilon_2$ with $\lambda' < \lambda$, resp. $\mu' < \mu$. In the first case, $\lambda' <_\nu \lambda$ (cf. (3.5)), hence $\lambda'^2\alpha_1 <_\nu \lambda^2\alpha_1 \cong_\nu \mu^2\alpha_2$, and

$$q(x') \leq q(\lambda'\varepsilon_1 + \mu\varepsilon_2) = \mu^2\alpha_2 < e\mu^2\alpha_2 = q(x).$$

In the second case, $q(x') < q(x)$ for the same reason. Thus x is q-minimal. □

Assume that (q, b) is a quadratic pair on the free binary module $V := R\varepsilon_1 + R\varepsilon_2$. We search for all q-minimal vectors in V having full support.

Let $\alpha_1 := q(\varepsilon_1)$, $\alpha_2 := q(\varepsilon_2)$, $\beta := b(\varepsilon_1, \varepsilon_2)$, and $x = x_1\varepsilon_1 + x_2\varepsilon_2$ with $x_1 \neq 0$, $x_2 \neq 0$. Then

$$q(x) = \alpha_1 x_1^2 + \beta x_1 x_2 + \alpha_2 x_2^2. \qquad (**)$$

Looking at the ν-dominant terms in the sum $(**)$ we will run through several cases and will find out easily when x is q-minimal.

0) Assume that $\alpha_1 x_1^2$ (or $\alpha_2 x_2^2$) is the only ν-dominant term. Then $q(x) = q(x_1\varepsilon_1)$ or $q(x) = q(x_2\varepsilon_2)$. Clearly x is not q-minimal.

1) Assume that both $\alpha_1 x_1^2$ and $\alpha_2 x_2^2$ are ν-dominant. If, say, $\alpha_1 x_1^2$ is ghost, then $q(x) = q(x_1\varepsilon_1)$ again, and x is not q-minimal. If both $\alpha_1 x_1^2$ and $\alpha_2 x_2^2$ are tangible, then for a vector $x' = x_1'\varepsilon_1 + x_2'\varepsilon_2 < x$ either $x_1' < x_1$ or $x_2' < x_2$, which implies $x_1' <_\nu x_1$ or $x_2' <_\nu x_2$, since both x_1', x_2' are tangible. We conclude that $q(x') < q(x)$. Thus x is q-minimal iff $\alpha_1, \alpha_2, x_1, x_2$ are all tangible.
2) Assume that $\alpha_1 x_1^2 \cong_\nu \beta x_1 x_2 > \alpha_2 x_2^2$. Then $q(x) = e\alpha_1 x_1^2 = e\beta x_1 x_2 \in \mathcal{G}$. If $\alpha_1 x_1^2 \in \mathcal{G}$, then choosing $x_1' \in \mathcal{T}$ with $ex_1' = x_1$ we obtain a vector $x' = x_1'\varepsilon_1 + x_2\varepsilon_2 < x$ with $q(x') = \alpha_1' x_1^2 + \beta x_1' x_2 = q(x)$, and so x is not q-minimal.
   Assume now that $\alpha_1 x_1^2 \in \mathcal{T}$. If $x' = x_1'\varepsilon_1 + x_2'\varepsilon_2 < x$, then either $x_1' < x_1$, $x_2' \le x_2$, or $x_1' = x_1$, $x_2' < x_2$. If $x_1' < x_1$, then $x_1' <_\nu x_1$, whence $\alpha_1 x_1'^2 <_\nu \alpha_1 x_1^2$ $\beta x_1' x_2 <_\nu \beta x_1 x_2$, and we see that $q(x') < q(x)$. But if $x_1' = x_1$, $x_2' < x_2$, $ex_2' = x_2$, and $\beta \in \mathcal{G}$, then $q(x') = q(x)$, while if $\beta \in \mathcal{T}$ this cannot happen. We conclude that x is q-minimal iff α_1, β, x_1 are all tangible.
3) Analogously, if $\alpha_2 x_2^2 \cong_\nu \beta x_1 x_2 > \alpha_1 x_1^2$, then x is q-minimal iff α_2, β, x_2 are all tangible.
4) Assume that $\alpha_1 x_1^2 <_\nu \beta x_1 x_2$ and $\alpha_2 x_2^2 <_\nu \beta x_1 x_2$. Now $q(x) = \beta x_1 x_2$. Arguing as in Case 3), we see that, when $\beta \in \mathcal{G}$ then x is q-minimal iff $x_1 \in \mathcal{T}$ and $x_2 \in \mathcal{T}$, while when $\beta \in \mathcal{T}$, then x is q-minimal iff $x_1 \in \mathcal{T}$ or $x_2 \in \mathcal{T}$. Putting everything together, x is q-minimal iff at most one of the elements β, x_1, x_2 is ghost.

Summarizing we obtain

Theorem 4.6. *Assume that V is free with base $\varepsilon_1, \varepsilon_2$ and $x = x_1\varepsilon_1 + x_2\varepsilon_2$ with $x_1 \neq 0$, $x_2 \neq 0$. Let $q = \left[\begin{smallmatrix}\alpha_1 & \beta \\ & \alpha_2\end{smallmatrix}\right]$. Then x is q-minimal exactly in the following cases:*

1) $\alpha_1 x_1^2 \cong_\nu \alpha_2 x_2^2 \ge_\nu \beta x_1 x_2$ *and* $\alpha_1, \alpha_2, x_1, x_2 \in \mathcal{T}$;
2) $\alpha_i x_i^2 \cong_\nu \beta x_i x_j >_\nu \alpha_j x_j^2$ *for* $(i,j) = (1,2)$ *or* $(i,j) = (2,1)$;
3) $\beta x_1 x_2 >_\nu \alpha_1 x_1^2 + \alpha_2 x_2^2$ *and at most one of the elements* β, x_1, x_2 *is ghost.*

Comment 4.7. *In Cases 2), 3) we have $\alpha_1 x_1^2 \cdot \alpha_2 x_2^2 <_\nu (\beta x_1 x_2)^2$, whence $\alpha_1\alpha_2 <_\nu \beta^2$, while in Case 1) we have $\beta^2 \le_\nu \alpha_1\alpha_2$. Thus q is quasilinear on $R\varepsilon_1 + R\varepsilon_2$ in Case 1).*

Concerning g-anisotropic vectors we note the following immediate consequence of Theorem 4.6.

Corollary 4.8. *We still assume that $x = x_1\varepsilon_1 + x_2\varepsilon_2$ and $q = \left[\begin{smallmatrix}\alpha_1 & \beta \\ & \alpha_2\end{smallmatrix}\right]$. Then x is q-minimal and g-anisotropic iff β, x_1, x_2 are tangible and $\alpha_1 x_1^2 + \alpha_2 x_2^2 <_\nu \beta x_1 x_2$.*

4.2. **q-minimal vectors with large support.** Again we assume that R is a tangible supertropical semiring, $\mathcal{G}$ is a cancellative monoid, V is a free R-module with base $(\varepsilon_i \mid i \in I)$, and $q : V \to R$ is a quadratic form. For later use, we adopt the following notation.

Notation 4.9. *Let $x = \sum_{i\in I} x_i\varepsilon_i \in V$. For $J \subset I$, we put*

$$x(J) := \sum_{i\in J} x_i\varepsilon_i.$$

If $J = \{i\}$ or $J = \{i,j\}$, $i \neq j$, we write respectively $x(i), x(i,j)$ for short, instead of $x(\{i\})$, $x(\{i,j\})$.

Assume now that $I = \{1, \dots, n\}$ with $n = 3$ or $n = 4$, and that $x \in V$ is a vector of "full support," i.e.,

$$x = \sum_{i=1}^{n} x_i \varepsilon_i, \quad \text{all} \quad x_i \neq 0.$$

We choose a companion b of q, and then have a presentation

$$q(x) = \sum_{i=1}^{n} \alpha_i x_i^2 + \sum_{i<j} \beta_{ij} x_i x_j. \tag{4.1}$$

We ask, under which conditions is x q-minimal, and then we search for possibilities to write x as the supremum $y \vee z$ of two q-minimal vectors $y, z \in V$ of small support, i.e., $|\operatorname{supp}(y)| \leq 2$, $|\operatorname{supp}(z)| \leq 2$.

As in §4.1, we look for the ν-dominant terms in the sum (4.1). If there is only one dominant term, $\alpha_i x_i^2$ or $\beta_{ij} x_i x_j$, then $q(x) = q(x(i))$ or $q(x) = q(x(i,j))$, and so x is not q-minimal. Hence, we may assume that there are at least two dominant terms, implying $q(x) \in \mathcal{G}$. Furthermore, *we assume that all ν-dominant terms are tangible*, since otherwise again $q(x) = q(x(J))$ for some $J \subsetneqq I$.

We first study the case $n = 3$ and run through several subcases, as follows:

A) Assume that a ν-dominant term $\alpha_i x_i^2$ occurs in (4.1). Then, if x is q-minimal, there is exactly one other dominant term $\beta_{jk} x_j x_k$, and (i, j, k) is a permutation of $(1, 2, 3)$, since otherwise again $q(x) = q(x(J))$ for some $J \subsetneqq I$. We have

$$x = x(i) \vee x(j,k),$$

and $q(x(i)) = \alpha_i x_i^2 \in \mathcal{T}$, yielding

$$q(x(j,k)) = \alpha_j x_j^2 + \beta_{jk} x_j x_k + \alpha_k x_k^2 \in \mathcal{T}.$$

It follows that

$$\alpha_j x_j^2 + \alpha_k x_k^2 <_\nu \beta_{jk} x_j x_k,$$

and we read off from Theorem 4.6 that $x(j,k)$ is q-minimal. By Proposition 4.5.a, $x(i)$ also is q-minimal.

Note furthermore that

$$b(x(i), x(j,k)) <_\nu q(x(i)) \cong_\nu q(x).$$

Assume now that all the ν-dominant terms in the sum (4.1) are of the form $\beta_{ij} x_i x_j$, $1 \leq i < j \leq 3$. We then have to distinguish between two subcases.

B) Exactly two of the terms $\beta_{ij} x_i x_j$ are ν-dominant.

C) All three such terms are ν-dominant.

In Case B there is a permutation (i, j, k) of $(1, 2, 3)$ such that

$$q(x) \cong_\nu \beta_{ij} x_i x_j \cong_\nu \beta_{ik} x_i x_k >_\nu \beta_{jk} x_j x_k, \tag{4.2}$$

while in Case C we have

$$q(x) \cong_\nu \beta_{12} x_1 x_2 \cong_\nu \beta_{13} x_1 x_3 \cong_\nu \beta_{23} x_2 x_3. \tag{4.3}$$

In both cases $q(x) >_\gamma \alpha_i x_i^2$ for all $i \in I$. It follows by Corollary 4.8 in Case B that both vectors $x(i,j)$ and $x(i,k)$ are g-anisotropic and q-minimal, while in Case C all three vectors

$x(1,2)$, $x(1,3)$, $x(2,3)$ have these properties. Due to our knowledge of all ν-dominant terms in the sum (4.1), we see that in Case B

$$b(x(j), x(k)) <_\nu q(x(i,j)) \cong_\nu q(x(i,k)) \cong_\nu q(x),$$

while in Case C for every 2-element subset $\{r, s\}$ of I we have $b(x(r), x(s)) \in \mathcal{T}$ and

$$b(x(r), x(s)) \cong_\nu q(x(r,s)) \cong_\nu q(x).$$

(Observe that $b(\varepsilon_i, \varepsilon_i) \le_\nu \alpha_i$, cf. [6, Ineq. (1.9)].)

D) We turn to the case $n = 4$, which is easier. Assume that x is q-minimal. Then we have exactly two ν-dominant terms in the sum (4.1), $\beta_{ij}x_ix_j$, $\beta_{k\ell}x_kx_\ell$, with $\{i, j\}$ disjoint from $\{k, \ell\}$, since otherwise there would exist a set $S \subsetneqq I$ with $q(x(S)) = q(x)$. Moreover, these terms are tangible.

