

# PERFECT ISOMETRIES AND MURNAGHAN-NAKAYAMA RULES

OLIVIER BRUNAT AND JEAN-BAPTISTE GRAMAIN

ABSTRACT. This article is concerned with perfect isometries between blocks of finite groups. Generalizing a method of Enguehard to show that any two  $p$ -blocks of (possibly different) symmetric groups with the same weight are perfectly isometric, we prove analogues of this result for  $p$ -blocks of alternating groups (where the blocks must also have the same sign when  $p$  is odd), of double covers of alternating and symmetric groups (for  $p$  odd, and where we obtain crossover isometries when the blocks have opposite signs), of complex reflection groups  $G(d, 1, n)$  (for  $d$  prime to  $p$ ), of Weyl groups of type  $B$  and  $D$  (for  $p$  odd), and of certain wreath products. In order to do this, we need to generalize the theory of blocks, in a way which should be of independent interest.

## 1. INTRODUCTION

Perfect isometries, introduced by M. Broué in [1], are the shadow, at the level of characters, of very deep structural correspondences between blocks of finite groups (such as derived equivalences, or splendid equivalences). The existence of such equivalences is at the heart of Broué's Abelian Defect Conjecture, which predicts that any  $p$ -block of a finite group  $G$  with abelian defect group  $P$  and its Brauer correspondent in  $N_G(P)$  are derived equivalent.

Recently, there has been considerable progress in the construction of equivalences between blocks, especially using a method, introduced and developed by J. Chuang and R. Rouquier in [5], and based on  $\mathfrak{sl}_2$ -categorification. As a consequence of their work, they show that two  $p$ -blocks of (possibly different) symmetric groups with isomorphic defect groups are splendidly equivalent; see [5, Theorem 7.2]. This explains the result [7, Theorem 11] of M. Enguehard, which is an analogue of [5, Theorem 7.2], but at the level of characters, that is, the existence of Broué perfect isometries between such blocks.

For  $p$ -blocks of (possibly different) double covers of the symmetric and alternating groups, it has been conjectured by M. Schaps and R. Kessar that, with some additional assumptions, a similar result holds. There are partial results in this direction, for example [13], [14] and [16]. However, even at the level of characters, the existence of perfect isometries between these  $p$ -blocks was yet unproved.

This article discusses perfect isometries. Besides suggesting the existence of a derived equivalence between blocks, any perfect isometry between two  $p$ -blocks of finite groups provides an isomorphism between their centres, and an isomorphism between the Grothendieck groups of their module categories. In particular, perfectly isometric  $p$ -blocks have the same numbers of ordinary and of modular characters,

and their Cartan matrices and decomposition matrices have the same invariant factors.

Furthermore, the weaker version of Broué’s Abelian Defect Conjecture (that is, Broué’s conjecture at the level of characters) gives, in the abelian defect case, deep insight into more numerical conjectures, such as the Alperin, Knörr-Robinson, Alperin-McKay and Dade conjectures (see for example [6]).

In this paper, we generalize Enguehard’s method (see [7]) based on the Murnaghan-Nakayama rule in the symmetric group (which gives a way to compute iteratively the values of irreducible complex characters). We will prove that similar results hold for many classes of groups where some analogues of the Murnaghan-Nakayama rule are available.

For this, we extract the properties of the Murnaghan-Nakayama rule needed in Enguehard’s method, which we axiomatize in the concept of an *MN-structure* for a finite group. In some cases, for example when the analogue of the Murnaghan-Nakayama rule for the considered groups do not give information on the whole group, but only on certain conjugacy classes (this happens for the double covers of the symmetric and alternating groups), we need to replace the set of  $p$ -regular elements of the group by an arbitrary union of conjugacy classes. We then develop a generalized modular theory, and define generalized blocks and generalized perfect isometries. Note that the notion of generalized blocks and generalized perfect isometries introduced by B. Külshammer, J. B. Olsson and G. R. Robinson in [15] is not exactly the same as ours. In some way, our notion is more general, because any Külshammer-Olsson-Robinson isometry or Broué isometry is a generalized perfect isometry in our sense.

The article is organized as follows. In Section 2 we generalize the theory of blocks of characters. Note that §2.1 is of independent interest, because it in particular gives a natural framework to use the techniques of the usual modular  $p$ -block theory for the theory of Külshammer, Olsson and Robinson. The main result of this section (Theorem 2.10) provides the bridge necessary to compare blocks and spaces of class functions of (possibly distinct) groups which have similar MN-structures. This *combinatorification* of the ideas in [7] can in turn be used to exhibit perfect isometries between blocks of these groups (see Corollary 2.17 and Theorem 2.20).

The remaining sections are devoted to describing MN-structures in several families of finite groups, and using our methods to build explicitly perfect isometries between their blocks.

More precisely, we prove in Section 3 that two  $p$ -blocks of (possibly different) alternating groups with same weight, (and the same *signature type* when  $p$  is odd) are perfectly isometric (see Theorems 3.9, 3.10 and 3.11).

Then, in Section 4, we study the case of spin blocks of the double covers of the symmetric and alternating groups, and we prove the perfect isometry version of the Kessar-Schaps conjecture. We show that, when  $p$  is odd, any two spin  $p$ -blocks with the same weight and sign are perfectly isometric (see Theorem 4.21 and Corollary 4.22). As is to be expected in these groups (see [14]), we also obtain *crossover* isometries, relating a  $p$ -block in “the symmetric case” to a  $p$ -block in “the alternating case”. Note that, in the proof of these results, even though the isometries we obtain are Broué isometries, we crucially need the generalized theory introduced in Section 2.

In the last section, we examine the case of certain wreath products. Applying our method, we give in §5.2 and §5.3 a new and more uniform construction of the isometries appearing in Broué's Abelian Defect Conjecture for symmetric groups, isometries introduced by M. Osima, and the generalized perfect isometry considered in [2] in order to show the existence of  $p$ -basic sets for the alternating group (see Theorem 5.1, Theorem 5.3 and Corollary 5.2). Even though these results are not new, they give explicit isometries, and considerably simplify the calculations (for example, note that the initial proof of Rouquier [25] of Broué's perfect isometries Conjecture for symmetric groups (see [25]) is not constructive, and is based on a strong result of Fong and Harris in [8] on perfect isometries in wreath products).

In §5.4, we apply our method to  $p$ -blocks of complex reflection groups  $G(d, 1, n)$  with  $d$  prime to  $p$ , and obtain in Theorem 5.4 an analogue of Enguehard's result for these groups. In particular, this gives the result for  $p$ -blocks (with  $p$  odd) of (possibly different) Weyl groups of type  $B$  (see Corollary 5.6). In §5.5, we also prove the result for  $p$ -blocks (with  $p$  odd) of (possibly different) Weyl groups of type  $D$  (Theorems 5.7 and 5.8). All of these are new results.

Finally, in §5.6, we give an analogue of the generalized perfect isometry of [2, Theorem 3.6] for  $p$ -blocks (with  $p$  odd) of alternating groups (see Theorem 5.12). In a certain sense (see Example 5.9), this is a natural analogue of Osima's isometry for alternating groups. When the  $p$ -block of the alternating group has abelian defect, our result gives an alternative proof of Broué's perfect isometries Conjecture first obtained by Fong and Harris in [9] (see Corollary 5.13).

We hope that our results, and in particular the fact that the Broué perfect isometries constructed here are explicit, will help to prove that the corresponding  $p$ -blocks are in fact derived equivalent.

## 2. GENERALITIES

In this section,  $G$  denotes a finite group and  $\mathcal{C}$  a set of conjugacy classes of  $G$ . We set

$$(1) \quad C = \bigcup_{c \in \mathcal{C}} c.$$

We write  $\text{Irr}(G)$  for the set of irreducible characters of  $G$  over the complex field  $\mathbb{C}$ , and  $\langle \cdot, \cdot \rangle_G$  for the usual hermitian product on  $\mathbb{C}\text{Irr}(G)$ . For  $x \in G$ , we denote by  $x^G$  the conjugacy class of  $x$  in  $G$ . Define  $\text{res}_C : \mathbb{C}\text{Irr}(G) \rightarrow \mathbb{C}\text{Irr}(G)$  by setting, for any class function  $\varphi \in \mathbb{C}\text{Irr}(G)$ ,

$$\text{res}_C(\varphi)(g) = \begin{cases} \varphi(g) & \text{if } g \in C, \\ 0 & \text{otherwise.} \end{cases}$$

For  $B \subseteq \mathbb{C}\text{Irr}(G)$ , we set  $B^C = \{\text{res}_C(\chi) \mid \chi \in B\}$ .

**2.1. Generalized modular theory.** Let  $b$  be a  $\mathbb{Z}$ -basis of the  $\mathbb{Z}$ -module  $\mathbb{Z}\text{Irr}(G)^C$ . For every  $\chi \in \text{Irr}(G)$ , there are uniquely determined integers  $d_{\chi\varphi}$  such that

$$(2) \quad \text{res}_C(\chi) = \sum_{\varphi \in b} d_{\chi\varphi} \varphi.$$

We denote by  $b^\vee$  the dual basis of  $b$  with respect to  $\langle \cdot, \cdot \rangle_G$ , i.e. the unique  $\mathbb{C}$ -basis  $b^\vee = \{\Phi_\varphi \mid \varphi \in b\}$  of  $\mathbb{C}\text{Irr}(G)^C$  such that  $\langle \Phi_\varphi, \vartheta \rangle = \delta_{\vartheta\varphi}$  for all  $\vartheta \in b$ .

**Proposition 2.1.** *Let  $\mathcal{C}$  be a set of conjugacy classes of  $G$ . Suppose that  $b$  is a  $\mathbb{Z}$ -basis of  $\mathbb{Z}\text{Irr}(G)^\mathcal{C}$ , and denote by  $b^\vee = \{\Phi_\varphi \mid \varphi \in b\}$  the dual basis of  $b$  with respect to  $\langle \cdot, \cdot \rangle_G$  (as above). Then:*

(i) *For every  $\varphi \in b$ , we have*

$$(3) \quad \Phi_\varphi = \sum_{\chi \in \text{Irr}(G)} d_{\chi\varphi} \chi = \sum_{\chi \in \text{Irr}(G)} d_{\chi\varphi} \text{res}_C(\chi),$$

where the  $d_{\chi\varphi}$ 's are the integers defined in Equation (2).

(ii) *We have*

$$\mathbb{Z}\text{Irr}(G) \cap \mathbb{Z}\text{Irr}(G)^\mathcal{C} = \mathbb{Z}b^\vee.$$

*Proof.* Let  $\varphi \in b$ . We have  $\Phi_\varphi \in \mathbb{C}\text{Irr}(G)^\mathcal{C}$ . It follows that  $\langle \Phi_\varphi, \chi \rangle_G = \langle \Phi_\varphi, \text{res}_C(\chi) \rangle_G$  for all  $\chi \in \text{Irr}(G)$ . Using Equation (2), we deduce that

$$\begin{aligned} \langle \Phi_\varphi, \chi \rangle_G &= \sum_{\vartheta \in b} d_{\chi\vartheta} \langle \Phi_\varphi, \vartheta \rangle_G \\ &= d_{\chi\varphi}. \end{aligned}$$

This proves (i).

By (i), we clearly have  $\mathbb{Z}b^\vee \subseteq \mathbb{Z}\text{Irr}(G) \cap \mathbb{Z}\text{Irr}(G)^\mathcal{C}$ . Conversely, suppose that  $\psi$  is a generalized character vanishing on the elements  $x$  such that  $x^G \notin \mathcal{C}$ . Then

$$\psi = \sum_{\varphi \in b} \langle \psi, \varphi \rangle_G \Phi_\varphi.$$

Since  $b \subset \mathbb{Z}\text{Irr}(G)^\mathcal{C}$ , for every  $\varphi \in b$ , there are integers  $a_{\varphi\chi}$  (not necessarily unique) such that

$$\varphi = \sum_{\chi \in \text{Irr}(G)} a_{\varphi\chi} \text{res}_C(\chi).$$

Define

$$\psi_\varphi = \sum_{\chi \in \text{Irr}(G)} a_{\varphi\chi} \chi \in \mathbb{Z}\text{Irr}(G).$$

Then  $\text{res}_C(\psi_\varphi) = \varphi$ . Moreover,  $\psi \in \mathbb{Z}\text{Irr}(G)^\mathcal{C}$ . It follows that

$$\langle \psi, \varphi \rangle_G = \langle \psi, \psi_\varphi \rangle_G,$$

which is an integer because  $\psi \in \mathbb{Z}\text{Irr}(G)$  and (ii) holds.  $\square$

Now, we introduce a graph as follows. The vertex set is  $\text{Irr}(G)$  and two vertices  $\chi$  and  $\chi'$  are linked by an edge, if there is  $\varphi \in b$  such that  $d_{\chi\varphi} \neq 0$  and  $d_{\chi'\varphi} \neq 0$ . The connected components of this graph are called the  $\mathcal{C}$ -blocks of  $G$ .

**Remark 2.2.** Note that the  $\mathcal{C}$ -blocks of  $G$  depend on the choice of the  $\mathbb{Z}$ -basis  $b$  of  $\mathbb{Z}\text{Irr}(G)^\mathcal{C}$ .

If  $B$  is a union of  $\mathcal{C}$ -blocks of  $G$ , we write  $\text{Irr}(B)$  for the subset of  $\text{Irr}(G)$  corresponding to the vertices of  $B$ , and  $b_B$  for the set of elements of  $b$  which give edges in  $B$ . We set  $b_B^\vee = \{\Phi_\varphi \mid \varphi \in b_B\}$ . Note that  $b_B^\vee$  is the dual basis of  $b_B$  (when  $b_B$  is viewed as a basis of the  $\mathbb{C}$ -vector space  $\mathbb{C}b_B$ ) with respect to  $\langle \cdot, \cdot \rangle_G$ .

We may (and do) order the elements of  $\text{Irr}(G)$  and  $b$  in such a way that, if the rows and columns of  $D = (d_{\chi\varphi})_{\chi \in \text{Irr}(G), \varphi \in b}$  are ordered correspondingly, then  $D$  is a block-diagonal matrix, and each (diagonal) block  $D_B$  of  $D$  corresponds to a  $\mathcal{C}$ -block  $B$  of  $G$ .

**Corollary 2.3.** *With the notation as above, for every  $\mathcal{C}$ -block  $B$  of  $G$ , we have  $\Phi_\varphi = \sum_{\chi \in \text{Irr}(B)} d_{\chi\varphi} \chi$  for all  $\varphi \in b_B$ , and*

$$\mathbb{Z} \text{Irr}(B) \cap \mathbb{Z} \text{Irr}(G)^{\mathcal{C}} = \mathbb{Z} b_B^\vee.$$

**Corollary 2.4.** *With the above notation, let  $\chi, \psi \in \text{Irr}(G)$  and  $\varphi, \vartheta \in b$  be such that  $\langle \varphi, \vartheta \rangle_G \neq 0$  and  $d_{\chi\varphi} \neq 0 \neq d_{\psi\vartheta}$ . Then  $\chi$  and  $\psi$  lie in the same  $\mathcal{C}$ -block.*

*Proof.* Let  $\varphi, \theta \in b$ . By Proposition 2.1(i), we have

$$\begin{aligned} \delta_{\varphi\theta} = \langle \Phi_\varphi, \theta \rangle_G &= \sum_{\chi \in \text{Irr}(G)} d_{\chi\varphi} \langle \text{res}_{\mathcal{C}}(\chi), \theta \rangle_G \\ &= \sum_{\chi \in \text{Irr}(G)} \left( \sum_{\eta \in b} d_{\chi\varphi} d_{\chi\eta} \right) \langle \eta, \theta \rangle_G \\ &= \sum_{\eta \in b} \left( \sum_{\chi \in \text{Irr}(G)} d_{\chi\varphi} d_{\chi\eta} \right) \langle \eta, \theta \rangle_G. \end{aligned}$$

Now, if we write  $K = (\langle \varphi, \theta \rangle_G)_{\varphi, \theta \in b}$ , then the preceding equation gives  $I = {}^t D D K$ . Thus,  $K$  is invertible and  $K^{-1} = {}^t D D$ . Furthermore,  $D$  is a block-diagonal matrix. Hence,  $K^{-1}$  also has a block-diagonal structure. More precisely, the blocks of  $K^{-1}$  are the  ${}^t D_B D_B$ 's for all  $\mathcal{C}$ -blocks  $B$  of  $G$ . It follows that  $K$  has the same block-diagonal structure as  $K^{-1}$ . In particular, if  $\langle \varphi, \theta \rangle_G \neq 0$ , then  $\varphi$  and  $\theta$  lie in the same  $\mathcal{C}$ -block of  $G$ .

Our assumption that  $\langle \varphi, \vartheta \rangle_G \neq 0$  therefore implies that  $\vartheta$  and  $\varphi$  lie in a common  $\mathcal{C}$ -block  $B$  of  $G$ . By the definition of  $\mathcal{C}$ -blocks, this means that  $\varphi$  and  $\vartheta$  correspond to some subsets  $c_\varphi$  and  $c_\vartheta$  of edges in a connected component  $B$  of the graph previously introduced. Moreover,  $\chi$  (respectively  $\psi$ ) is a vertex of some edge in  $c_\varphi$  (respectively in  $c_\vartheta$ ), because  $d_{\chi\varphi} \neq 0$  (respectively  $d_{\psi\vartheta} \neq 0$ ). Therefore  $\chi, \psi \in B$ .  $\square$

**2.2. MN-Restriction.** We fix a set of  $G$ -conjugacy classes  $\mathcal{C}$  and a union of  $\mathcal{C}$ -blocks  $B$  of  $G$ , and denote by  $C$  the corresponding set of elements as in Equation (1).

**Definition 2.5.** *We say that  $G$  has an MN-structure with respect to  $\mathcal{C}$  and  $B$ , if the following properties hold.*

1. *There is a subset  $S \subseteq G$  containing 1 and stable under  $G$ -conjugation.*
2. *There is a bijection between a subset  $A \subseteq S \times C$  and  $G$  (the image of  $(x_S, x_C) \in A$  will be denoted by  $x_S \cdot x_C$ ), such that for  $(x_S, x_C) \in A$*

$${}^g(x_S \cdot x_C) = ({}^g x_S) \cdot ({}^g x_C) \quad \text{and} \quad x_S \cdot x_C = x_S x_C = x_C x_S.$$

*Moreover, for all  $x_S \in S$  and  $x_C \in C$ , we have  $(x_S, 1) \in A$  and  $(1, x_C) \in A$ .*

3. *For  $x_S \in S$ , there is a subgroup  $G_{x_S} \leq C_G(x_S)$  such that*

$$G_{x_S} \cap C = \{x_C \in C \mid (x_S, x_C) \in A\}.$$

*For  $x_S \in S$ , we denote by  $\mathcal{C}_{x_S}$  the set of  $G_{x_S}$ -conjugacy classes of  $G_{x_S} \cap C$ .*

4. *For  $x_S \in S$ , there is a union of  $\mathcal{C}_{x_S}$ -blocks  $B_{x_S}$  of  $G_{x_S}$  and a homomorphism  $r^{x_S} : \mathbb{C} \text{Irr}(B) \rightarrow \mathbb{C} \text{Irr}(B_{x_S})$  satisfying*

$$r^{x_S}(\chi)(x_C) = \chi(x_S \cdot x_C) \quad \text{for all } \chi \in \mathbb{C} \text{Irr}(B) \text{ and } (x_S, x_C) \in A.$$

*Moreover, we assume that  $G_1 = G$ ,  $B_1 = B$  and  $r^1 = \text{id}$ .*

In the rest of this subsection, we suppose that  $G$  has an MN-structure. For  $x_S \in S$ , we define a homomorphism  $d_{x_S} : \mathbb{C} \text{Irr}(B) \rightarrow \mathbb{C} \text{Irr}(B_{x_S})^{\mathcal{C}_{x_S}}$  by setting

$$(4) \quad d_{x_S}(\chi) = \text{res}_C \circ r^{x_S}(\chi) \quad \text{for } \chi \in \text{Irr}(B).$$

The  $\mathbb{C}$ -basis of  $\mathbb{C} \text{Irr}(B_{x_S})^{\mathcal{C}_{x_S}}$  used to define the union of  $\mathcal{C}_{x_S}$ -blocks  $B_{x_S}$  of  $G_{x_S}$  (see Remark 2.2) is denoted by  $b_{x_S}$ , and we write

$$e_{x_S} : \mathbb{C} b_{x_S}^\vee \rightarrow \mathbb{C} \text{Irr}(B)$$

for the adjoint map of  $d_{x_S}$  with respect to  $\langle \cdot, \cdot \rangle_G$ .

Take any  $y = y_S \cdot y_C \in G$  with  $y_S \in x_S^G$ . For any  $t \in G$  such that  ${}^t y_S = x_S$ , one has  ${}^t y_C \in G_{x_S}$  by Definition 2.5(3), and the set  $X$  of elements  ${}^t y_C$  with  $t \in G$  such that  ${}^t y_S = x_S$  is stable under  $G_{x_S}$ -conjugation (because  $G_{x_S} \subseteq C_G(x_S)$ ). We denote by  $\mathcal{E}_{y_S, y_C}^{x_S}$  a set of representatives of the  $G_{x_S}$ -classes of  $X$ .

**Lemma 2.6.** *With the notation above, for any  $\phi \in \mathbb{C} b_{x_S}^\vee$  and  $(y_S, y_C) \in A$ , we have*

$$e_{x_S}(\phi)(y_S \cdot y_C) = \begin{cases} |C_G(y_S \cdot y_C)| \sum_{x_C \in \mathcal{E}_{y_S, y_C}^{x_S}} \frac{\phi(x_C)}{|C_{G_{x_S}}(x_C)|} & \text{if } y_S^G = x_S^G, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* We denote by  $1_{G, x}$  the indicator function of the conjugacy class of  $x$  in  $G$ . We have

$$\begin{aligned} e_{x_S}(\phi)(y_S \cdot y_C) &= |C_G(y_S \cdot y_C)| \langle e_{x_S}(\phi), 1_{G, y_S \cdot y_C} \rangle_G \\ &= |C_G(y_S \cdot y_C)| \langle \phi, d_{x_S}(1_{G, y_S \cdot y_C}) \rangle_{G_{x_S}}, \end{aligned}$$

because  $d_{x_S}$  and  $e_{x_S}$  are adjoint. Moreover, by Definition 2.5(3), we deduce that

$$d_{x_S}(1_{G, y_S \cdot y_C}) = \sum_{x_C \in \mathcal{E}_{y_S, y_C}^{x_S}} 1_{G_{x_S}, x_C}$$

if  $y_S^G = x_S^G$  and 0 otherwise. This implies in particular that  $e_{x_S}(\phi)(y_S \cdot y_C) = 0$  whenever  $y_S^G \neq x_S^G$ . Now, suppose that  $y_S^G = x_S^G$ . Then

$$\begin{aligned} e_{x_S}(\phi)(y_S \cdot y_C) &= |C_G(y_S \cdot y_C)| \sum_{x_C \in \mathcal{E}_{y_S, y_C}^{x_S}} \langle \phi, 1_{G_{x_S}, x_C} \rangle_{G_{x_S}} \\ &= |C_G(y_S \cdot y_C)| \sum_{x_C \in \mathcal{E}_{y_S, y_C}^{x_S}} \frac{\phi(x_C)}{|C_{G_{x_S}}(x_C)|}, \end{aligned}$$

as required.  $\square$

By Definition 2.5(1),  $S = \bigcup_{\lambda \in \Lambda} \lambda$ , where each  $\lambda \in \Lambda$  is a conjugacy class of  $G$ . For each  $\lambda \in \Lambda$ , we choose a representative  $x_\lambda \in \lambda$ , and we let  $G_\lambda = G_{x_\lambda}$ ,  $B_\lambda = B_{x_\lambda}$ ,  $r^\lambda = r^{x_\lambda}$ ,  $\mathcal{C}_\lambda = \mathcal{C}_{x_S}$  and  $d_\lambda = d_{x_\lambda}$ .

Let  $g = g_S \cdot g_C \in G$ . In the following, we say that  $g$  is of type  $\lambda$  if  $g_S \in \lambda$ . Furthermore, we set  $\mathcal{E}_{g_S, g_C}^\lambda = \mathcal{E}_{g_S, g_C}^{x_\lambda}$ .

Now, we set

$$d_G : \mathbb{C} \text{Irr}(B) \rightarrow \bigoplus_{\lambda \in \Lambda} \mathbb{C} \text{Irr}(B_\lambda)^{\mathcal{C}_\lambda}, \quad \chi \mapsto \sum_{\lambda \in \Lambda} d_\lambda(\chi).$$

For  $\lambda \in \Lambda$ , we define  $l_\lambda : \mathbb{C} \text{Irr}(G_\lambda)^{c_\lambda} \rightarrow \mathbb{C} \text{Irr}(G)$  by setting

$$(5) \quad l_\lambda(\psi)(g) = \begin{cases} \frac{1}{|\mathcal{E}_{g_S, g_C}^\lambda|} \sum_{x_C \in \mathcal{E}_{g_S, g_C}^\lambda} \psi(x_C) & \text{if } g_S^G = \lambda \\ 0 & \text{otherwise} \end{cases},$$

and put

$$l_G : \bigoplus_{\lambda \in \Lambda} \mathbb{C} \text{Irr}(G_\lambda)^{c_\lambda} \rightarrow \mathbb{C} \text{Irr}(G), \quad \sum_{\lambda \in \Lambda} \psi_\lambda \mapsto \sum_{\lambda \in \Lambda} l_\lambda(\psi_\lambda).$$

**Remark 2.7.** Let  $\psi \in \mathbb{C} \text{Irr}(G_\lambda)^{c_\lambda}$ , and suppose that  $\psi$  is constant on  $\mathcal{E}_{x_\lambda, y}^\lambda$  for every  $y \in C \cap G_\lambda$ . Then  $l_\lambda(\psi)(g) = 0$  except when  $g_S \in \lambda$ . In this case, we have  $l_\lambda(\psi)(g) = \psi(x_C)$ , where  $x_C$  is any element of  $\mathcal{E}_{g_S, g_C}^\lambda$ .

**Lemma 2.8.** *The homomorphism  $d_G$  is injective, and the map  $l_G \circ d_G$  is the identity on  $\mathbb{C} \text{Irr}(B)$ .*

*Proof.* Let  $x \in G$ . Then by Definition 2.5(2),  $x$  is  $G$ -conjugate to  $x_\lambda \cdot x_C$  for some  $\lambda \in \Lambda$  and  $x_C \in C$ , and for any  $\chi \in \mathbb{C} \text{Irr}(B)$ , we then have

$$\begin{aligned} \chi(x) = \chi(x_\lambda x_C) &= r^\lambda(\chi)(x_C) && \text{(by Definition 2.5(4))} \\ &= \text{res}_C(r^\lambda(\chi))(x_C) \\ &= d_\lambda(\chi)(x_C) && \text{(by definition of } d_\lambda). \end{aligned}$$

Now, fix  $\chi \in \mathbb{C} \text{Irr}(B)$  such that  $d_G(\chi) = 0$ . Then, for every  $\lambda' \in \Lambda$ , we have  $d_{\lambda'}(\chi) = 0$ . In particular,  $d_\lambda(\chi)(x_C) = 0$ , and it follows that  $\chi(x) = 0$ . Thus  $d_G$  is injective.

Note that

$$l_G \circ d_G = \sum_{\lambda' \in \Lambda} l_{\lambda'} \circ d_{\lambda'}.$$

Hence, for every  $\chi \in \mathbb{C} \text{Irr}(B)$ , we have

$$l_G \circ d_G(\chi)(x) = \sum_{\lambda' \in \Lambda} l_{\lambda'} \circ d_{\lambda'}(\chi)(x).$$

By Equation (5), if  $\lambda' \neq \lambda$ , then  $l_{\lambda'}(d_{\lambda'}(\chi))(x) = 0$ . On the other hand, since  $d_\lambda(\chi)$  is constant on  $\mathcal{E}_{x_\lambda, y}^\lambda$  for any  $y \in C \cap G_\lambda$ , Remark 2.7 implies

$$l_\lambda \circ d_\lambda(\chi)(x) = l_\lambda \circ d_\lambda(\chi)(x_\lambda x_C) = d_\lambda(\chi)(x_C) = \text{res}_C(r^\lambda(\chi))(x_C) = \chi(x),$$

as required.  $\square$

For  $\lambda \in \Lambda$ , we set  $b_\lambda = b_{x_\lambda}$  and  $e_\lambda = e_{x_\lambda}$ . The dual of  $\bigoplus_{\lambda \in \Lambda} \mathbb{C} \text{Irr}(B_\lambda)^{c_{x_S}}$  is  $\bigoplus_{\lambda \in \Lambda} \mathbb{C} b_\lambda^\vee$  and the homomorphism

$$e_G : \bigoplus_{\lambda \in \Lambda} \mathbb{C} b_\lambda^\vee \rightarrow \mathbb{C} \text{Irr}(B), \quad \sum_{\lambda \in \Lambda} \phi_\lambda \mapsto \sum_{\lambda \in \Lambda} e_\lambda(\phi_\lambda)$$

is the adjoint of  $d_G$ .

**Remark 2.9.** Write  $\mathbb{Z}b_\lambda(\mathcal{C})$  for the submodule of  $\mathbb{Z}b_\lambda$  consisting of class functions constant on  $\mathcal{E}_{x_\lambda, y}^\lambda$  for any  $y \in C \cap G_\lambda$ . Let  $K = \text{rk}_{\mathbb{Z}}(\mathbb{Z}b_\lambda(\mathcal{C}))$ . By the invariant factor decomposition theorem, there are a  $\mathbb{Z}$ -basis  $\mathfrak{b}_\lambda = \{b_1, \dots, b_N\}$  of  $\mathbb{Z}b_\lambda$  and positive integers  $m_1, \dots, m_K$  such that  $m_1 | m_2 | \dots | m_K$  and  $\{m_1 b_1, \dots, m_K b_K\}$  is a  $\mathbb{Z}$ -basis of  $\mathbb{Z}b_\lambda(\mathcal{C})$ . Let  $1 \leq i \leq K$  and  $y \in C \cap G_\lambda$ . Then for any  $t \in \mathcal{E}_{x_\lambda, y}^\lambda$ , one has  $m_i b_i(t) = m_i b_i(y)$  because  $m_i b_i \in \mathbb{Z}b_\lambda(\mathcal{C})$ . Since  $m_i \neq 0$ , we deduce that

$b_i(t) = b_i(y)$ . Thus,  $b_i \in \mathbb{Z}b_\lambda(\mathcal{C})$  and  $m_i = 1$ . In the following, we will write  $\mathfrak{b}_\lambda^{\mathcal{C}} = \{b_1, \dots, b_K\}$  for the  $\mathbb{Z}$ -basis of  $\mathbb{Z}b_\lambda(\mathcal{C})$  coming from such a construction.

**2.3. Isometries.** Let  $G$  and  $G'$  be two finite groups. We fix  $\mathcal{C}$  (respectively  $\mathcal{C}'$ ) a set of conjugacy classes of  $G$  (respectively  $G'$ ), and  $B$  (respectively  $B'$ ) a union of  $\mathcal{C}$ -blocks of  $G$  (respectively  $\mathcal{C}'$ -blocks of  $G'$ ). As above, we write

$$C = \bigcup_{c \in \mathcal{C}} c \quad \text{and} \quad C' = \bigcup_{c' \in \mathcal{C}'} c'.$$

We consider the isomorphism

$$\Theta : \begin{cases} \mathbb{C} \text{Irr}(B) \otimes \mathbb{C} \text{Irr}(B') & \longrightarrow & \text{End}(\mathbb{C} \text{Irr}(B), \mathbb{C} \text{Irr}(B')) \\ \sum_{\chi, \chi'} \chi \otimes \chi' & \longmapsto & (\varphi \mapsto \sum_{\chi, \chi'} \langle \varphi, \bar{\chi} \rangle_G \chi') \end{cases}$$

Note that, if we write  $\widehat{f} = \Theta^{-1}(f)$  for any  $f \in \text{End}(\mathbb{C} \text{Irr}(B), \mathbb{C} \text{Irr}(B'))$ , then

$$(6) \quad \widehat{f} = \sum_{i=1}^r e_i^{\vee} \otimes f(e_i),$$

where  $e = (e_1, \dots, e_r)$  is any  $\mathbb{C}$ -basis of  $\mathbb{C} \text{Irr}(B)$  with dual basis  $e^{\vee} = (e_1^{\vee}, \dots, e_r^{\vee})$  with respect to  $\langle \cdot, \cdot \rangle_G$ .

**Theorem 2.10.** *Let  $G$  and  $G'$  be two finite groups. Suppose that*

- (1) *The group  $G$  (respectively  $G'$ ) has an MN-structure with respect to  $\mathcal{C}$  and  $B$  (respectively  $\mathcal{C}'$  and  $B'$ ). We keep the same notation as above, and the object relative to  $G'$  are denoted with a ‘prime’.*
- (2) *Assume there are subsets  $\Lambda_0 \subseteq \Lambda$  and  $\Lambda'_0 \subseteq \Lambda'$  such that :*
  - (a) *For every  $\lambda \in \Lambda$  with  $\lambda \notin \Lambda_0$  (respectively  $\lambda' \in \Lambda'$  with  $\lambda' \notin \Lambda'_0$ ), we have  $r^\lambda = r^{\lambda'} = 0$ .*
  - (b) *There is a bijection  $\sigma : \Lambda_0 \rightarrow \Lambda'_0$  with  $\sigma(\{1\}) = \{1\}$  and for  $\lambda \in \Lambda_0$ , an isometry  $I_\lambda : \mathbb{C} \text{Irr}(B_\lambda) \rightarrow \mathbb{C} \text{Irr}(B'_{\sigma(\lambda)})$  such that*

$$I_\lambda \circ r^\lambda = r'^{\sigma(\lambda)} \circ I_{\{1\}}.$$

- (3) *For  $\lambda \in \Lambda_0$ , we have  $I_\lambda(\mathbb{C}b_\lambda^{\vee}) = \mathbb{C}b'_{\sigma(\lambda)}{}^{\vee}$ . We write  $J_\lambda = I_\lambda|_{\mathbb{C}b_\lambda^{\vee}}$ .*

Then for all  $x \in G$ ,  $x' \in G'$ , we have

$$(7) \quad \widehat{I}_{\{1\}}(x, x') = \sum_{\lambda \in \Lambda_0} \sum_{\phi \in \mathfrak{b}_\lambda^{\mathcal{C}}} \overline{e_\lambda(\Phi_\phi)(x)} l'_{\sigma(\lambda)}(J_\lambda^{*-1}(\phi))(x'),$$

where  $\mathfrak{b}_\lambda$  and  $\mathfrak{b}_\lambda^{\mathcal{C}}$  are as in Remark 2.9, and  $\mathfrak{b}_\lambda^{\vee} = \{\Phi_\phi \mid \phi \in \mathfrak{b}_\lambda\}$  is the dual basis of  $\mathfrak{b}_\lambda$  as in §2.1.

*Proof.* First, we remark that, for  $\lambda \in \Lambda_0$ , the adjoint of the inclusion  $i : \mathbb{C}b_\lambda^{\vee} \rightarrow \mathbb{C} \text{Irr}(B_\lambda)$  is  $i^* = \text{res}_C$ . Moreover, Hypothesis (3) implies that the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{C}b_\lambda^{\vee} & \xrightarrow{J_\lambda} & \mathbb{C}b'_{\sigma(\lambda)}{}^{\vee} \\ \downarrow i & & \downarrow i \\ \mathbb{C} \text{Irr}(B_\lambda) & \xrightarrow{I_\lambda} & \mathbb{C} \text{Irr}(B_{\sigma(\lambda)}) \end{array}$$



Dualizing, we obtain the following commutative diagram:

$$\begin{array}{ccc} \mathbb{C}b_\lambda & \xleftarrow{J_\lambda^*} & \mathbb{C}b'_{\sigma(\lambda)} \\ \text{res}_C \uparrow & & \uparrow \text{res}_{C'} \\ \mathbb{C}\text{Irr}(B_\lambda) & \xleftarrow{I_\lambda^{-1}} & \mathbb{C}\text{Irr}(B_{\sigma(\lambda)}) \end{array}$$

(The bottom arrow is indeed  $I_\lambda^{-1}$  because we identified  $\mathbb{C}\text{Irr}(B_\lambda)$  and  $\mathbb{C}\text{Irr}(B_{\sigma(\lambda)})$  with their duals.) Thus, we have  $\text{res}_C \circ I_\lambda^{-1} = J_\lambda^* \circ \text{res}_{C'}$ , which implies that  $J_\lambda^{*-1} \circ \text{res}_C = \text{res}_{C'} \circ I_\lambda$ , and we obtain

$$(8) \quad \begin{aligned} J_\lambda^{*-1} \circ \text{res}_C \circ r^\lambda &= \text{res}_{C'} \circ I_\lambda \circ r^\lambda \\ J_\lambda^{*-1} \circ d_\lambda &= \text{res}_{C'} \circ r'^{\sigma(\lambda)} \circ I_{\{1\}} \\ J_\lambda^{*-1} \circ d_\lambda &= d'_{\sigma(\lambda)} \circ I_{\{1\}}, \end{aligned}$$

where the second equality comes from Hypothesis (2).

Write  $\mathbb{Z}b_\lambda(\mathcal{C})$  as in Remark 2.9. We have  $d_\lambda(\mathbb{C}\text{Irr}(B)) \subset \mathbb{C}b_\lambda(\mathcal{C})$ . Define  $\mathfrak{b}_\lambda, \mathfrak{b}_\lambda^{\mathcal{C}}$  as in Remark 2.9, and set  $V_\lambda = l_\lambda(\mathbb{C}\mathfrak{b}_\lambda^{\mathcal{C}})$  and  $V_{\lambda'} = l'_{\lambda'}(\mathbb{C}\mathfrak{b}'_{\lambda'}^{\mathcal{C}})$ .

Now, the assumption (2.a) implies that  $d_G = \sum_{\lambda \in \Lambda_0} d_\lambda$  and if we again write  $l_G$  for the restriction of  $l_G$  to  $\bigoplus_{\lambda \in \Lambda_0} \mathbb{C}\text{Irr}(G_\lambda)^{\mathcal{C}}$  to simplify the notation, then Lemma 2.8 gives that  $l_G \circ d_G$  is the identity on  $\mathbb{C}\text{Irr}(B)$  (the same is true for  $l'_{G'} \circ d'_{G'}$ ). In particular,  $l_\lambda$  is surjective. Furthermore, for  $\lambda' \in \Lambda_0$  such that  $\lambda \neq \lambda'$ , one has  $d_{\lambda'} \circ l_\lambda(\phi) = 0$  for all  $\phi \in \mathbb{C}b_\lambda(\mathcal{C})$ . Indeed, for every  $x \in C \cap G_{\lambda'}$ , one has

$$d_{\lambda'} \circ l_\lambda(\phi)(x) = \text{res}_C \circ r^{\lambda'}(l_\lambda(\phi))(x) = r^{\lambda'}(l_\lambda(\phi)(x)) = l_\lambda(\phi)(x_{\lambda'} \cdot x) = 0,$$

because  $\lambda \neq \lambda'$ . It follows that  $\mathbb{C}\text{Irr}(B) = \bigoplus_{\lambda \in \Lambda_0} V_\lambda$  (the same is true for  $\mathbb{C}\text{Irr}(B')$ ). Thus, by Equation (8), the following diagram is commutative:

$$(9) \quad \begin{array}{ccc} \mathbb{C}\text{Irr}(B) & \xrightarrow{I_{\{1\}}} & \mathbb{C}\text{Irr}(B') \\ \downarrow d_G & & \downarrow d'_{G'} \\ \bigoplus_{\lambda \in \Lambda_0} \mathbb{C}b_\lambda & \xrightarrow{\bigoplus J_\lambda^{*-1}} & \bigoplus_{\lambda \in \Lambda_0} \mathbb{C}b'_{\sigma(\lambda)} \\ \downarrow l_G & & \downarrow l'_{G'} \\ \bigoplus_{\lambda \in \Lambda_0} V_\lambda & \xrightarrow{I_{\{1\}}} & \bigoplus_{\lambda \in \Lambda_0} V'_{\sigma(\lambda)} \end{array}$$

Let  $\lambda \in \Lambda_0$  and  $\phi \in \mathbb{C}b_\lambda(\mathcal{C})$ . Then  $d_\lambda \circ l_\lambda(\phi) \in \mathbb{C}b_\lambda(\mathcal{C})$ . Let  $x \in G_\lambda$ . If  $x \notin C$ , then  $\phi(x) = 0 = d_\lambda \circ l_\lambda(\phi)(x)$ . Assume that  $x \in C$ . Then  $x \in G_\lambda \cap C$ . So, by Definition 2.5(3),  $(x_\lambda, x) \in A$ , and by Definition 2.5(4) and Remark 2.7 we have

$$d_\lambda \circ l_\lambda(\phi)(x) = r^\lambda \circ l_\lambda(\phi)(x) = l_\lambda(\phi)(x_\lambda \cdot x).$$

Therefore, Remark 2.7 gives  $d_\lambda \circ l_\lambda(\phi)(x) = \phi(x)$ . So, this proves that for every  $\phi \in \mathbb{C}b_\lambda(\mathcal{C})$ , we have

$$(10) \quad d_\lambda \circ l_\lambda(\phi) = \phi.$$

Consider

$$e = \bigcup_{\lambda \in \Lambda_0} \{l_\lambda(\phi) \mid \phi \in \mathfrak{b}_\lambda^{\mathcal{C}}\}.$$

By Equation (10), for  $\lambda \in \Lambda_0$ , the family  $\{l_\lambda(\phi) \mid \phi \in \mathfrak{b}_\lambda^{\mathcal{C}}\}$  is linearly independent, and since  $l_\lambda$  is surjective, it is a basis of  $V_\lambda$ . Hence, it follows that  $e$  is a basis of  $\mathbb{C} \text{Irr}(B)$ .

Now, we claim that

$$e^\vee = \bigcup_{\lambda \in \Lambda_0} \{e_\lambda(\Phi_\phi) \mid \phi \in \mathfrak{b}_\lambda^{\mathcal{C}}\}.$$

Indeed, if  $\lambda, \mu \in \Lambda_0$  with  $\lambda \neq \mu$ , then for any  $\vartheta \in \mathfrak{b}_\lambda^{\mathcal{C}}$  and  $\phi \in \mathfrak{b}_\mu^{\mathcal{C}}$ , we have

$$\langle e_\lambda(\Phi_\vartheta), l_\mu(\phi) \rangle_G = \frac{1}{|G|} \sum_{g \in G} e_\lambda(\Phi_\vartheta)(g) \overline{l_\mu(\phi)(g)} = 0,$$

by Equation (5) and Lemma 2.6. Furthermore, if  $\phi, \varphi \in \mathfrak{b}_\lambda^{\mathcal{C}}$ , then Equation (10) gives

$$\langle e_\lambda(\Phi_\varphi), l_\lambda(\phi) \rangle_G = \langle \Phi_\varphi, d_\lambda \circ l_\lambda(\phi) \rangle_{G_\lambda} = \langle \Phi_\varphi, \phi \rangle_{G_\lambda} = \delta_{\varphi\phi},$$

and the result follows. Thus, writing  $\widehat{I}_{\{1\}}$  with respect to the basis  $e$ , we obtain

$$\widehat{I}_{\{1\}} = \sum_{\lambda \in \Lambda_0} \sum_{\phi \in \mathfrak{b}_\lambda} \overline{e_\lambda(\Phi_\phi)} \otimes l'_{\sigma(\lambda)}(J_\lambda^{*-1}(\phi)),$$

as required.  $\square$

**Remark 2.11.** Note that the assumption (2) of the theorem implies that the assumption (3) of the theorem holds for  $\lambda = \{1\}$ . Indeed, for  $\phi \in \mathbb{C} \text{Irr}(B)$ , we have  $\phi \in \mathbb{C} \mathfrak{b}_{\{1\}}^\vee$  if and only if  $\text{res}_{\overline{C}}(\phi) = 0$ , where  $\overline{C} = G \setminus C$ . However,  $x \in G$  lies in  $\overline{C}$  if and only if its type  $\lambda$  is non-trivial. Thus, Definition 2.5(4) implies that  $\phi \in \mathbb{C} \mathfrak{b}_{\{1\}}^\vee$  if and only if  $r^\lambda(\phi) = 0$  for all  $\lambda \neq \{1\}$ . Let  $\phi \in \mathbb{C} \mathfrak{b}_{\{1\}}^\vee$ . Then for any  $\{1\} \neq \lambda \in \Lambda_0$ ,

$$r'^{\sigma(\lambda)}(I_{\{1\}}(\phi)) = I_\lambda(r^\lambda(\phi)) = 0.$$

Since  $\sigma$  is a bijection with  $\sigma(\{1\}) = \{1\}$ , we deduce that  $I_{\{1\}}(\phi) \in \mathbb{C} \mathfrak{b}_{\sigma(\lambda)}^\vee$ . To obtain the reverse inclusion, we apply this argument to  $I_{\{1\}}^{-1}$ .

In particular, if for any  $\lambda \in \Lambda_0$ , the group  $G_\lambda$  has an MN-structure with respect to  $C \cap G_\lambda$  and  $B_\lambda$ , then the assumption (3) of Theorem 2.10 is automatically satisfied.

**Remark 2.12.** Suppose that  $(I_\lambda : \mathbb{C} \text{Irr}(B_\lambda) \rightarrow \mathbb{C} \text{Irr}(B'_{\sigma(\lambda)}))_{\lambda \in \Lambda_0}$  are isometries such that properties (1), (2) and (3) of Theorem 2.10 hold. Then  $I_\lambda^{-1} : \mathbb{C} \text{Irr}(B'_{\sigma(\lambda)}) \rightarrow \mathbb{C} \text{Irr}(B_\lambda)$  also satisfies the hypotheses of the theorem (for  $\sigma^{-1} : \Lambda'_0 \rightarrow \Lambda_0$ ). Moreover, writing  $\widehat{I}$  with respect to the self-dual  $\mathbb{C}$ -basis  $\text{Irr}(B)$  of  $\mathbb{C} \text{Irr}(B)$ , we have

$$(11) \quad \widehat{I} = \sum_{\chi \in \text{Irr}(B)} \overline{\chi} \otimes I(\chi).$$

It follows that

$$\widehat{I} = \sum_{\chi' \in \text{Irr}(B')} \overline{I^{-1}(\chi')} \otimes \chi' = \text{conj} \left( \sum_{\chi' \in \text{Irr}(B')} I^{-1}(\chi') \otimes \overline{\chi'} \right) = \text{conj} \left( \widehat{I}^{-1} \circ \tau \right),$$

where  $\tau : G \times G' \rightarrow G' \times G$ ,  $(x, x') \mapsto (x', x)$  and  $\text{conj}$  denotes the complex conjugation.

**2.4. Generalized perfect isometries.** An isometry  $I : \mathbb{C} \text{Irr}(B) \rightarrow \mathbb{C} \text{Irr}(B')$  with respect to the scalar products  $\langle \cdot, \cdot \rangle_G$  and  $\langle \cdot, \cdot \rangle_{G'}$  is said to be a generalized perfect isometry if  $I(\mathbb{Z} \text{Irr}(B)) = \mathbb{Z}(\text{Irr}(B'))$  and

$$(12) \quad I \circ \text{res}_C = \text{res}_{C'} \circ I.$$

**Remark 2.13.** Following Külshammer, Olsson and Robinson (see [15]), we say that an isometry  $I : \mathbb{C} \text{Irr}(B) \rightarrow \mathbb{C} \text{Irr}(B')$  is a KOR-isometry if  $I(\mathbb{Z} \text{Irr}(B)) = \mathbb{Z} \text{Irr}(B')$  and for all  $\chi, \psi \in \text{Irr}(B)$ , one has

$$\langle \text{res}_C(\chi), \text{res}_C(\psi) \rangle_G = \langle \text{res}_{C'}(I(\chi)), \text{res}_{C'}(I(\psi)) \rangle_{G'}.$$

Note that the argument in the proof of [2, Proposition 2.2] shows that the KOR-isometries are precisely the isometries that satisfy Equation (12). For the convenience of the reader, we now prove this fact. Before this, we recall that the notion of blocks in [15] is not the same as ours. The KOR-blocks are the equivalence classes for the equivalence relation on  $\text{Irr}(G)$  obtained by extending by transitivity the relation defined by  $\langle \text{res}_C(\chi), \text{res}_C(\psi) \rangle_G \neq 0$ . First, we will show that  $\text{Irr}(B)$  is a union of KOR-blocks. Since the KOR-blocks are a partition of  $\text{Irr}(G)$ , it is clear that  $\text{Irr}(B)$  is contained in a union of KOR-blocks. It is sufficient to show that if  $\chi \in \text{Irr}(B)$  and  $\psi \in \text{Irr}(G)$  are such that  $\langle \text{res}_C(\chi), \text{res}_C(\psi) \rangle_G \neq 0$ , then  $\psi \in \text{Irr}(B)$ . Let  $\chi \in \text{Irr}(B)$  and  $\psi \in \text{Irr}(G)$  be such that  $\langle \text{res}_C(\chi), \text{res}_C(\psi) \rangle_G \neq 0$ , that is

$$\sum_{\varphi, \vartheta \in b} d_{\chi\varphi} d_{\psi\vartheta} \langle \varphi, \vartheta \rangle_G \neq 0.$$

In particular, there exists some  $\varphi, \vartheta \in b$  such that  $d_{\chi\varphi} d_{\psi\vartheta} \langle \varphi, \vartheta \rangle_G \neq 0$ . Hence,  $d_{\chi\varphi} \neq 0 \neq d_{\psi\vartheta}$  and  $\langle \varphi, \vartheta \rangle_G \neq 0$ . Thanks to Corollary 2.4, we conclude that  $\psi$  lies in the  $\mathcal{C}$ -block of  $\chi$ .

Now suppose  $I : \mathbb{C} \text{Irr}(B) \rightarrow \mathbb{C} \text{Irr}(B')$  is a generalized perfect isometry. Let  $\chi, \psi \in \text{Irr}(B)$ . Then

$$\langle \text{res}_{C'}(I(\chi)), \text{res}_{C'}(I(\psi)) \rangle_{G'} = \langle I(\text{res}_C(\chi)), I(\text{res}_C(\psi)) \rangle_G = \langle \text{res}_C(\chi), \text{res}_C(\psi) \rangle_G,$$

because  $I$  is an isometry.

Conversely, assume that  $I$  is a KOR-isometry. Let  $\chi \in \text{Irr}(B)$ . We have

$$\begin{aligned} I(\text{res}_C(\chi)) &= I \left( \sum_{\psi \in \text{Irr}(G)} \langle \text{res}_C(\chi), \psi \rangle_G \psi \right) \\ &= \sum_{\psi \in \text{Irr}(B)} \langle \text{res}_C(\chi), \text{res}_C(\psi) \rangle_G I(\psi) \\ &= \sum_{\psi \in \text{Irr}(B)} \langle \text{res}_{C'}(I(\chi)), \text{res}_{C'}(I(\psi)) \rangle_{G'} I(\psi) \\ &= \sum_{\psi \in \text{Irr}(B)} \langle \text{res}_{C'}(I(\chi)), I(\psi) \rangle_{G'} I(\psi) \\ &= \text{res}_{C'}(I(\chi)), \end{aligned}$$

proving the claim.

**Proposition 2.14.** *Suppose that  $\{B_i \mid 1 \leq i \leq r\}$  is the set of KOR-blocks of  $G$  with respect to a set of classes  $\mathcal{C}$ . Then there is a  $\mathbb{Z}$ -basis  $b$  of  $\mathbb{Z} \text{Irr}(G)^{\mathcal{C}}$  such that the  $B_i$ 's are the  $\mathcal{C}$ -blocks of  $G$  with respect to  $b$ .*

*Proof.* By definition of the KOR-blocks, the sets  $\text{Irr}(B_i)^{\mathcal{C}}$  and  $\text{Irr}(B_j)^{\mathcal{C}}$  for  $i \neq j$  are orthogonal with respect to  $\langle \cdot, \cdot \rangle_G$ , implying that

$$\mathbb{Z} \text{Irr}(G)^{\mathcal{C}} = \bigoplus_{i=1}^r \mathbb{Z} \text{Irr}(B_i)^{\mathcal{C}}.$$

Choose any  $\mathbb{Z}$ -basis  $b_i$  of  $\mathbb{Z} \text{Irr}(B_i)^{\mathcal{C}}$  and write  $b_i^{\vee}$  for the dual basis of  $b_i$  in the  $\mathbb{C}$ -space  $\mathbb{C} \text{Irr}(B_i)^{\mathcal{C}}$  with respect to  $\langle \cdot, \cdot \rangle_G$ . Define  $b = b_1 \cup \dots \cup b_r$ . Then  $b$  is a  $\mathbb{Z}$ -basis of  $\mathbb{Z} \text{Irr}(G)^{\mathcal{C}}$ . Moreover, since  $b_i^{\vee} \subseteq \mathbb{C} \text{Irr}(B_i)^{\mathcal{C}}$ , and since the KOR-blocks are orthogonal, we deduce that  $b^{\vee} = b_1^{\vee} \cup \dots \cup b_r^{\vee}$  is the dual basis of  $b$ . Now, for  $\varphi \in b_i$ , we have

$$\Phi_{\varphi} = \text{res}_C(\Phi_{\varphi}) = \sum_{j=1}^r \sum_{\chi \in \text{Irr}(B_j)} d_{\chi\varphi} \text{res}_C(\chi) = \sum_{\chi \in \text{Irr}(B_i)} d_{\chi\varphi} \text{res}_C(\chi),$$

because  $\Phi_{\varphi} \in \mathbb{C} \text{Irr}(B_i)^{\mathcal{C}}$ . Hence, for  $\chi' \notin \text{Irr}(B_i)$ , we have

$$d_{\chi'\varphi} = \langle \Phi_{\varphi}, \chi' \rangle_G = \sum_{\chi \in \text{Irr}(B_i)} d_{\chi\varphi} \langle \text{res}_C(\chi), \text{res}_C(\chi') \rangle_G = 0.$$

This proves that  $B_i$  is a union of  $\mathcal{C}$ -blocks. Furthermore, we have seen in Remark 2.13 that conversely, the  $\mathcal{C}$ -blocks are unions of KOR-blocks. The result follows.  $\square$

**Proposition 2.15.** *Let  $I : \mathbb{C} \text{Irr}(B) \rightarrow \mathbb{C} \text{Irr}(B')$  be an isometry, and assume that  $I(\mathbb{Z} \text{Irr}(B)) = \mathbb{Z} \text{Irr}(B')$ . The following assertions are equivalent*

- (i)  *$I$  is a generalized perfect isometry.*
- (ii) *If  $\widehat{I}(x, y) \neq 0$ , then either  $(x, y) \in C \times C'$ , or  $(x, y) \in \overline{C} \times \overline{C}'$ , where  $\overline{C} = G \setminus C$  and  $\overline{C}' = G \setminus C'$ .*

*Proof.* Suppose that  $I$  is a generalized perfect isometry. Note that  $\mathbb{C} \text{Irr}(B)^{C^{\perp}} = \mathbb{C} \text{Irr}(B)^{\overline{C}}$  and

$$(13) \quad \mathbb{C} \text{Irr}(B) = \mathbb{C} \text{Irr}(B)^C \oplus \mathbb{C} \text{Irr}(B)^{\overline{C}}.$$

Moreover, for any  $\phi \in \mathbb{C} \text{Irr}(B')$ , there is  $\chi \in \mathbb{C} \text{Irr}(B)$  such that  $I(\chi) = \phi$  (because  $I$  is an isometry). Thanks to Equation (12), we have  $\text{res}_{C'}(\phi) = I(\text{res}_C(\chi))$ . Hence, the restriction  $I : \mathbb{C} \text{Irr}(B)^C \rightarrow \mathbb{C} \text{Irr}(B')^{C'}$  is surjective, and yet bijective (because  $I$  is injective). Since  $I$  is an isometry, we have

$$I((\mathbb{C} \text{Irr}(B)^C)^{\perp}) = I(\mathbb{C} \text{Irr}(B)^C)^{\perp} = (\mathbb{C} \text{Irr}(B')^{C'})^{\perp}.$$

It follows that

$$(14) \quad I(\mathbb{C} \text{Irr}(B)^{\overline{C}}) = \mathbb{C} \text{Irr}(B')^{\overline{C}'}$$

Now, we choose a  $\mathbb{C}$ -basis  $b$  of  $\mathbb{C} \text{Irr}(B)^C$  with dual basis  $b^{\vee}$  and a  $\mathbb{C}$ -basis  $\overline{b}$  of  $\mathbb{C} \text{Irr}(B)^{\overline{C}}$  with dual basis  $\overline{b}^{\vee}$ . Therefore, thanks to Equation (13),  $b \cup \overline{b}$  is a  $\mathbb{C}$ -basis of  $\mathbb{C} \text{Irr}(B)$  with dual basis  $b^{\vee} \cup \overline{b}^{\vee}$ . Writing  $\widehat{I}$  with respect to this basis, we obtain

$$(15) \quad \widehat{I} = \sum_{\alpha \in b} \overline{\alpha}^{\vee} \otimes I(\alpha) + \sum_{\beta \in \overline{b}} \overline{\beta}^{\vee} \otimes I(\beta).$$

Now, let  $(x, y) \in C \times \overline{C}'$ . Then Equation (15) gives  $\widehat{I}(x, y) = \sum_{\alpha \in b} \overline{\alpha}^{\vee}(x) I(\alpha)(y)$ . But  $I(\alpha) \in \mathbb{C} \text{Irr}(B')^{C'}$ , implying that  $I(\alpha)(y) = 0$ . Hence,  $\widehat{I}(x, y) = 0$ .

For  $(x, y) \in \overline{C} \times C'$ , we similarly conclude that  $\widehat{I}(x, y) = 0$  using Equations (15) and (14). This proves that (i) implies (ii).

Conversely, assume that (ii) holds. For  $y \in G'$ , we write  $\widehat{I}_y : G \rightarrow \mathbb{C}$ ,  $x \mapsto \widehat{I}(x, y)$ . This is a class function on  $G$ . We now write  $\widehat{I}$  with respect to the  $\mathbb{C}$ -basis  $\text{Irr}(B)$ . Thus, Equation (11) implies that for  $\chi \in \text{Irr}(G)$  and  $y \in G'$ , we have

$$\begin{aligned} I(\chi)(y) &= \sum_{\theta \in \text{Irr}(B)} I(\theta)(y) \langle \bar{\theta}, \bar{\chi} \rangle_G \\ &= \langle \widehat{I}_y, \bar{\chi} \rangle_G \\ &= \frac{1}{|G|} \sum_{x \in G} \widehat{I}(x, y) \chi(x). \end{aligned}$$

In particular, for any  $\psi \in \mathbb{C} \text{Irr}(G)$  and  $y \in G'$ , we have

$$(16) \quad I(\psi)(y) = \frac{1}{|G|} \sum_{x \in G} \widehat{I}(x, y) \psi(x).$$

Let  $\chi \in \mathbb{C} \text{Irr}(B)$  and  $y \in G'$ . Applying Equation (16) to  $\text{res}_C(\chi)$ , we obtain

$$I(\text{res}_C(\chi))(y) = \frac{1}{|G|} \sum_{x \in G} \widehat{I}(x, y) \text{res}_C(\chi)(x) = \frac{1}{|G|} \sum_{x \in C} \widehat{I}(x, y) \chi(x).$$

Suppose that  $y \in \overline{C}'$ . Then  $\widehat{I}(x, y) \neq 0$  only if  $x \in \overline{C}$  and the second equality gives  $I(\text{res}_C(\chi))(y) = 0$ . Otherwise, if  $y \in C'$ , then  $\widehat{I}(x, y) = 0$  for  $x \notin C$ . In particular,  $\frac{1}{|G|} \sum_{x \in C} \widehat{I}(x, y) \chi(x)$  is equal to  $I(\chi)(y) = \text{res}_{C'}(I(\chi))(y)$ . This proves that  $I$  satisfies Equation (12), whence is a generalized perfect isometry.  $\square$

**Remark 2.16.** Note that Equation (13) applied to  $B'$  and Equation (14) imply that

$$\text{res}_{\overline{C}'} \circ I = I \circ \text{res}_{\overline{C}}.$$

**Corollary 2.17.** *Let  $G$  and  $G'$  be two finite groups. We assume that Hypotheses (1), (2) and (3) of Theorem 2.10 are satisfied, and we keep the same notation. If  $I_{\{1\}}(\mathbb{Z} \text{Irr}(B)) = \mathbb{Z} \text{Irr}(B')$ , then  $I_{\{1\}}$  is a generalized perfect isometry.*

*Proof.* Let  $(x, x') \in G \times G'$ . Write  $\mu$  and  $\mu'$  for the type of  $x$  and  $x'$ . Suppose that  $(x, x') \notin C \times C'$  and  $(x, x') \notin \overline{C} \times \overline{C}'$ . Then either  $\mu = \{1\}$  and  $\mu' \neq \{1\}$ , or  $\mu \neq \{1\}$  and  $\mu' = \{1\}$ . Since  $\sigma(\{1\}) = \{1\}$ , we deduce that  $\mu' \neq \sigma(\mu)$ . Thanks to Lemma 2.6 and Equation (5), we have for every  $\lambda \in \Lambda_0$  and  $\phi \in \mathfrak{b}_\lambda^C$ , either  $e_\lambda(\Phi_\phi)(x) = 0$ , or  $l'_{\sigma(\lambda)}(J_\lambda^{*-1}(\phi))(x') = 0$ . In particular, Equation (7) gives  $\widehat{I}_{\{1\}}(x, x') = 0$ , and the result follows from Proposition 2.15.  $\square$

**2.5. Broué's isometries.** In this subsection, we fix a prime number  $p$  and assume that  $C$  and  $C'$  are the sets of  $p$ -regular elements (that is, elements whose order is prime to  $p$ ) of  $G$  and  $G'$ , respectively. Let  $B$  and  $B'$  be a union of  $p$ -blocks of  $G$  and  $G'$ . Denote by  $(K, \mathcal{R}, k)$  a splitting  $p$ -modular system for  $G$  and  $G'$ . Let  $I : \mathbb{C} \text{Irr}(B) \rightarrow \mathbb{C} \text{Irr}(B')$  be an isometry such that  $I(\mathbb{Z} \text{Irr}(B)) = \mathbb{Z} \text{Irr}(B')$  and  $\widehat{I}$  defined in Equation (6) is perfect, that is

- (i) For every  $(x, x') \in G \times G'$ ,  $\widehat{I}(x, x')$  lies in  $|C_G(x)|\mathcal{R} \cap |C_{G'}(x')|\mathcal{R}$ .
- (ii)  $\widehat{I}$  satisfies property (ii) of Proposition 2.15.

We call such an isometry a Broué isometry.

**Remark 2.18.** In fact, the perfect character  $\mu : G \times G' \rightarrow \mathbb{C}$  defined by Broué in [1] is not exactly  $\widehat{I}$ , but  $\mu(x, x') = \widehat{I}(x^{-1}, x')$ . However, since the sets of  $p$ -regular and  $p$ -singular elements are stable under  $g \mapsto g^{-1}$ , it follows that  $\mu$  is perfect if and only if  $\widehat{I}$  is perfect.

**Remark 2.19.** Since the set of irreducible Brauer characters  $\text{IBr}_p(G)$  is a  $\mathbb{Z}$ -basis of  $\mathbb{Z}\text{Irr}(G)^C$  which satisfies the conclusion of Proposition 2.14, Remark 2.13 and Proposition 2.15 imply that a Broué isometry is a perfect generalized isometry (in our sense and in the sense of Külshammer, Olsson and Robinson).

**Theorem 2.20.** *Assume that  $G$  and  $G'$  are two finite groups and that  $C$  and  $C'$  are the sets of  $p$ -regular elements of  $G$  and  $G'$ , respectively. Suppose that the following three conditions are satisfied:*

- (i) *The hypotheses of Theorem 2.10 hold.*
- (ii) *For any  $\lambda \in \Lambda_0$ , we have  $I_\lambda(\mathbb{Z}\text{Irr}(B_\lambda)) = \mathbb{Z}\text{Irr}(B'_{\sigma(\lambda)})$ .*
- (iii) *For every  $g = g_S \cdot g_C \in G$  and  $g' = g'_{S'} \cdot g'_{C'} \in G'$ ,  $p$  does not divide  $|\mathcal{E}_{g_S, g_C}^\lambda|$  for any  $\lambda \in \Lambda$ , and  $p$  does not divide  $|\mathcal{E}_{g'_{S'}, g'_{C'}}^{\lambda'}|$  for any  $\lambda' \in \Lambda'$ .*

*Then  $I_{\{1\}}$  is a Broué isometry.*

*Proof.* By Remark 2.19 and Corollary 2.17,  $I_{\{1\}}$  satisfies Property (ii). We thus only prove Property (i). For  $\lambda \in \Lambda_0$ , we take  $b_\lambda = \text{IBr}_p(B_\lambda)$ . In particular,  $\mathbb{Z}b_\lambda^\vee$  is the set of projective characters of  $B_\lambda$ . By Assumption (ii), and Hypothesis (iii) of Theorem 2.10, since  $I_\lambda$  is injective, one has

$$I_\lambda(\mathbb{Z}\text{Irr}(B_\lambda) \cap \mathbb{C}b_\lambda^\vee) = I_\lambda(\mathbb{Z}\text{Irr}(B_\lambda)) \cap I_\lambda(\mathbb{C}b_\lambda^\vee) = \mathbb{Z}\text{Irr}(B'_{\sigma(\lambda)}) \cap \mathbb{C}b'_{\sigma(\lambda)}.$$

Hence, Corollary 2.3 gives  $J_\lambda(\mathbb{Z}b_\lambda^\vee) = \mathbb{Z}b'_{\sigma(\lambda)}{}^\vee$ , and it follows that  $J_\lambda^{*-1}(\phi) \in \mathbb{Z}\text{IBr}_p(B'_{\sigma(\lambda)})$  for all  $\phi \in \mathbb{Z}b_\lambda$ . Now, let  $\mathfrak{b}_\lambda$  and  $\mathfrak{b}_\lambda^C$  be as in Remark 2.9. Let  $\phi \in \mathfrak{b}_\lambda^C$ . Let  $g \in G$  and  $g' \in G'$ . Write  $g = g_S \cdot g_C$  and  $g' = g'_{S'} \cdot g'_{C'}$ , and assume that  $g$  is of type  $\mu$ . Then by Lemma 2.6, one has  $e_\lambda(\Phi_\phi)(g) = 0$  for  $\lambda \neq \mu$  and  $e_\mu(\Phi_\phi)(g) = |\text{C}_G(g)| \sum_{x_C \in \mathcal{E}_{g_S, g_C}^\mu} \frac{\Phi_\phi(x_C)}{|\text{C}_{G_\mu}(x_C)|}$ . Furthermore, thanks to Equation (5) and the fact that  $p$  does not divide  $|\mathcal{E}_{g'_{S'}, g'_{C'}}^{\sigma(\mu)}|$ , we deduce that  $l'_{\sigma(\mu)}(J_\mu^{*-1}(\phi))(g') \in \mathcal{R}$ . Now, by Equation (7) and Theorem 2.10, we obtain

$$\frac{\widehat{I}_{\{1\}}(g, g')}{|\text{C}_G(g)|} = \sum_{\phi \in \mathfrak{b}_\lambda^C} \sum_{x_C \in \mathcal{E}_{g_S, g_C}^\mu} \frac{\overline{\Phi_\phi(x_C)}}{|\text{C}_{G_\mu}(x_C)|_p} \cdot \frac{l'_{\sigma(\mu)}(J_\mu^{*-1}(\phi))(g')}{|\text{C}_{G_\mu}(x_C)|_{p'}} \in \mathcal{R},$$

because  $1/|\text{C}_{G_\mu}(x_C)|_{p'} \in \mathcal{R}$ , and  $\Phi_\phi(x_C)/|\text{C}_{G_\mu}(x_C)|_p \in \mathcal{R}$  by [20, 2.21]. Similarly, using Remark 2.12, we deduce that  $\widehat{I}_{\{1\}}(g, g')/|\text{C}_{G'}(g')| \in \mathcal{R}$ , as required.  $\square$

**Remark 2.21.** The proof of Theorem 2.20 shows that the condition (iii) can be replaced by  $I_\lambda(\mathbb{C}\text{Irr}(B_\lambda)(C)) = \mathbb{C}\text{Irr}(B_{\sigma(\lambda)})(C')$  for any  $\lambda \in \Lambda$ , where similarly to Remark 2.9,  $\mathbb{C}\text{Irr}(B_\lambda)(C)$  denotes the set of class functions of  $\mathbb{C}\text{Irr}(B_\lambda)$  constant on  $\mathcal{E}_{x_\lambda, y}^\lambda$  for any  $y \in C \cap G_\lambda$ . Indeed, with this assumption, we have  $J_\lambda^{*-1}(\mathbb{Z}b_\lambda(C)) \subseteq \mathbb{Z}b_{\sigma(\lambda)}(C')$ , and Remark 2.7 gives that  $l'_{\sigma(\lambda)}(J_\lambda^{*-1}(\phi))(g') \in \mathcal{R}$  for any  $\phi \in \mathfrak{b}_\lambda^C$  and  $g' \in G'$ .

## 3. ALTERNATING GROUPS

Let  $n$  be a positive integer and  $p$  be a prime. We denote by  $\mathcal{P}_n$  (or  $\mathcal{P}$ ) the set of partitions of  $n$ , by  $\mathcal{O}_n$  (or  $\mathcal{O}$ ) the set of partitions of  $n$  whose parts are odd, and by  $\mathcal{D}_n$  (or  $\mathcal{D}$ ) the set of partitions of  $n$  whose parts are distinct. We also write  $\mathcal{OD}_n$  (or  $\mathcal{OD}$ ) for  $\mathcal{O}_n \cap \mathcal{D}_n$  (respectively  $\mathcal{O} \cap \mathcal{D}$ ). Moreover, for  $\lambda = (\lambda_1, \dots, \lambda_r) \in \mathcal{P}$ , we write  $|\lambda| = \sum \lambda_i$  and  $\ell(\lambda) = r$ .

**3.1. Notation.** For any  $\lambda \in \mathcal{P}_n$ , we write  $\chi_\lambda$  for the corresponding irreducible character of  $\mathfrak{S}_n$ , and  $\lambda^*$  for the conjugate partition of  $\lambda$ . It is well-known that  $\chi_{\lambda^*} = \chi_\lambda \otimes \varepsilon$ , where  $\varepsilon$  denotes the sign character of  $\mathfrak{S}_n$ . The character  $\chi_\lambda$  is called self-conjugate when  $\lambda = \lambda^*$ . We denote by  $\mathcal{SC}_n$  (or  $\mathcal{SC}$ ) the set of self-conjugate partitions of  $n$ . For any  $\lambda \in \mathcal{SC}_n$ , write  $\bar{\lambda} \in \mathcal{OD}_n$  for the partition whose parts are the diagonal hook lengths of  $\lambda$  (see [22, p. 4] for the definition of a hook and its hook length. Recall that with the notation of [22], a diagonal hook is an  $(i, i)$ -hook for some  $i$ ), and define the map

$$(17) \quad a : \mathcal{SC}_n \longrightarrow \mathcal{OD}_n, \quad \lambda \mapsto \bar{\lambda}.$$

We remark that  $a$  is bijective, and that  $a^{-1}(\lambda)$  is the self-conjugate partition whose diagonal hooks have lengths the parts of  $\lambda$ .

Now, recall that  $\text{Res}_{\mathcal{A}_n}^{\mathfrak{S}_n}(\chi_\lambda)$  is irreducible if and only if  $\lambda$  is a non self-conjugate partition (i.e.  $\lambda \neq \lambda^*$ ). In this case,  $\chi_\lambda$  and  $\chi_{\lambda^*}$  restrict to the same irreducible character, which we denote by  $\rho_\lambda$ . Otherwise, when  $\lambda = \lambda^*$ , the restriction of  $\chi_\lambda$  to  $\mathcal{A}_n$  is the sum of two irreducible characters  $\rho_\lambda^-$  and  $\rho_\lambda^+$ . Moreover, the conjugacy class of  $\mathfrak{S}_n$  labeled by  $a(\lambda)$  splits into two classes  $a(\lambda)^\pm$  of  $\mathcal{A}_n$ , and following [12, Theorem 2.5.13], the notation can be chosen such that  $\rho_\lambda^\pm(a(\lambda)^+) = x_\lambda \pm y_\lambda$  and  $\rho_\lambda^\pm(a(\lambda)^-) = x_\lambda \mp y_\lambda$  with

$$(18) \quad x_\lambda = \frac{1}{2}(-1)^{\frac{n-k}{2}} \quad \text{and} \quad y_\lambda = \frac{1}{2}\sqrt{(-1)^{\frac{n-k}{2}} h_1 \cdots h_k},$$

where  $a(\lambda) = (h_1 > h_2 > \cdots > h_k)$ . Note that  $x_\lambda = \chi_\lambda(a(\lambda))/2$ , and if  $x \in \mathcal{A}_n$  does not belong to the class of  $\mathfrak{S}_n$  parametrized by  $a(\lambda)$ , then  $\rho_\lambda^+(x) = \rho_\lambda^-(x) = \chi_\lambda(x)/2$ .

Let  $q$  be a positive integer. To any  $\lambda \in \mathcal{P}_n$ , we associate its  $q$ -core  $\lambda_{(q)}$  and its  $q$ -quotient  $\lambda^{(q)} = (\lambda^1, \dots, \lambda^q)$ ; see for example [22, p. 17]. Recall that the map

$$(19) \quad \lambda \mapsto (\lambda_{(q)}, \lambda^{(q)})$$

is bijective. Define

$$(20) \quad \lambda^{(q)*} = ((\lambda^q)^*, \dots, (\lambda^1)^*).$$

Then by [22, Proposition 3.5], the  $q$ -core and  $q$ -quotient of  $\lambda^*$  are  $\lambda_{(q)}^*$  and  $\lambda^{(q)*}$  respectively. In particular,

$$(21) \quad \lambda = \lambda^* \iff \lambda_{(q)}^* = \lambda_{(q)} \quad \text{and} \quad \lambda^{(q)*} = \lambda^{(q)}.$$

**3.2.  $p$ -blocks of  $\mathcal{A}_n$ .** The ‘‘Nakayama Conjecture’’ asserts that two irreducible characters lie in the same  $p$ -block of  $\mathfrak{S}_n$  if and only if the partitions labeling them have the same  $p$ -core; see [12, Theorem 6.1.21]. Hence, the  $p$ -blocks of  $\mathfrak{S}_n$  are labeled by the  $p$ -cores of partitions of  $n$ . Such  $p$ -cores are called  $p$ -cores of  $n$  (or of  $\mathfrak{S}_n$ ). For a  $p$ -core  $\gamma$  of  $n$ , we denote by  $B_\gamma$  the corresponding  $p$ -block of  $\mathfrak{S}_n$ . Moreover, we define the  $p$ -weight of  $\gamma$  (or of  $B_\gamma$ ) by setting  $w = (n - |\gamma|)/p$ .

Let  $\gamma$  be a  $p$ -core of  $n$ . Then  $\gamma^*$  is also a  $p$ -core of  $n$ , and  $\text{Irr}(B_{\gamma^*}) = \{\chi_{\lambda^*} \in \text{Irr}(\mathfrak{S}_n) \mid \lambda_{(p)} = \gamma^*\} = \text{Irr}(B_{\gamma})^*$ .

If  $\gamma \neq \gamma^*$ , then  $\text{Irr}(B_{\gamma}) \cap \text{Irr}(B_{\gamma^*}) = \emptyset$  and  $\text{Irr}(B_{\gamma})$  contains no self-conjugate character. In particular, the  $p$ -blocks  $B_{\gamma}$  and  $B_{\gamma^*}$  cover a unique  $p$ -block  $b_{\gamma, \gamma^*}$  of  $\mathcal{A}_n$ , which is such that  $\text{Irr}(b_{\gamma, \gamma^*}) = \{\rho_{\lambda} \in \text{Irr}(\mathcal{A}_n) \mid \lambda_{(p)} = \gamma\} = \{\rho_{\lambda} \in \text{Irr}(\mathcal{A}_n) \mid \lambda_{(p)} = \gamma^*\}$ .

Assume instead that  $\gamma = \gamma^*$ . Suppose that  $w > 0$ . By Equation (19), there is a partition  $\lambda$  of  $n$  with  $p$ -core  $\gamma$  and  $p$ -quotient  $((w), \emptyset, \dots, \emptyset)$ . Furthermore,  $\chi_{\lambda} \neq \chi_{\lambda^*}$  by Equation (21) and  $\chi_{\lambda} \in \text{Irr}(B_{\gamma})$ . Hence,  $\chi_{\lambda}$  restricts irreducibly to  $\mathcal{A}_n$ , and [20, Theorem 9.2] implies that  $B_{\gamma}$  covers a unique  $p$ -block  $b_{\gamma}$  of  $\mathcal{A}_n$ .

If, on the other hand,  $w = 0$ , then  $\text{Irr}(B_{\gamma}) = \{\chi_{\gamma}\}$  has defect zero. If  $n \leq 1$ , then  $\mathcal{A}_n = \mathfrak{S}_n = \{1\}$ , and  $\rho_{\gamma} = \chi_{\gamma}$  is the trivial character. The case  $n = 2$  does not occur, because there are no self-conjugate partitions of size 2. If  $n \geq 3$ , then  $\{\rho_{\gamma}^+\}$  and  $\{\rho_{\gamma}^-\}$  are  $p$ -blocks of defect zero of  $\mathcal{A}_n$ .

**3.3. Broué perfect isometries.** Let  $q$  be a positive integer. For  $\lambda \in \mathcal{P}_n$ , we denote by  $M_q(\lambda)$  the set of  $\mu \in \mathcal{P}_{n-q}$  such that  $\mu$  is obtained from  $\lambda$  by removing a  $q$ -hook. (The definition of  $q$ -hooks, and the process to remove a  $q$ -hook from a partition, is for example given in [22, p. 4, 5, 6]). Note that, if  $\mu \in M_q(\lambda)$ , then  $\mu^* \in M_q(\lambda^*)$ .

For  $\mu \in M_q(\lambda)$ , we denote by  $c_{\mu}^{\lambda}$  the  $q$ -hook of  $\lambda$  such that  $\mu$  is obtained from  $\lambda$  by removing  $c_{\mu}^{\lambda}$ . Define

$$(22) \quad \alpha_{\mu}^{\lambda} = (-1)^{L(c_{\mu}^{\lambda})},$$

where  $L(c_{\mu}^{\lambda})$  denotes the leglength of  $c_{\mu}^{\lambda}$  (see for example [22, p. 4] for the definition of the leglength of a hook).

**Lemma 3.1.** *If  $q$  is an odd integer, then  $\alpha_{\mu}^{\lambda} = \alpha_{\mu^*}^{\lambda^*}$ .*

*Proof.* First, note that  $c_{\mu^*}^{\lambda^*} = (c_{\mu}^{\lambda})^*$ . In particular, the leg of  $c_{\mu^*}^{\lambda^*}$  is the arm of  $c_{\mu}^{\lambda}$ . Hence,  $L(c_{\mu}^{\lambda}) + L(c_{\mu^*}^{\lambda^*}) = q - 1$ . Since  $q$  is odd, the result follows.  $\square$

**Lemma 3.2.** *Assume that  $q$  is odd, and that  $\lambda = \lambda^*$ . The set  $M_q(\lambda)$  contains a self-conjugate partition if and only if  $q \in \{\bar{\lambda}_1, \dots, \bar{\lambda}_k\}$ . In this case,  $M_q(\lambda)$  contains a unique self-conjugate partition  $\mu$ , and  $\mu$  is such that  $\bar{\mu} = \bar{\lambda} \setminus \{q\}$ .*

*Proof.* Since  $\lambda = \lambda^*$ , it follows from Equation (21) that  $\lambda^i = (\lambda^{q-i+1})^*$ , where  $\lambda^{(q)} = (\lambda^1, \dots, \lambda^q)$  is the  $q$ -quotient of  $\lambda$ . Moreover, by [22, Theorem 3.3] the multipartitions of  $n - q$  obtained from  $\lambda^{(q)}$  by removing any 1-hook are the  $q$ -quotients of partitions of  $M_q(\lambda)$ . In particular,  $\mu \in M_q(\lambda)$  is self-conjugate if and only if  $\mu^i = \lambda^i$  for  $1 \leq i \leq (q-1)/2$  and  $\mu^{(q+1)/2}$  is a self-conjugate partition obtained from  $\lambda^{(q+1)/2}$  by removing a 1-hook. However, when we remove a 1-hook from a self-conjugate partition, the resulting partition is never a self-conjugate partition, except if the removing box is a diagonal 1-hook. We now conclude with the argument of the proof of [2, 3.4].  $\square$

Assume  $\lambda = \lambda^*$ . In the case that  $M_q(\lambda)$  contains a (unique) self-conjugate partition  $\mu$ , then we write  $\mu_{\lambda} = \mu$  (which is well-defined by Lemma 3.2). Let  $\mu, \mu' \in M_q(\lambda)$ . We write  $\mu \sim \mu'$  if and only if  $\mu' = \mu^*$ , and we denote by  $M'_q(\lambda)$  a set of representatives modulo  $\sim$ . Finally, for  $\lambda \in \mathcal{P}_n$ , we set  $\alpha(\lambda) = 1$  if  $\lambda \neq \lambda^*$  and  $\alpha(\lambda) = \frac{1}{2}$  otherwise.



**Theorem 3.3.** Let  $q$  an odd integer and  $\lambda \in \mathcal{P}_n$ . If  $\lambda \neq \lambda^*$ , then we write  $\rho_\lambda = \rho_\lambda^+ = \rho_\lambda^-$ . Let  $\sigma$  be a  $q$ -cycle with support  $\{n - q + 1, \dots, n\}$ . Then for  $\epsilon \in \{\pm 1\}$  and  $g \in \mathcal{A}_{n-q}$ , we have

$$\rho_\lambda^\epsilon(\sigma g) = \sum_{\substack{\mu \in M_q(\lambda) \\ \mu \neq \mu^*}} a(\rho_\lambda^\epsilon, \rho_\mu) \rho_\mu(g) + \sum_{\substack{\mu \in M_q(\lambda) \\ \mu = \mu^*}} (a(\rho_\lambda^\epsilon, \rho_\mu^+) \rho_\mu^+(g) + a(\rho_\lambda^\epsilon, \rho_\mu^-) \rho_\mu^-(g)),$$

where the complex numbers  $a(\rho_\lambda^\epsilon, \rho_\mu^\eta)$  ( $\eta \in \{\pm 1\}$ ) are defined as follows.

- If  $\mu^* \neq \mu$  and  $\mu^* \in M_q(\lambda)$ , then  $a(\rho_\lambda^\epsilon, \rho_\mu) = \alpha(\lambda)(\alpha_\mu^\lambda + \alpha_{\mu^*}^\lambda)$ .
- If  $\mu^* \neq \mu$  and  $\mu^* \notin M_q(\lambda)$ , then  $a(\rho_\lambda^\epsilon, \rho_\mu) = \alpha(\lambda)\alpha_\mu^\lambda$ .
- If  $\mu^* = \mu$  and  $\mu \neq \mu_\lambda$ , then  $a(\rho_\lambda^\epsilon, \rho_\mu^\eta) = \alpha(\lambda)\alpha_\mu^\lambda$ .
- If  $\mu^* = \mu$  and  $\mu = \mu_\lambda$ , then  $a(\rho_\lambda^\epsilon, \rho_{\mu_\lambda}^\eta) = \frac{1}{2} \left( \alpha_{\mu_\lambda}^\lambda + \epsilon \eta \sqrt{(-1)^{(q-1)/2} q} \right)$ .

*Proof.* This is a consequence of Clifford theory and the Murnaghan-Nakayama formula in  $\mathfrak{S}_n$ . We only prove the last case for  $\epsilon = +1$ . Assume that  $\lambda = \lambda^*$  and that  $\sigma g$  has cycle type  $\bar{\lambda}$ . By Lemma 3.2,  $g \in \overline{\mu_\lambda}^\pm$ . Now, if  $\bar{\lambda} = (h_1 > h_2 > \dots > h_k)$  then

$$\begin{aligned} \rho_\lambda^+(\sigma g) &= \frac{1}{2} \left( \chi_\lambda(\sigma g) \pm \sqrt{(-1)^{\frac{n-k}{2}} h_1 \cdots h_k} \right), \\ &= \sum_{\substack{\{\mu, \mu^*\} \subset M_q(\lambda) \\ \mu \neq \mu^*}} \frac{1}{2} (\alpha_\mu^\lambda + \alpha_{\mu^*}^\lambda) \rho_\mu(g) + \sum_{\substack{\mu \in M_q(\lambda) \\ \mu^* \notin M_q(\lambda) \\ \mu \neq \mu^*}} \frac{1}{2} \alpha_\mu^\lambda \rho_\mu(g) \\ &\quad + \sum_{\substack{\mu \in M_q(\lambda) \\ \mu = \mu^*}} \frac{1}{2} \alpha_\mu^\lambda (\rho_\mu^+(g) + \rho_\mu^-(g)) \pm \frac{1}{2} \sqrt{(-1)^{\frac{n-k}{2}} h_1 \cdots h_k}, \end{aligned}$$

If we write  $\bar{\lambda} \setminus \{q\} = (h'_1 > \dots > h'_{k-1})$ , then

$$\begin{aligned} \sqrt{(-1)^{\frac{n-k}{2}} h_1 \cdots h_k} &= \sqrt{(-1)^{\frac{q-1}{2}} q} \cdot \sqrt{(-1)^{\frac{(n-q)-(k-1)}{2}} h'_1 \cdots h'_{k-1}}, \\ &= \sqrt{(-1)^{\frac{q-1}{2}} q} \cdot (\rho_{\mu_\lambda}^+(g) - \rho_{\mu_\lambda}^-(g)). \end{aligned}$$

The result follows.  $\square$

**Remark 3.4.** In the last proof, when  $\lambda = \lambda^*$  and  $\mu \in \mathcal{P}_{n-q}$  is not self-conjugate and satisfies  $\{\mu, \mu^*\} \subset M_q(\lambda)$ , then  $a(\rho_\lambda^\epsilon, \rho_\mu) = \alpha_\mu^\lambda$ , because, by Lemma 3.1,  $\alpha_\mu^\lambda = \alpha_{\mu^*}^\lambda$ .

For  $q_1, q_2$  multiples of  $p$ , we define

$$(23) \quad M_{q_1, q_2}(\lambda) = \{\mu \in M_{q_2}(\nu) \mid \nu \in M_{q_1}(\lambda)\},$$

and  $M'_{q_1, q_2}(\lambda)$  denotes a set of representatives modulo  $\sim$ , where  $\sim$  is defined in a similar way as before Theorem 3.3. Moreover, for  $\mu \in M_{q_1, q_2}(\lambda)$ , we denote by  $\mathcal{P}_{\lambda \rightsquigarrow \mu}^{q_1, q_2}$  the set all of pairs  $(c_\nu^\lambda, c_\mu^\nu)$ , where  $\nu \in M_{q_1}(\lambda)$  and  $\mu \in M_{q_2}(\nu)$ .

**Theorem 3.5.** Assume that  $q_1$  and  $q_2$  are even multiples of  $p$ . Let  $\sigma = \sigma_1 \sigma_2$  be such that  $\sigma_i$  is a  $q_i$ -cycle (for  $1 \leq i \leq 2$ ), and the supports of  $\sigma_1$  and  $\sigma_2$  are  $\{n - q_1 - q_2 + 1, \dots, n - q_2\}$  and  $\{n - q_2 + 1, \dots, n\}$ , respectively. Then for  $\epsilon \in \{\pm 1\}$  and  $g \in \mathcal{A}_{n-q_1-q_2}$ ,

$$\rho_\lambda^\epsilon(\sigma g) = \sum_{\substack{\mu \in M'_{q_1, q_2}(\lambda) \\ \mu \neq \mu^*}} a(\rho_\lambda^\epsilon, \rho_\mu) \rho_\mu(g) + \sum_{\substack{\mu \in M_{q_1, q_2}(\lambda) \\ \mu = \mu^*}} (a(\rho_\lambda^\epsilon, \rho_\mu^+) \rho_\mu^+(g) + a(\rho_\lambda^\epsilon, \rho_\mu^-) \rho_\mu^-(g)),$$

where the complex numbers  $a(\rho_\lambda^\xi, \rho_\mu^\eta)$  ( $\eta \in \{\pm 1\}$ ) are defined as follows: if  $\mu^* \neq \mu$  and  $\mu^* \in M_{q_1, q_2}(\lambda)$ , then

$$a(\rho_\lambda^\xi, \rho_\mu) = \alpha(\lambda) \left( \sum_{(c_\nu^\lambda, c_\mu^\nu) \in \mathcal{P}_{\lambda \rightsquigarrow \mu}^{q_1, q_2}} (-1)^{L(c_\nu^\lambda) + L(c_\mu^\nu)} + \sum_{(c_\nu^\lambda, c_{\mu^*}^\nu) \in \mathcal{P}_{\lambda \rightsquigarrow \mu^*}^{q_1, q_2}} (-1)^{L(c_\nu^\lambda) + L(c_{\mu^*}^\nu)} \right).$$

In all other cases, one has

$$a(\rho_\lambda^\xi, \rho_\mu^\eta) = \alpha(\lambda) \sum_{(c_\nu^\lambda, c_\mu^\nu) \in \mathcal{P}_{\lambda \rightsquigarrow \mu}^{q_1, q_2}} (-1)^{L(c_\nu^\lambda) + L(c_\mu^\nu)}.$$

*Proof.* Apply twice the Murnaghan-Nakayama formula in  $\mathfrak{S}_n$  and conclude with Clifford theory.  $\square$

Let  $q$  be an integer. For  $\lambda \in \mathcal{P}_n$  and  $\mu \in M_q(\lambda)$ , we introduce the relative  $q$ -sign  $\delta_q(\lambda, \mu) = \delta_q(\lambda)\delta_q(\mu)$  as in [19, p. 62], where  $\delta_q(\lambda)$  is the  $q$ -sign of  $\lambda$  (see [19, §2]).