Arguing as above we conclude easily that there is a partition $I = J\dot\cup K$ with $|J| = |K| = 2$, such that $x(J)$ and $x(K)$ are g-anisotropic and q-minimal with

$$q(x(J)) \cong_\nu q(x(K)) \cong_\nu q(x),$$

while $q(x(S)) <_\nu q(x)$ for all other subsets S of I with $|S| \le 2$. Also for any two different subsets S, T of I with $|S| \le 2$, $|T| \le 2$, including $S = J$, $T = K$, we have

$$b(x(S), x(T)) <_\nu q(x).$$

Summarizing the essentials of this analysis, we obtain

Theorem 4.10. *Assume that x is q-minimal and* $\mathrm{supp}(x) = I = \{1, \dots, n\}$ *with $n \ge 3$. Then x is g-isotropic and exactly one of the following four cases holds:*

A) $n = 3$. *There is a unique partition $I = J\dot\cup K$ with $|J| = 1$, $|K| = 2$, both $x(J)$, $x(K)$ g-anisotropic and q-minimal, and $q(x(J)) \cong_\nu q(x(K)) \cong_\nu q(x)$.*

B) $n = 3$. *There are exactly two 2-element subsets J and K of I with $x(J)$, $x(K)$ g-anisotropic and q-minimal and $q(x(J)) \cong_\nu q(x(K)) \cong_\nu q(x)$.*

C) $n = 3$. *For any 2-element subset J of I, the vector $x(J)$ is q-minimal and g-anisotropic and $q(x(J)) \cong_\nu q(x)$. Thus the properties listed in B) hold for any two 2-element subsets J, K of I.*

D) $n = 4$. *There are exactly two 2-element subsets J and K of I such that $x(J)$, $x(K)$ are g-anisotropic, q-minimal and*

$$q(x(J)) \cong_\nu q(x(K)) \cong_\nu q(x).$$

J and K are disjoint.

In all four cases, we have $I = J \cup K$, whence $x = x(J) \vee x(K)$ for the sets J, K from above. Moreover, in Cases A and D,

$$b(x(J), x(K)) <_\nu q(x). \tag{4.4}$$

In Case B,

$$b(x(J), x(K)) = q(x), \tag{4.5}$$

whereas

$$b(x(J \setminus K), x(K \setminus J)) \cong_\nu q(x). \tag{4.6}$$

In Case C, (4.5) *holds for any two different 2-element subsets J, K of I, and moreover*

$$b(x(J \setminus K), x(K \setminus J)) \cong_\nu q(x), \quad b(x(J \setminus K), x(K \setminus J) \in \mathcal{T}. \tag{4.7}$$

As before we assume that V is free with base $(\varepsilon_i \mid i \in I)$, $I = \{1, \dots, n\}$, with $n = 3$ or 4. Given two g-anisotropic q-minimal vectors $y, z \in V$ of small support, we now ask for conditions under which the vector $x := y \vee z$ is q-minimal and has full support I. In view of Theorem 4.10, we will be content to assume from the outset that

$$b(y,z) \leq_\nu q(y) \cong_\nu q(z). \tag{4.8}$$

A satisfactory converse to Theorem 4.10 in the cases A) and B) runs as follows:

Theorem 4.11. *Assume that $y, z \in V$ are g-anisotropic and q-minimal, and furthermore that $y \vee z$ has full support I, and*

$$b(y,z) <_\nu q(y) \cong_\nu q(z). \tag{4.9}$$

Assume finally that $n = 3$ with $|\operatorname{supp}(y)| = 1$ and $|\operatorname{supp}(z)| = 2$, or $n = 4$ with $|\operatorname{supp}(y)| = |\operatorname{supp}(z)| = 2$. Then $x := y \vee z$ is q-minimal.

Proof. We have $\operatorname{supp}(y) \cup \operatorname{supp}(z) = I$, which forces $\operatorname{supp}(y) \cap \operatorname{supp}(z) = \emptyset$.

a) Assume first that $n = 3$. After a permutation of the ε_i, we may assume that

$$y = y_1\varepsilon_1, \qquad z = z_2\varepsilon_2 + z_3\varepsilon_3,$$

and then have $x = \sum_1^3 x_i\varepsilon_i$ with

$$x_1 = y_1, \qquad x_2 = z_2, \qquad x_3 = z_3.$$

It follows from Proposition 4.5.a and Corollary 4.8 that $\alpha_1 x_1^2 = q(y) \in \mathcal{T}$ and

$$\alpha_2 x_2^2 + \alpha_3 x_3^2 <_\nu \beta_{23} x_2 x_3 = q(z) \in \mathcal{T}. \tag{4.10}$$

Thus $x_1, x_2, x_3, \alpha_1, \beta_{23}$ are all tangible. Furthermore, by assumption (4.9),

$$\beta_{11} x_1^2 + \beta_{12} x_1 x_2 + \beta_{13} x_1 x_3 <_\nu \alpha_1 x_1^2 \cong_\nu \beta_{23} x_2 x_3. \tag{4.11}$$

Here $\beta_{11} = b(\varepsilon_1, \varepsilon_1) \leq_\nu \alpha_1$ (cf. [6, Ineq. (1.9)]). It follows that

$$q(x) = \alpha_1 x_1^2 + \beta_{23} x_2 x_3 = eq(y) = eq(z).$$

Given $x' = \sum_1^3 x_i'\varepsilon_i < x$, we want to prove that $q(x') < q(x)$. It suffices to consider the cases $x_1' < x_1$, $x_2' = x_2$, $x_3' = x_3$, and $x_1' < x_1$, $x_2' < x_2$, $x_3' = x_3$. Notice that $x_i' < x_i$ implies $x_i' <_\nu x_i$ since all x_i are tangible.

In the first case $\beta_{23} x_2' x_3' = \beta_{23} x_2 x_3$, and we learn from (4.10) and (4.11) that in the sum

$$\sum_1^3 \alpha_i x_i'^2 + \sum_{i<j} \beta_{ij} x_i' x_j' = q(x')$$

there is only one ν-dominant term $\beta_{23} x_2 x_3$, which is tangible. Thus

$$q(x') = \beta_{23} x_2 x_3 \in \mathcal{T}, \quad \text{and} \quad q(x') \cong_\nu q(x).$$

Since $q(x)$ is ghost, this implies $q(x') < q(x)$. In the second case where $x_2' < x_2$, we can argue in the same way, now obtaining $q(x') = \alpha_1 x_2^2 \in \mathcal{T}$ and then $q(x') < q(x)$. Thus x is indeed q-minimal.

b) Now let $n = 4$. We may assume that $\operatorname{supp}(y) = \{1, 2\}$ and $\operatorname{supp}(z) = \{3, 4\}$, whence

$$y = y_1\varepsilon_1 + y_2\varepsilon_2, \qquad z = z_3\varepsilon_3 + z_4\varepsilon_4,$$

and $x = \sum_1^4 x_i \varepsilon_i$ with

$$x_1 = y_1, \qquad x_2 = y_1, \qquad x_3 = z_3, \qquad x_4 = z_4.$$

Trivially $y = x(1,2)$, $z = x(3,4)$. We infer from Corollary 4.8 that

$$\alpha_1 x_1^2 + \alpha_2 x_2^2 <_\nu \beta_{12} x_1 x_2 = q(y) \in \mathcal{T}, \tag{4.12}$$

$$\alpha_3 x_3^2 + \alpha_4 x_4^2 <_\nu \beta_{34} x_3 x_4 = q(z) \in \mathcal{T}, \tag{4.13}$$

and furthermore from Condition (4.7) that

$$\beta_{13} x_1 x_3 + \beta_{14} x_1 x_4 + \beta_{23} x_2 x_3 + \beta_{24} x_2 x_4 <_\nu q(y) \cong_\nu q(z).$$

Let $x' < x$, and assume w.l.o.g. that exactly one coordinate $x_i' < x_i$, say $x_1' < x_1$, which implies $x_1' <_\nu x_1$. If $q(x') = q(x)$ held, then

$$q(x') = \beta_{12} x_1' x_2 + \beta_{34} x_3 x_4 = \beta_{34} x_3 x_4.$$

But then $q(x')$ is tangible, while $q(x)$ is ghost. This contradiction proves that $q(x') < q(x)$, and we conclude that x is q-minimal. □

If $n = 3$ and $|\operatorname{supp}(y)| = |\operatorname{supp}(z)| = 2$, then the naive converse to Theorem 4.10, analogous to Theorem 4.11 but with condition (4.9) replaced by (4.8), does not hold, as the following example shows.

Example 4.12. *Let $y = y_1\varepsilon_1 + y_2\varepsilon_2$ and $z = z_1\varepsilon_1 + z_3\varepsilon_3$ with $y_1, y_2, z_1, z_3 \in \mathcal{T}$ and $ey_1 = ez_1$, $ey_2 = ez_3$, but $y_1 \neq z_1$. Then*

$$x := y \vee z = x_1\varepsilon_1 + x_2\varepsilon_2 + x_3\varepsilon_3$$

with

$$x_1 = ey_1, \qquad x_2 = y_2, \qquad x_3 = z_3.$$

Assume furthermore that

1) $\beta_{12}, \beta_{13} \in \mathcal{T}$,
2) $\alpha_1 y_1^2 + \alpha_2 y_2^2 <_\nu \beta_{12} y_1 y_2 \in \mathcal{T}$,
3) $\alpha_1 z_1^2 + \alpha_3 z_3^2 <_\nu \beta_{13} z_1 z_3$.

Both y and z are q-minimal and g-anisotropic by Corollary 4.8, and

$$q(y) = \beta_{12} y_1 y_2 \cong_\nu \beta_{13} z_1 z_3 = q(z).$$

Since $\beta_{11} := b(\varepsilon_1, \varepsilon_1) \leq_\nu \alpha_1$ and $ey_1 = ez_1$, we have

$$\beta_{11} y_1 z_1 = b(y_1\varepsilon_1, z_1\varepsilon_1) \leq_\nu \alpha_1 y_1^2 \cong_\nu \alpha_1 z_1^2$$

and we conclude that

$$b(y,z) = \beta_{11} y_1 z_1 + \beta_{12} z_1 y_2 + \beta_{13} y_1 z_3 = eq(y) = eq(z).$$

Thus Condition (4.8) *holds. We have $x = y + z$, whence*

$$q(x) = q(y) + q(z) + b(y,z) = eq(y).$$

Let now $x' := y_1\varepsilon_1 + y_2\varepsilon_2 + z_3\varepsilon_3$. Then $x' < x$, but

$$q(x') \geq \beta_{12} y_1 y_2 + \beta_{13} y_1 z_3 = eq(y).$$

Thus $q(x') = q(x)$. This proves that x is not q-minimal.

The vector $x = y \vee z$ in Theorem 4.11 obviously satisfies $y = x(J)$, $z = x(K)$ with $J := \mathrm{supp}(y)$, $K := \mathrm{supp}(z)$, while for the vector $y \vee z$ in Example 4.12 this does not hold. If we insist on the property $y = x(J)$, $z = x(K)$, then we also obtain the following converse of Theorem 4.10 in Cases B) and D):

Theorem 4.13. *Let $n = 3$. Assume that $y, z \in V$ are g-anisotropic and q-minimal with respective support J, K such that $|J| = 2$, $|K| = 2$, and $J \cup K = I$, whence $J \cap K$ is a singleton. Assume that $y(J \cap K) = z(J \cap K)$ and furthermore that either*

$$b(y(J \setminus K), z(K \setminus J)) <_\nu q(y) \cong_\nu q(z); \tag{4.14}$$

or

$$b(y(J \setminus K), z(K \setminus J)) \in \mathcal{T}, \quad b(y(J \setminus K), z(K \setminus J)) \cong_\nu q(y) \cong_\nu q(z). \tag{4.15}$$

Then $x := y \vee z$ is q-minimal and, of course, $x(J) = y$, $x(K) = z$.