**Lemma 3.6.** *Assume that  $q$  is odd. For any  $\lambda \in \mathcal{P}_n$ , one has  $\delta_q(\lambda) = \delta_q(\lambda^*)$ .*

*Proof.* Let  $k$  be the  $q$ -weight of  $\lambda$ . We construct a sequence of  $q$ -hooks  $c_1, \dots, c_k$  by choosing  $c_1$  to be a  $q$ -hook of  $\lambda$  and  $c_i$  to be a  $q$ -hook of  $\lambda \setminus \{c_1, \dots, c_{i-1}\}$  for  $2 \leq i \leq k$ , such that  $\lambda \setminus \{c_1, \dots, c_k\} = \lambda_{(q)}$ . Note that  $c_1^*, \dots, c_k^*$  is a sequence of  $q$ -hooks from  $\lambda^*$  to  $\lambda_{(q)}^*$ . So, by [19, Corollary 2.3],  $\delta_q(\lambda) = \delta_q(\lambda, \lambda^{(q)})$  and [19, Proposition 2.2] yields

$$(24) \quad \delta_q(\lambda) = (-1)^{L(c_1) + \dots + L(c_k)} \quad \text{and} \quad \delta_q(\lambda^*) = (-1)^{L(c_1^*) + \dots + L(c_k^*)}.$$

Now, by the argument of Lemma 3.1, we deduce that  $L(c_i) \equiv L(c_i^*) \pmod{2}$  for  $1 \leq i \leq k$ , because  $q$  is odd. The result follows.  $\square$

Let  $\gamma$  and  $\gamma'$  be two self-conjugate  $p$ -cores of  $\mathfrak{S}_n$  and  $\mathfrak{S}_m$  of the same  $p$ -weight  $w > 0$ . We denote by  $b_\gamma$  and  $b_{\gamma'}$  the corresponding  $p$ -blocks of  $\mathcal{A}_n$  and  $\mathcal{A}_m$ , respectively. Let  $\lambda \in \mathcal{P}_n$  be such that  $\lambda_{(p)} = \gamma$ . By Equation (19), there is a unique partition  $\Psi(\lambda) \in \mathcal{P}_m$  such that  $\Psi(\lambda)_{(p)} = \gamma'$  and  $\Psi(\lambda)^{(p)} = \lambda^{(p)}$ . In particular, if we denote by  $f$  the canonical bijection between the set of hooks of length divisible by  $p$  in  $\lambda$  and the set of hooks in  $\lambda^{(p)}$  as in [19, Proposition 3.1], then for any integer  $q$  divisible by  $p$  and  $\mu \in M_q(\lambda)$ , we have

$$(25) \quad f(c_\mu^\lambda) = f\left(c_{\Psi(\mu)}^{\Psi(\lambda)}\right),$$

where  $\Psi : \mathcal{P}_{n-q} \rightarrow \mathcal{P}_{m-q}$  is defined as above. Moreover, [19, Corollary 3.4] gives

$$(26) \quad (-1)^{L(c_\mu^\lambda)} = (-1)^{L(f(c_\mu^\lambda))} \delta_p(\lambda, \mu).$$

**Lemma 3.7.** *Let  $\lambda$  and  $\Psi(\lambda)$  be as above. For any multiple  $q$  of  $p$  and  $\mu \in M_q(\lambda)$  such that  $\mu \neq \mu^*$ , we have  $\Psi(\mu) \neq \Psi(\mu^*)$ . Moreover,  $\mu^* \in M_q(\lambda)$  if and only if  $\Psi(\mu^*) \in M_q(\Psi(\lambda))$ . In this case,  $\Psi(\mu^*) = \Psi(\mu)^*$ .*

*Proof.* This is a consequence of [19, Proposition 3.1] and of [22, Proposition 3.5].  $\square$

**Proposition 3.8.** *Assume  $p$  is odd and keep the notation as above. We have*

$$\delta_p(\lambda)\delta_p(\mu)a\left(\rho_\lambda^{\varepsilon\delta_p(\lambda)}, \rho_\mu^{\eta\delta_p(\mu)}\right) = \delta_p(\Psi(\lambda))\delta_p(\Psi(\mu))a\left(\rho_{\Psi(\lambda)}^{\varepsilon\delta_p(\Psi(\lambda))}, \rho_{\Psi(\mu)}^{\eta\delta_p(\Psi(\mu))}\right).$$

*Proof.* Let  $\mathcal{E}_\gamma$  be the set of partitions of  $n$  with  $p$ -core  $\gamma$ . Since  $\gamma$  is self-conjugate, by Equation (21),  $\lambda \in \mathcal{E}_\gamma$  is self-conjugate if and only if its  $p$ -quotient is self-conjugate. The same holds for  $\gamma'$  and  $\Psi(\lambda) \in \mathcal{E}_{\gamma'}$ . In particular, for  $\lambda \in \mathcal{E}_\gamma$ , we have

$$\alpha(\lambda) = \alpha(\Psi(\lambda)).$$

Let  $q_1$  and  $q_2$  be even multiples of  $p$ . With the notation of Theorem 3.5, for any partition  $\mu \in M_{q_1, q_2}(\lambda)$ , if  $(c_\nu^\lambda, c_\mu^\nu) \in \mathcal{P}_{\lambda \rightsquigarrow \mu}^{q_1, q_2}$ , then Equations (25) and (26) give

$$\begin{aligned} \delta_p(\lambda, \mu)(-1)^{L(c_\nu^\lambda) + L(c_\mu^\nu)} &= \delta_p(\lambda, \mu)(-1)^{L(f(c_\nu^\lambda)) + L(f(c_\mu^\nu))} \delta_p(\lambda, \nu) \delta_p(\nu, \mu) \\ &= \delta_p(\lambda) \delta_p(\mu) (-1)^{L(f(c_\nu^\lambda)) + L(f(c_\mu^\nu))} \delta_p(\lambda) \delta_p(\nu) \delta_p(\nu) \delta_p(\mu) \\ &= (-1)^{L(f(c_\nu^\lambda)) + L(f(c_\mu^\nu))} \\ &= (-1)^{L(f(c_{\Psi(\nu)}^{\Psi(\lambda)})) + L(f(c_{\Psi(\mu)}^{\Psi(\nu)}))} \\ &= \delta_p(\Psi(\lambda), \Psi(\mu)) (-1)^{L(c_{\Psi(\nu)}^{\Psi(\lambda)}) + L(c_{\Psi(\mu)}^{\Psi(\nu)})}. \end{aligned}$$

Now, using Lemmas 3.6 and 3.7, we deduce that, if  $\mu \neq \mu^*$  and  $\mu^* \in M_{q_1, q_2}(\lambda)$ , then

$$\delta_p(\lambda, \mu) a(\rho_\lambda^{\pm \delta_p(\lambda)}, \rho_\mu) = \delta_p(\Psi(\lambda), \Psi(\mu)) a(\rho_{\Psi(\lambda)}^{\pm \delta_p(\Psi(\lambda))}, \rho_{\Psi(\mu)}).$$

**Note :** Lemma 3.6 is used only when we apply the above computations to evaluate  $\delta_p(\lambda, \mu)(-1)^{L(c_\nu^\lambda) + L(c_\mu^\nu)}$ ; at the second line, we get a term  $\delta_p(\mu) \delta_p(\mu^*)$ , which is thus 1. In the same way, a term  $\delta_p(\Psi(\mu)) \delta_p(\Psi(\mu)^*)$  disappears at the end.

We conclude similarly for the other cases appearing in Theorem 3.5 and for the coefficients appearing in Theorem 3.3, except for  $\lambda = \lambda^*$  and  $\mu = \mu_\lambda$ . In this last case, first note that  $\Psi(\mu) = \mu_{\Psi(\lambda)}$ . Moreover,

$$\begin{aligned} a(\rho_\lambda^\xi, \rho_{\mu_\lambda}^\eta) &= \frac{1}{2} \left( \alpha_{\mu_\lambda}^\lambda + \epsilon \eta \sqrt{(-1)^{(q-1)/2} q} \right) \\ &= \delta_p(\lambda) \delta_p(\mu_\lambda) \delta_p(\Psi(\lambda)) \delta_p(\Psi(\mu_\lambda)) \frac{1}{2} \left( \alpha_{\Psi(\mu_\lambda)}^{\Psi(\lambda)} \right. \\ &\quad \left. + \delta_p(\lambda) \delta_p(\mu_\lambda) \delta_p(\Psi(\lambda)) \delta_p(\Psi(\mu_\lambda)) \epsilon \eta \sqrt{(-1)^{(q-1)/2} q} \right) \\ &= \delta_p(\lambda) \delta_p(\Psi(\lambda)) \delta_p(\mu_\lambda) \delta_p(\Psi(\mu_\lambda)) a \left( \rho_{\Psi(\lambda)}^{\epsilon \delta_p(\lambda) \delta_p(\Psi(\lambda))}, \rho_{\Psi(\mu_\lambda)}^{\eta \delta_p(\mu_\lambda) \delta_p(\Psi(\mu_\lambda))} \right). \end{aligned}$$

as required.  $\square$

**Theorem 3.9.** *Let  $p$  be an odd prime. Assume that  $\gamma$  and  $\gamma'$  are self-conjugate  $p$ -cores of  $\mathfrak{S}_n$  and  $\mathfrak{S}_m$  respectively, and of same  $p$ -weight  $w > 0$ . Let  $b_\gamma$  and  $b_{\gamma'}$  be the corresponding  $p$ -blocks of  $\mathcal{A}_n$  and  $\mathcal{A}_m$ . Define, for all  $\lambda \in \mathcal{E}_\gamma$  and  $\epsilon \in \{\pm 1\}$ ,*

$$(27) \quad I : \mathbb{C} \text{Irr}(b_\gamma) \rightarrow \mathbb{C} \text{Irr}(b_{\gamma'}), \quad \rho_\lambda^\epsilon \mapsto \delta_p(\lambda) \delta_p(\Psi(\lambda)) \rho_{\Psi(\lambda)}^{\epsilon \delta_p(\lambda) \delta_p(\Psi(\lambda))}.$$

*Then  $I$  is a Broué perfect isometry.*

*Proof.* First, we prove that  $\mathcal{A}_n$  has an MN-structure. Let  $S$  be the set of elements of  $\mathcal{A}_n$  with cycle decomposition  $\sigma_1 \cdots \sigma_r$  (where we only indicate non-trivial cycles) such that each  $\sigma_i$  is a cycle of length divisible by  $p$ . We remark that when  $\sigma_i$  has even length, there is  $j \neq i$  such that  $\sigma_j$  has even length (because  $\sigma_1 \cdots \sigma_r \in \mathcal{A}_n$ ). Moreover,  $S$  contains 1 and is stable by  $\mathcal{A}_n$ -conjugation. Let  $C$  be the set of  $p$ -regular elements of  $\mathcal{A}_n$ . Now take any  $\sigma \in \mathcal{A}_n$ . Using the cycle decomposition of  $\sigma$ , there are unique elements  $\sigma_S \in S$  and  $\sigma_C \in C$  with disjoint support such that

$\sigma = \sigma_S \sigma_C = \sigma_C \sigma_S$ . In particular, Definition 2.5(1) and (2) hold. Denote by  $J$  the support of  $\sigma_S$ ,  $\bar{J} = \{1, \dots, n\} \setminus J$  and define  $G_{\sigma_S} = \mathcal{A}_{\bar{J}}$ . Then  $G_{\sigma_S}$  satisfies Definition 2.5(3). Write  $\Omega$  for the set of partitions of the form  $p \cdot \beta$  such that

- There is some  $i \leq n$  such that  $p \cdot \beta$  is a partition of  $i$ .
- The number of even parts of  $\beta$  is even. In particular, we choose the notation such that  $\beta = (\beta_1, \dots, \beta_k)$  with  $|\beta| = \beta_1 + \dots + \beta_k$  and there is  $1 \leq r \leq k$  with  $\beta_i$  even for  $1 \leq i \leq r$  and  $\beta_i$  odd for  $i > r$ .

Note that each partition of  $\Omega$  labels either one  $\mathcal{A}_n$ -conjugacy class of  $S$  or two classes. Denote by  $\Lambda$  the set of parameters for the  $\mathcal{A}_n$ -classes of  $S$  obtained by this process. The elements of  $\Lambda$  will be denoted  $p \cdot \hat{\beta}$ , with  $\hat{\beta} = \beta$  when  $p \cdot \beta \in \Omega$  labels a unique class of  $S$ , and  $\hat{\beta} \in \{\beta^+, \beta^-\}$  when  $p \cdot \beta$  labels two classes of  $S$ . The notation is chosen as in Equation (18). For  $p \cdot \hat{\beta} \in \Lambda$ , we assume that the representative  $\sigma_{\hat{\beta}} = \sigma_{\hat{\beta}_1} \cdots \sigma_{\hat{\beta}_k}$  of the  $\mathcal{A}_n$ -class labeled by  $p \cdot \hat{\beta}$  in  $\mathcal{A}_n$  satisfies that the cycle  $\sigma_{\hat{\beta}_i}$  has support  $\{1 + \sum_{j < i} p\beta_j, \dots, \sum_{j \leq i} p\beta_j\}$ . Moreover, when  $p \cdot \beta$  labels two classes of  $\mathcal{A}_n$ , we assume that  $\sigma_{\beta_i^+} = \sigma_{\beta_i^-}$  for every  $2 \leq i \leq k$ , and  $\sigma_{\beta_1^+}$  and  $\sigma_{\beta_1^-}$  are representatives of the  $\mathcal{A}_{p \cdot \beta_1}$ -classes labeled by  $p \cdot \beta_1^+$  and  $p \cdot \beta_1^-$ , respectively. In particular,  $\sigma_{\hat{\beta}_i}$  has length  $p \cdot \beta_i$  and the support of  $\sigma_{\hat{\beta}}$  is  $\{1, \dots, p|\beta|\}$ . Hence,  $G_{\sigma_{\hat{\beta}}} = \mathcal{A}_{n-p|\beta|}$ .

Now, we denote by  $\Omega_0$  the subset of partitions  $p \cdot \beta \in \Omega$  such that  $|\beta| \leq w$ , and by  $\Lambda_0$  the corresponding subset of  $\Lambda$ . For  $p \cdot \beta \in \Omega_0$ , define  $r^{\hat{\beta}} : \mathbb{C}\text{Irr}(b_\gamma) \rightarrow \mathbb{C}\text{Irr}(b_\gamma(\mathcal{A}_{n-p|\beta|}))$  by applying iteratively Theorem 3.3 with  $\sigma = \sigma_{\hat{\beta}_i}$  when  $\beta_i$  is odd and Theorem 3.5 with  $\sigma = \sigma_{\hat{\beta}_i} \sigma_{\hat{\beta}_{i+1}}$  when  $\beta_i$  and  $\beta_{i+1}$  are even. By Theorems 3.3 and 3.5, Definition 2.5(4) holds. This proves that  $\mathcal{A}_n$  has an MN-structure with respect to  $b_\gamma$  and the set of  $p$ -regular elements of  $\mathcal{A}_n$ . Let  $\lambda \in \mathcal{E}_\gamma$ . Then  $r^{\hat{\beta}}(\rho_\lambda^\pm)(g) = \rho_\lambda^\pm(\sigma_{\hat{\beta}}g)$ , and for  $p \cdot \hat{\beta} \in \Lambda \setminus \Lambda_0$ , the Murnaghan-Nakayama rule in  $\mathfrak{S}_n$  and Clifford theory imply that  $\rho_\lambda^\pm(\sigma_{\hat{\beta}}g) = 0$  except, maybe, when  $\lambda = \lambda^*$  and  $\sigma_{\hat{\beta}}g$  is in the class of  $\mathfrak{S}_n$  labeled by  $a(\lambda)$ . In this last case,  $\lambda$  has more than  $w$  diagonal hooks with length divisible by  $p$ , contradicting the fact that  $\lambda$  has  $p$ -weight  $w$ . This proves that, if  $p \cdot \hat{\beta} \notin \Lambda_0$ , then  $r^{\hat{\beta}} = 0$ .

We define similarly an MN-structure for  $\mathcal{A}_m$  with respect to  $b_{\gamma'}$  and the set of  $p$ -regular elements of  $\mathcal{A}_m$ . We denote by  $\Omega'$ ,  $\Omega'_0$ ,  $\Lambda'$  and  $\Lambda'_0$  the corresponding sets. Note that  $\Omega_0 = \Omega'_0$ .

There are two cases to consider. First, assume that  $|\Lambda_0| = |\Lambda'_0|$ . In fact, this case occurs if and only if  $\Lambda_0 = \Lambda'_0$ , because  $\Omega_0 = \Omega'_0$ .

Now, we will prove that Theorem 2.10(2) holds. Let  $p \cdot \hat{\beta} \in \Lambda_0$ . Write  $\beta = (\beta_1, \dots, \beta_k)$  and  $r$  as above. Set  $q_i = p|\beta_i|$  for  $1 \leq i \leq k$ . For  $i > r$ , write  $x_i = q_i$ , and for  $1 \leq i \leq r/2$ , write  $x_i = \{q_{2i-1}, q_{2i}\}$ . We also set  $s = n - r/2$ . For  $1 \leq i \leq s$ , define  $M_{x_1, \dots, x_i}(\lambda) = \{\mu \in M_{x_i}(\nu) \mid \nu \in M_{x_1, \dots, x_{i-1}}(\lambda)\}$  (recall that  $M_{x_i}(\nu)$  is defined as in Equation (23) when  $x_i$  has two elements).

Let  $\theta \in \text{Irr}(b_\gamma)$ . There are  $\lambda \in \mathcal{E}_\gamma$  and  $\epsilon \in \{\pm 1\}$  such that  $\theta = \rho_\lambda^{\epsilon \delta_p(\lambda)}$  (with the convention, as above, that if  $\lambda \neq \lambda^*$ , then  $\rho_\lambda^+ = \rho_\lambda = \rho_{\lambda^-}$ ). Then we set  $\delta_p(\theta) = \delta_p(\lambda)$  and  $\Psi(\theta) = \rho_{\Psi(\lambda)}^{\epsilon \delta_p(\Psi(\lambda))} \in \text{Irr}(b_{\gamma'})$ . We have

$$(28) \quad r^{\hat{\beta}}(\theta) = \sum_{\vartheta \in \text{Irr}(b_{\gamma'}(n-p|\beta|))} a(\theta, \vartheta) \vartheta,$$

where  $b_\gamma(n-p|\beta|)$  denotes the union of  $p$ -blocks of  $\mathcal{A}_{n-p|\beta|}$  covered by the  $p$ -block  $B_\gamma$  of  $\mathfrak{S}_{n-p|\beta|}$  labeled by  $\gamma$ . By §3.2,  $b_\gamma(n-p|\beta|)$  is a  $p$ -block of  $\mathcal{A}_{n-p|\beta|}$ , except when  $|\gamma| > 2$  and  $|\beta| = w$ . In this last case, it is a union of two  $p$ -blocks  $\{\rho_\gamma^+\}$  and  $\{\rho_\gamma^-\}$  of defect zero. Similarly, we denote by  $b_{\gamma'}(m-p|\beta|)$  the union of  $p$ -blocks of  $\mathcal{A}_{m-p|\beta|}$  covered by the  $p$ -block of  $B_{\gamma'}$  of  $\mathfrak{S}_{m-p|\beta|}$  labeled by  $\gamma'$ . Define  $I_\beta : \mathbb{C} \text{Irr}(b_\gamma(n-p|\beta|)) \rightarrow \mathbb{C} \text{Irr}(b_{\gamma'}(m-p|\beta|))$  as in Equation (27). Note that

$$(29) \quad a(\theta, \vartheta) = \sum_{\vartheta_1, \dots, \vartheta_{s-1}} a(\vartheta_0, \vartheta_1) a(\vartheta_1, \vartheta_2) \cdots a(\vartheta_{s-1}, \vartheta_s),$$

where  $\vartheta_0 = \theta$ ,  $\vartheta_s = \vartheta$ , and the sum runs over the set of  $\vartheta_1, \dots, \vartheta_{s-1}$  such that for each  $1 \leq i \leq s$ , there are  $\mu_i \in M_{x_1, \dots, x_i}(\lambda)$  and  $\epsilon_i \in \{\pm 1\}$  such that  $\vartheta_i = \rho_{\mu_i}^{\epsilon_i \delta_p(\mu_i)}$ . Since

$$\delta_p(\theta) \delta_p(\vartheta) = (\delta_p(\theta) \delta_p(\vartheta_1)) \cdot (\delta_p(\vartheta_1) \delta_p(\vartheta_2)) \cdots (\delta_p(\vartheta_{s-1}) \delta_p(\vartheta)),$$

and thanks to Proposition 3.8, we deduce that

$$(30) \quad \begin{aligned} \delta_p(\theta) \delta_p(\vartheta) a(\theta, \vartheta) &= \sum \delta_p(\vartheta_0) \delta_p(\vartheta_1) a(\vartheta_0, \vartheta_1) \cdots \\ &\quad \cdots \delta_p(\vartheta_{s-1}) \delta_p(\vartheta_s) a(\vartheta_{s-1}, \vartheta_s) \\ &= \sum \delta_p(\Psi(\vartheta_0)) \delta_p(\Psi(\vartheta_1)) a(\Psi(\vartheta_0), \Psi(\vartheta_1)) \cdots \\ &\quad \cdots \delta_p(\Psi(\vartheta_{s-1})) \delta_p(\Psi(\vartheta_s)) a(\Psi(\vartheta_{s-1}), \Psi(\vartheta_s)) \\ &= \delta_p(\Psi(\theta)) \delta_p(\Psi(\vartheta)) a(\Psi(\theta), \Psi(\vartheta)). \end{aligned}$$

In particular, one has

$$\delta_p(\theta) \delta_p(\Psi(\theta)) a(\Psi(\theta), \Psi(\vartheta)) = \delta_p(\vartheta) \delta_p(\Psi(\vartheta)) a(\theta, \vartheta),$$

and it follows that

$$(31) \quad \begin{aligned} r^{\widehat{\beta}}(I(\theta)) &= \delta_p(\theta) \delta_p(\Psi(\theta)) r^{\widehat{\beta}}(\Psi(\theta)), \\ &= \sum_{\vartheta \in \text{Irr}(b_\gamma(n-p|\beta|))} \delta_p(\theta) \delta_p(\Psi(\theta)) a(\Psi(\theta), \Psi(\vartheta)) \Psi(\vartheta), \\ &= \sum_{\vartheta \in \text{Irr}(b_\gamma(n-p|\beta|))} \delta_p(\vartheta) \delta_p(\Psi(\vartheta)) a(\theta, \vartheta) \Psi(\vartheta), \\ &= \sum_{\vartheta \in \text{Irr}(b_\gamma(n-p|\beta|))} a(\theta, \vartheta) I_{\widehat{\beta}}(\vartheta), \\ &= I_{\widehat{\beta}}(r^{\widehat{\beta}}(\theta)). \end{aligned}$$

**Note :** Assume  $H$  is a normal subgroup of  $G$  and the MN-structure of  $H$  comes from Clifford theory. Then the  $\mathcal{E}_{x_\lambda, y}^\lambda$  for  $G$  have all size 1 and the  $\mathcal{E}_{x_\lambda, y}^\lambda$  for  $H$  have size dividing this size for  $G$  multiplied by  $[G : H]$ . Since here this index is 2, and since  $p$  is odd, condition (iii) of Theorem 2.20 holds.

Note that the groups  $G_{\sigma_{\widehat{\beta}}}$  and  $G'_{\sigma'_{\widehat{\beta}}}$  have an MN-structure with respect to

$$(b_\gamma(n-p|\beta|), C \cap G_{\sigma_{\widehat{\beta}}}) \quad \text{and} \quad (b'_{\gamma'}(m-p|\beta|), C \cap G'_{\sigma'_{\widehat{\beta}}}),$$

respectively. Applying the previous computations to  $G_{\sigma_{\widehat{\beta}}} = \mathcal{A}_{n-p|\beta|}$  and  $G'_{\sigma'_{\widehat{\beta}}} = \mathcal{A}_{m-p|\beta|}$ , we conclude that the condition (2) of Theorem 2.10 holds for  $I_{\widehat{\beta}}$ . Now, Remark 2.11 gives the condition (3) of Theorem 2.10 for  $I$ . On the other hand, by

construction, assumption (ii) of Theorem 2.20 holds. The result then follows from Theorems 2.10 and 2.20.

Assume now that  $|\Lambda_0| \neq |\Lambda'_0|$ . Without loss of generality, we can suppose that  $|\Lambda_0| > |\Lambda'_0|$ . This means that  $\Lambda'_0 = \Omega_0$ . Since  $\Lambda_0 \leq \Omega_0$ , there is  $p \cdot \beta \in \Omega_0$  such that  $(p \cdot \beta, 1^{|\gamma|}) \in \mathcal{D}_n \cap \mathcal{O}_n$ . This  $p \cdot \beta$  also belongs to  $\Lambda'_0$  and since  $\Lambda_0 = \Omega_0$ ,  $p \cdot \beta$  labels a unique conjugacy class of  $\mathcal{A}_m$ , *i.e.*  $(p \cdot \beta, 1^{|\gamma|}) \notin \mathcal{D}_m \cap \mathcal{O}_m$ . This happens if and only if  $|\gamma'| \geq 2$ . (In fact,  $|\gamma'| \geq 3$  because  $\gamma'$  is self-conjugate.) Since  $\gamma'$  is self-conjugate, it labels two irreducible characters  $\rho_{\gamma'}^+$  and  $\rho_{\gamma'}^-$  of  $\mathcal{A}_{m-pw}$ . Similarly,  $\Lambda_0 \neq \Omega_0$  implies that  $|\gamma| \leq 1$ . Note that in this case, although  $\gamma$  is self-conjugate, the restriction of  $\chi_\gamma$  from  $\mathfrak{S}_1$  (or  $\mathfrak{S}_0$ ) to  $\mathcal{A}_1$  (or  $\mathcal{A}_0$ ) is irreducible (because it is the trivial character of the trivial group). Let  $p \cdot \beta$  be in  $\Omega_0$ .

Suppose that  $|\beta| < w$  or that  $|\beta| = w$  and  $(p \cdot \beta, 1^{|\gamma|}) \notin \mathcal{D}_n \cap \mathcal{O}_n$ . Then  $p \cdot \beta \in \Lambda_0$ . The same computation as in Equation (31) gives

$$(32) \quad r^{\widehat{\beta}} \circ I = I_\beta \circ r^{\widehat{\beta}}.$$

Suppose now that  $|\beta| = w$  and  $(p \cdot \beta, 1^{|\gamma|}) \in \mathcal{D}_n \cap \mathcal{O}_n$ . Then  $p \cdot \beta$  parametrizes two classes of  $S$  and one class of  $S'$ . Moreover,  $|\text{Irr}(b_\gamma(n - pw))| = 1$  and  $|\text{Irr}(b_{\gamma'}(m - pw))| = 2$ . Denote by  $G_{\beta+}$  and  $G_{\beta-}$  two copies of the trivial group, and set  $\text{Irr}(G_{\beta\pm}) = \{1_{\beta\pm}\}$ . In particular,  $r^{\beta\pm}(\mathbb{C}b_\gamma) = \mathbb{C}\text{Irr}(G_{\beta\pm})$ . Define  $I_\beta : \mathbb{C}\text{Irr}(G_{\beta+}) \oplus \mathbb{C}\text{Irr}(G_{\beta-}) \rightarrow \mathbb{C}\text{Irr}(b_{\gamma'}(m - wp))$  by setting

$$(33) \quad I_\beta(1_{\beta+}) = \rho_{\gamma'}^+ \quad \text{and} \quad I_\beta(1_{\beta-}) = \rho_{\gamma'}^-.$$

Let  $\kappa$  be the self-conjugate partition of  $n$  whose diagonal hook lengths are the parts of the partition  $(p \cdot \beta, 1^{|\gamma|})$ . By [2, 3.4], the  $p$ -quotient of  $\kappa$  satisfies  $\kappa^i = \emptyset$  if  $i \neq (p+1)/2$  and  $\kappa^{(p+1)/2} = \beta_0$ , where  $\beta_0$  is the partition of  $w$  such that  $a(\beta_0) = \beta$ . By the definition of  $\Psi$ , the partition  $\Psi(\kappa)$  of  $m$  has the same  $p$ -quotient as  $\kappa$ . Thus, the proof of [2, 3.4] also implies that  $\Psi(\kappa)$  has the same diagonal hook lengths divisible by  $p$  as  $\kappa$ , and the other diagonal hooks of  $\Psi(\kappa)$  have  $p'$ -length. In particular,  $a(\Psi(\kappa))$  has  $p \cdot \beta$  as a subpartition (corresponding exactly to those of the parts of  $a(\Psi(\kappa))$  that are divisible by  $p$ ). On the other hand, the  $\mathfrak{S}_m$ -class labeled by  $a(\Psi(\kappa))$  splits into two  $\mathcal{A}_m$ -classes with representatives  $\sigma'_\beta \sigma^+$  and  $\sigma'_\beta \sigma^-$ , where the cycle type of  $\sigma'_\beta$  is  $p \cdot \beta$ , and the  $p$ -regular elements  $\sigma^+$  and  $\sigma^-$  are representatives of the split classes of  $\mathcal{A}_{m-p|\beta|}$  labeled by  $a(\gamma')^+$  and  $a(\gamma')^-$ , respectively.

Let  $\mu_1 = \kappa$ ,  $\mu_{\ell(\beta)} = \gamma$  and the  $\mu_i$ 's be partitions such that  $\mu_1 \rightsquigarrow \mu_2 \rightsquigarrow \dots \rightsquigarrow \mu_{\ell(\beta)}$ , where  $\mu_i$  is obtained from  $\mu_{i-1}$  by removing the diagonal hook of length  $p\beta_i$ . Since  $L(c_{\mu_i}^{\mu_i-1}) = L(c_{\Psi(\mu_i)}^{\Psi(\mu_i-1)})$  for every  $1 \leq i \leq \ell(\beta) - 1$ , Equations (25) and (26) give  $\delta_p(\mu_i, \mu_{i+1}) = \delta_p(\Psi(\mu_i), \Psi(\mu_{i+1}))$  and it follows from [19, Corollary 2.3] that

$$(34) \quad \delta_p(\kappa) = \prod_{i=1}^{\ell(\beta)-1} \delta_p(\mu_i, \mu_{i+1}) = \prod_{i=1}^{\ell(\beta)-1} \delta_p(\Psi(\mu_i), \Psi(\mu_{i+1})) = \delta_p(\Psi(\kappa)).$$

Now, by [7, Theorem 11], we have

$$\begin{aligned}
r^\beta(\chi_{\Psi(\kappa)}) &= r^\beta(\delta_p(\kappa)\delta_p(\Psi(\kappa))\chi_{\Psi(\kappa)}) \\
&= r^\beta \circ I(\chi_\kappa) \\
&= I \circ r^\beta(\chi_\kappa) \\
&= \chi_\kappa(x_\beta)I(1_{\{1\}}) \\
&= \chi_\kappa(x_\beta)\chi_{\gamma'},
\end{aligned}$$

where  $x_\beta$  denotes a representative of the  $\mathfrak{S}_n$ -class labeled by  $(p \cdot \beta, 1^{|\gamma|})$ . Furthermore, Clifford theory gives

$$(35) \quad r^\beta(\rho_{\Psi(\kappa)}^+) + r^\beta(\rho_{\Psi(\kappa)}^-) = \chi_\kappa(x_\beta)(\rho_{\gamma'}^+ + \rho_{\gamma'}^-),$$

For  $1 \leq i \leq \ell(\beta)$ , write  $q_i = p\beta_i$ . Then we have

$$(-1)^{\frac{1}{2}(n - \ell((p \cdot \beta, 1^{|\gamma|})))} = (-1)^{\frac{1}{2}(pw - \ell(\beta))}$$

and the product of the parts of  $(p \cdot \beta, 1^{|\gamma|})$  is  $q_1 \dots q_{\ell(\beta)}$ . Thus, Theorem 3.3 gives

$$(36) \quad \begin{aligned} r^\beta(\rho_{\Psi(\kappa)}^+) - r^\beta(\rho_{\Psi(\kappa)}^-) &= \sqrt{(-1)^{\frac{1}{2}\sum(q_i-1)}q_1 \dots q_{\ell(\beta)}}(\rho_{\gamma'}^+ - \rho_{\gamma'}^-) \\ &= 2y_\kappa(\rho_{\gamma'}^+ - \rho_{\gamma'}^-), \end{aligned}$$

because  $\sum(q_i - 1) = pw - \ell(\beta)$ . So, we deduce from Equations (35) and (36) that

$$(37) \quad r^\beta(\rho_{\Psi(\kappa)}^\epsilon) = (x_\kappa + \epsilon y_\kappa)\rho_{\gamma'}^+ + (x_\kappa - \epsilon y_\kappa)\rho_{\gamma'}^-.$$

Furthermore, one has

$$(38) \quad r^{\beta^+}(\rho_\kappa^\epsilon) = (x_\kappa + \epsilon y_\kappa)1_{\beta^+} \quad \text{and} \quad r^{\beta^-}(\rho_\kappa^\epsilon) = (x_\kappa - \epsilon y_\kappa)1_{\beta^-}.$$

Hence, Equations (33), (34), (37) and (38) give

$$I_\beta(r^{\beta^+}(\rho_\kappa^\epsilon) + r^{\beta^-}(\rho_\kappa^\epsilon)) = r^\beta(I(\rho_\kappa^\epsilon)).$$

Let now  $\lambda \neq \kappa$  be with  $p$ -core  $\gamma$ . Since  $\rho_\lambda^\epsilon(\sigma_{\beta^\pm}) = \alpha(\lambda)\chi_\lambda(x_\beta)$ , we derive from [7, Theorem 11] and Clifford theory that  $I_\beta(r^{\beta^+}(\rho_\lambda^\epsilon) + r^{\beta^-}(\rho_\lambda^\epsilon)) = r^\beta(I(\rho_\lambda^\epsilon))$ . Finally, we obtain

$$(39) \quad I_\beta \circ (r^{\beta^+} + r^{\beta^-}) = r^\beta \circ I.$$

Using Equations (32) and (39), Remark 2.11 holds. Hence, the condition (2.b) of Theorem 2.10 is automatic for  $I_{\widehat{\beta}}$  with  $|\beta| < w$ , and is true for  $I_\beta$  with  $|\beta| = w$  (because the characters  $1_{\beta^\pm}$  and  $\rho_{\gamma'}^\pm$  have defect zero). We remark that in the last case, with the notation of Theorem 2.10, one has  $J_\beta^{*-1} = I_\beta$ .

Write  $A$  and  $B$  for the sets of partitions  $\beta \in \Omega_0$  such that  $\beta = \widehat{\beta}$  and  $\beta \neq \widehat{\beta}$ , respectively. Now, following step by step the proof of Theorem 2.10, we obtain

$$(40) \quad \begin{aligned} \widehat{I}(x, x') &= \sum_{\beta \in A} \sum_{\phi \in \mathfrak{b}_\beta} \overline{e_\beta(\Phi_\phi)(x)} l'_\beta(J_\beta^{*-1}(\phi))(x') \\ &+ \sum_{\beta \in B} \sum_{\delta \in \{+, -\}} \overline{e_{\beta^\delta}(1_{\beta^\delta})(x)} l'_\beta(J_\beta^{*-1}(1_{\beta^\delta}))(x'), \end{aligned}$$

where  $\mathfrak{b}_\beta$  (for  $\beta \in A$ ) is the basis constructed from the set of irreducible Brauer characters in the  $p$ -block  $b_\gamma(n - p|\beta|)$  as in Remark 2.9. Since an analogue of Remark 2.12 holds, we conclude as in Theorem 2.20.  $\square$

**Theorem 3.10.** *Let  $p$  be an odd prime. Assume that  $\gamma$  and  $\gamma'$  are non self-conjugate  $p$ -cores of  $\mathfrak{S}_n$  and  $\mathfrak{S}_m$  respectively, of same  $p$ -weight  $w > 0$ . Let  $b_{\gamma, \gamma^*}$  and  $b_{\gamma', \gamma'^*}$  be the corresponding  $p$ -blocks of  $\mathcal{A}_n$  and  $\mathcal{A}_m$ . Let*

$$I : \mathbb{C} \text{Irr}(b_{\gamma, \gamma^*}) \rightarrow \mathbb{C} \text{Irr}(b_{\gamma', \gamma'^*}), \quad \rho_\lambda \mapsto \delta_p(\lambda) \delta_p(\Psi(\lambda)) \rho_{\Psi(\lambda)}.$$

*Then  $I$  is a Broué perfect isometry.*

*Proof.* The proof is similar to that of Theorem 3.9. We use the same MN-structure. In a sense, this case is easier, because every irreducible character in  $\text{Irr}(b_{\gamma, \gamma^*})$  is the restriction of a character of  $\mathfrak{S}_n$ . Hence, the Murnaghan-Nakayama rule for  $\mathfrak{S}_n$  directly gives the result.  $\square$

**Theorem 3.11.** *Let  $\gamma$  and  $\gamma'$  be 2-blocks of  $\mathcal{A}_n$  and  $\mathcal{A}_m$  of the same positive weight. Then  $I$  defined as in Equation (27) is a Broué perfect isometry.*

*Proof.* The MN-structure is defined as in the case where  $p$  is odd, and one always has that  $\Lambda_0 = \Omega_0 = \Lambda'_0$ . Note that  $I$  satisfies the assumption of Remark 2.21. Only the situation of Theorem 3.5 occurs. The result of Proposition 3.8 still holds, but the simplifications explained in the note within the proof are different. For any 2-hook  $c$ , one has  $L(c) + L(c^*) \equiv 1 \pmod{2}$ . In particular, for any  $\mu \in M_{q_1, q_2}(\lambda)$ , we deduce from Equation (24) that

$$\delta_2(\mu) \delta_2(\mu^*) = (-1)^r = \delta_2(\Psi(\mu)) \delta_2(\Psi(\mu)^*),$$

where  $r$  is the number of 2-hooks to remove from  $\mu$  to get to  $\mu_{(2)}$  (this is also the number of 2-hooks we have to remove from  $\Psi(\mu)$  to obtain  $\Psi(\mu)_{(2)}$ ). The rest of the proof is similar to that of Theorem 3.9.  $\square$

#### 4. DOUBLE COVERING GROUPS OF THE SYMMETRIC AND ALTERNATING GROUPS

In this section, we will consider the double covering group  $\tilde{\mathfrak{S}}_n$  (for a positive integer  $n$ ) of the symmetric group  $\mathfrak{S}_n$  defined by

$$\tilde{\mathfrak{S}}_n = \langle z, t_i, 1 \leq i \leq n-1 \mid z^2 = 1, t_i^2 = z, (t_i t_{i+1})^3 = z, (t_i t_j)^2 = z \ (|i-j| \geq 2) \rangle.$$

The group  $\tilde{\mathfrak{S}}_n$  and its representation theory were first studied by I. Schur in [26], and, unless otherwise specified, we always refer to [26] for details or proofs.

We recall that we have the following exact sequence

$$1 \rightarrow \langle z \rangle \rightarrow \tilde{\mathfrak{S}}_n \rightarrow \mathfrak{S}_n \rightarrow 1.$$

We denote by  $\theta : \tilde{\mathfrak{S}}_n \rightarrow \mathfrak{S}_n$  the natural projection. Note that for every  $\sigma \in \mathfrak{S}_n$ , we have  $\theta^{-1}(\sigma) = \{\tilde{\sigma}, z\tilde{\sigma}\}$ , where  $\tilde{\sigma} \in \tilde{\mathfrak{S}}_n$  is such that  $\theta(\tilde{\sigma}) = \sigma$ .

If we set

$$\tilde{\mathcal{A}}_n = \theta^{-1}(\mathcal{A}_n),$$

then  $\tilde{\mathcal{A}}_n$  is the double covering group of the alternating group  $\mathcal{A}_n$ .

Throughout this section, we fix an odd prime  $p$ .



**4.1. Conjugacy classes and spin characters of  $\tilde{\mathfrak{S}}_n$ .** If  $x, y \in \tilde{\mathfrak{S}}_n$  are  $\tilde{\mathfrak{S}}_n$ -conjugate, then  $\theta(x)$  and  $\theta(y)$  are  $\mathfrak{S}_n$ -conjugate. Let  $\sigma, \tau \in \mathfrak{S}_n$ . Choose  $\tilde{\sigma}, \tilde{\tau} \in \tilde{\mathfrak{S}}_n$  such that  $\theta(\tilde{\sigma}) = \sigma$  and  $\theta(\tilde{\tau}) = \tau$ . Suppose that  $\sigma$  and  $\tau$  are  $\mathfrak{S}_n$ -conjugate. Then  $\tilde{\tau}$  is  $\tilde{\mathfrak{S}}_n$ -conjugate to  $\tilde{\sigma}$  or to  $z\tilde{\sigma}$  (possibly both). Hence, each conjugacy class  $C$  of  $\mathfrak{S}_n$  gives rise to either one or two conjugacy classes of  $\tilde{\mathfrak{S}}_n$ , according to whether  $\tilde{\sigma}$  and  $z\tilde{\sigma}$  are conjugate or not (here,  $\sigma$  lies in  $C$  and  $\tilde{\sigma}$  is as above). In the first case, we say that the class is *non-split*, and, in the second case, that it is *split*. The split classes of  $\tilde{\mathfrak{S}}_n$  are characterized as follows. Recall that the conjugacy classes of  $\mathfrak{S}_n$  are labeled by the set  $\mathcal{P}_n$  of partitions of  $n$ . Write  $\mathcal{O}_n$  for the set of  $\pi \in \mathcal{P}_n$  such that all parts of  $\pi$  have odd length, and  $\mathcal{D}_n$  for the set of  $\pi \in \mathcal{P}_n$  with distinct parts. The partitions in  $\mathcal{D}_n$  are called *bar partitions*. Denote by  $\mathcal{D}_n^+$  (respectively  $\mathcal{D}_n^-$ ) the subset of  $\mathcal{D}_n$  consisting of all partitions  $\pi \in \mathcal{D}_n$  such that the number of parts of  $\pi$  with an even length is even (respectively odd). Schur proved (see [26, §7])

**Proposition 4.1.** *The split conjugacy classes of  $\tilde{\mathfrak{S}}_n$  are those classes  $C$  such that  $\theta(C)$  is labeled by  $\mathcal{O}_n \cup \mathcal{D}_n^-$ .*

We set  $s_i = (i, i+1) \in \mathfrak{S}_n$ . Then for every  $1 \leq i \leq n-1$ , we have  $\theta(t_i) = s_i$ . For  $\pi = (\pi_1, \dots, \pi_k) \in \mathcal{P}_n$ , write  $s_\pi$  for a representative of the class of  $\mathfrak{S}_n$  labeled by  $\pi$ . If  $s_\pi = s_{\pi_1} \cdots s_{\pi_k}$  is the cycle decomposition (with disjoint supports) of  $s_\pi$ , then we assume that the support of  $s_{\pi_i}$  is

$$(41) \quad \left\{ 1 + \sum_{j < i} \pi_j, \dots, \sum_{j \leq i} \pi_j \right\}.$$

Now, for any  $\pi \in \mathcal{P}_n$ , we make the same choice of Schur [26, §11] for a representative  $t_\pi \in \tilde{\mathfrak{S}}_n$  such that  $\theta(t_\pi) = s_\pi$ . So, when  $\pi \in \mathcal{O}_n \cup \mathcal{D}_n^-$ , the elements  $t_\pi$  and  $zt_\pi$  are representatives of the two split classes of  $\tilde{\mathfrak{S}}_n$  labeled by  $\pi$ . We denote by  $C_\pi^+$  (respectively  $C_\pi^-$ ) the conjugacy class of  $t_\pi$  (respectively  $zt_\pi$ ) in  $\tilde{\mathfrak{S}}_n$ . It will also sometimes be convenient to write  $t_\pi^+$  for  $t_\pi$ , and  $t_\pi^-$  for  $zt_\pi$ . When  $\pi \in \mathcal{P}_n \setminus (\mathcal{O}_n \cup \mathcal{D}_n^-)$ , the elements  $t_\pi$  and  $zt_\pi$  belong to the same conjugacy class  $C_\pi$  of  $\tilde{\mathfrak{S}}_n$ . In all cases, an element  $g$  (or an  $\tilde{\mathfrak{S}}_n$ -class  $C$ ) is said to be of *type*  $\pi$  if the  $\mathfrak{S}_n$ -class of  $\theta(g)$  (respectively of  $\theta(g)$  for any  $g \in C$ ) is labeled by  $\pi$ .

Note that if  $\pi = (\pi_1, \dots, \pi_k) \in \mathcal{P}_n$ , then the construction of [26, III p. 172] implies that

$$(42) \quad t_\pi = t_{\pi_1} \cdots t_{\pi_k}.$$

**Convention 4.2.** *Let  $\pi = (\pi_1, \dots, \pi_k) \in \mathcal{P}_n$ . In the following, we do not necessarily assume as usual that  $\pi_1 \geq \dots \geq \pi_k$ . Instead, we assume that the parts of  $\pi$  are ordered in such a way that  $\pi_1 \geq \dots \geq \pi_u$  and  $\pi_{u+1} \geq \dots \geq \pi_k$ , where  $u$  is such that  $\pi_{u+1}, \dots, \pi_k$  are exactly the odd parts of  $\pi$  which are divisible by  $p$  (if there is no such part in  $\pi$ , then  $u = k$ ).*

We are now interested in the set of irreducible complex characters of  $\tilde{\mathfrak{S}}_n$ . Any irreducible (complex) character of  $\tilde{\mathfrak{S}}_n$  with  $z$  in its kernel is simply lifted from an irreducible character of the quotient  $\mathfrak{S}_n$ . Any other irreducible character  $\xi$  of  $\tilde{\mathfrak{S}}_n$  is called a *spin character*, and it satisfies  $\xi(z) = -\xi(1)$ . In particular, for any spin

character  $\xi$  and any  $\pi \in \mathcal{P}_n$ , one has  $\xi(\tilde{z}t_\pi) = -\xi(t_\pi)$ , which implies that  $\xi$  vanishes on the non-split conjugacy classes of  $\tilde{\mathfrak{S}}_n$ .

Define  $\varepsilon = \text{sgn} \circ \theta$ , where  $\text{sgn}$  is the sign character of  $\mathfrak{S}_n$ . Note that  $\tilde{\mathcal{A}}_n = \ker(\varepsilon)$ . Then  $\varepsilon$  is a linear (irreducible) character of  $\tilde{\mathfrak{S}}_n$ , and for any spin character  $\xi$  of  $\tilde{\mathfrak{S}}_n$ ,  $\varepsilon \otimes \xi$  is a spin character (because  $\varepsilon \otimes \chi(z) = -\varepsilon \otimes \chi(1)$ ). A spin character  $\xi$  is said to be *self-associate* if  $\varepsilon\xi = \xi$ . Otherwise,  $\xi$  and  $\varepsilon\xi$  are called *associate characters*. It follows that, if  $\xi$  is self-associate and  $\pi \in \mathcal{D}_n^-$ , then  $\xi(t_\pi) = 0 = \xi(\tilde{z}t_\pi)$ .

In [26], Schur proved that the spin characters of  $\tilde{\mathfrak{S}}_n$  are, up to association, labeled by  $\mathcal{D}_n$ . More precisely, he showed that every  $\lambda \in \mathcal{D}_n^+$  indexes a self-associate spin character  $\xi_\lambda$ , and every  $\lambda \in \mathcal{D}_n^-$  a pair  $(\xi_\lambda^+, \xi_\lambda^-)$  of associate spin characters. In this case, we will sometimes write  $\xi_\lambda$  for  $\xi_\lambda^+$ , so that  $\xi_\lambda^- = \varepsilon\xi_\lambda$ .

For any partition  $\lambda = (\lambda_1, \dots, \lambda_k)$  of  $n$  (where we don't include 0 parts), we set  $|\lambda| = \sum \lambda_i$  and we define the *length*  $\ell(\lambda)$  of  $\lambda$  by  $\ell(\lambda) = k$ . If  $\lambda$  is furthermore a bar partition (i.e. if the parts of  $\lambda$  are pairwise distinct), then we set  $\sigma(\lambda) = (-1)^{|\lambda| - \ell(\lambda)}$ . With this notation, we then have (see e.g. [22, p. 45])

$$(43) \quad \lambda \in \mathcal{D}_n^{\sigma(\lambda)}.$$

If  $\sigma(\lambda) = 1$ , then  $\lambda$  is said to be *even*; otherwise, it is said to be *odd*.

Schur proved in [26] that, whenever  $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathcal{D}_n^-$ , the labeling can be chosen in such a way that, for any  $\pi \in \mathcal{D}_n^-$ , we have

$$(44) \quad \xi_\lambda^+(t_\pi) = \delta_{\pi\lambda} i^{\frac{n-k+1}{2}} \sqrt{\frac{\lambda_1 \dots \lambda_k}{2}}.$$

Writing  $z_\lambda$  for the product  $\lambda_1 \dots \lambda_k$ , we therefore have, for any  $\pi \in \mathcal{D}_n^-$ ,

$$(45) \quad \xi_\lambda^+(t_\pi) = \xi_\lambda^-(\tilde{z}t_\pi) = \delta_{\pi\lambda} i^{\frac{n-r+1}{2}} \sqrt{\frac{z_\lambda}{2}} \quad \text{and} \quad \xi_\lambda^+(\tilde{z}t_\pi) = \xi_\lambda^-(t_\pi) = -\delta_{\pi\lambda} i^{\frac{n-r+1}{2}} \sqrt{\frac{z_\lambda}{2}}.$$

Finally, for any  $\pi \in \mathcal{O}_n$ , we have

$$\xi_\lambda^+(t_\pi) = \xi_\lambda^-(t_\pi) \quad \text{and} \quad \xi_\lambda^+(\tilde{z}t_\pi) = \xi_\lambda^-(\tilde{z}t_\pi).$$

**4.2. Conjugacy classes and spin characters of  $\tilde{\mathcal{A}}_n$ .** We also write  $\theta : \tilde{\mathcal{A}}_n \rightarrow \mathcal{A}_n$  for the restriction of  $\theta$  to  $\tilde{\mathcal{A}}_n$ . As above, the type of  $g \in \tilde{\mathcal{A}}_n$  is the partition encoding the cycle structure of  $\theta(g)$ . As above, there is a notion of split classes with respect to  $\theta$ . Such classes will be called  $\mathcal{A}_n$ -split in the following. On the other hand, since  $\tilde{\mathcal{A}}_n$  is a subgroup of  $\tilde{\mathfrak{S}}_n$  with index 2, every  $\tilde{\mathfrak{S}}_n$ -class contained in  $\tilde{\mathcal{A}}_n$  is either a single  $\tilde{\mathcal{A}}_n$ -class or a union of two  $\tilde{\mathcal{A}}_n$ -classes. In the second case, the  $\tilde{\mathcal{A}}_n$ -classes will be called  $\tilde{\mathfrak{S}}_n$ -split classes. By [26, p. 176], we have

**Proposition 4.3.** *Assume  $n \geq 2$ . The  $\mathcal{A}_n$ -split classes of  $\tilde{\mathcal{A}}_n$  are the classes whose elements have type  $\pi \in \mathcal{D}_n^+ \cup \mathcal{O}_n$ .*

**Remark 4.4.** Let  $t \in \tilde{\mathcal{A}}_n$  be with support contained in  $X = \{k, k+1, \dots, k+l\}$  for some  $1 \leq k < n$  and  $1 \leq l$  with  $k+l \leq n$ . Let  $1 \leq i \leq n-1$  be such that  $\{i, i+1\} \cap X = \emptyset$ . Then  $t$  and  $t_i$  commute. Indeed, since  $\varepsilon(t) = 1$ , there are integers  $k \leq j_1, \dots, j_{2r} \leq k+l-1$  such that  $t = t_{j_1} \dots t_{j_{2r}}$ . Furthermore, we have  $|i - j_u| \geq 2$  for all  $1 \leq u \leq 2r$ . Hence,  ${}^i t_u = z t_u$  and  ${}^i t = z^{2r} t = t$ , as required.

Assume  $n \geq 2$ . Let  $\pi \in \mathcal{D}_n^+ \cup \mathcal{O}_n$ . If  $\pi \notin \mathcal{D}_n^+ \cap \mathcal{O}_n$ , then  $\pi$  labels two classes  $D_\pi^+$  and  $D_\pi^-$  of  $\tilde{\mathcal{A}}_n$ . We assume that  $t_\pi$  defined above lies in  $D_\pi^+$ , so that representatives for  $D_\pi^+$  and  $D_\pi^-$  are  $\tau_\pi^+ = t_\pi$  and  $\tau_\pi^- = zt_\pi$ .

Otherwise, if  $\pi \in \mathcal{D}_n^+ \cap \mathcal{O}_n$ , then  $\pi$  labels four  $\mathcal{A}_n$ -classes (that are  $\mathcal{A}_n$ -split and  $\tilde{\mathfrak{S}}_n$ -split simultaneously). Write  $\pi = (\pi_1, \dots, \pi_k)$  with respect to Convention 4.2, and  $o_{\pi_1} = z^{\frac{\pi_1^2-1}{8}} t_{\pi_1}$ . According to [26, footnote (\*), p. 179],  $o_{\pi_1}$  has odd order. Furthermore, we assume that the support of  $s_{\pi_j}$  is as in Equation (41). So,  $t_1 \in \tilde{\mathfrak{S}}_{\pi_1}$ . Since  $\pi_1 \in \mathcal{D}_{\pi_1}^+ \cap \mathcal{O}_{\pi_1}$ , the elements  $o_{\pi_1}$  and  ${}^{t_1}o_{\pi_1}$  are not  $\tilde{\mathcal{A}}_{\pi_1}$ -conjugate. Now, we get  $\tau_\pi^{++} = t_{\pi_k} \cdots t_{\pi_2} o_{\pi_1}$  and

$$(46) \quad \tau_\pi^{+-} = {}^{t_1}\tau_\pi^{++}, \quad \tau_\pi^{-+} = z\tau_\pi^{++} \quad \text{and} \quad \tau_\pi^{--} = z\tau_\pi^{+-}.$$

So,  $\tau_\pi^{++}$ ,  $\tau_\pi^{-+}$ ,  $\tau_\pi^{+-}$  and  $\tau_\pi^{--}$  belong to 4 distinct  $\tilde{\mathcal{A}}_n$ -classes, labeled  $D_\pi^{++}$ ,  $D_\pi^{-+}$ ,  $D_\pi^{+-}$  and  $D_\pi^{--}$  respectively. Note that  ${}^{t_1}t_{\pi_j} = t_{\pi_j}$  for  $j > 1$  by Remark 4.4. In particular, one has  $\tau_\pi^{+-} = t_{\pi_k} \cdots t_{\pi_2} {}^{t_1}o_{\pi_1}$ .

We can now describe the irreducible complex characters of  $\tilde{\mathcal{A}}_n$ . These are given by using Clifford's Theory between  $\tilde{\mathfrak{S}}_n$  and the subgroup  $\tilde{\mathcal{A}}_n$  of index 2. All the irreducible components of the restrictions to  $\tilde{\mathcal{A}}_n$  of non-spin characters of  $\tilde{\mathfrak{S}}_n$  have  $z$  in their kernel, whence are non-spin characters of  $\tilde{\mathcal{A}}_n$ . They are exactly the irreducible characters of  $\tilde{\mathcal{A}}_n$  lifted from those of  $\mathcal{A}_n$ .

We now turn to spin characters (which are the irreducible components of the restrictions to  $\tilde{\mathcal{A}}_n$  of the spin characters of  $\tilde{\mathfrak{S}}_n$ ).

First consider  $\lambda \in \mathcal{D}_n^-$ . Then  $\lambda$  labels two associated spin characters  $\xi_\lambda^+$  and  $\xi_\lambda^-$  of  $\tilde{\mathfrak{S}}_n$ , which have the same restriction to  $\tilde{\mathcal{A}}_n$ . The restriction

$$(47) \quad \zeta_\lambda = \text{Res}_{\tilde{\mathcal{A}}_n}^{\tilde{\mathfrak{S}}_n}(\xi_\lambda^+) = \text{Res}_{\tilde{\mathcal{A}}_n}^{\tilde{\mathfrak{S}}_n}(\xi_\lambda^-)$$

is an irreducible spin character of  $\tilde{\mathcal{A}}_n$ , and its only non-zero values are taken on elements of type belonging to  $\mathcal{O}_n$ .

Now consider  $\lambda \in \mathcal{D}_n^+$ . Then  $\lambda$  labels a single spin character  $\xi_\lambda$  of  $\tilde{\mathfrak{S}}_n$ , and  $\text{Res}_{\tilde{\mathcal{A}}_n}^{\tilde{\mathfrak{S}}_n}(\xi_\lambda) = \zeta_\lambda^+ + \zeta_\lambda^-$ , where  $\zeta_\lambda^+$  and  $\zeta_\lambda^-$  are two conjugate irreducible spin characters of  $\tilde{\mathcal{A}}_n$ . Throughout, the characters  $\zeta_\lambda^+$  and  $\zeta_\lambda^-$  are also called *associate characters*. These only differ on elements of type  $\lambda$ . Following Schur [26, p. 236], we have, writing  $\Delta_\lambda$  for the *difference character* of  $\xi_\lambda$  (which is not well-defined, but just up to a sign), that

$$(48) \quad \Delta_\lambda(t) = \begin{cases} \pm i^{\frac{n-\ell(\lambda)}{2}} \sqrt{z_\lambda} & \text{if } t \text{ has type } \lambda, \\ 0 & \text{otherwise,} \end{cases}$$

where  $z_\lambda$  is defined after Equation (44).

We will now make the notation precise. We distinguish two cases. Suppose first that  $\lambda \in \mathcal{D}_n^+ \setminus \mathcal{O}_n$ . Then  $\zeta_\lambda^+$  and  $\zeta_\lambda^-$  are completely defined by setting  $\Delta_\lambda = \zeta_\lambda^+ - \zeta_\lambda^-$  and  $\Delta_\lambda(\tau_\lambda^+) = i^{\frac{n-\ell(\lambda)}{2}} \sqrt{z_\lambda}$ , where  $\tau_\lambda^+$  is the representative of  $D_\lambda^+$  as above. Note that, using Equation (48) and  $\tau_\lambda^- = z\tau_\lambda^+$ , we deduce that  $\Delta_\lambda(\tau_\lambda^-) = -i^{\frac{n-\ell(\lambda)}{2}} \sqrt{z_\lambda}$ . Since, for  $\epsilon \in \{-1, 1\}$ ,

$$(49) \quad \zeta_\lambda^\epsilon = \frac{1}{2} \left( \text{Res}_{\tilde{\mathcal{A}}_n}^{\tilde{\mathfrak{S}}_n}(\xi_\lambda) + \epsilon \Delta_\lambda \right),$$

and  $\xi_\lambda(t_\lambda) = 0$  (because  $C_\lambda$  is a non-split class of  $\tilde{\mathfrak{S}}_n$ ), we obtain

$$(50) \quad \zeta_\lambda^+(\tau_\lambda^\pm) = \frac{1}{2}\Delta_\lambda(\tau_\lambda^\pm) \quad \text{and} \quad \zeta_\lambda^-(\tau_\lambda^\pm) = \frac{1}{2}\Delta_\lambda(\tau_\lambda^\mp) = -\frac{1}{2}\Delta_\lambda(\tau_\lambda^\pm).$$

And, on any element  $\sigma$  of type  $\pi \neq \lambda$  of  $\tilde{\mathcal{A}}_n$ , we have

$$(51) \quad \zeta_\lambda^+(\sigma) = \zeta_\lambda^-(\sigma) = \frac{1}{2}\xi_\lambda(\sigma).$$

Now suppose that  $\lambda \in \mathcal{D}_n^+ \cap \mathcal{O}_n$ . Again, we completely define  $\zeta_\lambda^+$  and  $\zeta_\lambda^-$  by setting  $\Delta_\lambda = \zeta_\lambda^+ - \zeta_\lambda^-$  and  $\Delta_\lambda(\tau_\lambda^{++}) = i^{\frac{n-\ell(\lambda)}{2}}\sqrt{z_\lambda}$ . Note that this does define  $\Delta_\lambda$ , and thus  $\zeta_\lambda^+$  and  $\zeta_\lambda^-$  by Equation (49), since we have  $\Delta_\lambda(\tau_\lambda^{-+}) = -\Delta_\lambda(\tau_\lambda^{++})$  because  $\tau_\lambda^{-+} = z\tau_\lambda^{++}$ , and  $\Delta_\lambda(\tau_\lambda^{+-}) = -\Delta_\lambda(\tau_\lambda^{++})$  by Clifford theory, because the elements  $\tau_\lambda^{+-}$  and  $\tau_\lambda^{++}$  are  $\tilde{\mathfrak{S}}_n$ -conjugate, and  $\zeta_\lambda^+$  and  $\zeta_\lambda^-$  are  $\tilde{\mathfrak{S}}_n$ -conjugate. Finally, on any element  $\sigma$  of type  $\pi \neq \lambda$  of  $\tilde{\mathcal{A}}_n$ , Equation (51) holds.

**4.3. Combinatorics of bar partitions.** We just saw that the spin characters of  $\tilde{\mathfrak{S}}_n$  and  $\tilde{\mathcal{A}}_n$  are labeled by the set  $\mathcal{D}_n$  of *bar partitions* of  $n$ . We now present some of the combinatorial notions and properties we will need to study the characters and blocks of these groups. For all of these, and unless otherwise specified, we refer to [22]. Note that, in this subsection and the next, where we only describe the standard combinatorics associated to bar partition and spin blocks, the parts of partitions and bar partitions are again ordered in decreasing order.

Let  $\lambda = (\lambda_1, \dots, \lambda_r) \in \mathcal{D}_n$  with  $\lambda_1 > \dots > \lambda_r > 0$ . For  $1 \leq i \leq r$ , consider the set

$$J_{i,\lambda} = (\{1, \dots, \lambda_i\} \cup \{\lambda_i + \lambda_j \mid j > i\}) \setminus \{\lambda_i - \lambda_j \mid j > i\}.$$

The multiset  $\mathcal{B}(\lambda) = \bigcup_{i=1}^r J_{i,\lambda}$  is the multiset of *bar lengths* of  $\lambda$ , which will play a role analogous to that played by hook lengths for partitions.

The *shifted tableau*  $S(\lambda)$  of  $\lambda$  is obtained from the usual Young diagram of  $\lambda$  by shifting the  $i$ th row  $i - 1$  positions to the right, and writing in the nodes of the  $i$ th row the elements of  $J_{i,\lambda}$  in decreasing order. The  $j$ th node in the  $i$ th row of  $S(\lambda)$  will be called the  $(i, j)$ -node of  $S(\lambda)$ . Write  $a_{i,j}$  for the integer lying in the  $(i, j)$ -node of  $S(\lambda)$ . As in the case of hooks, we can associate to this node a *bar*  $b_{i,j}$  of  $\lambda$  whose length is  $a_{i,j}$ . The construction goes as follows. If  $i + j \geq r + 2$ , then  $b_{i,j}$  is the usual  $(i, j)$ -hook of  $S(\lambda)$ . If  $i + j = r + 1$ , then  $b_{i,j}$  is the  $i$ th row of  $S(\lambda)$ . Finally, if  $i + j \leq r$ , then  $b_{i,j}$  is the union of the  $i$ th row together with the  $j$ th row of  $S(\lambda)$ . In all cases, one checks that  $a_{i,j}$  is exactly the number of nodes in  $b_{i,j}$ , and is therefore called the *bar length* of  $b_{i,j}$ . We can also define the *leg length*  $L(b_{i,j})$  of the bar  $b_{i,j}$  by setting

$$L(b_{i,j}) = \begin{cases} |\{k \mid \lambda_i > \lambda_k > \lambda_i - a_{i,j}\}| & \text{if } i + j \geq r + 1, \\ \lambda_{i+j} + |\{k \mid \lambda_i > \lambda_k > \lambda_{i+j}\}| & \text{if } i + j \leq r. \end{cases}$$

As for hooks, it is always possible to *remove* any bar  $b$  from  $S(\lambda)$ . If  $b$  has bar length  $a$ , then this operation produces a new bar partition, written  $\lambda \setminus b$ , of size  $n - a$ .

Let  $q$  be an odd integer. We call  $q$ -bar (respectively  $(q)$ -bar) any bar  $b$  of  $\lambda$  whose length is  $q$  (respectively divisible by  $q$ ). Note that, for any positive integer  $k$ , the removal of a  $kq$ -bar can be achieved by successively removing  $k$  bars of length  $q$  (this fails when  $q$  is even). By removing all the  $(q)$ -bars in  $\lambda$ , one obtains the  $\bar{q}$ -core  $\lambda_{(\bar{q})}$  of  $\lambda$ . One can show that  $\lambda_{(\bar{q})}$  is independent on the order in which one

removes  $q$ -bars from  $\lambda$ . In particular, the total number  $w_{\bar{q}}(\lambda)$  of  $q$ -bars to remove from  $\lambda$  to get to  $\lambda_{(\bar{q})}$  is uniquely defined by  $\lambda$  and  $q$ , and called the  $\bar{q}$ -weight of  $\lambda$ . Note that  $w_{\bar{q}}(\lambda)$  is also equal to the number of  $(q)$ -bars in  $\lambda$ .

It is also possible to define the  $\bar{q}$ -quotient  $\lambda^{(\bar{q})}$  of  $\lambda$ , which contains the information about all the  $(q)$ -bars in  $\lambda$  (see [22, p. 28]). We have  $\lambda^{(\bar{q})} = (\lambda^0, \lambda^1, \dots, \lambda^e)$ , where  $e = (q-1)/2$ ,  $\lambda^0$  is a bar partition, and the  $\lambda^i$ 's are partitions for  $1 \leq i \leq e$ . For any integer  $k$ , we define a  $k$ -bar  $b'$  of  $\lambda^{(\bar{q})} = (\lambda^0, \lambda^1, \dots, \lambda^e)$  to be either a  $k$ -bar of  $\lambda^0$ , or a  $k$ -hook of  $\lambda^i$  for some  $1 \leq i \leq e$ . The removal of  $b'$  from  $\lambda^{(\bar{q})}$  is then defined accordingly, and the resulting  $\bar{q}$ -quotient is denoted by  $\lambda^{(\bar{q})} \setminus b'$ . The leg length  $L(b')$  is also defined in a natural manner. We then have the following fundamental result (see [22, Proposition 4.2, Theorem 4.3])

**Theorem 4.5.** *Let  $q$  be an odd integer. Then a bar partition  $\lambda$  determines and is uniquely determined by its  $\bar{q}$ -core  $\lambda_{(\bar{q})}$  and its  $\bar{q}$ -quotient  $\lambda^{(\bar{q})}$ . Moreover, there is a canonical bijection  $g$  between the set of  $(q)$ -bars of  $\lambda$  and the set of bars of  $\lambda^{(\bar{q})}$ , such that, for each integer  $k$ , the image of a  $kq$ -bar of  $\lambda$  is a  $k$ -bar of  $\lambda^{(\bar{q})}$ . Furthermore, for the removal of corresponding bars, we have*

$$(\lambda \setminus b)^{(\bar{q})} = \lambda^{(\bar{q})} \setminus g(b).$$

Note that the above theorem provides a (canonical) bijection between the set of parts of  $\lambda$  with length divisible by  $q$  and the set of parts of  $\lambda^0$  (see [22, Corollary (4.6)]).

Theorem 4.5 also implies that the  $\bar{q}$ -weight  $w_{\bar{q}}(\lambda)$  of  $\lambda$  satisfies  $w_{\bar{q}}(\lambda) = \sum_{i=0}^e |\lambda^i|$  (we say that  $\lambda^{(\bar{q})}$  is a  $\bar{q}$ -quotient of size  $|\lambda^{(\bar{q})}| = w_{\bar{q}}(\lambda)$ ), and that  $|\lambda| = |\lambda_{(\bar{q})}| + qw_{\bar{q}}(\lambda)$  (see [22, Corollary 4.4]). In addition, if we write, in analogy with bar partitions,  $\sigma(\lambda^{(\bar{q})}) = (-1)^{|\lambda^{(\bar{q})}| - \ell(\lambda^0)} = (-1)^{w_{\bar{q}}(\lambda) - \ell(\lambda^0)}$ , then we obtain that

$$(52) \quad \sigma(\lambda) = \sigma(\lambda_{(\bar{q})})\sigma(\lambda^{(\bar{q})}).$$

When we introduce analogues of the Murnaghan-Nakayama rule for spin characters later on, we will also need to use the *relative sign* for bar partitions introduced by Morris and Olsson in [19]. Given an odd integer  $q$ , one can associate in a canonical way to each bar partition  $\lambda$  a sign  $\delta_{\bar{q}}(\lambda)$ . If  $\mu$  is a bar partition obtained from  $\lambda$  by removing a sequence of  $q$ -bars, then we define the *relative sign*  $\delta_{\bar{q}}(\lambda, \mu)$  by

$$(53) \quad \delta_{\bar{q}}(\lambda, \mu) = \delta_{\bar{q}}(\lambda)\delta_{\bar{q}}(\mu).$$

It is then possible to prove the following results (see [19, Proposition (2.5), Corollary (2.6), Corollary (3.8)]):

**Theorem 4.6.** *Let  $\lambda$  and  $\mu$  be bar partitions, and  $q$  be an odd integer.*

- (i) *If  $\mu$  is obtained from  $\lambda$  by removing a sequence of  $q$ -bars with leg lengths  $L_1, \dots, L_s$ , then*

$$\delta_{\bar{q}}(\lambda, \mu) = (-1)^{\sum_{i=1}^s L_i}.$$

*In particular, the parity of  $\sum_{i=1}^s L_i$  does not depend on the choice of  $q$ -bars being removed in going from  $\lambda$  to  $\mu$ .*

- (ii) *If  $\gamma$  is a  $\bar{q}$ -core, then  $\delta_{\bar{q}}(\gamma) = 1$ , so that*

$$\delta_{\bar{q}}(\lambda) = \delta_{\bar{q}}(\lambda, \lambda_{(\bar{q})}).$$

(iii) If  $b$  is a  $(q)$ -bar in  $\lambda$  and  $\mu = \lambda \setminus b$ , then

$$(-1)^{L(b)} = (-1)^{L(g(b))} \delta_{\bar{q}}(\lambda, \mu),$$

where  $g$  is the bijection introduced in Theorem 4.5.

**4.4. Spin blocks of  $\tilde{\mathfrak{S}}_n$  and  $\tilde{\mathcal{A}}_n$ ; Bijections.** We now describe the blocks of irreducible characters of  $\tilde{\mathfrak{S}}_n$  and  $\tilde{\mathcal{A}}_n$ , as well as bijections between them. Throughout this section, we assume that  $q$  is an odd prime (even though all the combinatorial arguments hold for any odd  $q$ ).

If  $B$  is any  $q$ -block of  $\tilde{\mathfrak{S}}_n$ , then  $B$  contains either no or only spin characters. In the former case,  $B$  coincides with a  $q$ -block of  $\mathfrak{S}_n$ ; in the latter, we say that  $B$  is a *spin block*. The distribution of spin characters into spin blocks was first conjectured by Morris. It was first proved by J. F. Humphreys in [11], then differently by M. Cabanes, who also determined the structure of the defect groups of spin blocks (see [3]).

Similarly, any  $q$ -block  $B^*$  of  $\tilde{\mathcal{A}}_n$  contains either no spin character, and coincides with a  $q$ -block of  $\mathcal{A}_n$ , or only spin characters, and is then called a spin block.

The spin blocks of  $\tilde{\mathfrak{S}}_n$  and  $\tilde{\mathcal{A}}_n$  are described by the following:

**Theorem 4.7.** *Let  $\chi$  and  $\psi$  be two spin characters of  $\tilde{\mathfrak{S}}_n$ , or two spin characters of  $\tilde{\mathcal{A}}_n$ , labeled by bar partitions  $\lambda$  and  $\mu$  respectively, and let  $q$  be an odd prime. Then  $\chi$  is of  $q$ -defect 0 (and thus alone in its  $q$ -block) if and only if  $\lambda$  is a  $\bar{q}$ -core. If  $\lambda$  is not a  $\bar{q}$ -core, then  $\chi$  and  $\psi$  belong to the same  $q$ -block if and only if  $\lambda_{(\bar{q})} = \mu_{(\bar{q})}$ .*

One can therefore define the  $\bar{q}$ -core of a spin block  $B$  and its  $\bar{q}$ -weight  $w_{\bar{q}}(B)$ , as well as its *sign*  $\sigma(B) = \sigma(\lambda_{(\bar{q})})$  (for any bar partition  $\lambda$  labeling some character  $\chi \in B$ ).

One sees that the spin  $q$ -blocks of positive weight (or defect) of  $\tilde{\mathfrak{S}}_n$  can be paired with those of  $\tilde{\mathcal{A}}_n$ . The spin characters in any such  $q$ -block of  $\tilde{\mathcal{A}}_n$  are exactly the irreducible components of the spin characters of a  $q$ -block of  $\tilde{\mathfrak{S}}_n$ .

We can now define bijections between different blocks of possibly different groups. Let  $w > 0$  be any integer, and let  $Q_w$  be the set of  $\bar{q}$ -quotients of size  $w$ . For any  $\bar{q}$ -core  $\gamma$ , we let  $E_{\gamma,w}$  be the set of bar partitions  $\lambda$  of length  $|\gamma| + qw$  with  $w_{\bar{q}}(\lambda) = w$  and  $\lambda_{(\bar{q})} = \gamma$ , and we denote by  $B_{\gamma,w}$  and  $B_{\gamma,w}^*$  the spin  $\bar{q}$ -blocks of  $\tilde{\mathfrak{S}}_{|\gamma|+qw}$  and  $\tilde{\mathcal{A}}_{|\gamma|+qw}$  respectively labeled by  $\gamma$ . Note that the characters in  $B_{\gamma,w}$  and those in  $B_{\gamma,w}^*$  are labeled by the partitions in  $E_{\gamma,w}$ . Note also that

$$\Psi_{\gamma}: \begin{cases} E_{\gamma,w} & \longrightarrow & Q_w \\ \lambda & \longmapsto & \lambda^{(\bar{q})} \end{cases}$$

is a bijection. It provides us with the following:

**Lemma 4.8.** *Let  $q$  be an odd prime,  $w > 0$  be any integer, and  $\gamma$  and  $\gamma'$  be any  $\bar{q}$ -cores. Define the bijection*

$$\Psi = \Psi_{\gamma'}^{-1} \circ \Psi_{\gamma}: E_{\gamma,w} \longrightarrow E_{\gamma',w}.$$

- (i) *If  $\sigma(\gamma) = \sigma(\gamma')$ , then  $\Psi$  induces bijections  $\tilde{\Psi}$  between  $B_{\gamma,w}$  and  $B_{\gamma',w}$ , and  $\tilde{\Psi}^*$  between  $B_{\gamma,w}^*$  and  $B_{\gamma',w}^*$ .*
- (ii) *If  $\sigma(\gamma) = -\sigma(\gamma')$ , then  $\Psi$  induces bijections  $\tilde{\Psi}$  between  $B_{\gamma,w}$  and  $B_{\gamma',w}^*$ , and  $\tilde{\Psi}^*$  between  $B_{\gamma,w}^*$  and  $B_{\gamma',w}$ .*

*Proof.* This follows easily from the definition of  $\Psi$  and formula (52), which gives that, for any  $\lambda \in E_{\gamma,w}$  and  $\mu \in E_{\gamma',w}$ , we have  $\sigma(\lambda) = \sigma(\gamma)\sigma(\Psi_\gamma(\lambda))$  and  $\sigma(\mu) = \sigma(\gamma')\sigma(\Psi_{\gamma'}(\mu))$ . We therefore have (taking  $\mu$  to be  $\Psi(\lambda)$ )

$$\sigma(\Psi(\lambda)) = \sigma(\gamma')\sigma(\Psi_{\gamma'}(\Psi(\lambda))) = \sigma(\gamma')\sigma(\Psi_{\gamma'}(\Psi_{\gamma'}^{-1}(\Psi_\gamma(\lambda)))) = \sigma(\gamma')\sigma(\Psi_\gamma(\lambda)),$$

so that  $\sigma(\Psi_\gamma(\lambda)) = \sigma(\gamma')\sigma(\Psi(\lambda))$  and, finally,

$$\sigma(\lambda) = \sigma(\gamma)\sigma(\gamma')\sigma(\Psi(\lambda)) \quad \text{for any } \lambda \in E_{\gamma,w}.$$

This means that, if  $\sigma(\gamma) = \sigma(\gamma')$ , then any  $\lambda \in E_{\gamma,w}$  labels the same numbers of spin characters in  $\tilde{\mathfrak{S}}_{|\gamma|+qw}$  and  $\tilde{\mathcal{A}}_{|\gamma|+qw}$  as  $\Psi(\lambda)$  does in  $\tilde{\mathfrak{S}}_{|\gamma'|+qw}$  and  $\tilde{\mathcal{A}}_{|\gamma'|+qw}$  respectively. If, on the other hand,  $\sigma(\gamma) = -\sigma(\gamma')$ , then  $\lambda$  labels the same numbers of spin characters in  $\tilde{\mathfrak{S}}_{|\gamma|+qw}$  and  $\tilde{\mathcal{A}}_{|\gamma|+qw}$  as  $\Psi(\lambda)$  does in  $\tilde{\mathcal{A}}_{|\gamma'|+qw}$  and  $\tilde{\mathfrak{S}}_{|\gamma'|+qw}$  respectively.

We obtain the following description for the bijections  $\tilde{\Psi}$  and  $\tilde{\Psi}^*$ :

$$\begin{aligned} & \text{(i) If } \sigma(\gamma) = \sigma(\gamma'), \text{ then, for any } \lambda, \mu \in E_{\gamma,w} \text{ with } \sigma(\lambda) = 1 \text{ and } \sigma(\mu) = -1, \\ \tilde{\Psi} : & \begin{cases} \xi_\lambda & \mapsto & \xi_{\Psi(\lambda)} \\ \{\xi_\mu^+, \xi_\mu^-\} & \mapsto & \{\xi_{\Psi(\mu)}^+, \xi_{\Psi(\mu)}^-\} \end{cases} \quad \text{and } \tilde{\Psi}^* : \begin{cases} \{\zeta_\lambda^+, \zeta_\lambda^-\} & \mapsto & \{\zeta_{\Psi(\lambda)}^+, \zeta_{\Psi(\lambda)}^-\} \\ \zeta_\mu & \mapsto & \zeta_{\Psi(\mu)} \end{cases} \\ & \text{(ii) If } \sigma(\gamma) = -\sigma(\gamma'), \text{ then, for any } \lambda, \mu \in E_{\gamma,w} \text{ with } \sigma(\lambda) = 1 \text{ and } \sigma(\mu) = -1, \\ \tilde{\Psi} : & \begin{cases} \xi_\lambda & \mapsto & \zeta_{\Psi(\lambda)} \\ \{\xi_\mu^+, \xi_\mu^-\} & \mapsto & \{\zeta_{\Psi(\mu)}^+, \zeta_{\Psi(\mu)}^-\} \end{cases} \quad \text{and } \tilde{\Psi}^* : \begin{cases} \{\zeta_\lambda^+, \zeta_\lambda^-\} & \mapsto & \{\xi_{\Psi(\lambda)}^+, \xi_{\Psi(\lambda)}^-\} \\ \zeta_\mu & \mapsto & \xi_{\Psi(\mu)} \end{cases} \end{aligned}$$

□

**4.5. Morris' Recursion Formula and MN-structures for  $\tilde{\mathfrak{S}}_n$  and  $\tilde{\mathcal{A}}_n$ .** First, for the convenience of the reader, we prove the following useful lemma.

**Lemma 4.9.** *Let  $\rho \in \mathfrak{S}_n$  be an element of odd order. Then the set  $\theta^{-1}(\rho)$  has an element of odd order.*

*Proof.* Let  $g \in \theta^{-1}(\rho)$ , so that  $\theta^{-1}(\rho) = \{g, zg\}$ , and let  $d$  be the order of  $\rho$ . Since  $\theta(g^d) = \theta(g)^d = \rho^d = 1$ , we obtain  $g^d \in \{1, z\}$ . If  $g^d = 1$ , then the order of  $g$  is odd. Otherwise,  $g^d = z$ , and  $(zg)^d = z^d g^d = z^2 = 1$  because  $d$  is odd. Thus  $zg$  has odd order, as required. □

In the following, if  $\rho \in \mathfrak{S}_n$  has odd order, then we denote by  $o_\rho$  the element of  $\theta^{-1}(\rho)$  with odd order.

A. O. Morris was the first to prove a recursion formula, similar to the Murnaghan-Nakayama Rule, for computing the values of spin characters of  $\tilde{\mathfrak{S}}_n$  (see [17] and [18]). This formula was then made more general by M. Cabanes in [3]. We have the following:

**Theorem 4.10.** [3, Theorem 20] *Let  $n \geq 2$  be an integer,  $q \in \{2, \dots, n\}$  be an odd integer, and  $\rho$  a  $q$ -cycle of  $\mathfrak{S}_n$  with support  $\{n - q + 1, \dots, n\}$ . Let  $\lambda$  be a bar partition of  $n$ . If  $\sigma(\lambda) = 1$ , then we write  $\xi_\lambda = \xi_\lambda^+ = \xi_\lambda^-$ . Then  $x = o_\rho$  satisfies  $C_{\tilde{\mathfrak{S}}_n}(x) = \tilde{\mathfrak{S}}_{n-q} \times \langle x \rangle$  and, for all  $g \in \tilde{\mathfrak{S}}_{n-q}$ ,*

$$(54) \quad \xi_\lambda^+(xg) = \sum_{\substack{\mu \in M_q(\lambda) \\ \sigma(\mu)=1}} a(\xi_\lambda^+, \xi_\mu) \xi_\mu(g) + \sum_{\substack{\mu \in M_q(\lambda) \\ \sigma(\mu)=-1}} (a(\xi_\lambda^+, \xi_\mu^+) \xi_\mu^+(g) + a(\xi_\lambda^+, \xi_\mu^-) \xi_\mu^-(g)),$$

where  $M_q(\lambda)$  is the set of bar partitions of  $n - q$  which can be obtained from  $\lambda$  by removing a  $q$ -bar, and  $a(\xi_\lambda^+, \xi_\mu^+)$ ,  $a(\xi_\lambda^+, \xi_\mu^-) \in \mathbb{C}^*$  are the following:

- if  $\sigma(\mu) = 1$ , then  $a(\xi_\lambda^+, \xi_\mu) = (-1)^{\frac{q^2-1}{8}} \alpha_\mu^\lambda$ ,
- if  $\sigma(\mu) = -1$  and  $\mu \neq \lambda \setminus \{q\}$ , then  $a(\xi_\lambda^+, \xi_\mu^+) = a(\xi_\lambda^+, \xi_\mu^-) = \frac{1}{2}(-1)^{\frac{q^2-1}{8}} \alpha_\mu^\lambda$ ,
- if  $\sigma(\mu) = -1$  and  $\mu = \lambda \setminus \{q\}$ , then  $a(\xi_\lambda^+, \xi_\mu^+) = \frac{1}{2}(-1)^{\frac{q^2-1}{8}} (\alpha_\mu^\lambda + i^{\frac{q-1}{2}} \sqrt{q})$   
and  $a(\xi_\lambda^+, \xi_\mu^-) = \frac{1}{2}(-1)^{\frac{q^2-1}{8}} (\alpha_\mu^\lambda - i^{\frac{q-1}{2}} \sqrt{q})$ .

The  $\alpha_\mu^\lambda$ 's are the coefficient found by Morris in his recursion formula (see [18, Theorem 2]). They are given by

$$\alpha_\mu^\lambda = (-1)^{L(b)} 2^{m(b)},$$

where  $L(b)$  is the leg length of the  $q$ -bar  $b$  removed from  $\lambda$  to get  $\mu$ , and

$$m(b) = \begin{cases} 1 & \text{if } \sigma(\lambda) = 1 \text{ and } \sigma(\mu) = -1, \\ 0 & \text{otherwise.} \end{cases}$$

**Remark 4.11.** Note that, with the notation of Theorem 4.10, since  $\xi_\lambda^-(xg) = \varepsilon \xi_\lambda^+(xg)$ , and since, with a slight abuse of notation,  $\varepsilon(xg) = \varepsilon(g)$  (as  $q$  is odd and  $x = o_\rho$ ), we can also write

$$(55) \quad \xi_\lambda^-(xg) = \sum_{\substack{\mu \in M_q(\lambda) \\ \sigma(\mu)=1}} a(\xi_\lambda^-, \xi_\mu) \xi_\mu(g) + \sum_{\substack{\mu \in M_q(\lambda) \\ \sigma(\mu)=-1}} a(\xi_\lambda^-, \xi_\mu^+) \xi_\mu^+(g) + a(\xi_\lambda^-, \xi_\mu^-) \xi_\mu^-(g),$$

where, whenever  $\sigma(\mu) = 1$ ,  $a(\xi_\lambda^-, \xi_\mu) = a(\xi_\lambda^+, \xi_\mu)$ , and, whenever  $\sigma(\mu) = -1$ ,  $a(\xi_\lambda^-, \xi_\mu^+) = a(\xi_\lambda^+, \xi_\mu^-)$  and  $a(\xi_\lambda^-, \xi_\mu^-) = a(\xi_\lambda^+, \xi_\mu^+)$ .

**Lemma 4.12.** Let  $q$  be an odd number, and  $a \in \tilde{\mathcal{A}}_n$  be of cycle type  $(q)$  such that  $\theta(a)$  has support  $I = \{n - q + 1, \dots, n\}$ . Let  $g$  and  $g'$  be in  $\tilde{\mathcal{A}}_{n-q}$  such that  $ag$  and  $ag'$  are  $\tilde{\mathcal{A}}_n$ -conjugate and in an  $\tilde{\mathcal{A}}_n$ -class labeled by  $\lambda \in D_n^+$ . Then  $g$  and  $g'$  have type  $\mu = \lambda \setminus \{q\} \in D_{n-q}^+$  and are  $\tilde{\mathcal{A}}_{n-q}$ -conjugate.

*Proof.* With the assumption, it is clear that  $g$  and  $g'$  have cycle type  $\mu$ . Let  $t \in \tilde{\mathcal{A}}_n$  be such that  ${}^t(ag) = ag'$ . Then  $\theta^{(t)}(\theta(a)\theta(g)) = \theta(a)\theta(g')$ . Since  $\theta^{(t)}\theta(a)$  is a  $q$ -cycle of  $\theta(a)\theta(g')$  and  $\theta(a)$  is the unique  $q$ -cycle of  $\theta(a)\theta(g')$  (because all the cycles of this element are distinct), it follows that  $\theta^{(t)}\theta(a) = \theta(a)$ . Thus,  $I$  is invariant by  $\theta(t)$ . Set  $v := \theta(t)|_I \in \mathfrak{S}_I$ . Then  $v \in C_{\mathfrak{S}_n}(\theta(a))$ , and since the cycle type of  $\theta(a)$  is odd and distinct in  $\mathcal{A}_I$ , one has  $C_{\mathfrak{S}_I}(\theta(a)) = C_{\mathcal{A}_I}(\theta(a))$ , and in particular,  $v \in \mathcal{A}_I$ . Now, let  $\tilde{v} \in \tilde{\mathcal{A}}_I$  be such that  $\theta(\tilde{v}) = v$ . By Remark 4.4, we have  $\tilde{v}g = g$ . Write  $w = t\tilde{v}^{-1} \in \tilde{\mathcal{A}}_{n-q}$ . Then  ${}^w a = a$  or  ${}^w a = za$ , and since  $a$  and  $za$  have distinct order, we deduce that  ${}^w a = a$ . It follows that  ${}^a w g = {}^w(ag) = {}^t(ag) = ag'$ , and  ${}^w g = g'$  with  $w \in \tilde{\mathcal{A}}_{n-q}$ , as required.  $\square$

**Remark 4.13.** Let  $q$  be an odd multiple of  $p$  and  $\lambda = (\lambda_1, \dots, \lambda_k = q) \in D_n^+$  be as in Convention 4.2 (in particular,  $q$  is the smallest odd part of  $\lambda$  divisible by  $p$ ). Assume that  $n - q \geq 2$  and write  $I = \{n - q + 1, \dots, n\}$ . Let  $\rho$  be the  $q$ -cycle of  $\mathfrak{S}_I$  with respect to the choice of representatives given before Equation (41). Denote by  $t_\rho$  the element of  $\tilde{\mathfrak{S}}_I$  such that  $\theta(t_\rho) = \rho$  with respect to the choice of Schur [26, §11], and write  $\mu = (\lambda_1, \dots, \lambda_{k-1}) \in \mathcal{D}_{n-q}^+$ . With the choice of Equation (46), one has  $\tau_\lambda^\pm = t_\rho \tau_\mu^\pm$  if  $\lambda \notin \mathcal{O}_n$  (note that  $\mu \notin \mathcal{O}_{n-q}$ ) and  $\tau_\lambda^{\pm\pm} = t_\rho \tau_\mu^{\pm\pm}$  if  $\lambda \in \mathcal{O}_n$  (in this case,  $\mu$  is automatically in  $\mathcal{O}_{n-q}$ ).



We now obtain an analogue of Theorem 4.10 for  $\tilde{\mathcal{A}}_n$ . Let  $q$  be an odd number such that  $n - q \geq 2$ . Let  $\rho$  and  $t_\rho$  be as in Remark 4.13. According to [26, footnote (\*), p. 179], recall that  $o_\rho = z^{\frac{q^2-1}{8}} t_\rho$ .