Proof. We may assume that $J = \{1, 2\}$, $K = \{1, 3\}$, and then have

$$y = y_1\varepsilon_1 + y_2\varepsilon_2, \qquad z = z_1\varepsilon_1 + z_3\varepsilon_3$$

with $y_1 = z_1$. Then $x = \sum_1^3 x_i\varepsilon_i$ with

$$x_1 = y_1 = z_1, \qquad x_2 = y_2, \qquad x_3 = z_3.$$

It follows from Corollary 4.8 that

(1) $\alpha_1 x_1^2 + \alpha_2 x_2^2 <_\nu \beta_{12}x_1x_2 = q(y) \in \mathcal{T}$,
(2) $\alpha_1 x_1^2 + \alpha_3 x_3^2 <_\nu \beta_{13}x_1x_3 = q(z) \in \mathcal{T}$.

Assume that $x' = \sum_1^3 x'_i\varepsilon_i$ is given with either

$$x'_1 < x_1, \quad x'_2 = x_2, \quad x'_3 = x_3 \quad \text{or}$$
$$x'_1 = x_1, \quad x'_2 < x_2, \quad x'_3 = x_3.$$

We will prove that $q(x') < q(x)$, and then will be done.

Taking into account that

$$b(y(J \setminus K), z(K \setminus J)) = b(y_2\varepsilon_2, z_3\varepsilon_3) = \beta_{23}x_2x_3,$$

we see that

(3) $\beta_{23}x_2x_3 <_\nu \beta_{12}x_1x_2 \cong_\nu \beta_{13}x_1x_3$,

while (4.15) says that

(4) $\beta_{23}x_2x_3 \in \mathcal{T}, \quad \beta_{23}x_2x_3 \cong_\nu \beta_{12}x_1x_2 \cong_\nu \beta_{13}x_1x_3$.

Assume that (3) holds. If $x'_1 < x_1$, then $x'_1 <_\nu x_1$ since $x_1 \in \mathcal{T}$ (cf. Proposition 3.2.b), and thus

$$\beta_{12}x'_1x_2 <_\nu \beta_{12}x_1x_2, \quad \beta_{13}x'_1x_3 <_\nu \beta_{13}x_1x_3.$$

It follows from (1), (2), (3) that $q(x') <_\nu q(x)$, whence $q(x') < q(x)$. If $x'_2 < x_2$, then $x'_2 <_\nu x_2$, and thus

$$\beta_{12}x_1x'_2 <_\nu \beta_{12}x_1x_2, \qquad \beta_{23}x'_2x_3 <_\nu \beta_{23}x_2x_3.$$

Now we conclude from (1), (2), (3) that

$$q(x') = \beta_{13}x_1x_3 \cong_\nu q(x).$$

But $q(x') \in \mathcal{T}$, $q(x) \in \mathcal{G}$, and so $q(x') < q(x)$ again.

Assume finally that (4) holds. If $x_1' < x_1$, we see by the same reasoning that

$$q(x') = \beta_{23}x_2x_3 \cong_\nu q(x),$$

while if $x_2' < x_2$ then

$$q(x') = \beta_{13}x_1x_3 \cong_\nu q(x).$$

In both cases $q(x') \in \mathcal{T}$, $q(x) \in \mathcal{G}$, and so $q(x') < q(x)$. This completes the proof that x is q-minimal. □

We supplement Theorems 4.10, 4.11, 4.13 with an observation on certain pairs of q-minimal vectors.

Theorem 4.14. *Assume that $x, y \in V$ are q-minimal vectors with $y < x$ and $q(y) \cong_\nu q(x)$. Let $J := \operatorname{supp}(y)$. Then $q(y) \in \mathcal{T}$, $q(x) \in \mathcal{G}$, and one of the following cases holds:*

1) $|\operatorname{supp}(y)| = |\operatorname{supp}(x)| = 1$, $x = ey$.
2) $|\operatorname{supp}(y)| = |\operatorname{supp}(x)| = 2$, $y < x < ey$.
3) $|\operatorname{supp}(y)| = 1$, $|\operatorname{supp}(x)| \geq 2$, $y = x(J)$.
4) $|\operatorname{supp}(y)| = 2$, $|\operatorname{supp}(x)| \geq 3$, $y = x(J)$.

Proof. We may assume that $\operatorname{supp}(x) = \{1, \dots, n\}$. We have $q(y) < q(x)$ because x is q-minimal. This forces $q(y) \in \mathcal{T}$, $q(x) \in \mathcal{G}$.

Assume that $n = 1$. Now $y = y_1\varepsilon_1$, $x = x_1\varepsilon_1$, and $\alpha_1 y_1^2 \in \mathcal{T}$, $e\alpha_1 y_1^2 = \alpha_1 y_1^2 \in \mathcal{T}$, $e\alpha_1 y_1^2 = \alpha_1 x_1^2$. This implies $x_1 = ey_1$, whence $x = ey$. Hence, we have settled Claim 1.

Suppose that $|J| = 1$, $n \geq 2$. We may assume that $J = \{1\}$. Now $y = y_1\varepsilon_1$, $\alpha_1 y_1^2 \in \mathcal{T}$ and $y_1 \leq x_1$, whence $\alpha_1 y_1^2 \leq \alpha_1 x_1^2$. Since $q(y) \cong_\nu q(x)$, the term $\alpha_1 x_1^2$ is ν-dominant in the sum

$$q(x) = \sum_1^n \alpha_i x_i^2 + \sum_{i<j} \beta_{ij} x_i x_j \tag{4.16}$$

Since x is q-minimal, this forces $\alpha_1 x_1^2 \in \mathcal{T}$ and then $\alpha_1 y_1^2 = \alpha_1 x_1^2$. We conclude that $y_1^2 = x_1^2$. Since $y_1 \leq x_1$, this implies $y_1 = x_1$, whence $y = x(1)$. This proves Claim 3.

Suppose that $|J| = 2$, $n \geq 2$. We may assume that $J = \{1, 2\}$. By Corollary 4.8,

$$\alpha_1 y_1^2 + \alpha_2 y_2^2 < \beta_{12} y_1 y_2 = q(y) \in \mathcal{T}.$$

It follows from $q(y) \cong_\nu q(x)$ and $y_1 \leq x_1$, $y_2 \leq x_2$ that $\beta_{12}x_1x_2$ is a ν-dominant term in the sum (4.16) and so $\beta_{12}x_1x_2 \cong_\nu \beta_{12}y_1y_2$, $\beta_{12}x_1x_2 \geq \beta_{12}y_1y_2$.

If $n > 2$, then the q-minimality of x forces $\beta_{12}x_1x_2 \in \mathcal{T}$, and we conclude from $y_1 \leq x_1$, $y_2 \leq x_2$ that $y_1 = x_1$, $y_2 = x_2$, i.e., $y = x(1, 2)$. This settles Claim 4.

If $n = 2$, we conclude from $q(y) < q(x)$ that $e\beta_{12}y_1y_2 = \beta_{12}x_1x_2$, and then that $y_1 \cong_\nu x_1$, $y_2 \cong_\nu x_2$, whence $ex = ey$. But $x \neq ey$, since the vector ey is not q-minimal. Thus either $x_1 = ey_1$, $x_2 = y_2$, or $x_1 = y_1$, $x_2 = ey_2$. We conclude that $y < x < ey$, which gives Claim 2. □

5. Explicit computation of q-minimal elements

Let q be a given quadratic form on a free R-module V. We choose a companion b of q. Let $c = q(x)$ where $x = \sum_{i \in I} x_i \varepsilon_i$ is fixed.

We define the set $\operatorname{Min}_q(x) = \{x' \leq x : x' \text{ is } q\text{-minimal and } q(x') = c\}$. We proved that $\operatorname{Min}_q(x) \neq \emptyset$ in Theorem 2.4 by general arguments. Now we describe this set explicitly.

Write
$$c = \sum_{i \in I} \alpha_i x_i^2 + \sum_{i<j} \beta_{ij} x_i x_j, \tag{5.1}$$
where $\alpha_i = q(\varepsilon_i)$ and $\beta_{ij} = b(\varepsilon_i, \varepsilon_j)$.

In view of Remark 2.1 we may assume that I is finite, so we take $I = \{1, \dots, n\}$. We write x^* for a typical element of $\mathrm{Min}_q(x)$. When $c \in \mathcal{T}$, (5.1) has only one ν-dominant summand, which we call h, which must equal c. If $h = \alpha_k x_k^2$ then we put $x^* = x_k \varepsilon_k$, whereas if $h = \beta x_k x_\ell$ we put $x^* = x_k \varepsilon_k + x_\ell \varepsilon_\ell$. Then $\mathrm{Min}_q(x) = \{x^*\}$, by Proposition 4.5. So **we assume from now on that** $c \in \mathcal{G}$, the far more challenging case.

Recall the notation $x(J) = \sum_{i \in J} x_i \varepsilon_i$ from Notation 4.9. Then
$$q(x(J)) = \sum_{i \in J} \alpha_i x_i^2 + \sum_{i<j,\ i,j \in J} \beta_{ij} x_i x_j \tag{5.2}$$

We are interested in the *minimal subsets* J of I for which $q(x(J)) = c$. For any such J we have $q(x') = c$ whenever $x(J) \leq x' \leq x$, but we shall see that one could also have $q(x') = c$ for some $x' < x(J)$.

Remark 5.1. *Since* $u \leq x(J)$ *forces* $\mathrm{supp}(u) \subset J$, *it is clear that* $\mathrm{Min}_q(x)$ *is the disjoint union of the sets* $\mathrm{Min}_q(x(J))$, *where* J *runs over all minimal sets* J *of* I *with* $q(x(J)) = c$,
$$\mathrm{Min}_q(x) = \dot{\bigcup_J} \mathrm{Min}_q(x(J)).$$
□

Now we pick one minimal set $J \subset I$ with $q(x(J)) = c$ and look for the q-minimal $x' < x(J)$ having q-value c. After a suitable permutation of $I = \{1, \dots, n\}$, we may assume that $J = \{1, \dots, m\}$ with $m \leq 4$.

It is now straightforward to determine the set $\mathrm{Min}_q(x)$ in each case by using repeatedly the following argument taken from §4.1 and §4.2, in similar situations: When $x' < x(J)$ and the coordinate x_i' of x' satisfies $x_i' < x_i \in \mathcal{T}$, then $x_i' <_\nu x_i$, so every term in (5.2) involving x_i decreases in ν-value when we replace x by x'. We distinguish six cases.

CASE 1: $m = 1$, $\alpha_1 x_1^2 = c$.

(a) Let $\alpha_1 \in \mathcal{G}$. If $x_1 \in \mathcal{G}$ then $x(J)$ is not q-minimal, but every vector $\lambda_1 \varepsilon_1$ with $\lambda_1 \in \mathcal{T}$, $e\lambda_1 = x_1$, is q-minimal, and so
$$\mathrm{Min}_q(x) = \{\lambda_1 \varepsilon_1 \mid \lambda_1 \in \mathcal{T}, e\lambda_1 = x_1\}.$$

(b) If $\alpha_1 \in \mathcal{T}$ then $x(J) = x_1 \varepsilon_1$ is q-minimal.

CASE 2: $m = 2$, $\alpha_1 x_1^2, \alpha_2 x_2^2 \in \mathcal{T}$ and $\alpha_1 x_1^2 \cong_\nu \alpha_2 x_2^2 \cong_\nu c$, $\beta_{12} x_1 x_2 <_\nu c$.

Now $x(J)$ is q-minimal.