**Theorem 4.14.** *Let  $q$  be an odd integer such that  $n - q \geq 2$ . We keep the notation of Remark 4.13 and we set  $x := o_\rho = z^{\frac{q^2-1}{8}} t_\rho$ . In particular,  $x \in \tilde{\mathcal{A}}_n$ , and  $C_{\tilde{\mathcal{A}}_n}(x) = \tilde{\mathcal{A}}_{n-q} \times \langle x \rangle$ . Assume that the choice for the labels of the classes (and thus for the labels of the characters by §4.2) are as in Remark 4.13. Take any  $\lambda \in \mathcal{D}_n$ ,  $\epsilon \in \{-1, 1\}$  and  $g \in \tilde{\mathcal{A}}_{n-q}$ . When  $\lambda \in \mathcal{D}_n^-$ , we set  $\zeta_\lambda^+ = \zeta_\lambda^- = \zeta_\lambda$ . Finally, let  $\pi$  be the cycle type of  $xg$ . Then, if  $\lambda \neq \pi$ , or if  $\lambda = \pi$  and  $q$  is the last part of  $\lambda$ , we have*

$$\zeta_\lambda^\epsilon(xg) = \sum_{\substack{\mu \in M_q(\lambda) \\ \sigma(\mu) = -1}} a(\zeta_\lambda^\epsilon, \zeta_\mu) \zeta_\mu(g) + \sum_{\substack{\mu \in M_q(\lambda) \\ \sigma(\mu) = 1}} (a(\zeta_\lambda^\epsilon, \zeta_\mu^+) \zeta_\mu^+(g) + a(\zeta_\lambda^\epsilon, \zeta_\mu^-) \zeta_\mu^-(g)),$$

where the coefficients are the following:

- if  $\lambda \in \mathcal{D}_n^-$ , then  $a(\zeta_\lambda, \zeta_\mu^\eta) = (-1)^{\frac{q^2-1}{8}} \alpha_\mu^\lambda$  for all  $\mu \in M_q(\lambda)$  and  $\eta \in \{-1, 1\}$ , where  $\alpha_\mu^\lambda$  is as in Theorem 4.10.
- if  $\lambda \in \mathcal{D}_n^+$ , then  $a(\zeta_\lambda^\epsilon, \zeta_\mu) = \frac{1}{2}(-1)^{\frac{q^2-1}{8}} \alpha_\mu^\lambda$  whenever  $\sigma(\mu) = -1$ , and  $a(\zeta_\lambda^\epsilon, \zeta_\mu^\eta) = \frac{1}{2}(-1)^{\frac{q^2-1}{8}} (\alpha_\mu^\lambda + \epsilon \eta i^{\frac{q-1}{2}} \sqrt{q})$  for  $\eta \in \{-1, 1\}$  whenever  $\sigma(\mu) = 1$ .

*Proof.* First assume that  $\lambda \in \mathcal{D}_n^-$ . Then, by Equation (47), and Clifford theory applied to Equation (54), we obtain the following. Whenever  $\sigma(\mu) = -1$ , we have

$$a(\zeta_\lambda, \zeta_\mu) = a(\xi_\lambda^+, \xi_\mu^+) + a(\xi_\lambda^+, \xi_\mu^-) = \frac{1}{2}(-1)^{\frac{q^2-1}{8}} \alpha_\mu^\lambda + \frac{1}{2}(-1)^{\frac{q^2-1}{8}} \alpha_\mu^\lambda = (-1)^{\frac{q^2-1}{8}} \alpha_\mu^\lambda,$$

and, whenever  $\sigma(\mu) = 1$ ,

$$a(\zeta_\lambda, \zeta_\mu^+) = a(\zeta_\lambda, \zeta_\mu^-) = a(\xi_\lambda^+, \xi_\mu) = (-1)^{\frac{q^2-1}{8}} \alpha_\mu^\lambda,$$

as required.

We now consider the case where  $\lambda \in \mathcal{D}_n^+$ . By Equation (49) and Clifford theory applied to Equation (54), we obtain

$$\begin{aligned} \zeta_\lambda^+(xg) &= \frac{1}{2} (\xi_\lambda(xg) + \Delta_\lambda(xg)) \\ &= \sum_{\substack{\mu \in M_q(\lambda) \\ \sigma(\mu) = 1}} \frac{a(\xi_\lambda, \xi_\mu)}{2} (\zeta_\mu^+(g) + \zeta_\mu^-(g)) + \frac{\Delta_\lambda(xg)}{2} \\ &\quad + \sum_{\substack{\mu \in M_q(\lambda) \\ \sigma(\mu) = -1}} \frac{a(\xi_\lambda, \xi_\mu^+) + a(\xi_\lambda, \xi_\mu^-)}{2} \zeta_\mu(g). \end{aligned} \tag{56}$$

We need to deal with the term  $\frac{\Delta_\lambda(xg)}{2}$ . Recall that this is 0 unless  $xg$  has cycle type  $\pi = \lambda$ . We start by noticing that, if  $xg$  does not have cycle type  $\lambda$ , then  $g$  does not have cycle type  $\mu$  for any  $\mu \in M_q(\lambda)$  with  $\sigma(\mu) = 1$ . Indeed, if  $\mu$  is obtained from  $\lambda$  by removing a bar  $b$  of odd length  $q$ , then, depending on the type of  $b$ , we have  $\ell(\mu) = \ell(\lambda)$ ,  $\ell(\mu) = \ell(\lambda) - 2$  or  $\ell(\mu) = \ell(\lambda) - 1$ . In the first two cases, we obtain  $\sigma(\mu) = (-1)^{n-q-\ell(\lambda)} = -\sigma(\lambda)$ . The last case can only happen if  $b$  is a part of  $\lambda$ , in which case  $\mu = \lambda \setminus \{q\}$  and  $\sigma(\mu) = \sigma(\lambda)$ . This has several consequences. The first is that  $\{\mu \in M_q(\lambda) \mid \sigma(\mu) = 1\}$  is either empty, or contains

only the partition  $\lambda \setminus \{q\}$ . This in turn implies that  $xg$  has cycle type  $\lambda$  if and only if  $\{\mu \in M_q(\lambda) \mid \sigma(\mu) = 1\} = \{\lambda \setminus \{q\}\}$  and  $g$  has cycle type  $\lambda \setminus \{q\}$ . Finally,  $\{\mu \in M_q(\lambda) \mid \sigma(\mu) = 1\}$  is empty if and only if  $\lambda$  does not have a part of length  $q$ , and, if this is the case, then  $\frac{\Delta_\lambda(xg)}{2} = 0$  for all  $g \in \tilde{\mathcal{A}}_{n-q}$ .

We therefore suppose that  $\lambda$  does have a part of length  $q$ , so that

$$\{\mu \in M_q(\lambda) \mid \sigma(\mu) = 1\} = \{\lambda \setminus \{q\}\}.$$

We will show that, if  $\mu = \lambda \setminus \{q\}$ , then, for all  $g \in \tilde{\mathcal{A}}_{n-q}$ , we have

$$(57) \quad \Delta_\lambda(xg) = (-1)^{\frac{q^2-1}{8}} i^{\frac{q-1}{2}} \sqrt{q} \Delta_\mu(g).$$

If  $g$  does not have cycle type  $\mu = \lambda \setminus \{q\}$ , then  $\Delta_\mu(g) = 0$ , and  $xg$  does not have cycle type  $\lambda$ , so that  $\Delta_\lambda(xg) = 0$  and Equation (57) holds.

Now, assume that  $g$  has cycle type  $\mu$ . Then  $xg$  has cycle type  $\pi = \lambda$ , so we assume furthermore that  $q$  is the last part of  $\lambda$ . If  $\lambda \in \mathcal{O}_n$  (resp.  $\lambda \notin \mathcal{O}_n$ ), then there are signs  $\delta$  and  $\eta$  such that  $xg$  is  $\tilde{\mathcal{A}}_n$ -conjugate to  $\tau_\lambda^{\delta\eta}$  (resp. to  $\tau_\lambda^\delta$ ). It follows that  $t_\rho z^{\frac{q^2-1}{8}} g$  and  $\tau_\lambda^{\delta\eta} = t_\rho \tau_\mu^{\delta\eta}$  (resp.  $\tau_\lambda^\delta = t_\rho \tau_\mu^\delta$ ) are  $\tilde{\mathcal{A}}_n$ -conjugate. By Lemma 4.12,  $z^{\frac{q^2-1}{8}} g$  is  $\tilde{\mathcal{A}}_{n-q}$ -conjugate to  $\tau_\mu^{\delta\mu}$  (resp. to  $\tau_\mu^\delta$ ), that is,  $g \in D_\mu^{(\frac{q^2-1}{8}\delta)\eta}$  (resp.  $g \in D_\mu^{\frac{q^2-1}{8}\delta}$ ).

Now, using the values and properties we gave for the difference characters, we obtain that, for  $xg \in D_\lambda^+$  (or similarly for  $xg \in D_\lambda^{++}$ ), we have

$$(58) \quad \begin{aligned} \Delta_\lambda(xg) = \Delta_\lambda(D_\lambda^+) &= i^{\frac{n-\ell(\lambda)}{2}} \sqrt{z_\lambda} \\ &= i^{\frac{n-q+q-\ell(\mu)-1}{2}} \sqrt{q} \sqrt{z_\mu} \\ &= i^{\frac{q-1}{2}} \sqrt{q} i^{\frac{n-q-\ell(\mu)}{2}} \sqrt{z_\mu} \\ &= i^{\frac{q-1}{2}} \sqrt{q} \Delta_\mu(D_\mu^+) \\ &= (-1)^{\frac{q^2-1}{8}} i^{\frac{q-1}{2}} \sqrt{q} \Delta_\mu(g). \end{aligned}$$

Using the property  $\Delta_\lambda(D_\lambda^-) = -\Delta_\lambda(D_\lambda^+)$ , and its analogues for the classes  $D_\lambda^{\pm\pm}$ , we easily deduce that Equation (57) does hold for all  $g \in \tilde{\mathcal{A}}_{n-q}$ .

Now, from Equations (56) and (57), and Theorem 4.10, we deduce that, for  $\mu \in M_q(\lambda)$ , if  $\sigma(\mu) = -1$ , then  $a(\zeta_\lambda^+, \zeta_\mu) = \frac{1}{2}(-1)^{\frac{q^2-1}{8}} \alpha_\mu^\lambda$ , and, if  $\sigma(\mu) = 1$ , then  $a(\zeta_\lambda^+, \zeta_\mu^+) = \frac{1}{2}(-1)^{\frac{q^2-1}{8}} (\alpha_\mu^\lambda + i^{\frac{q-1}{2}} \sqrt{q})$  and  $a(\zeta_\lambda^+, \zeta_\mu^-) = \frac{1}{2}(-1)^{\frac{q^2-1}{8}} (\alpha_\mu^\lambda - i^{\frac{q-1}{2}} \sqrt{q})$ .

Our analysis of the term  $\Delta_\lambda(xg)$  also yields a similar formula for  $\zeta_\lambda^-(xg)$ , and using Equation (55), we deduce the values of  $a(\xi_\lambda^-, \xi_\mu^\eta)$  for all  $\mu \in M_q(\lambda)$  and  $\eta \in \{-1, 1\}$ .  $\square$

**Remark 4.15.** Let  $n$  and  $q$  be as above. Assume  $n = q$  or  $n = q + 1$ . Then  $\tilde{\mathcal{A}}_{n-q} = \mathbb{Z}_2$  and the only spin character is the non-trivial character  $\varepsilon$  of  $\mathbb{Z}_2$ , labeled by  $\mu = \emptyset$  or  $\mu = (1)$  whenever  $n = q$  or  $n = q + 1$ . Set  $\pi = (q)$  if  $n = q$  and  $\pi = (q, 1)$  if  $n = q + 1$ . Then there are 4 classes of  $\tilde{\mathcal{A}}_n$  labeled by  $\pi$  with representatives  $\tau_\pi^{\pm\pm}$ . Write  $o_\pi^\pm = \tau_\pi^{\pm\pm}$ . Let  $k \in \{0, 1\}$  and  $\lambda \in \mathcal{D}_n$  be with  $\bar{q}$ -core  $\mu$ . If  $\lambda \in \mathcal{D}_n^-$ , then  $\zeta_\lambda(o_\pi^\pm z^k) = (-1)^{\frac{q^2-1}{8}} \alpha_\mu^\lambda \varepsilon(z^k)$ . If  $\lambda \in \mathcal{D}_n^+$  and  $\lambda \neq \pi$ , then

$\zeta_\lambda(o_\pi^\pm z^k) = \frac{1}{2}(-1)^{\frac{q^2-1}{8}} \alpha_\mu^\lambda \varepsilon(z^k)$ . Finally, for  $\delta, \eta \in \{-1, 1\}$ , one has

$$(59) \quad \zeta_\pi^\eta(o_\pi^\delta z^k) = \zeta_\pi^\eta(o_\pi^\delta) \varepsilon(z^k) = \frac{(-1)^{\frac{q^2-1}{8}}}{2} \left( \alpha_\mu^\pi + \eta \delta i^{\frac{q-1}{2}} \sqrt{q} \right) \varepsilon(z^k).$$

We define  $C_{\mathfrak{S}_n}$  to be the set of elements of  $\mathfrak{S}_n$  none of whose cycles has length an odd multiple of  $p$ . We then let

$$(60) \quad C_{\tilde{\mathfrak{S}}_n} = \theta^{-1}(C_{\mathfrak{S}_n}) \quad \text{and} \quad C_{\tilde{\mathcal{A}}_n} = C_{\tilde{\mathfrak{S}}_n} \cap \tilde{\mathcal{A}}_n.$$

Finally, we let  $\mathcal{C}_{\tilde{\mathfrak{S}}_n}$  and  $\mathcal{C}_{\tilde{\mathcal{A}}_n}$  be the sets of (respectively  $\tilde{\mathfrak{S}}_n$ - and  $\tilde{\mathcal{A}}_n$ -) conjugacy classes in  $C_{\tilde{\mathfrak{S}}_n}$  and  $C_{\tilde{\mathcal{A}}_n}$  respectively.

From now on, if  $G$  is a finite group and  $\mathcal{C}$  a union of conjugacy classes of  $G$ , then  $\mathcal{C}$ -blocks is meant in the sense of KOR-blocks (see Proposition 2.14).

We start by showing that the spin  $p$ -blocks of  $\tilde{\mathfrak{S}}_n$  (respectively  $\tilde{\mathcal{A}}_n$ ) are also  $\mathcal{C}_{\tilde{\mathfrak{S}}_n}$ -blocks (respectively  $\mathcal{C}_{\tilde{\mathcal{A}}_n}$ -blocks). Recall that the  $p$ -blocks of  $\tilde{\mathfrak{S}}_n$  are just the  $\tilde{\mathfrak{S}}_{n,p'}$ -blocks, where  $\tilde{\mathfrak{S}}_{n,p'}$  is the set of  $p$ -regular elements of  $\tilde{\mathfrak{S}}_n$ . Similarly, the  $p$ -blocks of  $\tilde{\mathcal{A}}_n$  are its  $\tilde{\mathcal{A}}_{n,p'}$ -blocks. Note that, by definition, we have  $\tilde{\mathfrak{S}}_{n,p'} \subset C_{\tilde{\mathfrak{S}}_n}$  and  $\tilde{\mathcal{A}}_{n,p'} \subset C_{\tilde{\mathcal{A}}_n}$ .

**Lemma 4.16.** *The  $p$ -blocks and  $\mathcal{C}_{\tilde{\mathfrak{S}}_n}$ -blocks of spin characters of  $\tilde{\mathfrak{S}}_n$  coincide, and the  $p$ -blocks and  $\mathcal{C}_{\tilde{\mathcal{A}}_n}$ -blocks of spin characters of  $\tilde{\mathcal{A}}_n$  coincide.*

*Proof.* Let  $\chi$  and  $\xi$  be a non-spin and a spin character of  $\tilde{\mathfrak{S}}_n$ , respectively. Since  $\chi$  is constant on the split classes, we deduce that  $\langle \chi, \xi \rangle_{C_{\tilde{\mathfrak{S}}_n}} = 0$ . Thus, spin and non-spin characters are never in the same  $\mathcal{C}_{\tilde{\mathfrak{S}}_n}$ -block.

Now, take any two spin characters  $\xi$  and  $\xi'$  of  $\tilde{\mathfrak{S}}_n$ , such that  $\xi' \notin \{\xi, \varepsilon\xi\}$ . Then the only elements of  $C_{\tilde{\mathfrak{S}}_n} \setminus \tilde{\mathfrak{S}}_{n,p'}$ , if any, on which  $\xi$  doesn't vanish belong to split conjugacy classes labeled by the partition labeling  $\xi$  (this is because any split conjugacy class of  $C_{\tilde{\mathfrak{S}}_n}$  labeled by a partition of  $\mathcal{O}_n$ , and thus without even cycles, must also belong to  $\tilde{\mathfrak{S}}_{n,p'}$ ). And since  $\xi' \notin \{\xi, \varepsilon\xi\}$ , we see that  $\xi'$  vanishes on these elements. In this case, we therefore have

$$(61) \quad \langle \xi, \xi' \rangle_{C_{\tilde{\mathfrak{S}}_n}} = \langle \xi, \xi' \rangle_{\tilde{\mathfrak{S}}_{n,p'}} = \langle \varepsilon\xi, \xi' \rangle_{\tilde{\mathfrak{S}}_{n,p'}} = \langle \varepsilon\xi, \xi' \rangle_{C_{\tilde{\mathfrak{S}}_n}}.$$

Assume that  $\xi' \notin \{\xi, \varepsilon\xi\}$  lies in the same  $p$ -block as  $\xi$ . Then there are distinct spin irreducible characters  $\xi' = \xi_1, \dots, \xi_s = \xi$  such that  $\xi_i \neq \xi$  for  $1 \leq i \leq s-1$  and  $\langle \xi_i, \xi_{i+1} \rangle_{\tilde{\mathfrak{S}}_{n,p'}} \neq 0$ . We can assume that  $\xi_i \neq \varepsilon\xi$  for all  $1 \leq i \leq s$ . Indeed, let  $2 \leq i \leq s$  be such that  $\xi_i = \varepsilon\xi$ . Since  $\langle \xi_{i-1}, \xi_i \rangle_{\tilde{\mathfrak{S}}_{n,p'}} = \langle \xi_{i-1}, \varepsilon\xi \rangle_{\tilde{\mathfrak{S}}_{n,p'}} = \langle \xi_{i-1}, \xi \rangle_{\tilde{\mathfrak{S}}_{n,p'}}$  by Equation (61), we can take  $s = i$ . We also can assume that  $\xi_{i+1} \neq \varepsilon\xi_i$  for all  $1 \leq i \leq s-1$ . Otherwise, if there is  $1 \leq i \leq s-1$  such that  $\xi_{i+1} = \varepsilon\xi_i$ , then  $i < s-1$ , and since  $\xi_{i+2} \notin \{\xi_i, \xi_{i+1}\} = \{\xi_i, \varepsilon\xi_i\}$ , we deduce from Equation (61) that  $\langle \xi_{i+1}, \xi_{i+2} \rangle_{\tilde{\mathfrak{S}}_{n,p'}} = \langle \xi_i, \xi_{i+2} \rangle_{\tilde{\mathfrak{S}}_{n,p'}}$  and we can remove  $\xi_{i+1}$  from the chain.

Hence, Equation (61) gives  $\langle \xi_i, \xi_{i+1} \rangle_{\tilde{\mathfrak{S}}_{n,p'}} = \langle \xi_i, \xi_{i+1} \rangle_{C_{\tilde{\mathfrak{S}}_n}}$  for all  $1 \leq i \leq s-1$ , and the characters  $\xi'$  and  $\xi$  lie in the same  $\mathcal{C}_{\tilde{\mathfrak{S}}_n}$ -block.

By a similar argument, Equation (61) implies that if  $\xi' \notin \{\xi, \varepsilon\xi\}$  lies in the same  $\mathcal{C}_{\tilde{\mathfrak{S}}_n}$ -block as  $\xi$ , then they are in the same  $p$ -block.

Furthermore, if  $\xi \neq \varepsilon\xi$ , then either  $\xi$  and  $\varepsilon\xi$  belong to the same  $p$ -block, or each is alone in its respective  $p$ -blocks. In the first case, the  $p$ -block which contains  $\xi$  and  $\varepsilon\xi$  also contains some spin character  $\xi'$  such that  $\xi' = \varepsilon\xi'$  (this follows from [21, (2.1)]). In particular, in the  $p$ -block of  $\xi$ , there is some irreducible  $\xi'' \notin \{\xi, \varepsilon\xi\}$  such that  $\langle \xi'', \xi \rangle_{\tilde{\mathfrak{S}}_{n,p'}} \neq 0$ . Thus, by Equation (61), we obtain  $\langle \xi'', \xi \rangle_{C_{\tilde{\mathfrak{S}}_n}} = \langle \xi'', \xi \rangle_{\tilde{\mathfrak{S}}_{n,p'}} = \langle \xi'', \varepsilon\xi \rangle_{C_{\tilde{\mathfrak{S}}_n}}$ , and  $\xi$  and  $\varepsilon\xi$  belong to the same  $C_{\tilde{\mathfrak{S}}_n}$ -block.

In the second case,  $\xi$  and  $\varepsilon\xi$  have  $p$ -defect zero. In particular,  $\xi$  and  $\varepsilon\xi$  are labeled by a  $\bar{p}$ -core  $\lambda \in D_n^-$ , and  $\lambda$  is  $p$ -regular. Hence they both vanish identically on  $p$ -singular elements, so also on  $C_{\tilde{\mathfrak{S}}_n} \setminus \tilde{\mathfrak{S}}_{n,p'}$ , and they are each alone in their respective  $C_{\tilde{\mathfrak{S}}_n}$ -blocks as well. The result follows.

For the case of  $\tilde{\mathcal{A}}_n$ , the argument is similar.  $\square$

We can now define on  $\tilde{\mathfrak{S}}_n$  and  $\tilde{\mathcal{A}}_n$  an MN-structure with respect to the set of spin  $p$ -blocks of  $\tilde{\mathfrak{S}}_n$  and  $\tilde{\mathcal{A}}_n$ , and the sets  $C_{\tilde{\mathfrak{S}}_n}$  and  $C_{\tilde{\mathcal{A}}_n}$  defined in Equation (60), respectively. For this, we define  $S_{\tilde{\mathfrak{S}}_n}$  to be the set of elements  $\sigma \in \tilde{\mathfrak{S}}_n$  all of whose cycles have length 1 or an odd multiple of  $p$ . By Lemma 4.9, we denote by  $o_\sigma$  the element of  $\tilde{\mathfrak{S}}_n$  of odd order such that  $\theta(o_\sigma) = \sigma$ , and we let

$$(62) \quad S_{\tilde{\mathfrak{S}}_n} = \{o_\sigma \mid \sigma \in S_{\tilde{\mathfrak{S}}_n}\} \quad \text{and} \quad S_{\tilde{\mathcal{A}}_n} = S_{\tilde{\mathfrak{S}}_n} \cap \tilde{\mathcal{A}}_n.$$

Note that, since  $p$  is odd, and since we only consider odd multiples of  $p$ , we have  $S_{\tilde{\mathcal{A}}_n} = S_{\tilde{\mathfrak{S}}_n} \cap \tilde{\mathcal{A}}_n = S_{\tilde{\mathfrak{S}}_n}$ .

**Proposition 4.17.** *Let  $n > 0$  be any integer, and  $p$  be an odd prime. Let  $Sp(\tilde{\mathfrak{S}}_n)$  and  $Sp(\tilde{\mathcal{A}}_n)$  be the sets of spin  $p$ -blocks of  $\tilde{\mathfrak{S}}_n$  and  $\tilde{\mathcal{A}}_n$  respectively. Then  $\tilde{\mathfrak{S}}_n$  has an MN-structure (as defined in Definition 2.5) with respect to  $C_{\tilde{\mathfrak{S}}_n}$  and  $Sp(\tilde{\mathfrak{S}}_n)$ , and  $\tilde{\mathcal{A}}_n$  has an MN-structure with respect to  $C_{\tilde{\mathcal{A}}_n}$  and  $Sp(\tilde{\mathcal{A}}_n)$ .*

*Proof.* First note that, by Lemma 4.16,  $Sp(\tilde{\mathfrak{S}}_n)$  and  $Sp(\tilde{\mathcal{A}}_n)$  are indeed unions of  $C_{\tilde{\mathfrak{S}}_n}$ -blocks and  $C_{\tilde{\mathcal{A}}_n}$ -blocks respectively.

To stick with the notation of Definition 2.5, we take  $G \in \{\tilde{\mathfrak{S}}_n, \tilde{\mathcal{A}}_n\}$ ,  $B = Sp(G)$ ,  $\mathcal{C} = C_G$  and  $S = S_G$  (as defined above). Properties 1 and 2 of Definition 2.5 are immediate consequences of the definition of  $S$  and  $\mathcal{C}$ . For  $x_S \in S$  and  $x_C \in C$ , we have  $(x_S, x_C) \in A$  if and only if the non-trivial cycles of  $\theta(x_S)$  and  $\theta(x_C)$  are disjoint (in particular,  $x_S$  and  $x_C$  commute).

Now take any  $x_S \in S$ . If  $x_S = 1$ , then  $G_1 = G$ ,  $B_1 = B$  and  $r^1 = \text{id}$  clearly satisfy Properties 3 and 4. If, on the other hand,  $x_S \neq 1$ , then, by definition of  $S$ , we have  $x_S = o_\sigma$  for some  $\sigma \in S_{\tilde{\mathfrak{S}}_n}$ . Write  $\sigma = \sigma_1 \cdots \sigma_k$ , where, for each  $1 \leq i \leq k$ ,  $\sigma_i$  is a  $q_i$ -cycle for some odd multiple  $q_i$  of  $p$ , and the  $\sigma_i$ 's are pairwise disjoint. In particular,  $\sigma_i \in S_{\tilde{\mathfrak{S}}_n}$  and, since  $\sigma_i \in \tilde{\mathcal{A}}_n$ , [26, III, p. 172] gives  $o_\sigma = o_{\sigma_1} \cdots o_{\sigma_k}$ , and  $C_G(x_S)$  has as a subgroup the group  $H = G_{x_S} \times \langle o_{\sigma_1} \rangle \times \cdots \times \langle o_{\sigma_k} \rangle$ , where  $G_{x_S} \cong \tilde{\mathfrak{S}}_{n - \sum_{i=1}^k q_i}$  if  $G = \tilde{\mathfrak{S}}_n$ , and  $G_{x_S} \cong \tilde{\mathcal{A}}_{n - \sum_{i=1}^k q_i}$  if  $G = \tilde{\mathcal{A}}_n$  (and with the convention that  $\tilde{\mathfrak{S}}_0 = \tilde{\mathcal{A}}_0 = \langle z \rangle$ ).

Property 3 now follows from the definition of  $A$  we gave above. Clearly, if  $x_C \in G_{x_S} \cap C$ , then the non-trivial cycles of  $\theta(x_C)$  and  $\theta(x_S)$  are disjoint, so that  $(x_S, x_C) \in A$ . Conversely, if  $(x_S, x_C) \in A$ , then one must have  $x_C \in C_G(x_S)$ . If  $x_C \in C_G(x_S) \setminus H$ , then  $\theta(x_C)$  must permute (non-trivially) the  $(p)$ -cycles of  $\theta(x_S)$ ; in particular, the non-trivial cycles of  $\theta(x_C)$  and  $\theta(x_S)$  cannot be disjoint, so that

$(x_S, x_C) \notin A$ . Hence, if  $(x_S, x_C) \in A$ , then necessarily  $x_C \in H$ . Now, in order for  $x_C$  to be disjoint from  $x_S$ , we see that one must have  $x_C \in G_{x_S}$ . This proves that  $G_{x_S} \cap C = \{x_C \in C \mid (x_S, x_C) \in A\}$ .

Finally, we obtain Property 4 by iterating Theorem 4.10 and Theorem 4.14. By considering (and removing) the ‘‘cycles’’  $o_{\sigma_i}$  ( $1 \leq i \leq k$ ) one at a time, and in increasing order of size, one sees that we can define  $r^{x_S}(\chi)$  for any spin character  $\chi \in B$ . By construction,  $r^{x_S}(\chi)$  does satisfy  $r^{x_S}(\chi)(x_C) = \chi(x_S \cdot x_C)$  for all  $(x_S, x_C) \in A$ . Taking  $B_{x_S}$  to be the set of spin characters of  $G_{x_S}$ , and extending  $r^{x_S}$  by linearity to  $\mathbb{C} \text{Irr}(B)$ , we obtain the result.  $\square$

**4.6. Broué perfect isometries.** Throughout this section, we denote by  $p$  an odd prime number.

Let  $n, q$  and  $\lambda$  be as in Theorem 4.10. Suppose furthermore that  $q$  is an odd multiple of  $p$ , and that  $w_{\bar{p}}(\lambda) > 0$ . Next consider any spin  $p$ -block  $B'$ , of  $\tilde{\mathfrak{S}}_m$  say, of the same weight and sign as the  $p$ -block  $B$  of  $\xi_\lambda^+$ , and the bijection  $\Psi$  described in Lemma 4.8. In particular,  $\Psi$  preserves the parity of bar partitions. Now, since  $q$  is a multiple of  $p$ , the removal of a  $q$ -bar can be obtained by removing a sequence of  $p$ -bars, and one sees from Theorem 4.5 that  $M_q(\Psi(\lambda)) = \Psi(M_q(\lambda))$ . This is a slight abuse of notation, as  $\Psi$  should only act on partitions of the same weight as  $\lambda$ , while the elements of  $M_q(\lambda)$  have a smaller weight. But we see that  $\Psi$  is compatible with the bijections  $g_\lambda$  and  $g_{\Psi(\lambda)}$  given by Theorem 4.5, since everything goes through the (common)  $\bar{p}$ -quotient of  $\lambda$  and  $\Psi(\lambda)$ . Also, thanks to Equation (52) one has  $\sigma(\Psi(\mu)) = \sigma(\mu)$  for any  $\mu \in M_q(\lambda)$ . We then have the following:

**Proposition 4.18.** *Let the notation be as above. For any  $\mu \in M_q(\lambda)$ , and for any  $\epsilon, \eta \in \{1, -1\}$ , we have*

$$\delta_{\bar{p}}(\lambda)\delta_{\bar{p}}(\mu)a(\xi_\lambda^{\epsilon\delta_{\bar{p}}(\lambda)}, \xi_\mu^{\eta\delta_{\bar{p}}(\mu)}) = \delta_{\bar{p}}(\Psi(\lambda))\delta_{\bar{p}}(\Psi(\mu))a(\xi_{\Psi(\lambda)}^{\epsilon\delta_{\bar{p}}(\Psi(\lambda))}, \xi_{\Psi(\mu)}^{\eta\delta_{\bar{p}}(\Psi(\mu))}).$$

*Proof.* Let  $\mu \in M_q(\lambda)$  be obtained by removing the  $q$ -bar  $b$  from  $\lambda$ . Then, by definition of  $\Psi$ , we see, using Theorem 4.5, that  $\Psi(\mu) \in M_q(\Psi(\lambda))$  is obtained by removing the  $q$ -bar  $\Psi(b)$  from  $\Psi(\lambda)$ .

We start by comparing  $\alpha_\mu^\lambda$  and  $\alpha_{\Psi(\mu)}^{\Psi(\lambda)}$ . By definition, we have

$$(63) \quad \alpha_\mu^\lambda = (-1)^{L(b)}2^{m(b)} \quad \text{and} \quad \alpha_{\Psi(\mu)}^{\Psi(\lambda)} = (-1)^{L(\Psi(b))}2^{m(\Psi(b))},$$

where

$$(64) \quad m(b) = \begin{cases} 1 & \text{if } \sigma(\lambda) = 1 \text{ and } \sigma(\mu) = -1, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$(65) \quad m(\Psi(b)) = \begin{cases} 1 & \text{if } \sigma(\Psi(\lambda)) = 1 \text{ and } \sigma(\Psi(\mu)) = -1, \\ 0 & \text{otherwise.} \end{cases}$$

And, since  $\Psi$  preserves the parity of partitions, we see that  $m(b) = m(\Psi(b))$ . Now  $L(b)$  is related to  $L(g_\lambda(b))$ , where  $g_\lambda$  is the bijection described in Theorem 4.5. Similarly,  $L(\Psi(b))$  is related to  $L(g_{\Psi(\lambda)}(\Psi(b)))$ , but, as we remarked above,  $g_{\Psi(\lambda)}(\Psi(b)) = g_\lambda(b)$ . We have, by Theorem 4.6(iii), applied to the  $(p)$ -bar  $b$ ,

$$(66) \quad (-1)^{L(b)} = (-1)^{L(g_\lambda(b))}\delta_{\bar{p}}(\lambda, \mu) = (-1)^{L(g_\lambda(b))}\delta_{\bar{p}}(\lambda)\delta_{\bar{p}}(\mu)$$

and similarly

$$(67) \quad (-1)^{L(\Psi(b))} = (-1)^{L(g_\lambda(b))} \delta_{\bar{p}}(\Psi(\lambda)) \delta_{\bar{p}}(\Psi(\mu)),$$

whence

$$(68) \quad \delta_{\bar{p}}(\lambda) \delta_{\bar{p}}(\mu) (-1)^{L(b)} = \delta_{\bar{p}}(\Psi(\lambda)) \delta_{\bar{p}}(\Psi(\mu)) (-1)^{L(\Psi(b))}.$$

If  $\sigma(\mu) = 1$ , then  $\xi_\mu^+ = \xi_\mu^- = \xi_\mu$  and  $a(\xi_\lambda^+, \xi_\mu) = a(\xi_\lambda^-, \xi_\mu) = (-1)^{\frac{q^2-1}{8}} \alpha_\mu^\lambda$ . We also have  $\sigma(\Psi(\mu)) = 1$ , so  $a(\xi_{\Psi(\lambda)}^+, \xi_{\Psi(\mu)}) = a(\xi_{\Psi(\lambda)}^-, \xi_{\Psi(\mu)}) = (-1)^{\frac{q^2-1}{8}} \alpha_{\Psi(\mu)}^{\Psi(\lambda)}$ . Thus Equation (68) immediately gives the result.

Suppose now that  $\sigma(\mu) = -1$ . Then, by Remark 4.11, we have  $a(\xi_\lambda^-, \xi_\mu^+) = a(\xi_\lambda^+, \xi_\mu^-)$  and  $a(\xi_\lambda^-, \xi_\mu^-) = a(\xi_\lambda^+, \xi_\mu^+)$ . We need to distinguish between the cases  $\mu = \lambda \setminus \{q\}$  and  $\mu \neq \lambda \setminus \{q\}$ . If  $\mu = \lambda \setminus \{q\}$ , this means  $b$  is a part of length  $q$  in  $\lambda$ . Then, in Theorem 4.5,  $g(b)$  must be a part of length  $q/p$  in the first (bar) partition of  $\lambda^{(\bar{q})}$  (see [22, Theorem (4.3)]), and  $\Psi(b)$  is then a part of length  $q$  in  $\Psi(\lambda)$ . We thus have  $\mu = \lambda \setminus \{q\}$  if and only if  $\Psi(\mu) = \Psi(\lambda) \setminus \{q\}$ .

Suppose first that  $\mu \neq \lambda \setminus \{q\}$ , so that  $\Psi(\mu) \neq \Psi(\lambda) \setminus \{q\}$ . Then, by Theorem 4.10 and Remark 4.11,

$$a(\xi_\lambda^+, \xi_\mu^+) = a(\xi_\lambda^+, \xi_\mu^-) = a(\xi_\lambda^-, \xi_\mu^+) = a(\xi_\lambda^-, \xi_\mu^-) = \frac{1}{2} (-1)^{\frac{q^2-1}{8}} \alpha_\mu^\lambda$$

and

$$a(\xi_{\Psi(\lambda)}^+, \xi_{\Psi(\mu)}^+) = a(\xi_{\Psi(\lambda)}^+, \xi_{\Psi(\mu)}^-) = a(\xi_{\Psi(\lambda)}^-, \xi_{\Psi(\mu)}^+) = a(\xi_{\Psi(\lambda)}^-, \xi_{\Psi(\mu)}^-) = \frac{1}{2} (-1)^{\frac{q^2-1}{8}} \alpha_{\Psi(\mu)}^{\Psi(\lambda)},$$

so that Equation (68) gives the result.

Suppose, finally, that  $(\sigma(\mu) = \sigma(\Psi(\mu)) = -1)$  and  $\mu = \lambda \setminus \{q\}$ , so that  $\Psi(\mu) = \Psi(\lambda) \setminus \{q\}$ . This is the only case which is not straightforward. By Theorem 4.10, we have

$$a(\xi_\lambda^+, \xi_\mu^+) = \frac{1}{2} (-1)^{\frac{q^2-1}{8}} (\alpha_\mu^\lambda + i^{\frac{q-1}{2}} \sqrt{q}) \quad \text{and} \quad a(\xi_\lambda^+, \xi_\mu^-) = \frac{1}{2} (-1)^{\frac{q^2-1}{8}} (\alpha_\mu^\lambda - i^{\frac{q-1}{2}} \sqrt{q})$$

(and similar expressions for  $a(\xi_{\Psi(\lambda)}^+, \xi_{\Psi(\mu)}^+)$  and  $a(\xi_{\Psi(\lambda)}^+, \xi_{\Psi(\mu)}^-)$ ). Since, by Remark 4.11,  $a(\xi_\lambda^-, \xi_\mu^+) = a(\xi_\lambda^+, \xi_\mu^-)$  and  $a(\xi_\lambda^-, \xi_\mu^-) = a(\xi_\lambda^+, \xi_\mu^+)$ , we deduce that, for any  $\epsilon \in \{1, -1\}$ ,

$$a(\xi_\lambda^{\epsilon \delta_{\bar{p}}(\lambda)}, \xi_\mu^{\delta_{\bar{p}}(\mu)}) = \frac{1}{2} (-1)^{\frac{q^2-1}{8}} (\alpha_\mu^\lambda + \epsilon \delta_{\bar{p}}(\lambda) \delta_{\bar{p}}(\mu) i^{\frac{q-1}{2}} \sqrt{q})$$

(and a similar expression for  $a(\xi_{\Psi(\lambda)}^{\epsilon \delta_{\bar{p}}(\Psi(\lambda))}, \xi_{\Psi(\mu)}^{\delta_{\bar{p}}(\Psi(\mu))})$ ). Multiplying by  $\delta_{\bar{p}}(\lambda) \delta_{\bar{p}}(\mu)$ , we obtain, using Equation (68),

$$\begin{aligned} \delta_{\bar{p}}(\lambda) \delta_{\bar{p}}(\mu) a(\xi_\lambda^{\epsilon \delta_{\bar{p}}(\lambda)}, \xi_\mu^{\delta_{\bar{p}}(\mu)}) &= \frac{1}{2} (-1)^{\frac{q^2-1}{8}} \left( \delta_{\bar{p}}(\lambda) \delta_{\bar{p}}(\mu) \alpha_\mu^\lambda + \epsilon i^{\frac{q-1}{2}} \sqrt{q} \right) \\ &= \frac{1}{2} (-1)^{\frac{q^2-1}{8}} \left( \delta_{\bar{p}}(\Psi(\lambda)) \delta_{\bar{p}}(\Psi(\mu)) \alpha_{\Psi(\mu)}^{\Psi(\lambda)} + \epsilon i^{\frac{q-1}{2}} \sqrt{q} \right) \\ &= \delta_{\bar{p}}(\Psi(\lambda)) \delta_{\bar{p}}(\Psi(\mu)) \frac{1}{2} (-1)^{\frac{q^2-1}{8}} \left( \alpha_{\Psi(\mu)}^{\Psi(\lambda)} + \epsilon \delta_{\bar{p}}(\Psi(\lambda)) \delta_{\bar{p}}(\Psi(\mu)) i^{\frac{q-1}{2}} \sqrt{q} \right), \end{aligned}$$

whence

$$\delta_{\bar{p}}(\lambda) \delta_{\bar{p}}(\mu) a(\xi_\lambda^{\epsilon \delta_{\bar{p}}(\lambda)}, \xi_\mu^{\delta_{\bar{p}}(\mu)}) = \delta_{\bar{p}}(\Psi(\lambda)) \delta_{\bar{p}}(\Psi(\mu)) a(\xi_{\Psi(\lambda)}^{\epsilon \delta_{\bar{p}}(\Psi(\lambda))}, \xi_{\Psi(\mu)}^{\delta_{\bar{p}}(\Psi(\mu))}).$$

Using Remark 4.11, this implies the last equality we have to prove:

$$\begin{aligned}
\delta_{\bar{p}}(\lambda)\delta_{\bar{p}}(\mu)a\left(\xi_{\lambda}^{\epsilon\delta_{\bar{p}}(\lambda)}, \xi_{\mu}^{-\delta_{\bar{p}}(\mu)}\right) &= \delta_{\bar{p}}(\lambda)\delta_{\bar{p}}(\mu)a\left(\xi_{\lambda}^{-\epsilon\delta_{\bar{p}}(\lambda)}, \xi_{\mu}^{\delta_{\bar{p}}(\mu)}\right) \\
&= \delta_{\bar{p}}(\Psi(\lambda))\delta_{\bar{p}}(\Psi(\mu))a\left(\xi_{\Psi(\lambda)}^{-\epsilon\delta_{\bar{p}}(\Psi(\lambda))}, \xi_{\Psi(\mu)}^{\delta_{\bar{p}}(\Psi(\mu))}\right) \\
&= \delta_{\bar{p}}(\Psi(\lambda))\delta_{\bar{p}}(\Psi(\mu))a\left(\xi_{\Psi(\lambda)}^{\epsilon\delta_{\bar{p}}(\Psi(\lambda))}, \xi_{\Psi(\mu)}^{-\delta_{\bar{p}}(\Psi(\mu))}\right).
\end{aligned}$$

□

Now we consider  $\bar{p}$ -cores  $\gamma$  and  $\gamma'$ , and a positive integer  $w$ . Let  $B$  and  $B'$  be the spin blocks of  $\tilde{\mathfrak{S}}_n$  and  $\tilde{\mathfrak{S}}_m$  of weight  $w$  and  $\bar{p}$ -core  $\gamma$  and  $\gamma'$  respectively, and let  $B^*$  and  $B'^*$  be the corresponding spin blocks of  $\tilde{\mathcal{A}}_n$  and  $\tilde{\mathcal{A}}_m$ . Suppose furthermore that  $\sigma(\gamma) = -\sigma(\gamma')$ , so that, with the notation of Lemma 4.8(ii),  $\Psi$  is a sign inverting bijection, and  $\tilde{\Psi}$  gives a bijection between  $B$  and  $B'^*$ .

**Proposition 4.19.** *Let the notation be as above, and assume  $m - q \geq 2$ . For any  $\lambda \in \mathcal{D}_n$  with  $\bar{p}$ -core  $\gamma$  and  $\mu \in M_q(\lambda)$ , and for any  $\eta, \epsilon \in \{1, -1\}$ , we have*

$$\delta_{\bar{p}}(\lambda)\delta_{\bar{p}}(\mu)a(\xi_{\lambda}^{\eta\delta_{\bar{p}}(\lambda)}, \xi_{\mu}^{\epsilon\delta_{\bar{p}}(\mu)}) = \delta_{\bar{p}}(\Psi(\lambda))\delta_{\bar{p}}(\Psi(\mu))a(\zeta_{\Psi(\lambda)}^{\eta\delta_{\bar{p}}(\Psi(\lambda))}, \zeta_{\Psi(\mu)}^{\epsilon\delta_{\bar{p}}(\Psi(\mu))}).$$

*Proof.* First, assume that  $\lambda \in \mathcal{D}_n^+$ . Then by Lemma 4.8(ii),  $\Psi(\lambda) \in \mathcal{D}_m^-$ . Furthermore, by Theorem 4.10, for any  $\mu \in M_q(\lambda)$ , we have  $a(\xi_{\lambda}, \xi_{\mu}) = (-1)^{\frac{q^2-1}{8}}\alpha_{\mu}^{\lambda}$  whenever  $\sigma(\mu) = 1$ , and  $a(\xi_{\lambda}, \xi_{\mu}^+) = a(\xi_{\lambda}, \xi_{\mu}^-) = \frac{1}{2}(-1)^{\frac{q^2-1}{8}}\alpha_{\mu}^{\lambda}$  whenever  $\sigma(\mu) = -1$ .

As previously, we see that  $\Psi$  is compatible with the bijections  $g_{\lambda}$  and  $g_{\Psi(\lambda)}$  given by Theorem [22, Theorem (4.3)], whence it gives a sign inverting bijection between  $M_q(\Psi(\lambda))$  and  $M_q(\lambda)$ . If  $\mu \in M_q(\lambda)$  is obtained by removing the  $q$ -bar  $b$  from  $\lambda$ , then  $\Psi(\mu) \in M_q(\Psi(\lambda))$  is obtained by removing the  $q$ -bar  $\Psi(b)$  from  $\Psi(\lambda)$ , so that we want to compare  $\alpha_{\mu}^{\lambda}$  and  $\alpha_{\Psi(\mu)}^{\Psi(\lambda)}$ . For this, we use Equations (63), (64) and (65). Since  $\Psi(\lambda) \in \mathcal{D}_m^-$ , we see that  $m(\Psi(b))$  is always 0, so that  $\alpha_{\Psi(\mu)}^{\Psi(\lambda)} = (-1)^{L(\Psi(b))}$ . And, since  $\lambda \in \mathcal{D}_n^+$ , we see that  $m(b) = 1$  and  $\alpha_{\mu}^{\lambda} = (-1)^{L(b)}2$  whenever  $\sigma(\mu) = -1$ , while  $m(b) = 0$  and  $\alpha_{\mu}^{\lambda} = (-1)^{L(b)}$  whenever  $\sigma(\mu) = 1$ .

As in the proof of Proposition 4.18,  $L(b)$  is related to  $L(g_{\lambda}(b))$ , and  $L(\Psi(b))$  to  $L(g_{\Psi(\lambda)}(\Psi(b)))$ , and  $g_{\Psi(\lambda)}(\Psi(b)) = g_{\lambda}(b)$ . Thus, using Equations (66) and (67), we see that Equation (68) holds. If  $\sigma(\mu) = 1$ , then  $\sigma(\Psi(\mu)) = -1$ , and we obtain

$$\begin{aligned}
\delta_{\bar{p}}(\lambda)\delta_{\bar{p}}(\mu)a(\xi_{\lambda}, \xi_{\mu}) &= (-1)^{\frac{q^2-1}{8}}\delta_{\bar{p}}(\lambda)\delta_{\bar{p}}(\mu)\alpha_{\mu}^{\lambda} \\
&= (-1)^{\frac{q^2-1}{8}}\delta_{\bar{p}}(\lambda)\delta_{\bar{p}}(\mu)(-1)^{L(b)} \\
&= (-1)^{\frac{q^2-1}{8}}\delta_{\bar{p}}(\Psi(\lambda))\delta_{\bar{p}}(\Psi(\mu))(-1)^{L(\Psi(b))} \\
&= (-1)^{\frac{q^2-1}{8}}\delta_{\bar{p}}(\Psi(\lambda))\delta_{\bar{p}}(\Psi(\mu))\alpha_{\Psi(\mu)}^{\Psi(\lambda)} \\
&= \delta_{\bar{p}}(\Psi(\lambda))\delta_{\bar{p}}(\Psi(\mu))a(\zeta_{\Psi(\lambda)}, \zeta_{\Psi(\mu)}).
\end{aligned}$$

If, on the other hand,  $\sigma(\mu) = -1$ , then  $\sigma(\Psi(\mu)) = 1$ , and we obtain

$$\begin{aligned}
\delta_{\bar{p}}(\lambda)\delta_{\bar{p}}(\mu)a(\xi_{\lambda}, \xi_{\mu}^{\pm}) &= (-1)^{\frac{q^2-1}{8}}\delta_{\bar{p}}(\lambda)\delta_{\bar{p}}(\mu)\frac{1}{2}\alpha_{\mu}^{\lambda} \\
&= (-1)^{\frac{q^2-1}{8}}\delta_{\bar{p}}(\lambda)\delta_{\bar{p}}(\mu)\frac{1}{2}2(-1)^{L(b)} \\
&= (-1)^{\frac{q^2-1}{8}}\delta_{\bar{p}}(\Psi(\lambda))\delta_{\bar{p}}(\Psi(\mu))(-1)^{L(\Psi(b))} \\
&= (-1)^{\frac{q^2-1}{8}}\delta_{\bar{p}}(\Psi(\lambda))\delta_{\bar{p}}(\Psi(\mu))\alpha_{\Psi(\mu)}^{\Psi(\lambda)} \\
&= \delta_{\bar{p}}(\Psi(\lambda))\delta_{\bar{p}}(\Psi(\mu))a(\zeta_{\Psi(\lambda)}, \zeta_{\Psi(\mu)}^{\pm}).
\end{aligned}$$

Assume now that  $\lambda \in \mathcal{D}_n^-$ . Then  $\Psi(\lambda) \in \mathcal{D}_m^+$ . Note that  $\lambda$  has a part of length  $q$ , if and only if  $\Psi(\lambda)$  has one. If this is the case, then  $\sigma(\lambda \setminus \{q\}) = -1$  and  $\{\mu \in M_q(\lambda) \mid \sigma(\mu) = -1\} = \{\lambda \setminus \{q\}\}$ . Otherwise,  $\{\mu \in M_q(\lambda) \mid \sigma(\mu) = -1\}$  is empty. By Theorem 4.10, we have  $a(\xi_\lambda^+, \xi_\mu) = (-1)^{\frac{q^2-1}{8}} \alpha_\mu^\lambda$  whenever  $\sigma(\mu) = 1$ , and  $a(\xi_\lambda^+, \xi_\mu^+) = \frac{1}{2}(-1)^{\frac{q^2-1}{8}} (\alpha_\mu^\lambda + i^{\frac{q-1}{2}} \sqrt{q})$  and  $a(\xi_\lambda^+, \xi_\mu^-) = \frac{1}{2}(-1)^{\frac{q^2-1}{8}} (\alpha_\mu^\lambda - i^{\frac{q-1}{2}} \sqrt{q})$ , whenever  $\sigma(\mu) = -1$  (and  $\mu = \lambda \setminus \{q\}$ ).

Furthermore, if  $\sigma(\mu) = 1$ , then  $a(\xi_\lambda^-, \xi_\mu) = a(\xi_\lambda^+, \xi_\mu)$ , and, if  $\sigma(\mu) = -1$ , then  $a(\xi_\lambda^-, \xi_\mu^+) = a(\xi_\lambda^+, \xi_\mu^-)$  and  $a(\xi_\lambda^-, \xi_\mu^-) = a(\xi_\lambda^+, \xi_\mu^+)$ .

As in the previous case,  $\Psi$  gives a sign inverting bijection between  $M_q(\lambda)$  and  $M_q(\Psi(\lambda))$ . If  $\mu \in M_q(\lambda)$  is obtained by removing the  $q$ -bar  $b$  from  $\lambda$ , then  $\Psi(\mu) \in M_q(\Psi(\lambda))$  is obtained by removing the  $q$ -bar  $\Psi(b)$  from  $\Psi(\lambda)$ . Note that  $\sigma(\lambda) = -1$  and  $\sigma(\Psi(\lambda)) = 1$ , and for  $\mu \in M_q(\lambda)$ , we have  $\sigma(\mu) = -\sigma(\Psi(\mu))$ . In particular, Equations (64) and (65) give  $m(b) = 0$ , and  $m(\Psi(b)) = 1$  whenever  $\sigma(\mu) = 1$ , and  $m(\Psi(b)) = 0$  otherwise.

If  $\sigma(\mu) = 1$ , then  $m(\Psi(b)) = 1$  and  $\sigma(\Psi(\mu)) = -1$ . Thus Theorem 4.10, Theorem 4.14, and Equations (63) and (68) give the result.

If  $\sigma(\mu) = -1$ , then  $m(\Psi(b)) = 0$  and  $\sigma(\Psi(\mu)) = 1$ . Thus, using Theorem 4.10, Theorem 4.14 and Equation (68), we conclude with a computation similar to that at the end of the proof of Proposition 4.18.  $\square$

**Remark 4.20.** Note that, with the notation of Remark 4.15 (in particular we have  $m \leq q + 1$  and  $\Psi(\mu) \in \{\emptyset, (1)\}$ ), if  $\Psi(\lambda) \neq \pi$  and if we set  $\zeta_{\Psi(\mu)}^+ = \zeta_{\Psi(\mu)}^- = \varepsilon$ , then Proposition 4.19 still holds.

We now can state the main result of this section. Let  $\gamma$  and  $\gamma'$  be two  $p$ -cores, and  $w$  be a positive integer. Write  $E_{\gamma,w}$ ,  $E_{\gamma',w}$  and  $\Psi : E_{\gamma,w} \rightarrow E_{\gamma',w}$  as in Lemma 4.8, and set  $n = |\gamma| + pw$  and  $m = |\gamma'| + pw$ . If  $\sigma(\gamma) = \sigma(\gamma')$ , then  $B_{\gamma,w}$  and  $B_{\gamma',w}$  denote the  $p$ -blocks of  $\bar{p}$ -weight  $w$  of  $G = \tilde{\mathfrak{S}}_n$  and  $G' = \tilde{\mathfrak{S}}_m$  corresponding to  $\gamma$  and  $\gamma'$  respectively. If  $\sigma(\gamma) = -\sigma(\gamma')$ , then  $B_{\gamma,w}$  and  $B_{\gamma',w}$  denote the  $p$ -blocks of  $\bar{p}$ -weight  $w$  of  $G = \tilde{\mathfrak{A}}_n$  and  $G' = \tilde{\mathfrak{S}}_m$  respectively. We write  $\text{Irr}(B_\gamma) = \{X_\lambda^\epsilon \mid \lambda \in E_{\gamma,w}, \epsilon \in \{-1, 1\}\}$  and  $\text{Irr}(B_{\gamma'}) = \{Y_\lambda^\epsilon \mid \lambda \in E_{\gamma',w}, \epsilon \in \{-1, 1\}\}$ , with the convention that, when  $X_\lambda$  or  $Y_\lambda$  are self-associate, we set  $X_\lambda^+ = X_\lambda^- = X_\lambda$  and  $Y_\lambda^+ = Y_\lambda^- = Y_\lambda$ .

**Theorem 4.21.** *Let  $p$  be an odd prime. We keep the notation as above. Then the isometry  $I : \mathbb{C} \text{Irr}(B_{\gamma,w}) \rightarrow \mathbb{C} \text{Irr}(B_{\gamma',w})$  defined by*

$$(69) \quad I \left( X_\lambda^{\epsilon \delta_{\bar{p}}(\lambda)} \right) = \delta_{\bar{p}}(\lambda) \delta_{\bar{p}}(\Psi(\lambda)) Y_{\Psi(\lambda)}^{\epsilon \delta_{\bar{p}}(\Psi(\lambda))},$$

where  $\lambda \in E_{\gamma,w}$  and  $\epsilon \in \{-1, 1\}$ , is a Broué perfect isometry.

*Proof.* Consider the map  $\hat{I}$  corresponding to  $I$  as in Equation (6). We will prove that  $\hat{I}$  satisfies Properties (i) and (ii) of a Broué isometry.

First, we use the MN-structures introduced in Proposition 4.17 for  $(C_G, B_{\gamma,w})$  and  $(C_{G'}, B_{\gamma',w})$ . Let  $S_G$  and  $S_{G'}$  be as in Equation (62). Write  $\Omega$  for the set of partitions  $\pi$  of  $i \leq n$  such that  $p$  divides each part of  $\pi$ . Note that  $\pi \in \Omega$  parametrizes one or two  $G$ -classes of elements of  $S_G$  (always one class when  $G = \tilde{\mathfrak{S}}_n$ , and two classes when  $G = \tilde{\mathfrak{A}}_n$  and  $\pi \in \mathcal{O}_n \cap \mathcal{D}_n$ ). In the case where  $\pi$  labels two classes, we denote the two parameters by  $\pi^\pm$ . Let  $\Lambda$  be the set of parameters obtained in this way. Then  $\Lambda$  labels the set of  $G$ -classes of  $S_G$ . We will now define



a precise set of representatives for these classes. Let  $\pi = (\pi_1, \dots, \pi_k) \in \Omega$ . Note that  $\pi$  and  $(\pi_1), \dots, (\pi_k)$  can all be viewed as labels of conjugacy classes of  $S_{\mathfrak{S}_n}$  (by completing the partitions with parts of length 1).

In particular, the element  $s_\pi = s_{\pi_1} \cdots s_{\pi_k}$  defined before Equation (41) is a representative for the class of  $S_{\mathfrak{S}_n}$  labeled by  $\pi$ , and  $s_\pi \in \mathfrak{S}_{|\pi|}$ . For all  $1 \leq i \leq k$ , denote by  $o_{\pi_i}$  the element of odd order such that  $\theta(o_{\pi_i}) = s_{\pi_i}$  (see Lemma 4.9) and write  $o_\pi = o_{\pi_1} \cdots o_{\pi_k}$ . Furthermore, for  $\pi \in \mathcal{O}_n \cap \mathcal{D}_n$ , we assume that the representatives of the two  $\mathcal{A}_n$ -classes labeled by  $\pi$  are  $s_{\pi^\pm} = s_{\pi_1^\pm} \cdots s_{\pi_{k-1}} s_{\pi_k}$  as in the proof of Theorem 3.9. If  $o_{\pi_1^\pm}$  denotes the elements of odd order satisfying  $\theta(o_{\pi_1^\pm}) = s_{\pi_1^\pm}$  (see Lemma 4.9), then we set  $o_{\pi^\pm} := o_{\pi_1^\pm} \cdots o_{\pi_{k-1}} o_{\pi_k}$ . Therefore, if  $G = \tilde{\mathfrak{S}}_n$ , then the set of  $o_\pi$  for  $\pi \in \Omega$  is a set of representatives of the  $G$ -classes of  $S_G$ . If  $G = \tilde{\mathcal{A}}_n$ , then the elements  $o_\pi$  (for  $\pi \in \Omega$  and  $\pi \notin \mathcal{O}_n \cap \mathcal{D}_n$ ) and  $o_{\pi^\pm}$  (for  $\pi \in \Omega \cap \mathcal{O}_n \cap \mathcal{D}_n$ ) form a system of representatives of the  $\tilde{\mathcal{A}}_n$ -classes of  $S_{\tilde{\mathcal{A}}_n}$ . Moreover, for any  $\hat{\pi} \in \Lambda$  (with  $\hat{\pi} \in \{\pi^+, \pi^-\}$  if  $\pi \in \mathcal{O}_n \cap \mathcal{D}_n$  and  $G = \tilde{\mathcal{A}}_n$ , and  $\hat{\pi} = \pi$  otherwise), we write  $G_{\hat{\pi}} = G_{o_{\hat{\pi}}}$  and  $r^{\hat{\pi}} = r^{o_{\hat{\pi}}}$ , where  $G_{o_{\hat{\pi}}}$  and  $r^{o_{\hat{\pi}}} : \mathbb{C} \text{Irr}(B_{\gamma, w}) \rightarrow \mathbb{C} \text{Irr}(B(G_\pi))$  are defined in Proposition 4.17 (here  $B(G_\pi)$  is one  $p$ -block or two  $p$ -blocks of  $G_\pi$ ).

Now, we define  $\Omega_0 = \{\pi \in \Omega \mid \sum \pi_i \leq pw\}$  and  $\Lambda_0$  the set of parameters  $\hat{\pi} \in \Lambda$  such that  $\pi \in \Omega_0$ . Then  $\Omega_0 = \Lambda_0$  whenever  $G = \tilde{\mathfrak{S}}_n$  or  $G = \tilde{\mathcal{A}}_n$  with  $n \notin \{pw, pw+1\}$ .

Similarly, we define  $\Omega', \Lambda', \Omega'_0$  and  $\Lambda'_0$  for  $G'$  and  $S_{G'}$ . Since  $G' = \tilde{\mathfrak{S}}_m$ , we have  $\Lambda' = \Omega'$  and  $\Lambda'_0 = \Omega'_0 = \Omega_0$ . We write  $o'_\pi$  for the representatives of the  $G'$ -classes of  $S_{G'}$  (as described above for  $G$ ) and, for  $\pi \in \Omega'$ , we define  $G'_\pi$  and  $r'^\pi : \mathbb{C} \text{Irr}(B_{\gamma', w}) \rightarrow \mathbb{C} \text{Irr}(B(G'_\pi))$  as above.

Using Theorems 4.10 and 4.14, we show that for any  $\pi \in \Omega \setminus \Omega_0$  or  $\pi \in \Omega' \setminus \Omega'_0$ , one has  $r^{\hat{\pi}} = 0$  and  $r'^{\hat{\pi}} = 0$ .

Now we suppose that  $\Lambda_0 = \Omega_0$ . Let  $\pi \in \Omega_0$ . If  $|\pi| < pw$ , then  $B(G_\pi)$  and  $B(G'_\pi)$  are just one  $p$ -block of  $G_\pi$  and  $G'_\pi$ , respectively. If  $|\pi| = pw$ , then  $B(G_\pi)$  and  $B(G'_\pi)$  are one  $p$ -block with defect zero whenever  $G = \tilde{\mathfrak{S}}_n$  and  $\sigma(\gamma) = 1$  or  $G = \tilde{\mathcal{A}}_n$  and  $\sigma(\gamma) = -1$ , or are the union of two  $p$ -blocks with defect zero otherwise. We define (and denote by the same symbol to simplify the notation)  $I : \text{Irr}(B(G_\pi)) \rightarrow \text{Irr}(B(G'_\pi))$  by Equation (69).

We assume that Convention 4.2 holds and that moreover, if  $\pi = (\pi_1, \dots, \pi_k) \in \mathcal{P}_n$  has an odd part divisible by  $p$ , then there is some  $r \leq r' \leq k$  such that  $\pi_j$  is divisible by  $p$  for all  $r' \leq j$ , and every odd  $\pi_j$  with  $j < r$  is prime to  $p$ . So, we can use Theorem 4.10 and 4.14 iteratively (see also Remark 4.13). Therefore, using Propositions 4.18 and 4.19, we show as in the proof of Theorem 3.9 (see Equations (29), (30) and (31)), that

$$(70) \quad I \circ r^\pi = r'^\pi \circ I.$$

Thus, Theorem 2.10 holds (see Remark 2.11).

Suppose, on the other hand, that  $\Lambda_0 \neq \Omega_0$ . Then  $G = \tilde{\mathcal{A}}_n$ ,  $n \in \{pw, pw+1\}$ , and  $G' = \tilde{\mathfrak{S}}_m$ . In particular,  $\sigma(\gamma) = 1$ . Let  $\pi = (\pi_1, \dots, \pi_l) \in \Omega_0$ . If  $\pi \notin \mathcal{O}_n \cap \mathcal{D}_n$ , then we are in the same situation as above, and Equation (70) holds. Suppose instead that  $\pi \in \mathcal{O}_n \cap \mathcal{D}_n$ . Then  $\pi$  labels two classes with representatives  $o_{\pi^+}$  and  $o_{\pi^-}$  of  $S_G$ , and  $G_{\pi^+}$  and  $G_{\pi^-}$  are two copies of  $\mathbb{Z}_2$ , whose only spin  $p$ -block has defect

zero, and consists of the (only) non-trivial character. Denote by  $\{\epsilon_+\}$  and  $\{\epsilon_-\}$  the spin  $p$ -blocks of  $G_{\pi^+}$  and  $G_{\pi^-}$ , respectively.

Now, even though  $\sigma(\gamma) = 1$ ,  $\gamma$  labels just one  $p$ -block of  $G_{\pi^+}$  (and of  $G_{\pi^-}$ ). Since  $\sigma(\gamma') = -\sigma(\gamma) = -1$ , it follows that  $\gamma'$  labels two  $p$ -blocks with defect zero of  $G'_\pi$ . In particular,  $\text{Irr}(B(G'_\pi))$  is the union of the  $p$ -blocks  $\{\xi_{\gamma'}^+\}$  and  $\{\xi_{\gamma'}^-\}$ .

We then define  $I_\pi : \mathbb{C}\{\epsilon_+\} \oplus \mathbb{C}\{\epsilon_-\} \longrightarrow \text{Irr}(B(G'_\pi))$  by setting  $I_\pi(\epsilon_+) = \xi_{\gamma'}^+$  and  $I_\pi(\epsilon_-) = \xi_{\gamma'}^-$ . Let  $\eta, \delta \in \{-1, 1\}$  and  $\lambda \in E_{\gamma, w}$ . We have  $r^{\pi^\delta}(\zeta_\lambda^\eta) = a(\zeta_\lambda^\eta, \epsilon_\delta)\epsilon_\delta$ , and iterating Theorem 4.14, we obtain

$$a(\zeta_\lambda^\eta, \epsilon_\delta) = \sum_{i_1, \dots, i_l} a(\zeta_{\lambda_{i_0}}^{\beta_{i_0}}, \zeta_{\lambda_{i_1}}^{\beta_{i_1}}) a(\zeta_{\lambda_{i_1}}^{\beta_{i_1}}, \zeta_{\lambda_{i_2}}^{\beta_{i_2}}) \cdots a(\zeta_{\lambda_{i_{l-1}}}^{\beta_{i_{l-1}}}, \zeta_{\lambda_{i_l}}^{\beta_{i_l}}),$$

where  $\lambda_{i_0} = \lambda$ ,  $\beta_{i_0} = \eta$ ,  $\zeta_{\lambda_{i_l}}^{\beta_{i_l}} = \epsilon_\delta$ ,  $\beta_{i_j} \in \{-1, 1\}$  for all  $1 \leq j \leq l-1$ , and  $\zeta_{\lambda_{i_j}}^{\beta_{i_j}}$  is obtained from  $\zeta_{\lambda_{i_{j-1}}}^{\beta_{i_{j-1}}}$  by removing a  $\pi_j$ -bar from  $\lambda_{i_{j-1}}$  for all  $1 \leq j \leq l$ . Similarly, we have  $r^{\pi^\pi}(\xi_{\Psi(\lambda)}^{\eta'}) = a(\xi_{\Psi(\lambda)}^{\eta'}, \xi_{\gamma'}^+) \xi_{\gamma'}^+ + a(\xi_{\Psi(\lambda)}^{\eta'}, \xi_{\gamma'}^-) \xi_{\gamma'}^-$  with

$$a(\xi_{\Psi(\lambda)}^{\eta'}, \xi_{\gamma'}^\delta) = \sum_{i_1, \dots, i_l} a(\xi_{\Psi(\lambda_{i_0})}^{\beta'_{i_0}}, \xi_{\Psi(\lambda_{i_1})}^{\beta'_{i_1}}) a(\xi_{\Psi(\lambda_{i_1})}^{\beta'_{i_1}}, \xi_{\Psi(\lambda_{i_2})}^{\beta'_{i_2}}) \cdots a(\xi_{\Psi(\lambda_{i_{l-1}})}^{\beta'_{i_{l-1}}}, \xi_{\Psi(\lambda_{i_l})}^{\beta'_{i_l}}),$$

where  $\beta'_{i_j} = \delta_{\bar{p}}(\lambda_{i_j}) \delta_{\bar{p}}(\Psi(\lambda_{i_j})) \beta_{i_j}$  for all  $0 \leq j \leq l-1$ ,  $\eta' = \beta'_{i_0}$ , and  $\beta'_{i_l} = \delta$ .

Write  $\mathfrak{f} = (\pi_1)$  if  $n = pw$  or  $\mathfrak{f} = (\pi_1, 1)$  if  $n = pw + 1$ . Note that  $\lambda_{i_{l-1}}$  is a partition of  $|\mathfrak{f}|$ , and if  $\lambda_{i_{l-1}} \neq \mathfrak{f}$ , then for  $1 \leq j \leq l$ , Theorem 4.19 and Remark 4.20 give that

$$a\left(\xi_{\Psi(\lambda_{i_{j-1}})}^{\beta'_{i_{j-1}}}, \xi_{\Psi(\lambda_{i_j})}^{\beta'_{i_j}}\right) = \delta_{\bar{p}}(\lambda_{i_{j-1}}) \delta_{\bar{p}}(\Psi(\lambda_{i_{j-1}})) \delta_{\bar{p}}(\lambda_{i_j}) \delta_{\bar{p}}(\Psi(\lambda_{i_j})) a\left(\zeta_{\lambda_{i_{j-1}}}^{\beta_{i_{j-1}}}, \zeta_{\lambda_{i_j}}^{\beta_{i_j}}\right),$$

and it follows that

$$(71) \quad a(\zeta_\lambda^\eta, \epsilon_\delta) = \delta_{\bar{p}}(\lambda) \delta_{\bar{p}}(\Psi(\lambda)) a(\xi_{\Psi(\lambda)}^{\eta \delta_{\bar{p}}(\lambda) \delta_{\bar{p}}(\Psi(\lambda))}, \xi_{\gamma'}^\delta).$$

In particular, one has

$$(72) \quad \begin{aligned} I_\pi \left( (r^{\pi^+} + r^{\pi^-})(\zeta_\lambda^\eta) \right) &= a(\zeta_\lambda^\eta, \epsilon_+) \xi_{\gamma'}^+ + a(\zeta_\lambda^\eta, \epsilon_-) \xi_{\gamma'}^- \\ &= \delta_{\bar{p}}(\lambda) \delta_{\bar{p}}(\Psi(\lambda)) r^{\pi^\pi} \left( \xi_{\Psi(\lambda)}^{\eta \delta_{\bar{p}}(\lambda) \delta_{\bar{p}}(\Psi(\lambda))} \right) \\ &= r^{\pi^\pi} (I(\zeta_\lambda^\delta)). \end{aligned}$$

Furthermore, Equation (59), Theorem 4.10 and a computation similar to that in the proof of Proposition 4.19 give

$$a(\zeta_{\mathfrak{f}}^\pm, \epsilon_\delta) = \delta_{\bar{p}}(\mathfrak{f}) \delta_{\bar{p}}(\Psi(\mathfrak{f})) a\left(\xi_{\Psi(\mathfrak{f})}^{\pm \delta_{\bar{p}}(\mathfrak{f}) \delta_{\bar{p}}(\Psi(\mathfrak{f}))}, \xi_{\gamma'}^\delta\right).$$

So, if  $\lambda_{i_{l-1}} = \mathfrak{f}$ , then Equation (71) and thus Equation (72) also hold. In summary, we have proved that

$$I_\pi \circ (r^{\pi^+} + r^{\pi^-}) = r^{\pi^\pi} \circ I.$$

Finally, by the argument of the proof of Theorem 3.9, we obtain for  $\widehat{I}$  a decomposition as in Equation (40).

We now prove that  $\widehat{I}$  satisfies property (ii) of a Broué isometry. Assume that  $x \in G$  is  $p$ -singular and  $x' \in G'$  is  $p$ -regular. If  $x \notin C_G$ , then  $\widehat{I}(x, x') = 0$  (see the proof of Corollary 2.17). Otherwise,  $x \in C_G$ , and without loss of generality, we can

assume that  $x = z^k t_\beta$  for some  $\beta \in \mathcal{P}_n$  and  $k \in \{0, 1\}$ . Note that  $z^k t_\beta \in C_G$  means that  $\beta$  has at least one part of length divisible by  $2p$ . In particular,  $\beta \notin \mathcal{O}_n$ . If  $\beta \notin D_n^-$  (when  $G = \tilde{\mathfrak{S}}_n$ ) or  $\beta \notin D_n^+$  (when  $G = \tilde{\mathfrak{A}}_n$ ), then Propositions 4.1 and 4.3 imply that  $X_\lambda^\pm(z^k t_\beta) = 0$  for all  $\lambda \in E_{\gamma, w}$ , and  $I(x, x') = 0$  by Equation (11). Hence, we can suppose that  $\beta \in \mathcal{D}_n^-$  if  $G = \tilde{\mathfrak{S}}_n$ , or that  $\beta \in \mathcal{D}_n^+$  if  $G = \tilde{\mathfrak{A}}_n$ . Therefore,  $X_\lambda^\pm(z^k t_\beta) \neq 0$  if and only if  $\lambda = \beta$ . Furthermore, if we write  $\beta^{(\bar{p})} = (\beta^0, \dots, \beta^{(p-1)/2})$  for the  $\bar{p}$ -quotient of  $\beta$ , then the parts of  $\beta$  divisible by  $p$  are the parts of  $p \cdot \beta^0$  (see [22, p. 27]). Hence, the definition of  $\Psi$  gives  $\beta^{(\bar{p})} = \Psi(\beta)^{(\bar{p})}$ , and  $\Psi(\beta)$  has non-trivial parts divisible by  $p$ . It follows that  $Y_{\Psi(\lambda)}^+(x') = Y_{\Psi(\lambda)}^-(x')$ , because  $x'$  is  $p$ -regular. Using Equation (11), we obtain

$$\widehat{I}(x, x') = \left( X_\beta^+(z^k t_\beta) + X_\beta^-(z^k t_\beta) \right) Y_{\Psi(\beta)}^+(x) = 0$$

by Equations (45) and (50). Note that we derive from Remark 2.12 and a similar computation that, if  $x$  is  $p$ -regular and  $x'$  is  $p$ -singular, then  $\widehat{I}(x, x') = 0$ .

Finally, we show that  $\widehat{I}$  satisfies property (i) of a Broué isometry. Note that the  $\mathcal{E}$  have size 1 or 2, and all the assumptions of Theorem 2.20 are satisfied.

First, we consider the case  $G = \tilde{\mathfrak{S}}_n$ . Take  $\Phi \in \mathbb{Z}\mathfrak{b}_{\gamma, w}^\vee$ , where  $\mathfrak{b}_{\gamma, w}$  is a  $\mathbb{Z}$ -basis of  $\mathbb{Z}\text{Irr}(B_{\gamma, w})^{C_{\tilde{\mathfrak{S}}_n}}$  as in Remark 2.9. By Corollary 2.3 and Proposition 4.1, for  $x \in \tilde{\mathfrak{A}}_n$ , we have  $\Phi(x) \neq 0$  only if  $x$  is  $p$ -regular. Thus, again by Corollary 2.3 (applied to  $\tilde{\mathfrak{A}}_n$  with respect to the set of  $p$ -regular elements),  $\text{Res}_{\tilde{\mathfrak{A}}_n}^{\tilde{\mathfrak{S}}_n}(\Phi)$  is a projective character of  $\tilde{\mathfrak{A}}_n$ . Let  $x$  be a  $p$ -regular element of  $\tilde{\mathfrak{A}}_n$ . In particular,  $x = x_{C_{\tilde{\mathfrak{S}}_n}}$ . Since  $\Phi(x) = \text{Res}_{\tilde{\mathfrak{A}}_n}^{\tilde{\mathfrak{S}}_n}(\Phi)(x)$ , it follows that  $\Phi(x)$  is the value of some projective character of  $\tilde{\mathfrak{A}}_n$ .

Let  $\pi \in \Omega_0$  and  $\phi \in \mathbb{Z}\mathfrak{b}_\pi$ , where  $\mathfrak{b}_\pi$  is a  $\mathbb{Z}$ -basis of  $\mathbb{Z}\text{Irr}(B_\pi)^{C_{\tilde{\mathfrak{S}}_n} \cap G_\pi}$  as in Remark 2.9. Now, we apply the previous computations to  $G_\pi$ ,  $G'_\pi$  and  $I_\pi$ . We conclude that the condition (2) of Theorem 2.10 holds for  $I_\pi$ . Hence, Remark 2.11 gives the condition (3) of Theorem 2.10 for  $I_\pi$  and we deduce as in the proof of Theorem 2.20 that  $J_\pi(\mathbb{Z}\mathfrak{b}_\pi^\vee) = \mathbb{Z}\mathfrak{b}'_\pi$ . Hence,  $J_\pi^{*-1}(\mathbb{Z}\mathfrak{b}_\pi) \subseteq \mathbb{Z}\mathfrak{b}'_\pi$ . Since  $\mathcal{E}$  have size 1 or 2 and  $p$  is odd, we have  $l'_\pi(J_\pi^{*-1}(\phi))(x') \in \mathcal{R}$  for all  $x' \in \tilde{\mathfrak{S}}_m$ . We conclude with the argument of the proof of Theorem 2.20 that  $\widehat{I}(x, x')/|C_G(x)| \in \mathcal{R}$ . Similarly, because of Remark 2.12, if  $x' \in \tilde{\mathfrak{A}}_m$ , then  $\widehat{I}(x, x')/|C_{G'}(x')| \in \mathcal{R}$ .

Assume now that  $x \notin \tilde{\mathfrak{A}}_n$ . By Equation (11) and Proposition 4.1,  $\widehat{I}(x, x') \neq 0$  only if  $x = z^u t_\beta$  and  $x' = z^v t'_{\Psi(\beta)}$  with  $\beta \in \mathcal{D}_n^-$  and  $u, v \in \{0, 1\}$ . In this case, Equation (44) gives

$$(73) \quad \widehat{I}(z^u t_\beta, z^v t'_{\beta'}) = \pm i^{\frac{n+m-\ell(\beta)-\ell(\beta')-2}{2}} \sqrt{\beta_1 \cdots \beta_k \beta'_1 \cdots \beta'_{k'}},$$

where  $\beta = (\beta_1, \dots, \beta_k)$  and  $\Psi(\beta) = (\beta'_1, \dots, \beta'_{k'})$ . However, we derive from the proof of [22, Theorem 4.3] that  $\nu_p(\beta_1 \cdots \beta_k) = p^{|\beta^0|} \nu_p(\text{prod}(\beta^0))$ , where  $\nu_p$  is the  $p$ -valuation,  $\beta^{(\bar{p})} = (\beta^0, \dots, \beta^{(p-1)/2})$  is the  $\bar{p}$ -quotient of  $\beta$ , and  $\text{prod}(\beta^0)$  is the product of the lengths of the parts of  $\beta^0$ . Hence,

$$(74) \quad \nu_p(\beta_1 \cdots \beta_k) = \nu_p(\beta'_1 \cdots \beta'_{k'}),$$

Furthermore,

$$|C_{\tilde{\mathfrak{S}}_n}(z^u t_\beta)| = 2 \prod_{i=1}^k \beta_i \quad \text{and} \quad |C_{\tilde{\mathfrak{S}}_m}(z^v t'_{\beta'})| = 2 \prod_{i=1}^{k'} \beta'_i,$$

because  $\beta, \beta' \in \mathcal{D}^-$ . By Equation (74), there are integers  $a$  and  $b$  prime to  $p$  such that  $\prod \beta_i = \nu_p(\beta_1 \cdots \beta_k) a$  and  $\prod \beta'_i = \nu_p(\beta'_1 \cdots \beta'_k) b$ . Therefore, Equation (73) implies that

$$\frac{\widehat{I}(z^u t_\beta, z^v t'_{\beta'})}{|C_{\tilde{\mathfrak{S}}_n}(z^u t_\beta)|} = \pm i^{\frac{n+m-\ell(\beta)-\ell(\beta')-2}{2}} \frac{\sqrt{ab}}{2a}.$$

Since  $\pm i^{\frac{n+m-\ell(\beta)-\ell(\beta')-2}{2}} \sqrt{ab} \in \mathcal{R}$  and  $2a$  is prime to  $p$ , we deduce that

$$\frac{\widehat{I}(z^u t_\beta, z^v t'_{\beta'})}{|C_{\tilde{\mathfrak{S}}_n}(z^u t_\beta)|} \in \mathcal{R}.$$

Similarly, we have  $\frac{\widehat{I}(z^u t_\beta, z^v t'_{\beta'})}{|C_{\tilde{\mathfrak{S}}_m}(z^v t'_{\beta'})|} \in \mathcal{R}$ .

Assume now that  $G = \tilde{\mathcal{A}}_n$ . Take  $\Phi \in \mathbb{Z} \mathfrak{b}_{\gamma,w}^\vee$ , where  $\mathfrak{b}_{\gamma,w}$  is a  $\mathbb{Z}$ -basis of  $\mathbb{Z} \text{Irr}(B_{\gamma,w})^{C_{\tilde{\mathcal{A}}_n}}$  as in Remark 2.9. By Corollary 2.3, there are integers  $a_\lambda$  (for  $\lambda \in E_{\gamma,w}$  with  $\sigma(\lambda) = -1$ ) and  $a_\lambda^\pm$  (for  $\lambda \in E_{\gamma,w}$  with  $\sigma(\lambda) = 1$ ) such that

$$(75) \quad \Phi = \sum_{\sigma(\lambda)=-1} a_\lambda \zeta_\lambda + \sum_{\sigma(\lambda)=1} (a_\lambda^+ \zeta_\lambda^+ + a_\lambda^- \zeta_\lambda^-),$$

and Clifford theory gives

$$(76) \quad \text{Ind}_{\tilde{\mathcal{A}}_n}^{\tilde{\mathfrak{S}}_n}(\Phi) = \sum_{\sigma(\lambda)=-1} a_\lambda (\xi_\lambda^+ + \xi_\lambda^-) + \sum_{\sigma(\lambda)=1} (a_\lambda^+ + a_\lambda^-) \xi_\lambda.$$

Let  $x$  be a  $p$ -regular element of  $\tilde{\mathcal{A}}_n$ . Assume  $x = z^k t_\beta$  with  $\beta \in \mathcal{O}_n$  and  $\beta \notin \mathcal{D}_n^+$ . In particular, one has  $x = x_{C_{\tilde{\mathcal{A}}_n}}$ , and for  $\lambda \in \mathcal{D}_n^+$ , we have

$$\xi_\lambda(x) = \zeta_\lambda^+(x) + \zeta_\lambda^-(x) = 2\zeta_\lambda^+(x) = 2\zeta_\lambda^-(x),$$

and it follows that

$$\begin{aligned} \text{Ind}_{\tilde{\mathcal{A}}_n}^{\tilde{\mathfrak{S}}_n}(\Phi)(x) &= \sum_{\sigma(\lambda)=-1} a_\lambda (\xi_\lambda^+(x) + \xi_\lambda^-(x)) + \sum_{\sigma(\lambda)=1} (a_\lambda^+ + a_\lambda^-) \xi_\lambda(x) \\ &= 2 \left( \sum_{\sigma(\lambda)=-1} a_\lambda \zeta_\lambda(x) + \sum_{\sigma(\lambda)=1} (a_\lambda^+ \zeta_\lambda^+(x) + a_\lambda^- \zeta_\lambda^-(x)) \right) \\ &= 2\Phi(x). \end{aligned}$$

By Equation (76), Proposition 4.1 and Corollary 2.3,  $\text{Ind}_{\tilde{\mathcal{A}}_n}^{\tilde{\mathfrak{S}}_n}(\Phi)$  is a projective character of  $\tilde{\mathfrak{S}}_n$  and hence  $\tilde{\mathcal{A}}_n$ . Thus,  $2\Phi(x)$  is the value of a projective character of  $\tilde{\mathcal{A}}_n$ , and we conclude as above, because 2 is not divisible by  $p$ .

Suppose now that  $x = z^u t_\beta$  with  $\beta \in \mathcal{O}_n \cap \mathcal{D}_n^+$ . By Lemma 4.9, we can assume that  $x = z^{u'} o_\beta$  for some non-negative integer  $u'$ . Write  $H$  for the centralizer of  $o_{\beta^\pm}$  in  $\tilde{\mathcal{A}}_n$ . Then  $H = \langle z \rangle \times \langle o_{\beta_1^\pm} \rangle \times \cdots \times \langle o_{\beta_k} \rangle$  contains no elements whose cycle structure has even parts. In particular,  $\text{Res}_H^{\tilde{\mathcal{A}}_n}(\Phi)$  is a projective character of  $H$ .

Since  $x \in H$ , it follows that  $\Phi(x)$  is the value of a projective character of  $H$ , and we again conclude with the same argument as above.

Finally, it remains to show the property for  $\beta \in \mathcal{D}_n^+$  and  $\beta \notin \mathcal{O}_n$ . However,  $\widehat{I}(z^{u'} o_\beta, x') \neq 0$  if and only if  $x' = z^v o'_{\Psi(\beta)}$  for some non-negative integer  $v$ . In particular, if  $\beta' := \Psi(\beta) \in \mathcal{D}_m^-$ , then

$$\widehat{I}(z^{u'} o_\beta, z^v o'_{\beta'}) = \pm \sqrt{2} i^{\frac{n+m-k-k'-1}{2}} \sqrt{\beta_1 \cdots \beta_k \beta'_1 \cdots \beta'_{k'}},$$

where  $\beta = (\beta_1, \dots, \beta_k)$  and  $\beta' = (\beta'_1, \dots, \beta'_{k'})$ . We conclude as above using Equation (74).  $\square$

**Corollary 4.22.** *If  $p$  is an odd prime, if  $B_{\gamma,w}$  and  $B_{\gamma',w}$  are  $p$ -blocks of  $\widetilde{\mathcal{A}}_n$  and  $\widetilde{\mathcal{A}}_m$  respectively, and if  $\sigma(\gamma) = \sigma(\gamma')$ , then the isometry  $I$  defined by Equation (69) is a Broué perfect isometry.*

*Proof.* Let  $\widetilde{\gamma}$  be any  $\bar{p}$ -core such that  $\sigma(\widetilde{\gamma}) = -\sigma(\gamma)$ . Denote by  $B_{\widetilde{\gamma},w}$  the  $p$ -block of  $\widetilde{\mathfrak{S}}_{|\widetilde{\gamma}|+pw}$  corresponding to  $\widetilde{\gamma}$ . Since  $\sigma(\gamma') = -\sigma(\widetilde{\gamma})$ , by Theorem 4.21, there are Broué perfect isometries  $I_1 : \text{Irr}(B_{\gamma,w}) \rightarrow \text{Irr}(B_{\widetilde{\gamma},w})$  and  $I_2 : \text{Irr}(B_{\gamma',w}) \rightarrow \text{Irr}(B_{\widetilde{\gamma},w})$ , defined by Equation (69). Furthermore, we have

$$I = I_2^{-1} \circ I_1,$$

which proves the result.  $\square$

## 5. SOME OTHER EXAMPLES

**5.1. Notation.** For any positive integers  $k$  and  $l$ , we denote by  $\mathcal{MP}_{k,l}$  the set of  $k$ -tuples of partitions  $(\mu_1, \dots, \mu_k)$  such that  $\sum |\mu_i| = l$ .

Let  $H$  be a finite group and  $w$  be a positive integer. We consider the wreath product  $G = H \wr \mathfrak{S}_w$ , that is, the semidirect product  $G = H^w \rtimes \mathfrak{S}_w$  where  $\mathfrak{S}_w$  acts on  $H^w$  by permutation. Write  $N = |\text{Irr}(H)|$  and  $\text{Irr}(H) = \{\psi_i \mid 1 \leq i \leq N\}$ , and denote by  $g_i$  ( $1 \leq i \leq N$ ) a system of representatives for the conjugacy classes of  $H$ .

The irreducible characters of  $G$  are parametrized by  $\mathcal{MP}_{N,w}$  as follows. For  $\mu = (\mu_1, \dots, \mu_N) \in \mathcal{MP}_{N,w}$ , consider the irreducible character  $\phi_\mu$  of  $\text{Irr}(H^w)$  given by

$$(77) \quad \phi_\mu = \prod_{i=1}^N \underbrace{\psi_i \otimes \dots \otimes \psi_i}_{|\mu_i| \text{ times}},$$

which, by [12, p.154], can be extended to an irreducible character  $\widehat{\phi}_\mu = \prod_{i=1}^N \widehat{\psi_i^{|\mu_i|}}$  of its inertia subgroup  $I_G(\phi_\mu) = \prod_{i=1}^N H \wr \mathfrak{S}_{|\mu_i|}$ . The irreducible character of  $G$  corresponding to  $\mu$  is then given by

$$\theta_\mu = \text{Ind}_{I_G(\phi_\mu)}^G \left( \prod_{i=1}^N \widehat{\psi_i^{|\mu_i|}} \otimes \chi_{\mu_i} \right),$$

where  $\chi_{\mu_i}$  denotes the irreducible character of  $\mathfrak{S}_{|\mu_i|}$  corresponding to the partition  $\mu_i$  of  $|\mu_i|$ .

Let  $(h_1, \dots, h_w; \sigma) \in G$  with  $h_1, \dots, h_w \in H$  and  $\sigma \in \mathfrak{S}_w$ . For any  $k$ -cycle  $\kappa = (j, \kappa j, \dots, \kappa^{k-1} j)$  in  $\sigma$ , we define the cycle product

$$g((h_1, \dots, h_w; \sigma); \kappa) = h_j h_{\kappa^{-1}j} \cdots h_{\kappa^{-(k-1)}j}.$$

If  $\sigma$  has cycle structure  $\pi$ , then we form the  $N$ -tuple of partitions  $(\pi_1, \dots, \pi_N)$  from  $\pi$ , where any cycle  $\kappa$  in  $\pi$  gives a cycle of the same length in  $\pi_i$  if  $g((h_1, \dots, h_w; \sigma); \kappa)$  is conjugate to  $g_i$  in  $H$ . The  $N$ -tuple

$$(78) \quad \mathfrak{s}(h_1, \dots, h_w; \sigma) = (\pi_1, \dots, \pi_N) \in \mathcal{MP}_{N,w}$$

describes the cycle structure of  $(h_1, \dots, h_w; \sigma)$ , and two elements of  $G$  are conjugate if and only if they have the same cycle structure (see [12, 4.2.8]). In particular, the conjugacy classes of  $G$  are labeled by  $\mathcal{MP}_{N,w}$ .