CASE 3: $m = 2$, $\alpha_1 x_1^2, \alpha_2 x_2^2 <_\nu c$, $\beta_{12} x_1 x_2 = c$;

(a) Let $\beta_{12} \in \mathcal{T}$. If $x_1, x_2 \in \mathcal{T}$, then $x(J)$ is q-minimal. Otherwise,
$$\mathrm{Min}_q(x) = \{\lambda_1 \varepsilon_1 + \lambda_2 \varepsilon_2 \mid \lambda_i \in \mathcal{T}, e\lambda_i = x_i\}.$$

(b) Let $\beta_{12} \in \mathcal{T}$. Since $\beta_{12} x_1 x_2 \in \mathcal{G}$, at least one of the coordinates x_1, x_2 is ghost. If $x_1 \in \mathcal{T}, x_2 \in \mathcal{G}$ or $x_1 \in \mathcal{G}, x_2 \in \mathcal{T}$ then $x(J)$ is q-minimal. If both x_1, x_2 are ghost then
$$\mathrm{Min}_q(x) = \{\lambda_1 \varepsilon_1 + \lambda_2 \varepsilon_2 \mid \lambda_1 \in \mathcal{T}, \lambda_2 \in \mathcal{G}, \text{ or } \lambda_1 \in \mathcal{G}, \lambda_2 \in \mathcal{T}, e\lambda_i = x_i\}.$$

CASE 4: $m = 2$, $\alpha_1 x_1^2, \beta_{12} x_1 x_2 \in \mathcal{T}$, $\alpha_1 x_1^2 \cong_\nu \beta_{12} x_1 x_2 \cong_\nu c$, $\alpha_2 x_2^2 <_\nu c$;

CASE 5: $m = 3$, $\alpha_1 x_1^2$ and $\beta_{23}x_2x_3$ are tangible with ν-value c, and all other terms in (5.2) have ν-value $< c$;

CASE 6: $m = 4$, $\beta_{12}x_1x_2$ and $\beta_{23}x_3x_4$ are tangible with ν-value c, and all other terms in (5.2) have ν-value $< c$.

In the three CASES 4-6 all coordinates x_i and relevant parameters α_i, β_{ij} are tangible, and so $x(J)$ is q-minimal.

So far we have proved the following (compare with Theorem 2.4):

Theorem 5.2. *Assume that x is an arbitrary vector in the free module V over a tangible supertropical semiring R, and let $c = q(x)$. Then:*

- $\mathrm{Min}_q(x) \neq \emptyset$.
- *If $c \in \mathcal{T} \cup \{0\}$ then $\mathrm{Min}_q(x) = \{x^*\}$, where $x^* = x(J)$ for some J of order ≤ 2.*
- *Let $u \leq x$ and $J := \mathrm{supp}(u)$. If $c \in \mathcal{G}$ and $u \in \mathrm{Min}_q(x)$, then $|J| \leq 4$. For "large support" $|J| \in \{3, 4\}$, we have $u = x(J)$. For "small support" $|J| \leq 2$, one could have $u < x(J)$, but always $u \leq x(J) \leq eu$.*

Note that $\mathrm{Min}_q(x)$ could be infinite, since there could exist infinitely many $a \in \mathcal{T}$ with $ea = x_i$. But this annoyance can be remedied by identifying order-associated vectors, i.e., passing to the equivalence relation $\approx$ introduced in §3.

In view of our previous assumptions ($\mathcal{T} \cdot \mathcal{T} \subset \mathcal{T}$, $\mathcal{G} \cdot \mathcal{G} \subset \mathcal{G}$), the equivalence $\approx$ is a congruence, i.e., respects multiplication and addition, and we remark that $y \approx z$ clearly implies $q(y) \approx q(z)$ (Proposition 3.6). A case by case inspection of our description above of $\mathrm{Min}_q(x)$ and of the description of all q-minimal vectors in §4 yields the following two results:

Theorem 5.3. *If $u \approx v$ and $u \leq x$, $v \leq x$, then u is q-minimal with q-value c iff v is q-minimal with value c. The set of all equivalence classes $\mathrm{Min}_q(x)/\approx$ is finite.*

Theorem 5.4. *Any vector order-associated to a q-minimal vector is q-minimal.*

Although both proofs are computational, there is one situation in which there is an easy conceptual proof. Define

$$\mathcal{T}_e = \{c \in \mathcal{T} \mid ec = e\}.$$

Then $ca \approx a$ for any $c \in \mathcal{T}_e$, so $\mathcal{T}_e$ acts on the equivalence classes. If $y = \sum_i y_i\varepsilon_i$ and $z = \sum_i z_i\varepsilon_i$ with $y_i = c_i z_i$ for $c_i \in \mathcal{T}_e$, then clearly $q(y) \approx q(z)$, so these two results follow easily whenever $a \approx b$ implies $b \in \mathcal{T}_e a$. In particular this is the case when R is a supertropical semifield.

6. Notions related to minimality

We now abandon the overall assumption in the preceding sections that both $\mathcal{T}$ and $\mathcal{G}$ are closed under multiplication and $\mathcal{G}$ is cancellative. At the moment R can be any supertropical semiring.

The following definition describes useful properties of elements of V, valid also for V non-free, which will be helpful in §7. R^* denotes the group of units of R.

Definition 6.1.

a) *We call a vector $x \in V$ **primitive** in V, if, for any $\lambda \in R$, $y \in V$,*

$$\lambda y = x \Rightarrow \lambda \in R^*. \tag{6.1}$$

b) *We call x **faithful** in V, if, for any $\lambda, \mu \in R$,*

$$\lambda x = \mu x \Rightarrow \lambda = \mu. \tag{6.2}$$

c) *We call x **basic** in V, if x is primitive in V, and, for any $z \in V$,*

$$z \leq x \Rightarrow \exists \lambda \in R \text{ with } z = \lambda x. \tag{6.3}$$

d) *We call x **faithfully basic** in V, if x is faithful and basic in V.*

e) *We call x **strictly basic** in V, if x is primitive in V, and if, for any $\alpha \in R$, $z \in V$,*

$$z \leq \alpha x \quad \Rightarrow \quad \exists \lambda \leq \alpha \text{ with } z = \lambda x. \tag{6.4}$$

When there is no danger of ambiguity, we omit the phrase "in V".

Proposition 6.2. *If x is a faithfully basic vector in V, then x is **indecomposable** in the following strong sense: For any $y, z \in V$ with $x = y + z$ either $y = x$ or $z = x$.*

Proof. There exist scalars $\lambda, \mu \in R$ with $y = \lambda x$, $z = \mu x$. This implies $x = (\lambda + \mu)x$, and then $\lambda + \mu = 1$, which forces $\lambda = 1$ or $\mu = 1$ (cf. (1.10)). □

Proposition 6.3. *Given a free R-module V, with R supertropical, and a vector $x \in V$, the following are equivalent:*

(1) *x is basic in V;*

(2) *x is a member of a base of V;*

(3) *x is primitive in V, and $|\operatorname{supp}(x)| = 1$.*

If (1)–(3) hold, then x is faithfully and strictly basic in V.

Proof. We choose a base $\{\varepsilon_i \mid i \in I\}$ of V.

The implications (2) ⇔ (3) are evident, as is the fact that (2) implies that x is faithfully and strictly basic in V. Trivially (3) ⇒ (1).

(1) ⇒ (2): Since x is primitive, certainly $x \neq 0$. Write $x = \sum_{i \in I} \lambda_i \varepsilon_i$. Then $\lambda_i \varepsilon_i \leq x$, and so $\lambda_i \varepsilon_i = \mu_i x$ for some $\mu_i \in R$. We obtain $x = \big(\sum_{i \in I} \mu_i\big)x$. Since x is primitive, $\sum_{i \in I} \mu_i \in R^*$. So $\sum_{i \in I} \mu_i$ has a unique ν-dominate term μ_k, and $\mu_k = \sum_{i \in I} \mu_i \in R^*$ by (1.10). Replacing ε_k by x and keeping the other ε_i, we obtain a new base of V. □

7. Quad(V) and its submodules QL(V) and Rig(V)

In this section, V is a free module over a supertropical semiring R, and $(\varepsilon_i \mid i \in I)$ is a fixed base of V.

Clearly the sets QL(V) and Rig(V), consisting of the quasilinear resp. rigid quadratic forms on V, are submodules of the R-module Quad(V) consisting of all quadratic forms on V, and (cf. [6, §4])

$$\operatorname{Quad}(V) = \operatorname{QL}(V) + \operatorname{Rig}(V), \tag{7.1}$$

$$\operatorname{QL}(V) \cap \operatorname{Rig}(V) = 0. \tag{7.2}$$

It will turn out that the R-modules QL(V) and Rig(V) are free, while Quad(V) most often is not free.

In [6, Eq. (8.1)], repeated above in §1.5, we defined a partial ordering on Quad(V) which we call the **function ordering** on Quad(V), and which we denote by $\leq$, as in [6, §8].

Since the function ordering is compatible with addition (and scalar multiplication), it is a refinement of the minimal ordering on $\mathrm{Quad}(V)$, which we denote by $\preceq$. Thus we use the following definition.

Definition 7.1. *Let $q, q' \in \mathrm{Quad}(V)$. Then, $q \leq q'$ iff $q(x) \leq q'(x)$ for all $x \in V$, while $q \preceq q'$ iff there exists a quadratic form $q_1 \in \mathrm{Quad}(V)$ with $q + q_1 = q'$.*

We define a quasilinear quadratic form d_i on V for every $i \in I$ by

$$d_i(x) = x_i^2 \tag{7.3}$$

and a rigid quadratic form h_{ij} for every $i < j \in I$ by

$$h_{ij}(x) = x_i x_j. \tag{7.4}$$

Here, as always, the x_i are the coordinates of the vector $x = \sum_{i\in I} x_i \varepsilon_i \in V$.

Proposition 7.2. $\mathrm{QL}(V)$ *is free with base* $(d_i | i \in I)$, *and* $\mathrm{Rig}(V)$ *is free with base* $(h_{ij} | i < j)$.

Proof. We read off from [6, Proposition 4.1] and [6, Scholium 4.7] that, if $\kappa \in \mathrm{QL}(V)$, then

$$\kappa = \sum_{i\in I} \kappa(\varepsilon_i) d_i, \tag{7.5}$$

and, if $\rho \in \mathrm{Rig}(V)$,

$$\rho = \sum_{i<j} \rho(\varepsilon_i + \varepsilon_j) h_{ij}. \tag{7.6}$$

Furthermore, if κ is a quasilinear form and $\kappa = \sum_{i\in I} \alpha_i d_i$ with $\alpha_i \in R$, then we obtain from $d_i(\varepsilon_i) = 1$ and $d_k(\varepsilon_i) = 0$ for $k \neq i$, that $\kappa(\varepsilon_i) = \alpha_i$. Also, if ρ is rigid and $\rho = \sum_{i<j} \alpha_{ij} h_{ij}$ for scalars $\alpha_{ij} \in R$, then we have $h_{ij}(\varepsilon_i + \varepsilon_j) = 1$, while $h_{k\ell}(\varepsilon_i + \varepsilon_j) = 0$ if $k < \ell$ and $k \neq i$ or $\ell \neq j$. This gives us $\alpha_{ij} = \rho(\varepsilon_i + \varepsilon_j)$. □

Proposition 7.3. *Both the functional ordering and the minimal ordering of* $\mathrm{Quad}(V)$ *restrict on* $\mathrm{Rig}(V)$ *and on* $\mathrm{QL}(V)$ *respectively to the minimal orderings of* $\mathrm{Rig}(V)$ *and* $\mathrm{QL}(V)$.

Proof. 1) Let $\rho_1, \rho_2 \in \mathrm{Rig}(V)$ and assume that $\rho_1(x) \leq \rho_2(x)$ for every $x \in V$. Then

$$\rho_1(\varepsilon_1 + \varepsilon_2) \leq \rho_2(\varepsilon_1 + \varepsilon_2).$$

By formula (7.6) we have

$$\rho_k = \sum_{i<j} \rho_k(\varepsilon_i + \varepsilon_j) h_{ij} \quad \text{for} \quad k = 1, 2.$$

It follows that $\rho_1 \leq \rho_2$ in the minimal ordering of $\mathrm{Rig}(V)$.