**5.2. Isometries between symmetric groups and natural subgroups.** Let  $n$  be a positive integer and  $p$  be a prime. We denote by  $\mathcal{P}_n$  the set of partitions of  $n$ . Write  $\chi_\lambda$  for the irreducible character of the symmetric group  $\mathfrak{S}_n$  corresponding to the partition  $\lambda \in \mathcal{P}_n$ . Recall that to every  $\lambda \in \mathcal{P}_n$ , we can associate its  $p$ -core  $\lambda_{(p)}$  and its  $p$ -quotient  $\lambda^{(p)} = (\lambda_1, \dots, \lambda_p)$  (see for example [22, p. 17]). Moreover, two irreducible characters  $\chi_\lambda$  and  $\chi_\mu$  lie in the same  $p$ -block if and only if  $\lambda$  and  $\mu$  have the same  $p$ -core. For  $B_\gamma$  the  $p$ -block of  $\mathfrak{S}_n$  corresponding to a fixed  $p$ -core  $\gamma$ , we define the  $p$ -weight  $w$  of  $B_\gamma$  by setting  $w = (n - |\gamma|)/p$ . Then  $\text{Irr}(B_\gamma)$  is parametrized by  $\mathcal{MP}_{p,w}$ . Now, we set  $G_{p,w} = (\mathbb{Z}_p \times \mathbb{Z}_{p-1}) \wr \mathfrak{S}_w$ . We recall that  $\text{Irr}(\mathbb{Z}_p \times \mathbb{Z}_{p-1}) = \{\psi_1, \dots, \psi_p\}$  with the following convention. If  $p$  is odd (respectively  $p = 2$ ), then put  $p^* = (p+1)/2$  (respectively  $p^* = 2$ ). Then we can choose the labeling such that  $\psi_i(1) = 1$  for  $i \neq p^*$  and  $\psi_{p^*}(1) = p-1$ . Fix now  $\eta$  and  $\omega$  generators of  $\mathbb{Z}_{p-1}$  and  $\mathbb{Z}_p$  respectively. Write  $g_i = \eta^i$  for  $1 \leq i \leq p-1$  and  $g_p = \omega$ . Then the elements  $g_i \in \mathbb{Z}_p \times \mathbb{Z}_{p-1}$  form a system of representatives for the conjugacy classes of  $\mathbb{Z}_p \times \mathbb{Z}_{p-1}$ . As explained in §5.1, the irreducible characters and conjugacy classes of  $G_{p,w}$  are labeled by  $\mathcal{MP}_{p,w}$ . As above, for  $\mu \in \mathcal{MP}_{p,w}$ , we write  $\theta_\mu$  for the corresponding irreducible character of  $G_{p,w}$ .

**Theorem 5.1.** *We keep the notation as above, and define the linear map  $I : \mathbb{C} \text{Irr}(B_\gamma) \rightarrow \mathbb{C} \text{Irr}(G_{p,w})$  by*

$$I(\chi_\lambda) = (-1)^{|\lambda_{p^*}|} \delta_p(\lambda) \theta_{\tilde{\lambda}^{(p)}},$$

where  $\tilde{\lambda}^{(p)}$  is obtained from the  $p$ -quotient  $\lambda^{(p)}$  of  $\lambda$  replacing  $\lambda_{p^*}$  by its conjugate, and  $\delta_p(\lambda)$  is the  $p$ -sign of  $\lambda$ . Then  $I$  is a generalized perfect isometry with respect to the  $p$ -regular elements of  $\mathfrak{S}_n$  and the set  $C'$  of elements of  $G_{p,w}$  with cycle structure  $\pi = (\pi_1, \dots, \pi_p)$  satisfying  $\pi_p = \emptyset$ .

*Proof.* Let  $S$  be the set of elements of  $\mathfrak{S}_n$  with cycle decomposition  $\sigma_1 \cdots \sigma_r$  (where we omit trivial cycles), such that  $\sigma_i$  is a  $q_i p$ -cycle for some positive integer  $q_i$ , and let  $C$  be the set of  $p$ -regular elements of  $\mathfrak{S}_n$ . The sets  $S$  and  $C$  are unions of  $\mathfrak{S}_n$ -conjugacy classes, and  $1 \in S$ . Moreover,  $\tau_1 \cdots \tau_k$  is the cycle decomposition of  $\tau \in C$  if and only if  $\tau_i$  has  $p'$ -length. Hence the cycle decomposition with disjoint support in  $\mathfrak{S}_n$  proves that (1), (2) and (3) of Definition 2.5 hold with  $G_{\sigma_S} = \mathfrak{S}_{\bar{J}}$  whenever  $\sigma = \sigma_S \sigma_C$  with  $\sigma_S \in S$  and  $\sigma_C \in C$ , and  $J$  is the support of  $\sigma_S$ . Denote by  $\Lambda$  the set of classes consisting of elements of  $S$  and define

$$\Gamma_0 = \bigcup_{b \leq w} \mathcal{P}_b.$$

Write  $\Lambda_0$  for the classes of  $S$  parametrized by  $p \cdot \Gamma_0$ . For each  $\beta \in \Gamma_0$ , we choose a representative  $s_\beta \in S$  in the class of  $\Lambda_0$  labeled by  $p \cdot \beta$  with support in  $\{n - p|\beta| + 1, \dots, n\}$ . Then  $G_{s_\beta} = \mathfrak{S}_{n-p|\beta|} \subseteq \mathbb{C}\mathfrak{S}_n(s_\beta)$ . Denote by  $\text{Irr}(B_\gamma(\mathfrak{S}_{n-p|\beta|}))$  the set of irreducible characters of  $\mathfrak{S}_{n-p|\beta|}$  labeled by partitions with  $p$ -core  $\gamma$ , and

define  $r^\beta : \mathbb{C} \text{Irr}(B_\gamma) \rightarrow \mathbb{C} \text{Irr}(B_\gamma(\mathfrak{S}_{n-p|\beta|}))$  by applying [12, 2.4.7] to the cycles of  $p \cdot \beta$ . Then  $\mathfrak{S}_n$  has an MN-structure with respect to  $C$  and  $B_\gamma$  in the sense of Definition 2.5.

Now, write  $S'$  (respectively  $C'$ ) for the set of elements of  $G_{p,w}$  with cycle structure  $(\pi_1, \dots, \pi_p) \in \mathcal{MP}_{p,b}$  for some  $b \leq w$  (respectively  $(\pi_1, \dots, \pi_p) \in \mathcal{MP}_{p,w}$ ), such that  $\pi_1 = \dots = \pi_{p-1} = \emptyset$  (respectively  $\pi_p = \emptyset$ ). In particular, the classes of  $S'$  are also parametrized by  $\Gamma_0$ . Let  $s'_\beta \in S'$  be with cycle structure  $(\emptyset, \dots, \emptyset, \beta)$  for  $\beta \in \Gamma_0$ . Assume that the support of  $s'_\beta$  is  $\{w - |\beta| + 1, \dots, w\}$ . Then  $G_{p,w-|\beta|}$  lies in  $C_{G_{p,w}}(s)$ , and we define  $r'^\beta : \mathbb{C} \text{Irr}(G_{p,w}) \rightarrow \mathbb{C} \text{Irr}(G_{p,w-|\beta|})$  by applying [24, Theorem 4.4] to the cycles of  $\beta$ . Then  $G_{p,w}$  has an MN-structure with respect to  $C'$  and  $\text{Irr}(G_{p,w})$ .

Let  $q = pa$ . Define the set  $M_a(\lambda^{(p)})$  of  $p$ -multipartitions of  $w - a$  obtained from  $\lambda^{(p)}$  by removing an  $a$ -hook. Recall that the canonical bijection  $f$  (defined in [19, Proposition 3.1]) induces a bijection  $M_q(\lambda) \rightarrow M_a(\lambda^{(p)})$ ,  $\mu \mapsto \mu^{(p)}$ . Write

$$\tilde{\theta}_{\lambda^{(p)}} = (-1)^{|\lambda_{p^*}|} \theta_{\tilde{\lambda}^{(p)}},$$

and assume  $\beta = (\beta_1)$ . Then

$$(79) \quad r^\beta \left( \tilde{\theta}_{\lambda^{(p)}} \right) = \sum_{\mu \in M_{p|\beta_1|}(\lambda)} \alpha'_\mu{}^\lambda \tilde{\theta}_{\mu^{(p)}},$$

where  $\alpha'_\mu{}^\lambda = (-1)^{L(f(c_\mu^\lambda))}$ . See the proof of [10, Proposition 3.8] for more details. For multiples  $q_1, \dots, q_k$  of  $p$ , define inductively the set  $M_{q_1, \dots, q_k}(\lambda)$  of partitions  $\mu$  of  $n - \sum q_i$  such that  $\mu \in M_{q_k}(\nu)$  for some  $\nu \in M_{q_1, \dots, q_{k-1}}(\lambda)$ . Let  $\beta = (\beta_1 \geq \dots \geq \beta_k) \in \Gamma_0$ . Applying recursively formula (79) to the cycles of  $\beta$ , we obtain

$$(80) \quad r^\beta \left( \tilde{\theta}_{\lambda^{(p)}} \right) = \sum_{\mu \in M_{p|\beta_1|, \dots, p|\beta_k|}(\lambda)} a'(\lambda, \mu) \tilde{\theta}_{\mu^{(p)}}.$$

Similarly, the Murnaghan-Nakayama rule in  $\mathfrak{S}_n$  gives

$$(81) \quad r^\beta (\chi_\lambda) = \sum_{\mu \in M_{p|\beta_1|, \dots, p|\beta_k|}(\lambda)} a(\lambda, \mu) \chi_\mu.$$

Now, with the above notation, Equation (26) gives  $\alpha'_\mu{}^\lambda = \delta_p(\lambda) \delta_p(\mu) \alpha'^\lambda_\mu$ , and by the same argument as in the proof of Theorem 3.9, we obtain

$$a(\lambda, \mu) = \delta_p(\lambda) \delta_p(\mu) a'(\lambda, \mu).$$

It follows that

$$\begin{aligned} r^\beta (I(\chi_\lambda)) &= \delta_p(\lambda) \sum_{\mu} a'(\lambda, \mu) \tilde{\theta}_{\mu^{(p)}}, \\ &= \sum_{\mu} \delta_p(\lambda) \delta_p(\mu) a'(\lambda, \mu) \delta_p(\mu) \tilde{\theta}_{\mu^{(p)}}, \\ &= \sum_{\mu} a(\lambda, \mu) I(\chi_\mu), \\ &= I(r^\beta (\chi_\lambda)). \end{aligned}$$

The result now follows from Corollary 2.17.  $\square$

**Corollary 5.2.** *Assume furthermore that  $p > w$ . Then the isometry defined in Theorem 5.1 is a Broué isometry. In particular, Broué's perfect isometry Conjecture holds for symmetric groups.*

*Proof.* We apply Theorem 2.20.  $\square$

**5.3. Osima's perfect isometry.** Using Theorem 2.10, we also can prove the following well-known result (see [15, Proposition 5.11]).

**Theorem 5.3.** *Let  $n$  be an integer and  $p \leq n$  be a prime. Let  $B$  be a  $p$ -block of  $\mathfrak{S}_n$  labeled by the  $p$ -core  $\gamma$ . Assume that  $B$  has weight  $w$ . Then the map defined by*

$$I(\chi_\lambda) = \delta_p(\lambda)\theta_{\lambda^{(p)}}$$

*between  $B$  and  $\text{Irr}(\mathbb{Z}_p \wr \mathfrak{S}_w)$  induces a generalized perfect isometry with respect to the  $p$ -regular elements of  $\mathfrak{S}_n$  and the set of elements  $x \in \mathbb{Z}_p \wr \mathfrak{S}_w$  with cycle structure  $g(x)$  satisfying  $g(x)_1 = \emptyset$  (here, the first coordinate of  $g(x)$  correspond to the trivial class).*

*Proof.* The proof is analogue to that of Theorem 5.1.  $\square$

**5.4. Isometries between blocks of wreath products.** In this section, we fix a positive integer  $l$  and a prime number  $p$  such that  $p$  does not divide  $l$ , and we consider the groups  $G_n = \mathbb{Z}_l \wr \mathfrak{S}_n$ , where  $n$  is any positive integer. Write  $\mathbb{Z}_l = \{\zeta_1, \zeta_2, \dots, \zeta_l\}$  and  $\text{Irr}(\mathbb{Z}_l) = \{\psi_1, \dots, \psi_l\}$ .

Following [23, Theorem 1], we recall that two irreducible characters  $\theta_\mu$  and  $\theta_{\mu'}$  corresponding to  $\mu = (\mu_1, \dots, \mu_l)$  and  $\mu' = (\mu'_1, \dots, \mu'_l)$  of  $G_n$  lie in the same  $p$ -block  $B$  if and only if, for every  $1 \leq i \leq l$ , the partitions  $\mu_i$  and  $\mu'_i$  have the same  $p$ -core  $\gamma_i$  and same  $p$ -weight  $b_i$ . The tuple  $b = (b_1, \dots, b_l)$  (respectively  $\gamma = (\gamma_1, \dots, \gamma_l)$ ) is called the  $p$ -weight of  $B$  (respectively the  $p$ -core of  $B$ ). We denote by  $\mathcal{E}_{\gamma, b}$  the set of  $l$ -multipartitions  $\mu = (\mu_1, \dots, \mu_l)$  such that  $(\mu_i)_{(p)} = \gamma_i$  and the  $p$ -weight of  $\mu_i$  is  $b_i$ .

**Theorem 5.4.** *Let  $n$  and  $m$  be any two positive integer. As above, we write  $\text{Irr}(G_n) = \{\theta_\mu; \mu \Vdash n\}$  and  $\text{Irr}(G_m) = \{\theta_{\mu'}; \mu' \Vdash m\}$  for the sets of irreducible characters of  $G_n$  and  $G_m$ . Let  $B$  and  $B'$  be two  $p$ -blocks of  $G_n$  and  $G_m$ , with  $p$ -cores  $\gamma = (\gamma_1, \dots, \gamma_l)$  and  $\gamma' = (\gamma'_1, \dots, \gamma'_l)$  respectively. Assume that  $B$  and  $B'$  have the same  $p$ -weight  $b = (b_1, \dots, b_l)$ . Define*

$$I(\theta_\mu) = \left( \prod_{i=1}^l \delta_p(\mu_i) \delta_p(\Psi(\mu_i)) \right) \theta_{\psi(\mu)},$$

*where  $\Psi$  is the map defined before Lemma 3.7,  $\psi(\mu) = (\Psi(\mu_1), \dots, \Psi(\mu_l))$ , and  $\delta_p(\mu_i)$  is the  $p$ -sign of  $\mu_i$ . Then  $I$  induces a Broué perfect isometry between  $B$  and  $B'$ .*

*Proof.* First, we notice that  $\psi(\mathcal{E}_{\gamma, b}) = \mathcal{E}_{\gamma', b}$ . Let  $g = (g_1, \dots, g_n; \sigma) \in G_n$ . Write  $\sigma = \sigma_S \sigma_C$ , where all the cycles of  $\sigma_S$  have length divisible by  $p$ , and  $\sigma_C$  is a  $p$ -regular element. Define  $g_S = (g_{S,1}, \dots, g_{S,n}; \sigma_S)$  (resp.  $g_C = (g_{C,1}, \dots, g_{C,n}; \sigma_C)$ ) by setting  $g_{S,i} = g_i$  (respectively  $g_{C,i} = g_i$ ) if  $i$  lies in the support of  $\sigma_S$  (respectively of  $\sigma_C$ ) and  $g_{S,i} = 1$  (respectively  $g_{C,i} = 1$ ) otherwise. Since  $\sigma_S$  and  $\sigma_C$  have disjoint supports, we have the unique decomposition

$$g = g_S g_C = g_C g_S.$$



Denote by  $S$  (respectively  $C$ ) the set of elements  $g = (t; \sigma)$  such that all the cycles of  $\sigma$  have length divisible by  $p$  (respectively prime to  $p$ ). Let  $\Lambda$  be the set of  $l$ -multipartitions  $(\pi_1, \dots, \pi_l)$  such that  $\pi_i \in p\mathcal{P}$  and  $\sum |\pi_i| \leq n$ . Let  $\boldsymbol{\pi} = (\pi_1, \dots, \pi_l) \in \Lambda$ . Denote by  $I$  the set of integers  $1 \leq i \leq l$  with  $\pi_i \neq \emptyset$ . For  $i \in I$ , write  $u_{ij} = \sum_{k < i} |\pi_k| + \sum_{r < j} \pi_{i,r}$ , where  $\pi_i = (\pi_{i,1}, \dots, \pi_{i,\ell(\pi_i)})$ . Consider now the  $\pi_{i,j}$ -cycle

$$\sigma_{ij} = (n - u + u_{ij} + 1, \dots, n - u + u_{ij} + \pi_{i,j}),$$

where  $u = \sum |\pi_i|$ . For  $1 \leq k \leq n$ , set  $t_{ij,k} = 1$  for  $k \neq n - u + u_{ij} + 1$  and  $t_{ij,n-u+u_{ij}+1} = \zeta_i$ . Write  $t_{ij} = (t_{ij,1}, \dots, t_{ij,n}; \sigma_{ij})$  and define

$$(82) \quad t_{\boldsymbol{\pi}} = \prod_{i \in I} \prod_{j=1}^{\ell(\pi_i)} t_{ij}.$$

Then by §5.1 and Equation (78), the elements  $t_{\boldsymbol{\pi}}$  with  $\boldsymbol{\pi} \in \Lambda$  form a set of representatives of the  $G_n$ -classes of  $S$ . Write  $G_{\boldsymbol{\pi}} = G_{n-\sum |\pi_i|}$ . Note that the support of  $\boldsymbol{\pi}$  is  $\{n - \sum |\pi_i| + 1, \dots, n\}$ , and  $G_{\boldsymbol{\pi}} \subseteq C_{G_n}(t_{\boldsymbol{\pi}})$ .

**Example 5.5.** For example, assume that  $l = 3$ ,  $n = 6$ , and  $p = 2$ . Write  $\zeta_1 = 1$  and consider  $\boldsymbol{\pi} = (\emptyset, (2), (2)) \in \Lambda$ . Then one has  $u = 4$ ,  $I = \{2, 3\}$ ,  $u_{21} = 0$ ,  $u_{31} = 2$ . So  $\sigma_{21} = (3\ 4)$ ,  $\sigma_{31} = (5\ 6)$  and

$$t_{21} = (1, 1, \zeta_2, 1, 1, 1; (3\ 4)) \quad \text{and} \quad t_{31} = (1, 1, 1, 1, \zeta_3, 1; (5\ 6)).$$

Finally,  $t_{\boldsymbol{\pi}} = t_{21}t_{31} = (1, 1, \zeta_2, 1, \zeta_3, 1; (3\ 4)(5\ 6))$  is a representative for the class of  $\mathbb{Z}_3 \wr \mathfrak{S}_6$  labeled by  $\boldsymbol{\pi}$ .

Assume that  $\boldsymbol{\pi} = (k) \in \Lambda$  (so that, in particular,  $k$  is divisible by  $p$ ). Then, for all  $x \in G_{\boldsymbol{\pi}}$  and  $\boldsymbol{\mu} \in \mathcal{E}_{\gamma,b}$ , [24, Theorem 4.4] gives

$$(83) \quad \theta_{\boldsymbol{\mu}}(t_{\boldsymbol{\pi}}x) = \sum_{s=1}^l \psi_s(\zeta_l) \sum_{\nu \in M_k(\mu_s)} (-1)^{L(c_{\nu}^{\mu_s})} \theta_{\boldsymbol{\mu}_s}(x),$$

where the partitions in  $\boldsymbol{\mu}_s$  are the same as those in  $\boldsymbol{\mu}$ , except the  $s$ -th one which is equal to  $\nu$ . Applying iteratively this process to the cycles of  $\boldsymbol{\pi}$ , we define a linear map  $r^{\boldsymbol{\pi}} : \mathbb{C} \text{Irr}(B) \rightarrow \mathbb{C} \text{Irr}(B_{n-\sum |\pi_i|})$ , where  $B_{n-\sum |\pi_i|}$  denotes the union of  $p$ -blocks of  $G_{n-\sum |\pi_i|}$  with  $p$ -core  $\gamma$  and  $p$ -weight  $(a_1, \dots, a_l)$  such that  $0 \leq a_i \leq b_i$  and  $\sum (b_i - a_i) = \sum |\pi_i|$ . In particular, we have  $r^{\boldsymbol{\pi}}(\theta_{\boldsymbol{\mu}})(x) = \theta_{\boldsymbol{\mu}}(t_{\boldsymbol{\pi}}x)$  for all  $x \in G_{n-\sum |\pi_i|}$ . This defines an MN-structure for  $G_n$  with respect to  $C$  and  $B$ .

Similarly, we define an MN-structure for  $G_m$  with respect to  $B'$  and the set of  $p$ -regular elements of  $G_m$ . Now, write  $w = \sum b_k$ , and denote by  $\Lambda_0$  the set of  $\boldsymbol{\pi} \in \Lambda$  such that  $\sum |\pi_k| \leq w$ . By [24, Theorem 4.4], we have  $r^{\boldsymbol{\pi}}(\theta_{\boldsymbol{\mu}}) = r^{\boldsymbol{\pi}}(\theta_{\Psi(\boldsymbol{\mu})}) = 0$  for every  $\boldsymbol{\mu} \in \mathcal{E}_{\gamma,b}$  and  $\boldsymbol{\pi} \in \Lambda \setminus \Lambda_0$ .

Let  $\pi \in \Lambda_0$  and  $c$  be a part of  $\pi_t$  of length  $k$ . Then, by [24, Theorem 4.4] (see also Equation (83)), we have

$$\begin{aligned}
r^c(I(\theta_\mu)) &= \prod_{i=1}^l \delta_p(\mu_i) \delta_p(\Psi(\mu_i)) \sum_{s=1}^l \psi_s(\zeta_t) \sum_{\nu \in M_k(\mu_s)} (-1)^{L(c_{\Psi(\nu)}^{\Psi(\mu_s)})} \theta_{\psi(\mu_s)} \\
&= \sum_{s=1}^l \psi_s(\zeta_t) \sum_{\nu \in M_k(\mu_s)} (-1)^{L(c_{\Psi(\nu)}^{\Psi(\mu_s)})} \delta_p(\mu_s) \delta_p(\Psi(\mu_s)) \\
(84) \quad &\quad \cdot \delta_p(\nu) \delta_p(\Psi(\nu)) I(\theta_{\mu_s}) \\
&= \sum_{s=1}^l \psi_s(\zeta_t) \sum_{\nu \in M_k(\mu_s)} (-1)^{L(c_{\nu}^{\mu_s})} I(\theta_{\mu_s}) \\
&= I(r^c(\theta_{\mu_s})).
\end{aligned}$$

Using the argument of the proof of Theorem 3.9 (see Equations (29), (30) and (31)), we conclude that  $r^\pi(I(\theta_\mu)) = I(r^\pi(\theta_\mu))$  for all  $\pi \in \Lambda_0$  and  $\mu \in \mathcal{E}_{\gamma,b}$ .

Hence, the hypotheses of Theorem 2.20 are satisfied, and the result holds.  $\square$

**Corollary 5.6.** *Let  $W_1$  and  $W_2$  be Coxeter groups of type  $B$ . Assume that  $p$  is odd. Then two  $p$ -blocks of  $W_1$  and  $W_2$  with the same  $p$ -weight are perfectly isometric (in the sense of Broué).*

*Proof.* This is a direct consequence of Theorem 5.4, noting that a Coxeter group of type  $B_n$  is isomorphic to  $\mathbb{Z}_2 \wr \mathfrak{S}_n$ .  $\square$

**5.5. Isometries between blocks of Weyl groups of type  $D$ .** Let  $n$  be a positive integer and let  $W$  be a Weyl group of type  $B_n$ . We keep the notation of §5.4. Let  $p$  be an odd prime number. We consider the linear character  $\alpha = \theta_{(\emptyset, (n))} \in \text{Irr}(W)$ , and denote by  $W'$  its kernel. Then  $W'$  is a Weyl group of type  $D_n$ , and one has that  $g \in W$  belongs to  $W'$  if and only if its cycle structure  $\mathfrak{s}(g) = (\pi_1, \pi_2)$  is such that  $\ell(\pi_2)$  is even. Furthermore, the  $W$ -class of such an element splits into two  $W'$ -classes if and only if  $\pi_2 = \emptyset$  and  $\pi_1$  has only parts of even length (i.e. if  $\pi_1 = 2 \cdot \pi$  for some partition  $\pi$  of  $n/2$ ); see [4, Proposition 25]. We fix representatives  $t_{(2 \cdot \pi, \emptyset)}^\pm$  for the  $W'$ -classes whose elements have cycle structure  $(2 \cdot \pi, \emptyset)$  as follows. If  $\pi = (\pi^1, \dots, \pi^r)$  then write  $u_i = \sum_{j < i} \pi^j$ ,  $\sigma_i = (u_i + 1 \cdots u_i + 2\pi^i)$ , and  $t_i = ((1, \dots, 1); \sigma_i)$ . In particular,  $t_1 \in B_{2\pi^1}$ . Let  $\rho \in B_{2\pi^1} \setminus D_{2\pi^1}$ . Set  $t_1^+ = t_1$  and  $t_1^- = \rho t_1 \rho^{-1}$ . Then  $t_1^+$  and  $t_1^-$  are representatives for the two split classes of  $D_{2\pi^1}$  labeled by  $((2\pi^1), \emptyset)$ . Now, define

$$(85) \quad t_{(2 \cdot \pi, \emptyset)}^\pm = t_1^\pm t_2 \cdots t_r.$$

Since  $\rho \in B_n \setminus D_n$ , and  $\rho$  commutes with  $t_2, \dots, t_r$  (because for  $2 \leq i \leq r$ , the supports of  $\rho$  and of  $t_i$  are disjoint), we deduce that  $t_{(2 \cdot \pi, \emptyset)}^- = \rho t_{(2 \cdot \pi, \emptyset)}^+ \rho^{-1}$ . Hence,  $t_{(2 \cdot \pi, \emptyset)}^\pm$  are representatives of the two split classes of  $D_n$  labeled by  $(2 \cdot \pi, \emptyset)$ .

For every 2-multipartition  $(\mu_1, \mu_2)$  of  $n$ , one has  $\alpha \otimes \theta_{(\mu_1, \mu_2)} = \theta_{(\mu_2, \mu_1)}$ . By Clifford theory, if  $\mu_1 \neq \mu_2$ , then  $\chi_{\mu_1, \mu_2} = \text{Res}_{W'}^W(\theta_{\mu_1, \mu_2}) = \text{Res}_{W'}^W(\theta_{\mu_2, \mu_1})$  is irreducible. If  $\mu = \mu_1 = \mu_2$ , then  $\text{Res}_{W'}^W(\theta_{\mu, \mu})$  splits into two irreducible characters  $\chi_{\mu, \mu}^+$  and  $\chi_{\mu, \mu}^-$  of  $W'$ , which we can label so that (see [24, Theorem 5.1])

$$(86) \quad \chi_{\mu, \mu}^\epsilon \left( t_{(2 \cdot \pi, \emptyset)}^\delta \right) = \frac{1}{2} \left( \theta_{(\mu, \mu)}(t_{(2 \cdot \pi, \emptyset)}^\delta) + \epsilon \delta 2^{\ell(\pi)} \chi_\mu(\pi) \right),$$

where  $\delta, \epsilon \in \{-1, 1\}$  and  $\chi_\mu$  is the character of the symmetric group  $\mathfrak{S}_{n/2}$  corresponding to  $\mu$ .

The  $p$ -blocks of  $W'$  can be described as follows. Let  $B_{\gamma_1, \gamma_2}^{(b_1, b_2)}$  be a  $p$ -block of  $W'$  labeled by the  $p$ -cores  $\gamma_1$  and  $\gamma_2$  and with  $p$ -weight  $(b_1, b_2)$ ; see §5.4. If  $(b_1, b_2) \neq (0, 0)$  or  $\gamma_1 \neq \gamma_2$ , then  $B_{\gamma_1, \gamma_2}^{(b_1, b_2)}$  contains characters that are not self-conjugate. By [20, Theorem 9.2],  $B_{\gamma_1, \gamma_2}^{(b_1, b_2)}$  covers a unique  $p$ -block  $b_{\gamma_1, \gamma_2}^{(b_1, b_2)}$  of  $W'$ . Furthermore, when  $\gamma_1 \neq \gamma_2$  or  $b_1 \neq b_2$ ,  $B_{\gamma_1, \gamma_2}^{(b_1, b_2)}$  and  $B_{\gamma_2, \gamma_1}^{(b_2, b_1)}$  contain no self-conjugate character, and  $b_{\gamma_1, \gamma_2}^{(b_1, b_2)} = b_{\gamma_2, \gamma_1}^{(b_2, b_1)}$  consists of the restrictions to  $W'$  of the irreducible characters lying in  $B_{\gamma_1, \gamma_2}^{(b_1, b_2)}$  and  $B_{\gamma_2, \gamma_1}^{(b_2, b_1)}$ . If  $(b_1, b_2) = (0, 0)$ , then  $B_{\gamma_1, \gamma_2}^{(0, 0)} = \{\theta_{(\gamma_1, \gamma_2)}\}$  has defect zero. If  $\gamma := \gamma_1 = \gamma_2$ , then  $b_\gamma^+ = \{\chi_{\gamma, \gamma}^+\}$  and  $b_\gamma^- = \{\chi_{\gamma, \gamma}^-\}$  are two distinct  $p$ -blocks of  $W'$  with defect zero, except when  $n = 0$ . In this last case,  $W = W' = \{1\}$  and  $\theta_{(\emptyset, \emptyset)} = \chi_{(\emptyset, \emptyset)}^+ = \chi_{(\emptyset, \emptyset)}^- = 1_{\{1\}}$ .

**Theorem 5.7.** *Assume  $p$  is odd. Let  $W'_1$  and  $W'_2$  be Coxeter groups of type  $D$ . Let  $b_{\gamma, \gamma}^{(b, b)}$  and  $b_{\gamma', \gamma'}^{(b, b)}$  be  $p$ -blocks of  $W'_1$  and  $W'_2$  with the same  $p$ -weight  $(b, b)$ . Then the isometry defined by*

$$I(\chi_{\mu_1, \mu_2}) = \left( \prod_{i=1}^2 \delta_p(\mu_i) \delta_p(\Psi(\mu_i)) \right) \chi_{\Psi(\mu_1), \Psi(\mu_2)} \quad \text{and} \quad I(\chi_{\mu, \mu}^\epsilon) = \chi_{\Psi(\mu), \Psi(\mu)}^{\epsilon \delta_p(\mu) \delta_p(\Psi(\mu))},$$

where the notation is as above, is a Broué perfect isometry between  $b_{\gamma, \gamma}^{(b, b)}$  and  $b_{\gamma', \gamma'}^{(b, b)}$ .

*Proof.* Assume that  $W'_1$  and  $W'_2$  are of type  $D_n$  and  $D_m$ , respectively. We denote by  $S$  and  $C$  the intersections of  $W'_1$  with the sets  $S$  and  $C$  defined in the proof of Theorem 5.4, and we write  $\Omega$  (respectively  $\Omega_0$ ) for the set of bipartitions  $\pi = (\pi_1, \pi_2)$  with  $\pi_1, \pi_2 \in p\mathcal{P}$  and  $\ell(\pi_2)$  even, such that  $|\pi_1| + |\pi_2| \leq n$  (respectively  $|\pi_1| + |\pi_2| \leq 2pb$ ). Denote by  $\Lambda$  the  $W'_1$ -classes of elements of  $S$ . Note that  $\Omega$  is the set of cycle type of the classes in  $\Lambda$ . Furthermore, we write  $\Lambda_0$  for the set of classes in  $\Lambda$  whose cycle type belong to  $\Omega_0$ . When  $n \neq 2pb$ , the set  $\Omega_0$  labels  $\Lambda_0$ . Otherwise, there are in  $\Omega_0$  elements  $\pi$  that parametrize two  $W'_1$ -classes denoted by  $\pi^+$  and  $\pi^-$ . In this case,  $\pi = (2 \cdot \pi, \emptyset) \in \Omega_0$  for some partition  $\pi$  of  $n/2$ , and we denote by  $t_\pi^+$  and  $t_\pi^-$  representatives for the split classes as in Equation (85). The two corresponding classes are denoted by  $\hat{\pi}^+$  and  $\hat{\pi}^-$ . So, when  $n = 2pb$ , the elements of  $\Lambda_0$  are denoted by  $\hat{\pi}$  with  $\hat{\pi} = \pi$  when  $\pi \in \Omega_0$  labels one class, and  $\hat{\pi} \in \{\pi^+, \pi^-\}$  otherwise. We also will write  $t_{\pi^+} = t_\pi^+$  and  $t_{\pi^-} = t_\pi^-$ . Finally, for  $\pi \in \Omega_0$ , we define  $G_{t_{\hat{\pi}}} = D_{n-|\pi_1|-|\pi_2|}$ .

We then take  $t_\pi$  as in Equation (82) for a representative of the class of  $S$  labeled by  $\pi \in \Omega_0$  whenever  $\hat{\pi} = \pi$ .

Assume that  $n$  is even. For any partition  $\mu$  of  $n/2$ , we write  $\Delta_\mu = \chi_{\mu, \mu}^+ - \chi_{\mu, \mu}^-$ . Let  $1 \leq k < n$ , and  $t = ((1, \dots, 1); \sigma) \in D_n$ , where  $\sigma = (n - k + 1 \dots n)$ . We will prove that

$$(87) \quad \Delta_\mu(tx) = 2 \sum_{\nu \in M_k(\mu)} (-1)^{L(c_\nu^\mu)} \Delta_\nu(x),$$

for all  $x \in D_{n-k}$ . Note that  $tx$  lies in a split class of  $D_n$  if and only if  $x$  lies in a split class of  $D_{n-k}$ . So, to prove Equation (87), we can assume that  $x$  lies in a split class of  $D_{n-k}$ . Suppose that  $tx$  is  $D_n$ -conjugate by  $g \in D_n$  to  $ty$  with  $y \in D_{n-k}$  (in particular,  $y$  lies in a split class of  $D_{n-k}$ ). Then  $x$  and  $y$  have the

same cycle type, so they are  $B_{n-k}$ -conjugate, say by  $g_0 \in B_{n-k}$ . Furthermore,  $g_0$  and  $t$  commute (because they have disjoint supports). It follows that  ${}^{g_0}(tx) = ty$ , and the set of elements that conjugate  $tx$  and  $ty$  is  $g_0 C_{B_n}(tx)$ . Furthermore, since  $tx$  lies in a split class, one has  $C_{B_n}(tx) = C_{D_n}(tx)$ . So, there is  $h \in C_{D_n}(tx)$  such that  $g = g_0 h$ . This proves that  $g_0 \in D_{n-k}$ . Hence  $x$  and  $y$  are  $D_{n-k}$ -conjugate, and Equation (87) now follows from Equation (86).

Furthermore, assume that  $k = n$ . Then Equation (86) gives  $\Delta_\mu(t_{(k)}^\delta) = \delta 2\chi_\mu((k))$  for  $\delta \in \{+, -\}$ . If  $\mu$  is not a hook, then  $M_k(\mu) = \emptyset$ . Otherwise,  $M_k(\mu) = \{\emptyset\}$ . Setting  $\Delta_\emptyset = 1$ , and using the Murnaghan-Nakayama rule for the symmetric group, we obtain

$$(88) \quad \Delta_\mu(t_\sigma^\delta) = 2 \sum_{\nu \in M_k(\mu)} (-1)^{L(c_\nu^\delta)} \Delta_\nu(1).$$

For  $\pi \in \Omega_0$ , we define  $r^{\widehat{\pi}}(\Delta_\mu)(x) = \Delta_\mu(t_{\widehat{\pi}}x)$  for all  $x \in G_{t_{\widehat{\pi}}}$ . Applying iteratively Equation (87) and Equation (88) to the parts of  $t_{\widehat{\pi}}$ , we obtain

$$(89) \quad r^{\widehat{\pi}}(\Delta_\mu) = 2^{\ell(\pi)} \sum_{\nu} a(\mu, \nu) \Delta_\nu,$$

where the coefficients are those appearing in Equation (81).

Now, for  $\boldsymbol{\mu} = (\mu_1, \mu_2) \in \mathcal{E}_{(\gamma, \gamma), (b, b)}$  with  $\mu_1 \neq \mu_2$ , we define  $r^{\widehat{\pi}}(\chi_{\mu_1, \mu_2})$  to be the restriction to  $W'_1$  of  $r^\pi(\theta_{(\mu_1, \mu_2)})$ , where  $r^\pi$  is the map defined in the proof of Theorem 5.4. For  $\boldsymbol{\mu} = (\mu, \mu) \in \mathcal{E}_{(\gamma, \gamma), (b, b)}$ , define

$$(90) \quad r^{\widehat{\pi}}(\chi_{\mu, \mu}^\epsilon) = \frac{1}{2} \left( \text{Res}_{W'_1}^{W_1}(r^\pi(\theta_{(\mu, \mu)})) + \epsilon r^{\widehat{\pi}}(\Delta_\mu) \right).$$

It is then straightforward to show that, if  $b_{\gamma, \gamma}(n - |\pi_1| - |\pi_2|)$  denotes the union of the  $p$ -blocks of  $G_{n - |\pi_1| - |\pi_2|}$  with  $p$ -core  $(\gamma, \gamma)$  and  $p$ -weights  $(b_1, b_2)$  such that  $0 \leq b_i \leq b$  and  $b_1 + b_2 = |\pi_1| + |\pi_2|$ , then the map  $r^{\widehat{\pi}} : \mathbb{C} \text{Irr}(b_{\gamma, \gamma}^{(b, b)}) \rightarrow \mathbb{C} \text{Irr}(b_{\gamma, \gamma}(n - |\pi_1| - |\pi_2|))$  defines an MN-structure for  $W'_1$  with respect to the set of  $p$ -regular elements and  $b_{\gamma, \gamma}^{(b, b)}$ . Similarly, we define an MN-structure for  $W'_2$  with respect to the set of  $p$ -regular elements of  $W'_2$  and  $b_{\gamma', \gamma'}^{(b, b)}$ . As we showed in the proof of Theorem 5.4, if  $\mu_1 \neq \mu_2$  and  $I$  is defined on  $\text{Irr}(b_{\gamma, \gamma}(n - |\pi_1| - |\pi_2|))$  by the same formula, then we have

$$(91) \quad I \left( r^{\widehat{\pi}}(\chi_{\mu_1, \mu_2}) \right) = r^{\widehat{\pi}} \left( I(\chi_{\mu_1, \mu_2}) \right).$$

For any  $\mu \neq \emptyset$  with  $p$ -core  $\gamma$ , one has

$$\Delta_\mu = \delta_p(\mu) \delta_p(\Psi(\mu)) \left( \chi_{\mu, \mu}^{\delta_p(\mu) \delta_p(\Psi(\mu))} - \chi_{\mu, \mu}^{-\delta_p(\mu) \delta_p(\Psi(\mu))} \right).$$

In particular,

$$(92) \quad I(\Delta_\mu) = \delta_p(\mu) \delta_p(\Psi(\mu)) \Delta_{\Psi(\mu)}.$$

Therefore, we deduce from the fact that  $I(\theta_{(\mu, \mu)}) = \theta_{\Psi(\mu), \Psi(\mu)}$  and Equations (90), (89), (25) and (26) that

$$(93) \quad I \left( r^{\widehat{\pi}}(\chi_{\mu, \mu}^\epsilon) \right) = r^{\widehat{\pi}} \left( I(\chi_{\mu, \mu}^\epsilon) \right).$$

Assume first that  $|\Lambda_0| = |\Lambda'_0|$ . Then Equations (91) and (93) hold and we derive from Theorem 2.20 (see also the note in the proof of Theorem 3.9) that  $I$  is a Broué perfect isometry.

Assume, on the other hand, that  $|\Lambda_0| > |\Lambda'_0|$ . In particular,  $n$  is divisible by  $2p$ ,  $\gamma = \emptyset$ , and  $\Lambda'_0 = \Omega_0$ . Let  $\pi \in \Omega_0$  be such that  $\widehat{\pi} = \pi$ . If we define  $I_\pi$  on  $G_{t_{\widehat{\pi}}}$  in the same way as  $I$ , then by Equations (91) and (93), we have  $I_{\widehat{\pi}} \circ r^{\widehat{\pi}} = r^{\widehat{\pi}} \circ I$ . Let now  $\pi = (2 \cdot \pi, \emptyset)$  be such that  $2|\pi| = n$ . Then  $\widehat{\pi} \in \{\pi^+, \pi^-\}$ , and  $G_{t_{\pi^+}}$  and  $G_{t_{\pi^-}}$  are two copies of the trivial group. We set  $\text{Irr}(G_{t_{\pi^+}}) = \{1_{\pi^+}\}$  and  $\text{Irr}(G_{t_{\pi^-}}) = \{1_{\pi^-}\}$ . Furthermore,  $\text{Irr}(b_{\gamma', \gamma'}(m-n)) = \{\chi_{\gamma', \gamma'}^+, \chi_{\gamma', \gamma'}^-\}$ . We define  $I_\pi : \mathbb{C}\text{Irr}(G_{t_{\pi^+}}) \oplus \mathbb{C}\text{Irr}(G_{t_{\pi^-}}) \rightarrow \mathbb{C}\text{Irr}(b_{\gamma', \gamma'}(m-n))$  by setting  $I_\pi(1_{\pi^\delta}) = \chi_{\gamma', \gamma'}^\delta$ . Note that

$$(94) \quad r^{\pi^\delta}(\chi_{\mu, \mu}^\epsilon) = \chi_{\mu, \mu}^\epsilon(t_{\pi^\delta}^\delta)1_{\pi^\delta} = \frac{1}{2} \left( \theta_{(\mu, \mu)}(t_{\pi^\delta}^\delta) + \epsilon \delta 2^{\ell(\pi)} a(\mu, \gamma) \right) 1_{\pi^\delta}.$$

Moreover, by Equation (89), one has

$$\begin{aligned} r^\pi(\delta_p(\mu)\delta_p(\Psi(\mu))\Delta_{\Psi(\mu)}) &= 2^{\ell(\pi)}\delta_p(\mu)\delta_p(\Psi(\mu))a(\Psi(\mu), \gamma')\Delta_{\gamma'} \\ &= 2^{\ell(\pi)}a(\mu, \gamma)\Delta_{\gamma'}, \end{aligned}$$

because  $\delta_p(\gamma) = \delta_p(\gamma') = 1$ . Write  $\epsilon_\mu = \delta_p(\mu)\delta_p(\Psi(\mu))$ , and note that  $r^\pi(\theta_{\mu, \mu}) = \theta_{\mu, \mu}(\pi)1_{\{1\}}$ ,  $I(r^\pi(\theta_{\mu, \mu})) = r^\pi(\theta_{\Psi(\mu), \Psi(\mu)})$ , and  $r^\pi \circ \text{Res}_{W_2'}^{W_2} = \text{Res}_{W_2'}^{W_2} \circ r^\pi$ . So we obtain

$$\begin{aligned} r^\pi(I(\chi_{\mu, \mu}^\epsilon)) &= r^\pi \left( \chi_{\Psi(\mu), \Psi(\mu)}^{\epsilon_\mu \epsilon} \right) \\ &= \frac{1}{2} \left( \text{Res}_{W_2'}^{W_2}(r^\pi(\theta_{\Psi(\mu), \Psi(\mu)})) + \epsilon_\mu \epsilon r^\pi(\Delta_{\Psi(\mu)}) \right) \\ &= \frac{1}{2} \left( \theta_{(\mu, \mu)}(t_\pi) + \epsilon 2^{\ell(\pi)} a(\mu, \gamma) \right) \chi_{\gamma', \gamma'}^+ \\ &\quad + \frac{1}{2} \left( \theta_{(\mu, \mu)}(t_\pi) - \epsilon 2^{\ell(\pi)} a(\mu, \gamma) \right) \chi_{\gamma', \gamma'}^- \\ &= \chi_{\mu, \mu}^\epsilon(t_\pi^+) \chi_{\gamma', \gamma'}^+ + \chi_{\mu, \mu}^\epsilon(t_\pi^-) \chi_{\gamma', \gamma'}^- \\ &= I_\pi \left( r^{\pi^+}(\chi_{\mu, \mu}^\epsilon) + r^{\pi^-}(\chi_{\mu, \mu}^\epsilon) \right). \end{aligned}$$

Now, assume that  $\mu_1 \neq \mu_2$ . Note that  $r^{\pi^\pm}(\chi_{\mu_1, \mu_2}) = \theta_{\mu_1, \mu_2}(\pi)1_{\pi^\pm}$ . Thus,

$$\begin{aligned} I_\pi(r^{\pi^+}(\chi_{\mu_1, \mu_2}) + r^{\pi^-}(\chi_{\mu_1, \mu_2})) &= \theta_{\mu_1, \mu_2}(\pi)(\chi_{\gamma', \gamma'}^+ + \chi_{\gamma', \gamma'}^-) \\ &= \theta_{\mu_1, \mu_2}(\pi) \text{Res}_{W_2'}^{W_2}(\theta_{\gamma', \gamma'}) \\ &= \text{Res}_{W_2'}^{W_2}(I(\theta_{\mu_1, \mu_2}(\pi)1_{\{1\}})) \\ &= \text{Res}_{W_2'}^{W_2}(I(r^\pi(\theta_{\mu_1, \mu_2}))) \\ &= \text{Res}_{W_2'}^{W_2}(r^\pi(I(\theta_{\mu_1, \mu_2}))) \\ &= r^\pi(I(\chi_{\mu_1, \mu_2})). \end{aligned}$$

Hence, we have

$$r^\pi \circ I = I_\pi \circ (r^{\pi^+} + r^{\pi^-}),$$

and we conclude as in the proof of Theorem 3.9.  $\square$

**Theorem 5.8.** *Assume  $p$  is odd. Let  $W_1'$  and  $W_2'$  be Coxeter groups of type  $D$ . Assume that  $\gamma_1 \neq \gamma_2$  and  $\gamma_1' \neq \gamma_2'$ , or  $(\gamma_1, \gamma_2) = (\gamma_1', \gamma_2')$  and  $b_1 \neq b_2$ . If the  $p$ -blocks  $b_{\gamma_1, \gamma_2}^{(b_1, b_2)}$  and  $b_{\gamma_1', \gamma_2'}^{(b_1, b_2)}$  have the same  $p$ -weight  $(b_1, b_2)$ , then they are perfectly isometric in the sense of Broué.*

*Proof.* The isometry is the restriction to  $\text{Irr}(b_{\gamma_1, \gamma_2}^{(b_1, b_2)})$  of that of Corollary 5.6.  $\square$

**5.6. Isometries between alternating groups and natural subgroups.** It would be interesting to give an analogue of Osima's perfect isometry between  $p$ -blocks of the alternating groups and the "alternating" subgroup of  $\mathbb{Z}_p \wr \mathfrak{S}_w$ . But such perfect isometries do not exist, as we can show in the following example.