Conversely, if $\rho_1 \leq \rho_2$ in the minimal ordering of $\mathrm{Rig}(V)$, then trivially $\rho_1 \leq \rho_2$ in the minimal ordering of $\mathrm{Quad}(V)$, and thus also in the function ordering of $\mathrm{Quad}(V)$.

2) Concerning $\mathrm{QL}(V)$, we can argue precisely in the same way using formula (7.5) instead of (7.6). □

Let $\mathfrak{D}$ and $\mathfrak{H}$ denote the sets of basic elements (cf. Definition 6.1) of the free R-modules $\mathrm{Rig}(V)$ and $\mathrm{QL}(V)$, respectively, i.e.,

$$\mathfrak{D} = \bigcup_{i\in I} R^* d_i, \qquad \mathfrak{H} = \bigcup_{i<j} R^* h_{ij}. \tag{7.7}$$

We will see that, under mild conditions on the supertropical semiring R, the union $\mathfrak{D} \cup \mathfrak{H}$ is precisely the set of all basic elements of $\mathrm{Quad}(V)$.

Definition 7.4. *We say that eR (or $\mathcal{G}$) is* ***multiplicatively unbounded****, if for any $x, y \in \mathcal{G}$ there exists some $z \in \mathcal{G}$ with $y < xz$.* [4]

Remark 7.5. *We note in passing that if eR is multiplicatively unbounded, then the semiring R has no zero divisors. Indeed, if x_1, x_2 are non-zero elements of R, then there exist $z_1, z_2 \in \mathcal{G}$ such that $(ex_1)z_1 > e$, $(ex_2)z_2 > e$ and so $e(x_1x_2)(z_1z_2) \geq e$, which implies $x_1x_2 \neq 0$.*

Example 7.6. *Assume that $\mathcal{G}$ is cancellative, and that for any $x \in \mathcal{G}$ there exists a unit u of eR with $u < x$. Then eR is multiplicatively unbounded. Indeed, given $x, y \in \mathcal{G}$ we have*

$$y = uu^{-1}y < x(u^{-1}y).$$

In particular, if eR is a semifield $\neq \{0, e\}$, then eR is multiplicatively unbounded.

Theorem 7.7. *Assume that eR is multiplicatively unbounded. Then every element of $\mathfrak{D} \cup \mathfrak{H}$ is faithfully and strictly basic (cf. Definition 6.1) in* $\mathrm{Quad}(V)$.

Proof. 1) If $p \in \mathfrak{D} \cup \mathfrak{H}$ then (6.2) holds for p in a submodule of $\mathrm{Quad}(V)$ (cf. Proposition 6.3), and so in $\mathrm{Quad}(V)$. Thus p is certainly faithful.

2) Let $p = d_i$ for some $i \in I$ or $p = h_{ij}$ for some $i < j$. There exists a vector $x \in V$ with $p(x) = 1$, namely, $x = \varepsilon_i$ if $p = d_i$ and $x = \varepsilon_i + \varepsilon_j$ if $p = h_{ij}$. This implies (6.1) for p in $\mathrm{Quad}(V)$. Indeed, if $p = \lambda q$ with $\lambda \in R$, $q \in \mathrm{Quad}(V)$, then $1 = p(x) = \lambda q(x)$, whence $\lambda \in R^*$. Thus p is primitive in $\mathrm{Quad}(V)$.

3) Let $p = h_{ij}$ for some $i < j$. We verify (6.4) for p. Assume that $q \in \mathrm{Quad}(V)$ and $q \preceq \alpha p$ with $\alpha \in R$. We have $\alpha h_{ij} = q + q_1$ with some $q_1 \in \mathrm{Quad}(V)$. Since $\mathrm{Rig}(V)$ is a lower set in $\mathrm{Quad}(V)$, in the function ordering and hence in the minimal ordering, this implies that $q, q_1 \in \mathrm{Rig}(V)$. Since h_{ij} is strictly basic in $\mathrm{Rig}(V)$ (cf. Proposition 6.3), we conclude that $q = \beta h_{ij}$ with $\beta \leq \alpha$ (and that $q_1 = \gamma h_{ij}$ with $\gamma \leq \alpha$). Thus p is strictly basic in $\mathrm{Quad}(V)$.

4) Let $p = d_i$ for some $i \in I$. We verify again (6.4) for p. Assume that $q \in \mathrm{Quad}(V)$ and $q \preceq \alpha p$ with $\alpha \in R$. We have coefficients α_k, $\beta_{k\ell} \in R$ such that for every $x \in V$

$$q(x) = \sum_{k \in I} \alpha_k x_k^2 + \sum_{k<\ell} \beta_{k\ell} x_k x_\ell \leq \alpha x_i^2. \tag{7.8}$$

Substituting $x = \varepsilon_k$ with $k \neq i$ into (7.8), we see that $\alpha_k = 0$ for every $k \neq i$. Substituting $\varepsilon_k + \varepsilon_\ell$ with $k < \ell$, $k \neq i$, $\ell \neq i$, gives $\beta_{k\ell} = 0$ for these k, ℓ. We finally substitute $\varepsilon_i + \lambda\varepsilon_k$ for any $k \neq i$, where $\lambda \in R$ varies. If $k > i$ we obtain

$$\alpha_i + \lambda\beta_{ik} \leq \alpha. \tag{7.9}$$

If $\beta_{ik} \neq 0$, there would exist some $\lambda \in R$ with $\lambda\beta_{ik} >_\nu \alpha$, because eR is multiplicatively unbounded. Thus $\beta_{ik} = 0$. If $k < i$, we conclude in the same way that $\beta_{ki} = 0$. Thus all $\beta_{k\ell}$ are zero, and $q = \alpha_i d_i$ with $\alpha_i \leq \alpha$, as desired. □

[4]Indeed, as it is formed, this very mild condition has a legitimate place in tropical algebra, although specialists in tropical geometry may find it too exotic. However, when dealing with theoretical structural studies, it provides a convenient "algebraic notation" in which the totally ordered monoid $\mathcal{G}$ (or eR, whose absorbing element is denoted 0) is written multiplicatively rather than additively by the use of "logarithmic notation". The latter notation is employed in most of our explicit computations.

Theorem 7.8. *Assume again that eR is multiplicatively unbounded. Then $\mathfrak{D} \cup \mathfrak{H}$ is the set of all basic elements of* $\mathrm{Quad}(V)$.

Proof. Let p be a basic element of $\mathrm{Quad}(V)$. If $p \in \mathrm{QL}(V)$ (resp. $p \in \mathrm{Rig}(V)$), then certainly p is basic in $\mathrm{QL}(V)$ (resp. in $\mathrm{Rig}(V)$). We conclude by Proposition 6.3 that $p \in \mathfrak{D}$ (resp. $p \in \mathfrak{H}$).

Assume now that $p \notin \mathrm{QL}(V) \cup \mathrm{Rig}(V)$. We write $p = q_1 + q_2$ with $q_1 \in \mathrm{QL}(V)$, $q_2 \in \mathrm{Rig}(V)$. We arrive at a contradiction as follows:

Since $p = q_1 + q_2$, we have scalars $\alpha_1, \alpha_2 \in R$ with $q_1 = \alpha_1 p$, $q_2 = \alpha_2 p$ (cf. (6.3)). Thus $\alpha_1 q_2 = \alpha_2 q_1 \in \mathrm{Rig}(V) \cap \mathrm{QL}(V) = \{0\}$. Since q_1 and q_2 are nonzero elements of free submodules of $\mathrm{Quad}(V)$, and R has no zero divisors (cf. Remark 7.5), we conclude from $\alpha_1 q_2 = 0$ and $\alpha_2 q_1 = 0$ that $\alpha_1 = \alpha_2 = 0$, which gives us the contradiction $q_1 = \alpha_1 p = 0$, $q_2 = \alpha_2 p = 0$. Thus $p \in \mathrm{QL}(V) \cup \mathrm{Rig}(V)$. □

Corollary 7.9. *If eR is mutiplicatively unbounded, then $\mathfrak{D} \cup \mathfrak{H}$ is the set of all primitive indecomposable elements of* $\mathrm{Quad}(V)$.

Proof. It follows from Theorem 7.8 and Proposition 6.2 that every element of $\mathfrak{D} \cup \mathfrak{H}$ is indecomposable. Conversely, if q is a nonzero element of $\mathrm{Quad}(V)$, we may write

$$q = \lambda_1 q_1 + \cdots + \lambda_n q_n$$

with $q_i \in \mathfrak{D} \cup \mathfrak{H}$, $\lambda_i \in R$, and $n \in \mathbb{N}$ as small as possible. If q is indecomposable, we have $n = 1$, whence $q = \lambda_1 q_1$. If q is also primitive, then λ_1 is a unit in R, and so $q \in \mathfrak{D} \cup \mathfrak{H}$. □

Proposition 7.10. *If the index set I has more than one element and eR is multiplicatively unbounded, then the R-module* $\mathrm{Quad}(V)$ *is not free.*

Proof. If $\mathrm{Quad}(V)$ were free, then $\mathfrak{B}_0 = \{d_i \mid i \in I\} \cup \{h_{ij} \mid i < j\}$ would be a base of $\mathrm{Quad}(V)$, as follows from Proposition 6.3 and Theorem 7.8. But, given two indices $i < j$, we have the relation

$$d_i + d_j = d_i + d_j + h_{ij}, \tag{7.10}$$

since $a^2 + b^2 = a^2 + b^2 + ab$ for any $a, b \in R$. Thus $\mathfrak{B}_0$ is certainly not a base of $\mathrm{Quad}(V)$. □

8. Uniqueness of $\mathfrak{D}$ and $\mathfrak{H}$

We assume throughout this section that R is supertropical, eR is multiplicatively unbounded, and, as before, V is a free R-module with base $(\varepsilon_i \mid i \in I)$.

It follows from Theorem 7.8, and as well from Corollary 7.9 that the set $\mathfrak{D} \cup \mathfrak{H}$ is uniquely determined by the R-module structure of $\mathrm{Quad}(V)$. We now start out to prove that the sets $\mathfrak{D}$ and $\mathfrak{H}$ individually have this property. We put

$$\mathfrak{D}_0 := \{d_i \mid i \in I\}, \qquad \mathfrak{H}_0 := \{h_{ij} \mid i < j\}. \tag{8.1}$$

Our argument is based on the following observation, which actually gives somewhat more than we need.

Lemma 8.1. *Assume that p, q, r are different elements of $\mathfrak{D}_0 \cup \mathfrak{H}_0$, and that there exist $\lambda, \mu, \rho \in R \setminus \{0\}$ with*

$$\rho r \leq \lambda p + \mu q. \tag{8.2}$$

Then there are indices $i, j \in I$ such that $p = d_i$, $q = d_j$, and $r = h_{ij}$.

Proof. We run through several cases for the pair p, q.

1) Assume that $p, q \in \mathfrak{H}_0$, $p = h_{\alpha\beta}$, $q = h_{\gamma\delta}$. Substituting ε_k into the relation (8.2) for any $k \in I$, we have $p(\varepsilon_k) = q(\varepsilon_k) = 0$, whence we obtain $r(\varepsilon_k) = 0$ for all $k \in I$. Thus $r \in \mathfrak{H}_0$, say $r = h_{ij}$. Substituting $\varepsilon_i + \varepsilon_j$ into (8.2), we conclude from $r(\varepsilon_i + \varepsilon_j) = 1$ that $p(\varepsilon_i + \varepsilon_j) \neq 0$ or $q(\varepsilon_i + \varepsilon_j) \neq 0$, say $p(\varepsilon_i + \varepsilon_j) \neq 0$. This forces $\{\alpha, \beta\} = \{i, j\}$, whence $h_{\alpha\beta} = h_{ij}$, contradicting $p \neq r$. Thus $p, q \in \mathfrak{H}_0$ is impossible.

2) Assume that $p \in \mathfrak{H}_0$, $q \in \mathfrak{D}_0$, say $p = h_{k\ell}$, $q = d_i$. Now $p(\varepsilon_j) = 0$ for all $j \in I$ and $q(\varepsilon_j) = 0$ for all $j \neq i$. By (8.2) we conclude that $\rho r(\varepsilon_j) = 0$ for all $j \neq i$. If $r \in \mathfrak{D}_0$ this forces $r = d_i$, contradicting $r \neq q$. Thus $r \in \mathfrak{H}_0$, say $r = h_{\alpha\beta}$. Observe that $r(\varepsilon_\alpha + \varepsilon_\beta) = 1$ but $p(\varepsilon_\alpha + \varepsilon_\beta) = 0$, since $h_{k\ell}$ is different from r. Suppose that $i \notin \{\alpha, \beta\}$. Then $q(\varepsilon_\alpha + \varepsilon_\beta) = 0$. Substituting $\varepsilon_\alpha + \varepsilon_\beta$ into (8.2) gives a contradiction. Thus $i \in \{\alpha, \beta\}$, and we may assume that $\alpha = i$. Let $c \in R$. We have $r(\varepsilon_\alpha + c\varepsilon_\beta) = c$, and again $p(\varepsilon_\alpha + c\varepsilon_\beta) = 0$, since $p = h_{k\ell}$ is different from $r = h_{i\beta}$. Substituting $\varepsilon_\alpha + c\varepsilon_\beta$ into (8.2) we obtain $\rho c \leq \mu$, and then $\rho c \leq_\nu \mu$ for all $c \in R$. This contradicts our assumption that eR is multiplicatively unbounded. Thus $p \in \mathfrak{H}_0$, $q \in \mathfrak{D}_0$ is impossible.

3) There remains the case that $p, q \in \mathfrak{D}_0$, say $p = d_i$, $q = d_j$. If $\alpha \in I$ and $\alpha \neq i, j$ then $p(\varepsilon_\alpha) = q(\varepsilon_\alpha) = 0$, and we conclude from (8.2) that $r(\varepsilon_\alpha) = 0$. If $r \in \mathfrak{D}_0$ this forces $r = d_i$ or $r = d_j$, contradicting $r \neq p$ and $r \neq q$. Thus $r \in \mathfrak{H}_0$, $r = h_{k\ell}$ for some $k \neq \ell$. If $\{i, j\} \cap \{k, \ell\} = \emptyset$ then $p(\varepsilon_k + \varepsilon_\ell) = q(\varepsilon_k + \varepsilon_\ell) = 0$ and $r(\varepsilon_k + \varepsilon_\ell) = 1$, in contradiction to (8.2). Thus $\{k, \ell\}$ meets $\{i, j\}$, say $i = k$, and (8.2) reads $\rho h_{i\ell} \leq \lambda d_i + \mu d_j$. Suppose that $j \neq \ell$. Substituting $x = \varepsilon_i + c\varepsilon_\ell$ with $c \in R$, we obtain $\rho c \leq \lambda$, and then $\rho c \leq_\nu \lambda$ for every $c \in R$. Since eR is multiplicatively unbounded, this is impossible, and thus $r = h_{ij}$. Indeed $h_{ij} \leq d_i + d_j$ by (7.10). □

Definition 8.2. *We call two elements x, y of an R-module* ***associated****, if there exists a unit μ of R with $\mu x = y$.*

(N.B. This makes sense for any semiring.)

Recall that we always assume that eR is multiplicatively unbounded.

Theorem 8.3. *$\mathfrak{H}$ is the set of all primitive indecomposable elements h in the R-module* $\mathrm{Quad}(V)$ *such that there exist two primitive irreducible elements p, q of* $\mathrm{Quad}(V)$*, which are not associated, such that $h \prec p + q$. If this holds then p and q can be chosen such that $h + p + q = p + q$.*

Proof. We know that $\mathfrak{D} \cup \mathfrak{H}$ is the set of all primitive indecomposable elements in $\mathrm{Quad}(V)$ (Corollary 7.9). If $h \in \mathfrak{H}$, whence $h = \mu h_{ij}$ for $i, j \in I$, $i \neq j$, and $\mu \in R^*$. Then, (7.10)

$$h + \mu d_i + \mu d_j = \mu d_i + \mu d_j,$$

and so $h \prec \mu d_i + \mu d_j$. Conversely, if $h \in \mathfrak{D} \cup \mathfrak{H}$ and $h \prec p + q$ with $p, q \in \mathfrak{D} \cup \mathfrak{H}$ and p, q not associated, then a fortiori $h \leq p + q$, and we conclude by Lemma 8.1 that $h \in \mathfrak{H}$. □

Corollary 8.4. *If α is an automorphism of the R-module* $\mathrm{Quad}(V)$*, i.e., a bijective R-linear map from* $\mathrm{Quad}(V)$ *to* $\mathrm{Quad}(V)$*, then α maps each set $\mathfrak{D}$ and $\mathfrak{H}$ bijectively to itself, inducing permutations of the sets of orbits $\mathfrak{D}/R^*$ and $\mathfrak{H}/R^*$ under the group of units R^*.*

Proof. α restricts to a bijection $\mathfrak{D} \cup \mathfrak{H} \to \mathfrak{D} \cup \mathfrak{H}$, since the set $\mathfrak{D} \cup \mathfrak{H}$ is completely determined by the R-module structure of $\mathrm{Quad}(V)$, due to Corollary 7.9 (or Theorem 7.8). The characterization of $\mathfrak{H}$ inside $\mathfrak{D} \cup \mathfrak{H}$ given in Theorem 8.3 implies that $\alpha(\mathfrak{H}) = \mathfrak{H}$. Since $\mathfrak{D}$

is the complement of $\mathfrak{H}$ in the set $\mathfrak{D} \cup \mathfrak{H}$, also $\alpha(\mathfrak{D}) = \mathfrak{D}$. It is now obvious that α induces permutations of the orbits sets $\mathfrak{D}/R^*$ and $\mathfrak{H}/R^*$. □

9. Comparing the minimal ordering with the function ordering on Quad(V)

Assume that V is a free R-module with base $\{\varepsilon_i \mid i \in I\}$ over a supertropical semiring R. In [6, §5] we introduced the function ordering on the R-module $\mathrm{Quad}(V)$ of quadratic forms on R (cf. [6, Eq. (5.14)]). Here in §7, we observed that this ordering is a refinement of the minimal ordering on $\mathrm{Quad}(V)$, but that both orderings coincide on the submodules $\mathrm{QL}(V)$ and $\mathrm{Rig}(V)$ of $\mathrm{Quad}(V)$ consisting of the quasilinear and rigid forms on V, respectively (Proposition 7.3).

We now continue the study of these orderings on $\mathrm{Quad}(V)$. We first show that the minimal ordering can be described in terms of the function ordering by using quasilinear parts and rigid complements as defined and studied in [6, §5] and here in §7. As in [6] we denote the set of rigid complements in any $q \in \mathrm{Quad}(V)$ [5] by $\mathrm{Rig}(q)$.

Lemma 9.1.

(a) *If q is a quadratic form on V then*

$$q_{\mathrm{QL}} = \sum_{i \in I} q(\varepsilon_i) d_i. \tag{9.1}$$

(b) *If q is a second quadratic form on V and $q \le q'$, then $q_{\mathrm{QL}} \le q'_{\mathrm{QL}}$.*

Proof. a): We choose a rigid complement ρ in q and then have

$$q(\varepsilon_i) = q_{\mathrm{QL}}(\varepsilon_i) + \rho(\varepsilon_i) = q_{\mathrm{QL}}(\varepsilon_i).$$

b): Now clear, since $q \le q'$ implies $q(\varepsilon_i) \le q'(\varepsilon_i)$. □

Lemma 9.2. *Assume that $\rho \in \mathrm{Rig}(V)$, $q \in \mathrm{Quad}(V)$ and $\rho \preceq q$. Then there exists a quadratic form $\rho' \in \mathrm{Rig}(q)$ with $\rho \preceq \rho'$.*

Proof. $q = q_1 + \rho$ with some $q_1 \in \mathrm{Quad}(V)$. It follows (cf. [6, Eq. (5.10)]) that

$$q_{\mathrm{QL}} = (q_1)_{\mathrm{QL}} + \rho_{\mathrm{QL}} = (q_1)_{\mathrm{QL}},$$

since $\rho_{\mathrm{QL}} = 0$. We choose a rigid complement ρ_1 of $(q_1)_{\mathrm{QL}}$ in q_1,

$$q_1 = (q_1)_{\mathrm{QL}} + \rho_1,$$

and obtain

$$q = q_1 + \rho = (q_1)_{\mathrm{QL}} + \rho_1 + \rho = (q)_{\mathrm{QL}} + \rho_1 + \rho.$$

Thus $\rho' := \rho_1 + \rho$ is a rigid complement of $(q)_{\mathrm{QL}}$ in q and $\rho \preceq \rho'$. □

Theorem 9.3. *Assume that q and q' are quadratic forms on V and that ρ is a rigid complement in q. The following are equivalent:*

(1) $q \preceq q'$.

(2) $q_{\mathrm{QL}} \le (q')_{\mathrm{QL}}$, *and there exists $\rho' \in \mathrm{Rig}(q')$ with $\rho \le \rho'$.*

[5] By which we mean the rigid complements of q_{QL} in q, cf. [6, Definition 4.3].

Proof. (2) ⇒ (1) : By Proposition 7.3, we conclude that $q_{\mathrm{QL}} \preceq (q')_{\mathrm{QL}}$ and $\rho \preceq \rho'$, whence

$$q = q_{\mathrm{QL}} + \rho \preceq (q')_{\mathrm{QL}} + \rho' = q'.$$

(1) ⇒ (2) : We have $q \leq q'$, whence $q_{\mathrm{QL}} \leq q'_{\mathrm{QL}}$ by Lemma 9.1, whence $q_{\mathrm{QL}} \preceq q'_{\mathrm{QL}}$ by Proposition 7.3. By our assumptions $\rho \preceq q \preceq q'$. Now Lemma 9.2, applied to ρ and q', tells us that there exists some $\rho' \in \mathrm{Rig}(q')$ with $\rho \preceq \rho'$. We conclude that

$$q = (q)_{\mathrm{QL}} + \rho \preceq (q')_{\mathrm{QL}} + \rho' = q'.$$

□

Corollary 9.4. *Assume that $q, q' \in \mathrm{Quad}(V)$ and $q \leq q'$. The following are equivalent:*

(1) $q \preceq q'$.

(2) $q_{\mathrm{QL}} \leq (q')_{\mathrm{QL}}$, *and for every $\rho \in \mathrm{Rig}(q)$ there exists $\rho' \in \mathrm{Rig}(q')$ with $\rho \leq \rho'$.*

(3) $q_{\mathrm{QL}} \leq (q')_{\mathrm{QL}}$, *and there exist quadratic forms $\rho \in \mathrm{Rig}(q)$, $\rho' \in \mathrm{Rig}(q')$ with $\rho \leq \rho'$.*

Proof. (1) ⇒ (2) : Already from $q \leq q'$ we obtain that $q_{\mathrm{QL}} \leq (q')_{\mathrm{QL}}$. Now we apply the implication (1) ⇒ (2) in the theorem.

(2) ⇒ (3) : Trivial.