**Example 5.9.** *Consider the principal 3-block  $b$  of  $\mathcal{A}_6$ . It contains 6 irreducible characters. Note that  $b$  is covered by the principal 3-block  $B$  of  $\mathfrak{S}_6$  (which has 3-weight 2 and contains 9 irreducible characters). Let  $G = \mathbb{Z}_3 \wr \mathfrak{S}_2$ . Then  $G$  has 9 irreducible characters and by Theorem 5.3,  $B$  and  $G$  are perfectly isometric. Now, viewing  $G$  as a subgroup of  $\mathfrak{S}_6$ , we can restrict the sign character  $\varepsilon : \mathfrak{S}_6 \rightarrow \{-1, 1\}$  to a linear character (also denoted by  $\varepsilon$ ) of  $G$ , whose kernel is the base group  $H = \mathbb{Z}_3^2$  of  $G$ . Define the regular elements of  $H$  to be the elements with cycle structure  $(\pi_1, \pi_2, \pi_3)$  and  $\pi_1 = \emptyset$ . These elements are the products of 2 disjoint 3-cycles contained in  $H$ , when  $H$  is viewed as a subgroup of  $\mathfrak{S}_6$ , and there are 4 such elements. Now, a straightforward computation gives that  $\langle \text{res}_{\text{reg}}(\chi), \text{res}_{\text{reg}}(1_H) \rangle \in \{-2/9, 1/9, 4/9\}$  for any  $\chi \in \text{Irr}(H)$ . So, we conclude by Remark 2.13 that  $\text{Irr}(H)$  forms a reg-block, and since  $\text{Irr}(H)$  has 9 elements,  $b$  and  $\text{Irr}(H)$  are not perfectly isometric.*

However, when we replace  $\mathbb{Z}_p \wr \mathfrak{S}_w$  by  $G_{p,w}$  (see §5.2 for the notation), we can show that the  $p$ -blocks of  $\mathcal{A}_n$  are perfectly isometric with the "alternating" subgroup of  $G_{p,w}$ . In a way, we prove in this section an analogue of Osima's isometries for the alternating groups.

Throughout, we keep the notation of §5.2, and view  $G_{p,w}$  as a subgroup of  $\mathfrak{S}_{pw}$ . Moreover, we assume that  $p$  is odd, so that, in particular,  $p^* = (p+1)/2$ . Furthermore, we view  $H = \mathbb{Z}_p \rtimes \mathbb{Z}_{p-1}$  as the normalizer of some Sylow  $p$ -subgroup of  $\mathfrak{S}_p$ , and denote by  $\varepsilon_H$  the restriction of the sign character  $\varepsilon_{\mathfrak{S}_p}$  to  $H$ . Note that only the irreducible character of degree  $p-1$  of  $H$  is  $\varepsilon_H$ -stable. So we choose the labeling of  $\text{Irr}(H) = \{\psi_1, \dots, \psi_p\}$  so that  $\psi_1 = \varepsilon_H$ ,  $\psi_p = 1_H$ , and  $\psi_i = \psi_{p+1-i} \otimes \varepsilon_H$  for any  $1 \leq i \leq p$  (in particular,  $\psi_{p^*}(1) = p-1$ ). Recall that  $\text{Irr}(G_{p,w})$  is labeled by  $\mathcal{MP}_{p,w}$ , and, with the above choices, for every  $\boldsymbol{\mu} = (\mu_1, \dots, \mu_p) \in \mathcal{MP}_{p,w}$ , we have (see [9, Proposition 4D])

$$(95) \quad \varepsilon_{\boldsymbol{\mu}} = \theta_{\boldsymbol{\mu}^*},$$

where  $\varepsilon$  again denotes the restriction of the sign character of  $\mathfrak{S}_{pw}$  to  $G_{w,p}$ , and  $\boldsymbol{\mu}^* = (\mu_p^*, \dots, \mu_1^*)$  is as in Equation (20). Define the "alternating" subgroup of  $G_{p,w}$  by setting

$$H_{p,w} = \ker(\varepsilon : G_{p,w} \rightarrow \{-1, 1\}).$$

Consider the set of partitions  $\mathcal{E}$  (respectively  $\mathcal{OD}$ ) all of whose parts have even length (respectively whose parts are distinct and of odd length). We recall that (see for example [9, Lemma 4E]) the set

$$(96) \quad \mathcal{T} = \{(\pi_1, \dots, \pi_p) \in \mathcal{MP}_{p,w} \mid \pi_{2i} = \emptyset, \pi_{2i+1} \in \mathcal{E}, \pi_p \in \mathcal{OD}\}$$

labels the set of splitting classes of  $G_{w,p}$  with respect to  $H_{w,p}$ . We will now give representatives for these classes. Let  $\boldsymbol{\pi} = (\pi_1, \dots, \pi_p) \in \mathcal{T}$ . For  $1 \leq i \leq p$ , write  $\pi_i = (\pi_{i,1}, \dots, \pi_{i,\ell(\pi_i)})$ , and assume that there is some integer  $1 \leq r_i \leq \ell(\pi_i)$  such that  $\pi_{i,j}$  is prime to  $p$  for all  $j < r_i$  and  $\pi_{i,j}$  is divisible by  $p$  for  $j \geq r_i$ . Let  $t_{\boldsymbol{\pi}}$  be the element of  $G_{p,w}$  obtained in the same way as in Equation (82). Let  $1 \leq i \leq p$

be such that  $\pi_i \neq \emptyset$ . Then with the notation of Equation (82),  $t_{i1} \in G_{p,K}$ , where  $K$  is the support of  $t_{i1}$  and  $G_{p,K} = (\mathbb{Z}_p \times \mathbb{Z}_{p-1}) \wr \mathfrak{S}_K$ . In particular, viewed as an element of  $G_{p,K}$ , the cyclic structure of  $t_{i1}$  is  $(\emptyset, \dots, \emptyset, (\pi_{i,1}), \emptyset, \dots, \emptyset)$ . Since  $\boldsymbol{\pi} \in \mathcal{T}$ , either  $i < p$  is odd and  $\pi_i \in \mathcal{E}$ , and so also  $(\pi_{i,1})$ , or  $i = p$  and  $(\pi_{i,1}) \in \mathcal{OD}_{\pi_{i,1}}$ . Hence,  $t_{i1}$  lies in a split class of  $G_{p,K}$ . Let  $\rho_K \in G_{p,K} \setminus H_{p,K}$ . Set  $t_{i1}^+ = t_{i1}$  and  $t_{i1}^- = \rho_K t_{i1}^+ \rho_K^{-1}$ . Write  $m$  for the minimum integer such that  $\pi_m \neq \emptyset$ . Using the notation of Equation (82), we define

$$(97) \quad r_{\boldsymbol{\pi}} = \left( \prod_{i \neq m} \prod_{j=1}^{\ell(\pi)} t_{ij} \right) \prod_{j \neq 1} t_{mj} \quad \text{and} \quad t_{\boldsymbol{\pi}}^{\pm} = r_{\boldsymbol{\pi}} t_{m1}^{\pm}.$$

Since  $\rho_K \notin H_{p,w}$  and the supports of  $\rho_K$  and  $r_{\boldsymbol{\pi}}$  are disjoint, the elements  $t_{\boldsymbol{\pi}}^+$  and  $t_{\boldsymbol{\pi}}^-$  are representatives for the two split classes of  $H_{p,w}$  labeled by  $\boldsymbol{\pi}$ .

Write  $\mathcal{S}$  for the set of  $\boldsymbol{\mu} \in \mathcal{MP}_{p,w}$  such that  $\boldsymbol{\mu}^* = \boldsymbol{\mu}$ . Now, following [9], we define an explicit bijection  $\mathbf{a} : \mathcal{S} \rightarrow \mathcal{T}$  as follows. Let  $\boldsymbol{\mu} = (\mu_1, \dots, \mu_p) \in \mathcal{S}$ . Then  $\mu_{p+1-i}^* = \mu_i$  for all  $1 \leq i \leq p$ . In particular,  $\mu_{p^*} = \mu_{p^*}^*$ . Write  $\mu_i = \prod_j j^{p^{ij}}$  for  $1 \leq i < p^*$ , and recall the definition of  $\mathbf{a}(\boldsymbol{\mu}) := (\pi_1, \dots, \pi_p) \in \mathcal{T}$  from [9] by setting  $\pi_p = a(\mu_{p^*})$ ,  $\pi_{2i-1} = \prod_j (2j)^{p^{ij}}$ , and  $\pi_{2i} = \emptyset$ , where  $a$  is the map defined in Equation (17). Then  $\mathbf{a}$  is a bijection. Indeed, if for  $(\pi_1, \dots, \pi_p) \in \mathcal{T}$ , we define  $\boldsymbol{\mu} = (\mu_1, \dots, \mu_p)$  by setting  $\mu_{p^*} = a^{-1}(\pi_p)$ ,  $\mu_i = \prod_j j^{p^{ij}}$  for  $1 \leq i < p^*$ , where  $\pi_{2i-1} = \prod_j (2j)^{p^{ij}}$ , and  $\mu_i = \mu_{p+1-i}^*$  for  $p^* < i \leq p$ , then the map  $(\pi_1, \dots, \pi_p) \mapsto (\mu_1, \dots, \mu_p)$  is the inverse map of  $\mathbf{a}$ .

**Lemma 5.10.** *The conjugacy class of  $G_{p,w}$  labeled by  $(\emptyset, \dots, \emptyset, 1^{w-k}, \beta) \in \mathcal{MP}_{p,w}$  (for  $k \leq w$ ) lies in  $H_{p,w}$  if and only if  $\beta$  has an even number of even parts.*

*Proof.* Because of [9, Equation (4.1)], for every  $(h_1, \dots, h_w; \sigma) \in G_{p,w}$  with  $h_i \in H$  and  $\sigma \in \mathfrak{S}_w$ , we have

$$(98) \quad \varepsilon(h_1, \dots, h_w; \sigma) = \varepsilon(\sigma) \prod_{i=1}^w \varepsilon_H(h_i).$$

Write  $\beta = (\beta_1, \dots, \beta_r)$ , and set  $\sigma_{\beta} = \sigma_1 \cdots \sigma_k$ , where  $\sigma_i$  is a cycle of length  $|\beta_i|$ . Let  $\{j_1, \dots, j_{|\beta_i|}\}$  be the support of  $\sigma_i$ . Define  $h_{j_1} = \omega$  (the element  $\omega$  is as in §5.2, *i.e.* a generator of the Sylow  $p$ -subgroup of the base group of the wreath product  $G_{p,w}$ ) and  $h_{j_l} = 1$  for  $2 \leq l \leq |\beta_i|$ . If  $l$  doesn't belong to the support of any  $\beta_i$  then put  $h_l = 1$ . Thus, the element  $x_{\beta} = (h_1, \dots, h_w; \sigma_{\beta})$  is a representative for the class of  $G_{p,w}$  labeled by  $(\emptyset, \dots, \emptyset, \beta)$ . By Equation (98),  $\varepsilon(h_1, \dots, h_w; \sigma_{\beta}) = 1$  if and only if  $\varepsilon(\sigma_{\beta}) = 1$  (because  $\varepsilon_H(\omega) = 1$ ), as required.  $\square$

By Equation (95) and Clifford Theory, if  $\boldsymbol{\mu} \notin \mathcal{S}$ , then the restriction  $\vartheta_{\boldsymbol{\mu}} = \text{Res}_{H_{p,w}}^{G_{p,w}}(\theta_{\boldsymbol{\mu}}) = \text{Res}_{H_{p,w}}^{G_{p,w}}(\theta_{\boldsymbol{\mu}^*})$  is irreducible. Otherwise, the restriction of  $\theta_{\boldsymbol{\mu}}$  splits into a sum of two irreducible characters of  $H_{p,w}$ , denoted  $\vartheta_{\boldsymbol{\mu}}^+$  and  $\vartheta_{\boldsymbol{\mu}}^-$ . In the last case, such a  $\theta_{\boldsymbol{\mu}}$  is called a split irreducible character of  $G_{p,w}$ .

Let  $\boldsymbol{\mu} = (\mu_1, \dots, \mu_p) \in \mathcal{S}$ . In order to distinguish  $\vartheta_{\boldsymbol{\mu}}^+$  and  $\vartheta_{\boldsymbol{\mu}}^-$ , we need to introduce some notation. We associate to  $\boldsymbol{\mu}$  two multipartitions  $\boldsymbol{\mu}' \in \mathcal{MP}_{p, w-|\mu_{p^*}|}$  and  $\boldsymbol{\mu}'' \in \mathcal{MP}_{p, |\mu_{p^*}|}$  by setting

$$\boldsymbol{\mu}' = (\mu_1, \dots, \mu_{(p-1)/2}, \emptyset, \mu_{(p+3)/2}, \dots, \mu_p) \quad \text{and} \quad \boldsymbol{\mu}'' = (\emptyset, \dots, \emptyset, \mu_{p^*}, \emptyset, \dots, \emptyset).$$

Moreover, to  $\mu'$  and  $\mu''$ , we associate subgroups as follows. Write  $E_{\mu'} = \{1, \dots, n - |\mu_{p^*}|\}$  and  $E_{\mu''} = \{n - |\mu_{p^*}| + 1, \dots, n\}$ , and define  $G_{\mu'} = H \wr \mathfrak{S}(E_{\mu'})$  and  $G_{\mu''} = H \wr \mathfrak{S}(E_{\mu''})$ . Note that  $\mu'$  and  $\mu''$  are self-conjugate.

In particular, by §5.1,  $\mu'$  and  $\mu''$  label split irreducible characters  $\theta_{\mu'}$  and  $\theta_{\mu''}$  of  $G_{\mu'}$  and  $G_{\mu''}$  respectively.

Since  $\mu''$  is self-conjugate,  $\mathbf{a}(\mu'')$  is a splitting class of  $G_{\mu''}$ , and thus labels two classes  $\mathbf{a}(\mu'')^{\pm}$  of  $H_{\mu''} = \ker(\varepsilon_{G_{\mu''}})$ . Now, we make the same choices for the labeling for the irreducible characters  $\vartheta_{\mu''}^{\pm}$  and for the classes  $\mathbf{a}(\mu'')^{\pm}$  of  $H_{\mu''}$  as in [9, Proposition 4F], so that yields

$$(99) \quad \left( \vartheta_{\mu''}^+ - \vartheta_{\mu''}^- \right) (g) = \begin{cases} \epsilon(\sqrt{\epsilon_p p})^d \sqrt{\epsilon_{\mu_{p^*}} \text{ph}(\mu_{p^*})} & \text{if } g \in \mathbf{a}(\mu'')^{\epsilon}, \\ 0 & \text{otherwise,} \end{cases}$$

where  $\epsilon \in \{\pm 1\}$ ,  $\epsilon_p = (-1)^{(p-1)/2}$ ,  $d$  is the number of parts of  $a(\mu_{p^*})$ ,  $\epsilon_{\mu_{p^*}} = (-1)^{(|\mu_{p^*}|-d)/2}$ , and  $\text{ph}(\mu_{p^*})$  denotes the product of the lengths of the parts of  $a(\mu_{p^*})$ .

Furthermore, fix any labeling for the irreducible characters  $\vartheta_{\mu'}^{\pm}$  of  $H_{\mu'} = \ker(\varepsilon_{G_{\mu'}})$ . Labelings for  $\mu'$  and  $\mu''$  being fixed as above, we can assume that the characters  $\vartheta_{\mu}^{\pm}$  are parametrized as in [9, Proposition 4H(ii)], and we always make this choice in the following. We can now show the following crucial result.

**Lemma 5.11.** *Let  $c$  be a cycle of odd length  $k \leq w$ . Let  $x = (t; \sigma) \in G_{p,w}$  have cycle structure  $(\emptyset, \dots, \emptyset, 1^{w-k}, (k))$ , and be such that  $\sigma = (w - k + 1, \dots, w)$ . Let  $\mu = (\mu_1, \dots, \mu_p) \in \mathcal{MP}_{p,w}$  be such that  $\mu = \mu^*$ . If  $c$  is a cycle of  $a(\mu_{p^*})$ , then for any  $g \in H_{p,w-k}$ , we have*

$$\left( \vartheta_{\mu}^+ - \vartheta_{\mu}^- \right) (xg) = \sqrt{(-1)^{(pk-1)/2} pk} \left( \vartheta_{\mu_c}^+ - \vartheta_{\mu_c}^- \right) (g),$$

where  $(\mu_c)_i = \mu_i$  if  $i \neq p^*$ , and  $(\mu_c)_{p^*}$  is obtained from  $\mu_{p^*}$  by removing the diagonal hook of length  $k$ .

*Proof.* By Lemma 5.10, one has  $x \in H_{p,w}$ . Furthermore,  $\vartheta_{\mu_c}^{\pm}$  are irreducible characters of  $H_{p,w-k}$ . Write  $\mu'$  and  $\mu''$  for the multipartitions associated to  $\mu$  as above. By construction, we have  $\mu'_c = \mu'$ , and  $\mu''_c$  is obtained from  $\mu''$  by removing the diagonal hook of length  $k$  (this is possible because  $c$  is a cycle of  $a(\mu_{p^*})$ ) at the  $p^*$ -coordinate.

Let  $g \in H_{p,w-k}$ . Then by [9, (i) of Proposition 4H], either  $(\vartheta_{\mu}^+ - \vartheta_{\mu}^-)(xg) = 0 = (\vartheta_{\mu_c}^+ - \vartheta_{\mu_c}^-)(g)$  (and the claim is true), or there are  $y \in H_{\mu'_c}$  and  $z \in H_{\mu''_c}$  such that  $g = yz = zy$  and  $\mathfrak{s}(y)_p = \emptyset$ . Since  $H_{\mu'_c} \subseteq H_{\mu''}$ , the elements  $x$  and  $z$  lie in  $H_{\mu''}$ . On the other hand,  $x$  commutes with  $z$  and with  $y$  (because these elements have disjoint supports). Hence, [9, Proposition 4H] implies that

$$(100) \quad \left( \vartheta_{\mu}^+ - \vartheta_{\mu}^- \right) (xg) = (\vartheta_{\mu'}^+ - \vartheta_{\mu'}^-)(y) (\vartheta_{\mu''}^+ - \vartheta_{\mu''}^-)(xz).$$

First, suppose that  $xz \in \mathbf{a}(\mu'')^{\epsilon}$ . Without loss of generality, in the writing of  $t_{\mathbf{a}(\mu'')}$  as in Equation (97), we can assume that  $x = t_{p\ell(\pi_p)}$ . Hence,  $t_{\mathbf{a}(\mu'')}^{\epsilon} = xt_{\mathbf{a}(\mu'_c)}^{\epsilon}$ . A similar argument to that after Equation (87) shows that  $xz \in \mathbf{a}(\mu'')^{\epsilon}$  if and only if  $z \in \mathbf{a}(\mu'_c)^{\epsilon}$ . Note that  $H_{\mu'_c} = H_{\mu'}$ ,  $\vartheta_{\mu'_c}^+ = \vartheta_{\mu'}^+$  and  $\vartheta_{\mu'_c}^- = \vartheta_{\mu'}^-$  (because  $\mu'_c = \mu$ ). Let  $d$  be the number of parts of  $a(\mu_{p^*})$ . Then  $a((\mu_c)_{p^*})$  has  $(d-1)$  parts. Moreover, one



has  $\epsilon_{\mu_{p^*}} = (-1)^{(k-1)/2} \epsilon_{(\mu_c)_{p^*}}$  and  $\text{ph}(\mu_{p^*}) = k \text{ph}((\mu_c)_{p^*})$ , so that Equations (99) and (100) give

$$\begin{aligned} (\vartheta_{\mu}^+ - \vartheta_{\mu}^-)(gx) &= (\vartheta_{\mu'}^+ - \vartheta_{\mu'}^-)(y) \epsilon(\sqrt{\epsilon_p p})^d \sqrt{\epsilon_{\mu_{p^*}} \text{ph}(\mu_{p^*})} \\ &= (\vartheta_{\mu'_c}^+ - \vartheta_{\mu'_c}^-)(y) \sqrt{\epsilon_p p k (-1)^{(k-1)/2} \epsilon(\sqrt{\epsilon_p p})^{d-1}} \\ &\quad \cdot \sqrt{\epsilon_{(\mu_c)_{p^*}} \text{ph}((\mu_c)_{p^*})} \\ &= \sqrt{\epsilon_p p k (-1)^{(k-1)/2}} (\vartheta_{\mu'_c}^+ - \vartheta_{\mu'_c}^-)(y) (\vartheta_{\mu''_c}^+ - \vartheta_{\mu''_c}^-)(z) \\ &= \sqrt{\epsilon_p p k (-1)^{(k-1)/2}} (\vartheta_{\mu_c}^+ - \vartheta_{\mu_c}^-)(g). \end{aligned}$$

Furthermore,

$$(-1)^{(pk-1)/2} = \left( (-1)^{(p-1)/2} \right)^k (-1)^{(k-1)/2} = \epsilon_p (-1)^{(k-1)/2},$$

because  $k$  is odd. The result follows.

Now, if  $xz \notin \mathbf{a}(\mu'')^{\pm}$ , then  $z \notin \mathbf{a}(\mu'_c)^{\pm}$ . We then have  $(\vartheta_{\mu''}^+ - \vartheta_{\mu''}^-)(xz) = 0 = (\vartheta_{\mu'_c}^+ - \vartheta_{\mu'_c}^-)(z) = 0$  by Equation (99), and Equation (100) gives

$$(\vartheta_{\mu}^+ - \vartheta_{\mu}^-)(gx) = 0 = (\vartheta_{\mu_c}^+ - \vartheta_{\mu_c}^-)(g).$$

This proves the result.  $\square$

For  $\lambda \neq \lambda^*$  and  $\mu \neq \mu^*$ , we write  $\rho_{\lambda}^+ = \rho_{\lambda}^- = \rho_{\lambda}$  and  $\vartheta_{\mu}^+ = \vartheta_{\mu}^- = \vartheta_{\mu}$ . Furthermore, an element  $h \in H_{p,w}$  is said regular if its cycle structure  $\mathfrak{s}(h)$  satisfies  $\mathfrak{s}(h)_p = \emptyset$ .

**Theorem 5.12.** *Let  $p$  be an odd prime number. Let  $\gamma$  be a self-conjugate  $p$ -core of  $\mathfrak{S}_n$  of  $p$ -weight  $w > 0$ . Denote by  $b_{\gamma}$  the corresponding  $p$ -block of  $\mathcal{A}_n$ . Then the linear map  $I : \mathbb{C} \text{Irr}(b_{\gamma}) \rightarrow \mathbb{C} \text{Irr}(H_{p,w})$  defined, for  $\epsilon \in \{\pm 1\}$  and  $\lambda$  with  $p$ -core  $\gamma$ , by*

$$I(\rho_{\lambda}^{\epsilon}) = (-1)^{|\lambda_{p^*}|} \delta_p(\lambda) \vartheta_{\tilde{\lambda}(p)}^{\epsilon \delta_p(\lambda)},$$

where the notation is as in Theorem 5.1, is a generalized perfect isometry with respect to the  $p$ -regular elements of  $\mathcal{A}_n$  and the regular elements of  $H_{p,w}$  (defined as above).

*Proof.* First, we consider the case  $n = pw$ . In this case, one has  $\gamma = \emptyset$ . Let  $S$  and  $C$  be the sets that define an MN-structure for the principal  $p$ -block of  $\mathcal{A}_{pw}$  with respect to the set of  $p$ -regular elements of  $\mathcal{A}_{pw}$ . We denote by  $\Omega_0$  and  $\Lambda_0$  the corresponding sets of partitions (see the proof of Theorem 3.9). Write  $S'$  and  $C'$  as in the proof of Theorem 5.1 (but for elements of  $H_{p,w}$ ). Then  $\Lambda_0$  labels the  $H_{p,w}$  classes of  $S'$  by  $p \cdot \hat{\beta} \in \Lambda_0 \mapsto t_{\hat{\beta}}$ , where by Equation (97),  $t_{\beta^{\pm}} = t_{(\emptyset, \dots, \emptyset, \beta)}^{\pm}$  and  $t_{\beta} = t_{(\emptyset, \dots, 1^{w-|\beta|}, \beta)}$ . Hence, if we set  $H_{\beta} = H_{p, w-|\beta|}$ , then  $H_{\beta}$  satisfies Definition 2.5(3).

Now, for every partition  $\lambda$  of  $pw$  with trivial  $p$ -core, and any  $\epsilon \in \{-1, 1\}$ , we define

$$\tilde{\vartheta}_{\tilde{\lambda}(p)}^{\epsilon} = (-1)^{|\lambda_{p^*}|} \vartheta_{\tilde{\lambda}(p)}^{\epsilon}.$$

Let  $p \cdot \beta \in \Omega_0$  be such that  $\beta = (\beta_1, \dots, \beta_k)$ . Assume that  $\beta_k$  is odd. Then  $t_{\hat{\beta}_k} \in H_{p,w}$ , and by Lemma 5.11,  $(\tilde{\vartheta}_{\tilde{\lambda}(p)}^+ - \tilde{\vartheta}_{\tilde{\lambda}(p)}^-)(t_{\hat{\beta}_k} g) = 0$  for  $g \in H_{p, w-|\beta_1|}$ , except

when  $(\lambda^{(p)})_{p^*}$  contains a diagonal hook  $c_k$  of length  $|\beta_k|$ . In this case, we have.

$$\begin{aligned}
(101) \quad & \left( \tilde{\vartheta}_{\lambda^{(p)}}^+ - \tilde{\vartheta}_{\lambda^{(p)}}^- \right) (t_{\hat{\beta}_k} g) = (-1)^{|\beta_0|} \sqrt{(-1)^{(p\beta_k-1)/2} p \beta_k} \left( \vartheta_{\lambda^{(p)}}^+ - \vartheta_{\lambda^{(p)}}^- \right) (s'_{\hat{\beta}_1} g) \\
& = -(-1)^{|\beta_0 \setminus \{c_k\}|} \sqrt{(-1)^{(p\beta_k-1)/2} p \beta_k} \left( \vartheta_{\lambda^{(p)} \setminus \{c_k\}}^+ - \vartheta_{\lambda^{(p)} \setminus \{c_k\}}^- \right) (g) \\
& = \sqrt{(-1)^{(p\beta_k-1)/2} p \beta_k} \left( \tilde{\vartheta}_{\lambda^{(p)} \setminus \{c_k\}}^- - \tilde{\vartheta}_{\lambda^{(p)} \setminus \{c_k\}}^+ \right) (g),
\end{aligned}$$

where  $\lambda^{(p)} \setminus \{c_k\}$  is the multipartition with the same parts as  $\lambda^{(p)}$ , except the  $p^*$ -part which is obtained from  $(\lambda^{(p)})_{p^*}$  by removing the diagonal hook of length  $\beta_k$ . Therefore, Equations (79), (101) and Clifford theory give, for  $\epsilon \in \{\pm 1\}$  and  $g \in H_{n-|\beta_k|}$ ,

$$\begin{aligned}
\tilde{\vartheta}_{\lambda^{(p)}}^\epsilon (t_{\hat{\beta}_k} g) &= \sum_{\substack{\mu \in M'_{\beta_k}(\lambda) \\ \mu \neq \mu^*}} b(\tilde{\vartheta}_{\lambda^{(p)}}^\epsilon, \tilde{\vartheta}_{\mu^{(p)}}) \tilde{\vartheta}_{\mu^{(p)}}(g) + \sum_{\substack{\mu \in M_{\beta_k}(\lambda) \\ \mu = \mu^*}} \left( b(\tilde{\vartheta}_{\lambda^{(p)}}^\epsilon, \tilde{\vartheta}_{\mu^{(p)}}^+) \tilde{\vartheta}_{\mu^{(p)}}^+(g) + \right. \\
& \quad \left. b(\tilde{\vartheta}_{\lambda^{(p)}}^\epsilon, \tilde{\vartheta}_{\mu^{(p)}}^-) \tilde{\vartheta}_{\mu^{(p)}}^-(g) \right),
\end{aligned}$$

where  $M_{\beta_k}(\lambda)$  and  $M'_{\beta_k}(\lambda)$  are defined as in §3.3, and the complex numbers  $b(\tilde{\vartheta}_{\lambda^{(p)}}^\epsilon, \tilde{\vartheta}_{\mu^{(p)}}^\eta)$  satisfy the following:

- If  $\mu^* \neq \mu$  and  $\mu^* \in M_{\beta_1}(\lambda)$ , then  $b(\tilde{\vartheta}_{\lambda^{(p)}}^\epsilon, \tilde{\vartheta}_{\mu^{(p)}}) = \alpha(\lambda)(\alpha'_{\mu^*} + \alpha'_{\mu^*})$  (see Equation (79) for the definition of  $\alpha'_{\mu^*}$ ).
- If  $\mu^* \neq \mu$  and  $\mu^* \notin M_{\beta_1}(\lambda)$ , then  $b(\tilde{\vartheta}_{\lambda^{(p)}}^\epsilon, \tilde{\vartheta}_{\mu^{(p)}}) = \alpha(\lambda)\alpha'_{\mu^*}$ .
- If  $\mu^* = \mu$  and  $\mu \neq \mu_\lambda$ , then  $b(\tilde{\vartheta}_{\lambda^{(p)}}^\epsilon, \tilde{\vartheta}_{\mu^{(p)}}^\eta) = \alpha(\lambda)\alpha'_{\mu^*}$ .
- If  $\mu^* = \mu$  and  $\mu = \mu_\lambda$ , then  $b(\tilde{\vartheta}_{\lambda^{(p)}}^\epsilon, \tilde{\vartheta}_{\lambda^{(p)} \setminus \{c_k\}}^\eta) = \frac{1}{2} \left( \alpha'_{\mu_\lambda} - \eta \epsilon \sqrt{(-1)^{(q-1)/2} q} \right)$ , where  $q = p\beta_k$ .

Note that, as in the proof of Theorem 5.1, we use that  $f$  induces a bijection between  $M_{p\beta_k}(\lambda)$  and  $M_{\beta_k}(\lambda^{(p)})$ .

Assume now that  $\beta_k$  and  $\beta_{k-1}$  are even. Let  $\mu \in M_{\beta_k, \beta_{k-1}}(\lambda)$ . We denote by  $b(\tilde{\vartheta}_{\lambda^{(p)}}^\epsilon, \tilde{\vartheta}_{\mu^{(p)}}^\eta)$  the hermitian product of the class function  $x \rightarrow \tilde{\vartheta}_{\lambda^{(p)}}^\epsilon (t_{\beta_k} t_{\beta_{k-1}} x)$  with  $\tilde{\vartheta}_{\mu^{(p)}}^\eta \in \mathbb{Z} \text{Irr}(H_{p, w - \beta_k - \beta_{k-1}})$ . Then, applying Equation (79) twice and Clifford theory, we obtain an analogue of Theorem 3.5. For  $\mu \in M_{\beta_k, \beta_{k-1}}(\lambda)$  or  $\mu \in M'_{\beta_k, \beta_{k-1}}(\lambda)$ , the coefficient  $b(\tilde{\vartheta}_{\lambda^{(p)}}^\epsilon, \tilde{\vartheta}_{\mu^{(p)}}^\eta)$  is obtained from  $a(\rho_\lambda^\epsilon, \rho_\mu^\eta)$  by replacing  $(-1)^{L(c_\mu^\nu)}$  and  $(-1)^{L(c_\nu^\lambda)}$  by  $(-1)^{L(f(c_\mu^\nu))}$  and  $(-1)^{L(f(c_\nu^\lambda))}$  respectively.

Now, as in the proof of Theorem 3.9, if we suppose that  $\beta$  is labeled such that there is some integer  $r$  with  $\beta_i$  even for  $i \leq r$  and  $\beta_i$  odd for  $i > r$ , then, applying iteratively the above process, as in the proof of Theorem 3.9, and using the fact that the  $\tilde{\vartheta}_{\lambda^{(p)}}^\epsilon$ 's give a basis of  $\mathbb{C} \text{Irr}(H_{p, w})$ , we can define a linear map  $r^{\hat{\beta}} : \mathbb{C} H_{p, w} \rightarrow \mathbb{C} H_{p, w - |\beta|}$  such that  $r^{\hat{\beta}}(\chi)(x) = \chi(t_{\hat{\beta}} x)$  for all  $\chi \in \mathbb{C} \text{Irr}(H_{p, w})$  and  $x \in H_{p, w - |\beta|}$ . In particular,  $\text{Irr}(H_{p, w})$  has an MN-structure in the sense of Definition 2.5 with respect to  $C'$ .

Let  $p \cdot \beta \in \Omega_0$ . We define  $I_{\widehat{\beta}} : \mathbb{C} \text{Irr}(b_\gamma(n-p|\beta|)) \rightarrow \mathbb{C}H_{p,w-|\beta|}$ , where  $b_\gamma(n-p|\beta|)$  is defined in §3.2, by setting

$$I_{\widehat{\beta}}(\rho_\mu^\eta) = (-1)^{|\mu_{p^*}|} \delta_p(\mu) \widetilde{\vartheta}_{\widehat{\mu}(p)}^{\eta \delta_p(\mu)(-1)^{\ell(\beta)}},$$

where  $\eta \in \{\pm 1\}$  and  $\mu$  is a partition of  $p(w-|\beta|)$  with  $p$ -core  $\gamma$ . Note that  $I_{\{1\}} = I$ .

Write  $b(\widetilde{\vartheta}_{\lambda(p)}^\epsilon, \widetilde{\vartheta}_{\mu(p)}^\eta) = \langle r^{\widehat{\beta}}(\widetilde{\vartheta}_{\lambda(p)}^\epsilon), \widetilde{\vartheta}_{\mu(p)}^\eta \rangle_{H_{p,w-|\beta|}}$ . If either  $\widehat{\beta} = \beta$  or  $\widehat{\beta} = \beta^\pm$  and  $\lambda \neq \kappa$  (where  $\kappa$  is the partition defined in the proof of Theorem 3.9), then a straightforward computation (see the proofs of Theorem 3.9 and of Theorem 5.1) gives

$$a(\rho_\lambda^\epsilon, \rho_\mu^\eta) = b\left(I\left(\widetilde{\vartheta}_{\lambda(p)}^\epsilon\right), I_{\widehat{\beta}}\left(\widetilde{\vartheta}_{\mu(p)}^\eta\right)\right).$$

Hence, the only case to consider is  $\beta = (\beta_1, \dots, \beta_k) \in \mathcal{OD}_w$  and  $\lambda = \kappa$ . Write  $(h_1, \dots, h_k)$  for the diagonal hooks of  $\kappa$  and assume that the hook length of  $h_i$  is  $p\beta_i$ . Furthermore, define  $\beta(0) = \{1\}$  and  $\beta(i) = \{\beta_1, \dots, \beta_i\}$  for  $1 \leq i \leq k$  (in particular,  $p \cdot \beta(i) \in \Omega_0$ ). Note that  $\ell(\beta(i)) = i$ .

Let  $1 \leq i \leq k$ . Write  $\nu = \kappa \setminus \{h_1, \dots, h_{i-1}\}$  and  $\mu = \kappa \setminus \{h_1, \dots, h_i\}$ . Therefore, if we set  $q = ph_i$ , then we have

$$\begin{aligned} b\left(I_{\widehat{\beta}(i-1)}(\rho_\nu^\epsilon), I_{\widehat{\beta}(i)}(\rho_\mu^\eta)\right) &= \delta_p(\nu)\delta_p(\mu)b\left(\widetilde{\vartheta}_{\nu(p)}^{\epsilon \delta_p(\nu)(-1)^{i-1}}, \widetilde{\vartheta}_{\mu(p)}^{\eta \delta_p(\mu)(-1)^i}\right), \\ &= \delta_p(\nu)\delta_p(\mu)\left(\alpha'_\mu{}^\nu - \epsilon\eta\delta_p(\nu)\delta_p(\mu)(-1)^{2i-1}\sqrt{(-1)^{(q-1)/2}q}\right), \\ &= \left(\alpha'_\mu{}^\nu + \epsilon\eta\sqrt{(-1)^{(q-1)/2}q}\right), \\ &= a(\rho_\nu^\epsilon, \rho_\mu^\eta). \end{aligned}$$

Thus, using an argument similar to Equations (29) and (30), we conclude that  $b(I(\rho_\kappa^\epsilon), I_{\widehat{\beta}}(\rho_\mu^\eta)) = a(\rho_\kappa^\epsilon, \rho_\mu^\eta)$ . It follows that

$$r^{\widehat{\beta}} \circ I = I_{\widehat{\beta}} \circ r^{\widehat{\beta}}$$

for every  $p \cdot \widehat{\beta} \in \Lambda_0$ , and Corollary 2.17 gives the result.

Now we return to the general case, that is,  $\gamma$  is any self-conjugate  $p$ -core of  $n$  with  $p$ -weight  $w$ . Let  $b'$  be the principal  $p$ -block of  $\mathcal{A}_{pw}$ . We consider  $I_n : \mathbb{C} \text{Irr}(b_\gamma) \rightarrow \mathbb{C} \text{Irr}(b')$  the perfect isometry obtained in Theorem 3.9 and  $I_{pw} : \mathbb{C} \text{Irr}(b') \rightarrow \mathbb{C} \text{Irr}(H_{p,w})$  the perfect isometry obtained in the first part of the proof. Then  $I_{pw} \circ I_n : \mathbb{C} \text{Irr}(b_\gamma) \rightarrow \mathbb{C} \text{Irr}(H_{p,w})$  is a perfect isometry. In order to prove the result, it is sufficient to show that  $I = I_{pw} \circ I_n$ . Let  $\lambda \in \mathcal{P}_n$  be such that  $\lambda^{(p)} = \gamma$  and  $\epsilon \in \{\pm 1\}$ . Then using that the  $p$ -quotient of  $\Psi(\lambda)$  is  $\lambda^{(p)}$ , we derive that

$$\begin{aligned} I_{pw} \circ I_n(\rho_\lambda^\epsilon) &= I_{pw}\left(\delta_p(\lambda)\delta_p(\Psi(\lambda))\rho_{\Psi(\lambda)}^{\epsilon \delta_p(\lambda)\delta_p(\Psi(\lambda))}\right), \\ &= \delta_p(\lambda)\delta_p(\Psi(\lambda))\delta_p(\Psi(\lambda))\widetilde{\vartheta}_{\lambda(p)}^{\epsilon \delta_p(\lambda)\delta_p(\Psi(\lambda))\delta_p(\Psi(\lambda))}, \\ &= \delta_p(\lambda)\widetilde{\vartheta}_{\lambda(p)}^{\epsilon \delta_p(\lambda)}, \\ &= I(\rho_\lambda^\epsilon), \end{aligned}$$

as required.  $\square$

**Corollary 5.13.** *With the assumptions of Theorem 5.12, and if furthermore  $w < p$ , then  $I$  is a Broué perfect isometry.*

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## REFERENCES

- [1] M. Broué. Isométries parfaites, types de blocs, catégories dérivées. *Astérisque*, (181-182):61–92, 1990.
- [2] O. Brunat and J. Gramain. A basic set for the alternating group. *J. Reine Angew. Math.*, 641:177–202, 2010.
- [3] M. Cabanes. Local structure of the  $p$ -blocks of  $\tilde{S}_n$ . *Math. Z.*, 198:519–543, 1988.
- [4] R. W. Carter. Conjugacy classes in the Weyl group. *Compositio Math.*, 25:1–59, 1972.
- [5] J. Chuang and R. Rouquier. Derived equivalences for symmetric groups and  $\mathfrak{sl}_2$ -categorification. *Ann. of Math. (2)*, 167(1), 2008.
- [6] C. W. Eaton. Perfect isometries and the Alperin-McKay conjecture. pages 49–64, 2007.
- [7] M. Enguehard. Isométries parfaites entre blocs de groupes symétriques. *Astérisque*, (181-182):157–171, 1990.
- [8] P. Fong and M. E. Harris. On perfect isometries and isotypies in finite groups. *Invent. Math.*, 114(1):139–191, 1993.
- [9] P. Fong and M. E. Harris. On perfect isometries and isotypies in alternating groups. *Trans. Amer. Math. Soc.*, 349(9):3469–3516, 1997.
- [10] J.-B. Gramain. On defect groups for generalized blocks of the symmetric group. *J. London Math. Soc.*, 78(1):155–171, 2008.
- [11] J. F. Humphreys. Blocks of projective representations of the symmetric groups. *J. London Math. Soc. (2)*, 32:441–452, 1986.
- [12] G. James and A. Kerber. *The representation theory of the symmetric group*, volume 16 of *Encyclopedia of Mathematics and its Applications*. Addison-Wesley Publishing Co., Reading, Mass., 1981.
- [13] R. Kessar. Blocks and source algebras for the double covers of the symmetric and alternating groups. *J. Algebra*, 186(3):872–933, 1996.
- [14] R. Kessar and M. Schaps. Crossover Morita equivalences for blocks of the covering groups of the symmetric and alternating groups. *J. Group Theory*, 9(6):715–730, 2006.
- [15] B. Külshammer, J. B. Olsson, and G. R. Robinson. Generalized blocks for symmetric groups. *Invent. Math.*, 151(3):513–552, 2003.
- [16] R. Leabovich and M. Schaps. Crossover morita equivalences of spin representations of the symmetric and alternating groups. *Preprint*, 2009.
- [17] A. O. Morris. The spin representation of the symmetric group. *Proc. London Math. Soc.*, 12(3):55–76, 1962.
- [18] A. O. Morris. The spin representation of the symmetric group. *Canad. J. Math.*, 17:543–549, 1965.
- [19] A. O. Morris and J. B. Olsson. On  $p$ -quotients for spin characters. *J. Algebra*, 119(1):51–82, 1988.
- [20] G. Navarro. *Characters and blocks of finite groups*, volume 250 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1998.
- [21] J. B. Olsson. On the  $p$ -blocks of symmetric and alternating groups and their covering groups. *J. Algebra*, 128:188–213, 1990.
- [22] J. B. Olsson. *Combinatorics and Representations of Finite Groups*. Vorlesungen aus dem Fachbereich Mathematik der Universität GH Essen, Helf 20, 1993.
- [23] M. Osima. On the representations of the generalized symmetric group. II. *Math. J. Okayama Univ.*, 6:81–97, 1956.
- [24] G. Pfeiffer. Character tables of weyl groups in gap. *Bayreuth. Math. Schr.*, 47:165–222, 1994.
- [25] R. Rouquier. Isométries parfaites dans les blocs à défaut abélien des groupes symétriques et sporadiques. *J. Algebra*, 168(2):648–694, 1994.
- [26] I. Schur. Über die Darstellung der symmetrischen und der alternierenden Gruppe durch gebrochene lineare Substitutionen. *J. Reine Angew. Math.*, 139:155–250, 1911.

UNIVERSITÉ PARIS-DIDEROT PARIS 7, INSTITUT DE MATHÉMATIQUES DE JUSSIEU – PARIS  
RIVE GAUCHE, UFR DE MATHÉMATIQUES, CASE 7012, 75205 PARIS CEDEX 13, FRANCE.  
*E-mail address:* `brunat@math.univ-paris-diderot.fr`

INSTITUTE OF MATHEMATICS, UNIVERSITY OF ABERDEEN, KING'S COLLEGE, FRASER NOBLE  
BUILDING, ABERDEEN AB24 3UE, UK  
*E-mail address:* `jbgramain@abdn.ac.uk`