(3) ⇒ (1) : By Proposition 7.3 we have $q_{\mathrm{QL}} \preceq (q')_{\mathrm{QL}}$ and $\rho \preceq \rho'$. Thus $q = q_{\mathrm{QL}} + \rho \preceq q'_{\mathrm{QL}} + \rho' = q'$. □

Given two quadratic forms q, q' on V, it is desirable to have an algorithm to determine whether or not $q \preceq q'$. Starting from Corollary 9.4, this problem can be reduced to the case that q and q' are binary forms as described below.

Notation 9.5. *Let J be a subset of the index set I.*

a) $V_J := \sum_{i \in J} R\varepsilon_i$ *is a free submodule of V. It comes with a natural linear projection $\pi_J : V \to V_J$, given by $\pi_J(\varepsilon_i) = \varepsilon_i$ for $i \in J$, and $\pi_J(\varepsilon_i) = 0$ for $i \in I \setminus J$. We also have the inclusion mapping $i_J : V_J \hookrightarrow V$, with $i_J(\varepsilon_i) = \varepsilon_i$ for every $i \in J$.*

b) *Any form $\vartheta \in \mathrm{Quad}(V_J)$ gives us a form*

$$\vartheta^I := \vartheta \circ \pi_J \in \mathrm{Quad}(V). \tag{9.2}$$

c) *Given $q \in \mathrm{Quad}(V)$, we define*

$$q_J := (q|V_J)^I = q \circ i_J \circ \pi_J \in \mathrm{Quad}(V). \tag{9.3}$$

Remarks 9.6.

a) *It is obvious that $(q_1 + q_2)_J = (q_1)_J + (q_2)_J$ for $q_1, q_2 \in \mathrm{Quad}(V)$, whence*

$$q \preceq q' \quad \Rightarrow \quad q_J \preceq (q')_J \tag{9.4}$$

for $q, q' \in \mathrm{Quad}(V)$. Furthermore, $(\vartheta_1 + \vartheta_2)^I = \vartheta_1^I + \vartheta_2^I$ for $\vartheta_1, \vartheta_2 \in \mathrm{Quad}(V_J)$, whence

$$\vartheta \preceq \vartheta' \quad \Rightarrow \quad \vartheta^I \preceq (\vartheta')^I \tag{9.5}$$

for $\vartheta, \vartheta' \in \mathrm{Quad}(V_J)$.

b) *It is easily seen that*

$$(q_J)_{\mathrm{QL}} = (q_{\mathrm{QL}})_J \tag{9.6}$$

for $q \in \mathrm{Quad}(V)$, and

$$(\vartheta^I)_{\mathrm{QL}} = (\vartheta_{\mathrm{QL}})^I \tag{9.7}$$

for $\vartheta \in \mathrm{Quad}(V_J)$. *Furthermore, if* $\rho \in \mathrm{Rig}(q)$ *then* $\rho_J \in \mathrm{Rig}(q_J)$*, and if* $\chi \in \mathrm{Rig}(\vartheta)$*, then* $\chi^I \in \mathrm{Rig}(\vartheta^I)$.

Proposition 9.7. *Let* $q \in \mathrm{Quad}(V)$. *Assume that* I *is finite, and that for every* $i < j$ *in* I *a form* $\rho_{ij} \in \mathrm{Rig}(q_{\{i,j\}})$ *is given. (We use Notation 9.5 for two-element subsets of* I*.) Then*

$$\sum_{i<j} \rho_{i,j} \in \mathrm{Rig}(q).$$

Proof. Define $\rho := \sum_{i<j} \rho_{i,j}$, noting that $\rho_{\{i,j\}} = \rho_{i,j}$. We choose a presentation

$$q = \sum_{i \in I} \alpha_i d_i + \sum_{i<j} \alpha_{i,j} h_{i,j}. \tag{9.8}$$

Then $q_{\{i,j\}} = \alpha_i d_i + \alpha_j d_j + \alpha_{i,j} h_{i,j}$. Let $\rho_{i,j} = \beta_{i,j} h_{i,j}$. The quasilinear part of $q_{\{i,j\}}$ is $\alpha_i d_i + \alpha_j d_j$, while the quasilinear part of q is $\sum\limits_{i \in I} \alpha_i d_i$. Our assumption that $\rho_{i,j} \in \mathrm{Rig}(q_{\{i,j\}})$ means that

$$\alpha_i d_i + \alpha_j d_j + \beta_{i,j} h_{i,j} = \alpha_i d_i + \alpha_j d_j + \alpha_{ij} h_{i,j}. \tag{9.9}$$

We now obtain

$$\sum_{i \in I} \alpha_i d_i + \sum_{i<j} \beta_{ij} h_{i,j} = \sum_{i \in I} \alpha_i d_i + \sum_{i<j} \alpha_{ij} h_{i,j} \tag{9.10}$$

by successively replacing every summand $\alpha_{ij} h_{i,j}$ on the right side of (9.10) by $\beta_{ij} h_{i,j}$, which is justified by (9.9). The relation (9.10) says that $\rho \in \mathrm{Rig}(q)$. □

Theorem 9.8. *Let* $q, q' \in \mathrm{Quad}(V)$. *Assume that* $|I| \geq 2$*, and that, for each* $i < j$ *in* I*,*

$$q_{\{i,j\}} \preceq q'_{\{i,j\}}. \tag{9.11}$$

Then $q \preceq q'$.

Proof. We first deal with the case that I is finite. From (9.11) we obtain, that for every $i < j$ in I,

$$(q_{\mathrm{QL}})_{\{i,j\}} = (q_{\{i,j\}})_{\mathrm{QL}} \preceq (q'_{\{i,j\}})_{\mathrm{QL}} = (q'_{\mathrm{QL}})_{\{i,j\}}$$

(cf. (9.6)), and hence $(q_{\mathrm{QL}})_{\{i,j\}} \leq (q'_{\mathrm{QL}})_{\{i,j\}}$. Writing $q_{\mathrm{QL}} = \sum\limits_i \alpha_i d_i$, $q'_{\mathrm{QL}} = \sum\limits_i \alpha'_i d_i$, this means

$$\alpha_i d_i + \alpha_j d_j \leq \alpha'_i d_i + \alpha'_j d_j.$$

Evaluation at ε_i gives us $\alpha_i \leq \alpha'_i$; this for every $i \in I$. We conclude that $q_{\mathrm{QL}} \preceq q'_{\mathrm{QL}}$.

For every $i < j$ in I we choose a form $\rho_{ij} \in \mathrm{Rig}(q_{\{i,j\}})$. Applying part (1) ⇒ (2) of Corollary 9.4, we see that there exists some ρ'_{ij} in $\mathrm{Rig}(q'_{\{i,j\}})$ with $\rho_{ij} \preceq \rho'_{ij}$. Now Proposition 9.7 gives us forms

$$\rho := \sum_{i<j} \rho_{ij} \in \mathrm{Rig}(q), \quad \rho' := \sum_{i<j} \rho'_{ij} \in \mathrm{Rig}(q').$$

From $\rho_{ij} \preceq \rho'_{ij}$ for all $i < j$ we obtain $\rho \preceq \rho'$, and then conclude by part (3) ⇒ (1) of Corollary 9.4 that $q \preceq q'$.

If I is infinite we choose a finite subset J of I such that

$$q = (q|V_J)^I, \qquad q' = (q'|V_J)^I.$$

As just proven, $q|V_J \preceq q'|V_J$, i.e., there exists some $\vartheta \in \mathrm{Quad}(V_J)$ with

$$(q|V_J) + \vartheta = (q'|V_J).$$

From this we conclude that $q + \vartheta^I = q'$. □

The task of deciding whether or not $q \preceq q'$ for forms q, q' on V with $q \leq q'$, can be reduced to the following special case by applying Theorem 9.8 and Corollary 9.4, cf. the proof of Corollary 9.11 below, where the reduction argument is detailed in a special case.

Problem 9.9. *Let V be free with base $\varepsilon_1, \varepsilon_2$. Assume that forms*

$$q = \begin{bmatrix} \alpha_1 & \alpha \\ & \alpha_2 \end{bmatrix}, \qquad \rho = \gamma h_{12} = \begin{bmatrix} 0 & \gamma \\ & 0 \end{bmatrix}$$

are given with $\rho \leq q$. Does $\rho \preceq q$?

Together, Lemma 9.2 and Corollary 9.4 tell us that $\rho \preceq q$ iff there exists a form $\rho' = \beta h_{12} \in \mathrm{Rig}(q)$ with $\rho \leq \rho'$. This means by [6, Proposition 4.6] that there exists some $\beta \in C_{12}(q)$ with $\gamma \leq \beta$, where $C_{12}(q)$ denotes the off-diagonal entry in the companion matrix $C(q)$, defined in [6, §6]. Thus the problem can be reformulated as follows:

Problem 9.9′ Given $\alpha_1, \alpha_2, \alpha \in R$, let $q := \left[\begin{smallmatrix} \alpha_1 & \alpha \\ & \alpha_2 \end{smallmatrix}\right]$, and assume that

$$\forall x_1, x_2 \in R : \quad \gamma x_1 x_2 \leq \alpha_1 x_1^2 + \alpha_2 x_2^2 + \alpha x_1 x_2. \tag{9.12}$$

Does there exist $\beta \in C_{12}(q)$ with $\gamma \leq \beta$?

In all the following we assume that R is a **nontrivial tangible supersemifield**, i.e. both $\mathcal{G}$ and $\mathcal{T}$ are groups, $e\mathcal{T} = \mathcal{G}$, and $\mathcal{G} \neq \{e\}$.

In order to settle the equivalent problems 9.9 and 9.9′ we borrow more terminology from [6]. We first choose a *full multiquadratic extension* $R^{1/2}$ of R. This is a tangible supertropical semifield $R^{1/2} \supset R$ such that for every $x \in \mathcal{G}$ there exists a unique $z \in \mathcal{G}^{1/2} := \mathcal{G}(R^{1/2})$ with $z^2 = x$, denoted by $\sqrt{x}$, and for every $x \in \mathcal{T}$ there exists an element $z \in \mathcal{T}^{1/2} := \mathcal{T}(R^{1/2})$, not necessarily unique, with $z^2 = x$. Such multiquadratic extension $R^{1/2}$ of R always exists, but most often are not unique, cf. [6, §7].

We say that a tangible supersemifield R is **dense** if the group $\mathcal{G}$ is densely ordered, i.e., for any $a < b$ in $\mathcal{G}$ there exists $c \in \mathcal{G}$ with $a < c < b$. Otherwise we call R **discrete**. When R is discrete, there exists a biggest element $x < e$ in $\mathcal{G}$. We chose some $\pi \in \mathcal{T}$ with $e\pi = x$, and call π a **prime element** of R. Then $\sqrt{e\pi}$ is the biggest element of $\mathcal{T}^{1/2}$ smaller then e. We choose an element of $\mathcal{T}^{1/2}$ with $z^2 = \pi$ and denote this element z by $\sqrt{\pi}$.

When $\alpha_1 = 0$ or $\alpha_2 = 0$, it will be easy to settle Problem 9.9′. But when $\alpha_1\alpha_2 \neq 0$, we need an elaborate case distinction. If $\alpha_1\alpha_2$ is a "ν-square", i.e., $\alpha_1\alpha_2 \cong_\nu \lambda^2$ for some $\lambda \in R$, we choose some $\xi \in \mathcal{T}$ with $\alpha_1\xi^2 \cong_\nu \alpha_2$. Otherwise we choose $\xi \in \mathcal{T}^{1/2}$ with $\alpha_1\xi^2 \cong_\nu \alpha_2$. Thus $\alpha_1\xi \cong_\nu \xi^{-1}\alpha_2$ (in $R^{1/2}$) in both cases. (If $\alpha_1, \alpha_2 \in \mathcal{T}$, we may think of $\xi\alpha_1$ as a sort of "tangible geometric mean" of α_1, α_2, since $e\xi\alpha_1 = \sqrt{e\alpha_1 \cdot e\alpha_2}$.)

If R is discrete and $\xi \notin \mathcal{T}$, we choose $\sigma, \tau \in \mathcal{T}$ with $e\tau < e\xi < e\sigma$ and with no element of $\mathcal{G}$ between $e\tau$ and $e\sigma$. In other terms, employing the prime element π of R,

$$\tau \cong_\nu \pi\sigma, \qquad \xi \cong_\nu \sqrt{\pi}\sigma.$$

Problem 9.9′ can now be settled by use of the determination of $C_{12}(q)$ in [6], cf. [6, Proposition 7.9, Theorem 7.12]. Perhaps surprisingly the answer most often is "Yes".

Theorem 9.10. *Assume that R is a nontrivial tangible supersemifield. The answer to Problem 9.9 is "Yes" except in the case that $\alpha_1, \alpha_2 \neq 0$, $\alpha^2 \leq \alpha_1\alpha_2$, R discrete, $\xi \notin \mathcal{T}$, and $\alpha_1 \in \mathcal{T}$ or $\alpha_2 \in \mathcal{T}$. Then the answer is "No" iff $\gamma \cong_\nu \sigma\alpha_1$. For every $x \in R$ with $\gamma \cong_\nu \sigma\alpha_1$, we have $\gamma h_{12} \leq q$, but the forms γh_{12} and q are not comparable in the minimal ordering.*

Proof. Instead of Condition (9.12) we most often employ the weaker condition

$$\forall \lambda \in \mathcal{T}: \quad \gamma \leq \alpha_1\lambda + \alpha + \alpha_2\lambda^{-1}, \tag{9.13}$$

which is obtained from (9.12) for $x_1, x_2 \in \mathcal{T}$ by dividing out x_1x_2 and taking $\lambda = \frac{x_1}{x_2}$. In several instances we will deduce from (9.13) that $\gamma \leq \alpha$, or that $\gamma \in C_{12}(q)$, and then of course the answer is "Yes". (Notice that $\alpha \in C_{12}(q)$.) We run through several cases.

a) Assume that $\alpha_1 = 0$ or $\alpha_2 = 0$, say $\alpha_2 = 0$. Then (9.13) reads

$$\forall \lambda \in \mathcal{T}: \quad \gamma \leq \alpha_1\lambda + \alpha. \tag{9.14}$$

If $\alpha_1 = 0$, it follows that $\gamma \leq \alpha$. If $\alpha_1 \neq 0$ and $\alpha \neq 0$, we can choose $\lambda \in \mathcal{T}$ with $\lambda <_\nu \frac{\alpha}{\alpha_1}$ and obtain again from (9.14) that $\gamma \leq \alpha$. Finally, if $\alpha_1 \neq 0$ and $\alpha = 0$, then (9.14) implies that $\gamma = 0$, hence $\gamma \leq \alpha$ again.

b) Assume that $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, $\alpha^2 >_\nu \alpha_1\alpha_2$. If $\alpha_1\alpha_2$ is a ν-square, we choose some $\xi \in \mathcal{T}$ with $\alpha_1\xi^2 \cong_\nu \alpha_2$. Otherwise we choose $\xi \in \mathcal{T}^{1/2}$ with $\alpha_1\xi^2 \cong_\nu \alpha_2$. Then

$$\alpha_1\xi^2 \cong_\nu \alpha_2, \qquad \alpha_1\xi \cong_\nu \alpha_2\xi^{-1} \qquad (\xi \in \mathcal{T}^{\frac{1}{2}}).$$

b.1) Assume first that $\xi \in \mathcal{T}$, i.e., $\alpha_1\alpha_2$ is a ν-square. Substituting $\lambda = \xi$ into (9.13) we obtain

$$\gamma \leq \xi\alpha_1 + \alpha + \xi^{-1}\alpha_2 = e\xi\alpha_1 + \alpha.$$

We have $\alpha^2 >_\nu \alpha_1^2\xi^2$, i.e., $\alpha >_\nu \xi\alpha_1$, and conclude that $\gamma \leq \alpha$.

b.2) Assume now that no $\lambda \in \mathcal{T}$ is ν-equivalent to ξ. If $\lambda >_\nu \xi$, then (9.13) reads

$$\gamma \leq \lambda\alpha_1 + \alpha. \tag{9.15}$$

If $\lambda <_\nu \xi$, then (9.13) reads

$$\gamma \leq \lambda^{-1}\alpha_1 + \alpha. \tag{9.16}$$

Assume first that R is dense. We can choose $\lambda \in \mathcal{T}$ with $\xi\alpha_1 <_\nu \lambda\alpha_1 <_\nu \alpha$, and obtain from (9.15) that $\gamma \leq \alpha$.

Now we deal with the case that R is discrete and $\xi \notin \mathcal{T}$. Assuming that $\alpha^2 \geq_\nu \pi^{-1}\alpha_1\alpha_2$, we have

$$\alpha_1\alpha_2 \cong_\nu \xi^2\alpha_1 \cong_\nu \pi\sigma^2\alpha_1.$$

If $\alpha >_\nu \sigma\alpha_1$, i.e., $\alpha^2 >_\nu \pi^{-1}\alpha_1\alpha_2$, we can insert $\lambda = \sigma$ into (9.15) and obtain $\gamma \leq \alpha$. If $\alpha \cong_\nu \sigma\alpha_1$, i.e., $\alpha^2 \cong \pi^{-1}\alpha_1\alpha_2$, then, inserting $\lambda = \sigma$ into (9.14), we obtain

$$\gamma \leq \sigma\alpha_1 + \alpha \cong_\nu \alpha.$$

Since $C_{12}(q)$ is closed under ν-equivalence, cf. [6, Theorem 7.12.a], we can choose $\beta = \sigma\alpha_1 + \alpha \in C_{12}(q)$ and then have $\gamma \leq \beta$.

This finishes the case $\alpha^2 >_\nu \alpha_1\alpha_2 \neq 0$.

c) From now on we assume that $\alpha_1\alpha_2 \neq 0$, $\alpha^2 \leq_\nu \alpha_1\alpha_2$. Now q is quasilinear, and (9.13) reads

$$\forall \lambda \in \mathcal{T}: \quad \gamma \leq \lambda\alpha_1 + \lambda^{-1}\alpha_2. \tag{9.17}$$

We have

$$\alpha \leq_\nu \xi\alpha_1 \cong_\nu \xi^{-1}\alpha_2 \qquad (\xi \in \mathcal{T}^{\frac{1}{2}}).$$

If $\xi \in \mathcal{T}$, we may insert $\lambda = \xi$ into (9.17) and obtain $\gamma \leq e\xi\alpha_1$. Now [6, Theorem 7.12] tells us that $\gamma \in C_{12}(q)$.

If R dense and $\xi \notin \mathcal{T}$, we may insert any $\lambda >_\nu \xi$, $\lambda \in \mathcal{T}$, into (9.17) and obtain $\gamma \leq \lambda\alpha_1$. Since $e\lambda$ may be arbitrary close to $e\xi$, we obtain $\gamma \leq_\nu \xi\alpha_1$ and then $\gamma <_\nu \xi\alpha_1$. Now [6, Theorem 7.12] tells us that $\gamma \in C_{12}(q)$ again.

d) We finally deal with the case that R discrete and $\xi \notin \mathcal{T}$. Inserting again any $\lambda \in \mathcal{T}$ with $\lambda >_\nu \xi$ into (9.17), we obtain $\gamma \leq e\sigma\alpha_1$. If $\alpha_1 \in \mathcal{G}$ and $\alpha_2 \in \mathcal{G}$ then $\gamma \in C_{12}(q)$ by [6, Theorem 7.12], and we are done. Assume now that $\alpha_1 \in \mathcal{T}$ or $\alpha_2 \in \mathcal{T}$. By [6, Theorem 7.12],
$$C_{12}(q) = [0, e\tau\alpha_1] = [0, e\sigma^{-1}\alpha_2].$$
If $\gamma \leq_\nu \tau\alpha_1$, then $\gamma \in C_{12}(q)$. Otherwise $\gamma \cong_\nu \sigma\alpha_1$, since $\gamma \leq e\sigma\alpha_1$, as stated above. Now $\gamma > \beta$ for every $\beta \in C_{12}(q)$.

e) Remaining in the case that R discrete and $\xi \notin \mathcal{T}$, let now $\gamma \in R$ be given with $\gamma \cong_\nu \sigma\alpha_1$, hence $\gamma > \beta$ for every $\beta \in C_{12}(q)$. We prove that Condition (9.12) holds, which now reads
$$\forall x_1, x_2 \in R: \quad \gamma x_1 x_2 \leq \alpha_1 x_1^2 + \alpha_2 x_2^2. \tag{9.18}$$
The inequality in (9.18) holds trivially if $x_1x_2 = 0$. Otherwise $x_1 \in \mathcal{T} \cup e\mathcal{T}$ and $x_2 \in \mathcal{T} \cup e\mathcal{T}$. For $x_1, x_2 \in \mathcal{T}$ condition (9.18) boils down to
$$\forall \lambda \in \mathcal{T}: \quad \gamma \leq \lambda\alpha_1 + \lambda^{-1}\alpha_2, \tag{9.19}$$
and this is clearly true since $\gamma \cong_\nu \sigma\alpha_1 \cong_\nu \tau^{-1}\alpha_2$. Since here only the ν-values of $\gamma, \alpha_1, \alpha_2$ matter, (9.18) remains valid if we replace the parameters $\gamma, \alpha_1, \alpha_2$ by $e\gamma, e\alpha_1, \alpha_2$, by $e\gamma, \alpha_1, e\alpha_2$, and by $e\gamma, e\alpha_1, e\alpha_2$. Thus (9.18) is valid for all $x_1, x_2 \in \mathcal{T} \cup e\mathcal{T}$. We conclude that indeed $\gamma h_{12} \leq q$, while *not* $\gamma h_{12} \prec q$.

□

Corollary 9.11. *Assume that R is a dense tangible supersemifield (and V is a free R-module). Then the function ordering and the minimal ordering on* $\mathrm{Quad}(V)$ *are the same.*

Proof. Let $q, q' \in \mathrm{Quad}(V)$ with $q \leq q'$. We want to prove that $q \preceq q'$. This is trivial if $|I| = 1$. Assume next that $|I| = 2$. We choose some $\rho \in \mathrm{Rig}(q)$. Then $\rho \leq q'$. Since R is dense, Theorem 9.10 tells us that there exists some $\rho' \in \mathrm{Rig}(q')$ with $\rho \preceq \rho'$. We conclude by Corollary 9.4 that $q \leq q'$.

Assume finally that $|I| > 2$. It follows from $q \leq q'$ that $q_{\{i,j\}} \leq q'_{\{i,j\}}$ for any $i < j$ in I, whence $q_{\{i,j\}} \preceq q'_{\{i,j\}}$, as we have shown. By Theorem 9.8 this implies $q \preceq q'$, as desired. □

Corollary 9.12. *Assume that $q \leq q'$, and that for every $i < j$ in I with $q'(\varepsilon_i) \neq 0$, $q'(\varepsilon_j) \neq 0$ either $q'(\varepsilon_i)$, $q'(\varepsilon_j) \in \mathcal{G}$, or $eq'(\varepsilon_i) = \lambda^2 eq'(\varepsilon_j)$ for some $\lambda \in \mathcal{G}$. Then $q \preceq q'$.*

Proof. Proceed as in the last proof, using the appropriate parts of Theorem 9.10. □

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Institute of Mathematics, University of Aberdeen, AB24 3UE, Aberdeen, UK.
E-mail address: `zzur@abdn.ac.uk; zzur@math.biu.ac.il`

Department of Mathematics, NWF-I Mathematik, Universität Regensburg 93040 Regensburg, Germany
E-mail address: `manfred.knebusch@mathematik.uni-regensburg.de`

Department of Mathematics, Bar-Ilan University, Ramat-Gan 52900, Israel
E-mail address: `rowen@math.biu.ac.il`